Coming down from infinity for some population models

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19th june. CIRM, Luminy, Probability and biological evolution.
Some motivations in population dynamics or genetics

Coming down from infinity"=" regulation for large initial population.

- The effect of the competition arising in a large population [think of trees having a huge number of seeds].
  The short time behavior of genealogies in large population (such as Lambda coalescent, see Aldous, Schweinsberg, Berestycki, Berestycki, Limic, ...).

- Minimal conditions for persistence in a varying environment (WIP with Sylvie Méléard), scaling limits of individual based models.

- Geometric convergence to stationary distribution, Uniqueness of Quasi-Stationary Distribution (see [Van Dorn, Cattiaux & al] ...).
  Speed of convergence to the QSD (see [Champagnat, Villemonais, 15]).
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- The effect of the *competition* arising in a large population [think of trees having a huge number of seeds].
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  Speed of convergence to the QSD (see [Champagnat, Villemonais, 15]).
Coming down from infinity for birth and death processes and competition for one specie [joint work with S. Méléard and M. Richard].

Comparing a stochastic process to a dynamical system with non-expansive vector field and coming down from infinity.

Some example in dimension 2: (stochastic) Lotka Volterra competition model.
Evolution of the population size \((X_t : t \geq 0)\) as a jump process:

\[
\begin{align*}
  k &\rightarrow k + 1 \quad \textit{birth} \quad \text{at rate } \lambda_k \\
  k &\rightarrow k - 1 \quad \textit{death} \quad \text{at rate } \mu_k
\end{align*}
\]

We work under the extinction condition [Karlin McGregor 57]

\[
\sum_{k \geq 1} \frac{1}{\lambda_k \pi_k} = \infty, \tag{1}
\]

where

\[
\begin{align*}
  \pi_1 &= \frac{1}{\mu_1} \quad \text{and for } k \geq 2, \quad \pi_k = \frac{\lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}.
\end{align*}
\]
Model

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\]
Coming down from infinity

Let $T_n = \inf\{ t \geq 0 : X_t = n \}$ and

$$S = \lim_{n \to \infty} \mathbb{E}_n(T_0) = \sum_{i \geq 1} \pi_i + \sum_{n \geq 1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i.$$  

Proposition

The process comes down from infinity, in the sense that

$$\exists m, t > 0 : \inf_{k \in \mathbb{N}} \mathbb{P}_k(T_m < t) > 0$$

iff

$$S < \infty.$$  

The weak limit of $\mathbb{P}_n$ in $\mathcal{P}(\mathbb{D}([0, \infty), \mathbb{N} \cup \{\infty\}))$ as $n \to \infty$ exists and is denoted by $\mathbb{P}_\infty$ and, as soon as the process comes down from infinity,

$$\forall t > 0 : X_t < \infty, \quad \text{while} \quad X_0 = \infty \quad \mathbb{P}_\infty \text{ a.s.}$$
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How does $X$ come down from infinity?

We assume that

$$\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \xrightarrow{n \to \infty} \alpha.$$ 

**Theorem**

i) If $\alpha > 0$ and $\lambda_n/\mu_n \to \ell \in [0, 1)$, then

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow{(d) n \to +\infty} \sum_{k \geq 0} \alpha (1 - \alpha)^k Z_k,$$

where $(Z_k)_k$ i.i.d. r.v. whose Laplace transform $G_{\ell,\alpha}$ is characterized by

$$\forall a > 0, \quad G_{\ell,\alpha}(a) \left[ \ell (1 - G_{\ell,\alpha}(a(1 - \alpha))) + 1 + a(1 - \ell (1 - \alpha)) \right] = 1.$$ 

ii) If $\alpha = 0$ (+$L^2$ assumption), then

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow{n \to \infty} 1 \quad \text{in } \mathbb{P}_\infty - \text{probability.}$$
How does $X$ come down from infinity?

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\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow{(d) \; n \to +\infty} \sum_{k \geq 0} \alpha (1 - \alpha)^k Z_k,
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$$

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A.s. convergence and central limit theorem under additional assumptions for $ii$).

**Proofs** relies on the decomposition of $T_n$ as the infinite sum of independent r.v.+
- Convergence of Laplace exponent as fixed point following proofs for continuous fractions for $i$).
- Klesov asymptotic results for sum of i.i.d. r.v. for $ii$).

**Examples**

- If $\mu_n = \exp(\beta n)$ and $\lambda_n/\mu_n \to \ell$, then $T_n/\mathbb{E}_\infty(T_n) \to Z_{\ell,1-\exp(-\beta)}$ in distribution.
- If $\mu_n = \exp(n/\log n) \log n$, then $T_n/\mathbb{E}_\infty(T_n) \to 1$ in $\mathbb{P}_\infty$ but not a.s.
- If $\mu_n = cn^\rho$ ($\rho > 1$) and $\lambda_n/\mu_n \to 0$, then the a.s. convergence and C.L.T. hold.
The speed of coming down from infinity

Define the speed

\[ v_t := \inf\{n \geq 0; \mathbb{E}_\infty(T_n) \leq t\} \]

Corollary

Assuming also that \( \limsup_{n \to \infty} \lambda_n/\mu_n < 1 \), then

\[ \frac{X_t}{v_t} \xrightarrow{t \downarrow 0} 1 \quad \text{in} \quad \mathbb{P}_\infty - \text{probability}. \]

Proof using the maximal height of the excursions of \( X \) during \([T_{n+1}, T_n)\) + inversion technic.

Example: \( \mu_n \sim cn^\varphi \), then a.s. convergence and C.L.T. for

\[ t^{1/(\varphi-1)} X_t \quad \text{as} \quad t \downarrow 0. \]

\( \varphi = 2, \lambda_k = 0 \) yields Aldous speed of coming down from infinity for Kingman Coalescent (or logistic pure death process).
The speed of coming down from infinity

Define the speed

$$v_t := \inf\{n \geq 0; \ E_\infty(T_n) \leq t\}$$

Corollary

Assuming also that $\limsup_{n \to \infty} \frac{\lambda_n}{\mu_n} < 1$, then

$$\frac{X_t}{v_t} \overset{t \downarrow 0}{\longrightarrow} 1 \quad \text{in} \quad \mathbb{P}_\infty - \text{probability}.$$

Proof using the maximal height of the excursions of $X$ during $[T_{n+1}, T_n)$ + inversion technic.

Example: $\mu_n \sim c n^\varphi$, then a.s. convergence and C.L.T. for

$$t^{1/(\varphi-1)} X_t \quad \text{as} \quad t \downarrow 0.$$
Random Perturbation of a dynamical system

Let $X$ be a càdlàg process on $E \subset \mathbb{R}^d$ such that

$$X_t = x_0 + \int_0^t \psi(X_s)\,ds + R_t,$$

where $\psi$ satisfies for each $x, y \in D \subset \mathbb{R}^d$ ($E \subset D$ and $D$ open),

$$(\psi(x) - \psi(y))(x - y) \leq L \| x - y \|^2_2 \quad [L \text{ non-expansivity}]$$

and

$$R_t = A_t + M_t^c + M_t^d \quad (R_0 = 0)$$

where $A_t$ is càdlàg adapted with finite variations, $M_t^c$ is a continuous local martingale and $M_t^d$ is a totally discontinuous local martingale.
Approximation by a dynamical system

\[ X_t = x_0 + \int_0^t \psi(X_s)\,ds + R_t \]

and \( x \) the dynamical system associated with \( \psi \)

\[ x_t = x_0 + \int_0^t \psi(x_s)\,ds \]

**Proposition**

As long as the dynamical system \( x_t \) is in \( D \) (i.e. for \( T \leq T_D(x_0) \)),

\[
\left\{ \sup_{t \leq T} \| X_t - x_t \|_2 \geq \epsilon \right\} \subset \left\{ T^R_L(\epsilon) \leq T \right\}
\]

where \( T^R_L(\epsilon) := \inf \left\{ t \geq 0 : \sup_{s \leq t} \| X_s - x_s \|_2 \leq \epsilon, \widetilde{R}_t \geq (\epsilon \exp(-2LT))^2 \right\} \)

and \( \widetilde{R}_t = 2 \int_0^t (X_s - x_s) \cdot dR_s + \| < M^c_t > \|_1 + \sum_{s \leq t} \| \Delta R_s \|_2^2 \).
Sketch of proof

Taking the $L^1$ norm of the quadratic variation of $X - x$ (or using Itô’s formula),

$$
\|X_t - x_t\|_2^2 = 2 \int_0^t (X_s - x_s)(\psi(X_s) - \psi(x_s))ds + 2 \int_0^t (X_{s-} - x_s).dR_s
$$

$$
+ \|<M_t^c>\|_1 + \sum_{s\leq t} \|X_s - X_{s-}\|_2^2.
$$

As $\psi$ is $L$ non-expansive on $D$, for each $t \leq T_D(x_0)$, noting $S_t = \sup_{s\leq t} \|X_s - x_s\|_2$,

$$
\|S_{t-}\leq\epsilon S^2_t \leq \|S_{t-}\leq\epsilon \left[ 2L \int_0^t \|X_s - x_s\|_2^2 ds + 2 \int_0^t (X_{s-} - x_s).dR_s 
$$

$$
+ \|<M_t^c>\|_1 + \sum_{s\leq t} \|X_s - X_{s-}\|_2^2 \right].
$$

+ Gronwall lemma to get $\|S_{t-}\leq\epsilon S^2_t < \epsilon$ for $t < T^R_L(\eta)$. 

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Sketch of proof

Taking the $L^1$ norm of the quadratic variation of $X - x$ (or using Itô’s formula),

$$\| X_t - x_t \|^2 = 2 \int_0^t (X_s - x_s). (\psi(X_s) - \psi(x_s)) ds + 2 \int_0^t (X_s - x_s). dR_s$$

$$+ \| < M^c_t > \|_1 + \sum_{s \leq t} \| X_s - X_s^- \|^2.$$

As $\psi$ is $L$ non-expansive on $D$, for each $t \leq T_D(x_0)$, noting $S_t = \sup_{s \leq t} \| X_s - x_s \|_2$,

$$\| S_{t^-} \leq \epsilon \| S_t^2 \leq \| S_{t^-} \leq \epsilon \left[ 2L \int_0^t \| X_s - x_s \|^2 ds + 2 \int_0^t (X_s - x_s). dR_s ight.$$

$$+ \| < M^c_t > \|_1 + \sum_{s \leq t} \| X_s - X_s^- \|^2 \right].$$

+ Gronwall lemma to get $\| S_{t^-} \leq \epsilon \| S_t^2 < \epsilon$ for $t < T^R_L(\eta)$. 
Inequality for martingale

Writing

\[ M_t = M_t^c + M_t^d \]

the local martingale and

\[ S_t = \sup_{s \leq t} \| X_s - x_s \|_2, \]

we obtain by Markov inequality and (Doob) maximal inequality for martingales

\[
\mathbb{P}_{x_0} (S_T > \epsilon) \\
\leq \frac{Ce^{4LT}}{\epsilon^2} \left[ \mathbb{E}_{x_0} \left( \left\| \int_0^T \mathbb{1}_{S_{s-} \leq \epsilon} \| dA_s \|_1 \right\|_2^2 \right) \right] \\
+ \mathbb{E}_{x_0} \left( \sup_{s \leq T} \mathbb{1}_{S_{s-} \leq \epsilon} \| < M_s^c > \|_1 \right) + \mathbb{E}_{x_0} \left( \sum_{s \leq T} \mathbb{1}_{S_{s-} \leq \epsilon} \| \Delta X_s \|_2^2 \right)
\]
Stochastic differential equations

\[ X = (X^i : i = 1 \ldots d) \in \mathbb{D}([0, \infty), E) \] satisfies

\[ X^i_t = x_0 + \int_0^t b^i(X_s) \, ds + \int_0^t \sigma^i(X_s) \, dB^i_s + \int_0^t \int \chi H^i(X_{s^-}, z) N(ds, dz) \]

where

- \( B \) is a \( d \) dimensional Brownian motion;
- \( N \) is a punctual Poisson measure independent of \( B \), with intensity \( dsq(dz) \) and \( \tilde{N} \) its compensated measure.

\[ X_t = x_0 + \int_0^t \psi(X_s) \, ds + M_t \]

where \( M \) is a local martingale given by

\[ M_t = \int_0^t \sigma(X_s) \, dB_s + \int_0^t \int \chi H(X_{s^-}, z) \tilde{N}(ds, dz) \]

and \( \langle M_t \rangle = \int_0^t \sigma(X_s)^2 \, ds + \int_0^t \int \chi H(X_{s^-}, z)^2 \, dsq(dz) \)
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M_t = \int_0^t \sigma(X_s) dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) \tilde{N}(ds, dz)
\]

and \( \langle M_t \rangle = \int_0^t \sigma(X_s)^2 ds + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z)^2 dsq(dz) \)
Let $F$ be a $C^2$ function such that its Jacobian matrix $J_F$ is invertible on $D$. We set

$$b_F(x) = b(x) + J_F(x)^{-1} \left( \int_X [F(x + H(x, z)) - F(x)] q(dz) \right)$$

and the associated flow $\phi_F$ par

$$\phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)).$$

and

$$V_F(x) = (J_F(x)\sigma(x))^2 + \int_X [F(x + K(x, z)) - F(x)]^2 q(dz).$$

is giving the bracket of the martingale part.
Approximation by a dynamical system

\[ \phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)) \]

\[ \psi_F = (J_F b_F) \circ F^{-1} = \left( J_F b(\cdot) + \int_X [F(\cdot + H(x, z)) - F(\cdot)] q(dz) \right) \circ F^{-1} \]

**Theorem**

We assume that \( \psi_F \) is \( L \) non-expansive on \( F(D) \) (+some technical assumption). Then, for all \( x_0 \in D \)

\[
\mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_D(x_0)} \| F(X_t) - F(\phi_F(x_0, t)) \|_2 > \epsilon \right) 
\leq \frac{Ce^{4LT}}{\epsilon^2} \int_0^T \left[ 1 + \bar{V}_{F, \epsilon}(x_0, s) \right] ds,
\]

where \( \bar{V}_{F, \epsilon}(x_0, s) = \sup_{x \in E} \| V_F(x) \|_1 \cdot \| F(x) - F(\phi_F(x_0, s)) \|_2 \leq \epsilon \)
\[ \phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)) \]

\[ \psi_F = (J_F b_F) \circ F^{-1} = \left( J_F b(.) + \int_X [F(.) + H(x, z)) - F(.)] q(dz) \right) \circ F^{-1} \]

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\]

*where* \( \bar{V}_{F,\epsilon}(x_0, s) = \sup_{x \in E} \frac{\| V_F(x) \|_1}{\| F(x) - F(\phi_F(x_0, s)) \|_2 \leq \epsilon} \).
Stochastically monotone model

**Definition**

For all \( x_0 \leq x_1, t \geq 0, a \in \mathbb{R} \),

\[
P_{x_0}(X_t \geq a) \leq P_{x_1}(X_t \geq a)
\]

**Examples**: birth and death process, \( \Lambda \) coalescent; random catastrophes IF the rate of catastrophe does not depend (or decreases) on the size of the population, diffusions ...

We also assume that \( F \) goes to \( \infty \) and \( b_F(x) \) is negative for \( x \) large enough.
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*For all* $x_0 \leq x_1$, $t \geq 0$, $a \in \mathbb{R}$,

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**Examples**: birth and death process, Λ coalescent; random catastrophes *IF* the rate of catastrophe does not depend (or decreases) on the size of the population, diffusions ... 

We also assume that $F$ goes to $\infty$ and $b_F(x)$ is negative for $x$ large enough.
Criteria for instantaneous coming down from infinity

Proposition (In progress.)

Assume that $X$ is stochastically monotone, $\psi_F$ is $L$ non expansive and

$$\int_0^\cdot \sup_{x_0 \in E} \bar{V}_F(x_0, s) < \infty.$$  

The sequence $\mathbb{P}_x$ converges weakly in $\mathcal{P}(\mathbb{D}_{EU\{\infty\}}([0, T]))$ as $x \to \infty$ ($x \in E$) to $\mathbb{P}_\infty$.

(i) If $\int_0^\infty \frac{1}{-b_F(x)} < +\infty$, then

$$\forall t > 0 : X_t < \infty \text{ and } \lim_{t \downarrow 0^+} F(X_t) - F(x_t) = 0 \quad \mathbb{P}_\infty \text{ a.s.}$$

(ii) Otherwise $\mathbb{P}_\infty(\forall t > 0 : X_t = +\infty) = 1$. 
Proposition (In progress.)

Assume that $X$ is stochastically monotone, $\psi_F$ is $L$ non expansive and

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The sequence $\mathbb{P}_x$ converges weakly in $\mathcal{P}(\mathbb{D}_{E \cup \{\infty\}}([0, T]))$ as $x \to \infty$ ($x \in E$) to $\mathbb{P}_\infty$.

(i) If $\int_{-\infty}^\infty \frac{1}{-b_F(x)} < +\infty$, then

$$\forall t > 0 : X_t < \infty \quad \text{and} \quad \lim_{t \downarrow 0^+} F(X_t) - F(x_t) = 0 \quad \mathbb{P}_\infty \ a.s.$$ 

(ii) Otherwise $\mathbb{P}_\infty (\forall t > 0 : X_t = +\infty) = 1$. 
Two examples

- **Λ coalescent.** $X=$number of blocks.

  \[ F(x) = \log(x), \quad \psi_F(x) \downarrow \quad V_F(x) \text{ bounded} \]

  and we recover [Berestycki, Berestycki, Limic 10]

  \[
  \lim_{t \downarrow 0} \log(X_t) - \log(v_t) = 0, \quad \text{i.e.} \quad \frac{X_t}{v_t} \to 1 \quad \mathbb{P}_\infty \text{ a.s.}
  \]

- **Birth and death processes.** $\mu_k = ck^\varrho$ ($\varrho > 1$), $\lambda_k - \lambda_n \leq C(k-n)$

  \[ F(x) = x^{1/2-\epsilon}, \quad \psi_F(x) \downarrow, \quad V_F(x) \sim x^{\varrho-2\epsilon}, \quad \phi(x_0, t) \leq c.t^{1/(1-\varrho)} \]

  and setting $\phi(\infty, t) = [c(\varrho - 1)t]^{1/(1-\varrho)}$

  \[
  \lim_{t \downarrow 0} X_t^{1/2-\epsilon} - \phi(\infty, t)^{1/2-\epsilon} = 0 \quad \mathbb{P}_\infty \text{ a.s.}
  \]

Possible extension to multiple births, random catastrophes, ... and (logistic) Feller diffusion...
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  and we recover [Berestycki, Berestycki, Limic 10]

  $$\lim_{t \downarrow 0} \log(X_t) - \log(\nu_t) = 0, \text{ i.e. } X_t/\nu_t \to 1 \quad P_\infty \text{ a.s.}$$

- Birth and death processes. $\mu_k = c k^\varrho \ (\varrho > 1), \lambda_k - \lambda_n \leq C(k - n)$

  $$F(x) = x^{1/2-\epsilon}, \quad \psi_F(x) \downarrow, \quad V_F(x) \sim x^{\varrho-2\epsilon}, \quad \phi(x_0, t) \leq c t^{1/(1-\varrho)}$$

  and setting $\phi(\infty, t) = [c(\varrho - 1)t]^{1/(1-\varrho)}$

  $$\lim_{t \downarrow 0} X_t^{1/2-\epsilon} - \phi(\infty, t)^{1/2-\epsilon} = 0 \quad P_\infty \text{ a.s.}$$

Possible extension to multiple births, random catastrophes, ... and (logistic) Feller diffusion...
Two examples

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  $$F(x) = \log(x), \quad \psi_F(x) \downarrow \quad V_F(x) \text{ bounded}$$

  and we recover [Berestycki, Berestycki, Limic 10]

  $$\lim_{t \downarrow 0} \log(X_t) - \log(v_t) = 0, \text{ i.e. } \frac{X_t}{v_t} \to 1 \quad \mathbb{P}_\infty \text{ a.s.}$$

- **Birth and death processes.** $\mu_k = ck^\varrho (\varrho > 1), \lambda_k - \lambda_n \leq C(k - n)$

  $$F(x) = x^{1/2-\epsilon}, \quad \psi_F(x) \downarrow, \quad V_F(x) \sim x^{\varrho - 2\epsilon}, \quad \phi(x_0, t) \leq c.t^{1/(1-\varrho)}$$

  and setting $\phi(\infty, t) = [c(\varrho - 1)t]^{1/(1-\varrho)}$

  $$\lim_{t \downarrow 0} X_t^{1/2-\epsilon} - \phi(\infty, t)^{1/2-\epsilon} = 0 \quad \mathbb{P}_\infty \text{ a.s.}$$

Possible extension to multiple births, random catastrophes, ... and (logistic) Feller diffusion...
Two dimensional competition Lotka Volterra diffusion

In progress

\[
\begin{align*}
    dX_t^1 &= X_t^1(\tau_1 - aX_t^1 - cX_t^2)\,dt + \sigma_1 \sqrt{X_t^1} \,dB_t^1 \\
    dX_t^2 &= X_t^2(\tau_2 - bX_t^2 - dX_t^1)\,dt + \sigma_2 \sqrt{X_t^2} \,dB_t^2
\end{align*}
\]

with intraspecific competition \(a, b > 0\) and interspecific competition \(c, d \geq 0\).

We compare this process to dynamical system whose flow \(\phi_F = \phi\) given by

\[
\begin{align*}
    (x_t^1)' &= x_t^1(\tau_1 - ax_t^1 - cx_t^2) \\
    (x_t^2)' &= x_t^2(\tau_2 - bx_t^2 - dx_t^1)
\end{align*}
\]
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\[ dX_t^1 = X_t^1 (\tau_1 - aX_t^1 - cX_t^2) dt + \sigma_1 \sqrt{X_t^1} dB_t^1 \]
\[ dX_t^2 = X_t^2 (\tau_2 - bX_t^2 - dX_t^1) dt + \sigma_2 \sqrt{X_t^2} dB_t^2 \]

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We compare this process to dynamical system whose flow \( \phi_F = \phi \) given by

\[ (x_t^1)' = x_t^1 (\tau_1 - ax_t^1 - cx_t^2) \]
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Approximation by the flow coming down from infinity

Note that each component of $X$ comes back to infinity and set

$$D_\epsilon = \{ x \in (0, \infty)^2 : x_1 \geq 2\epsilon, \ x_2 \geq 2\epsilon \}$$

and

$$d_\beta(x, y) = |x_1^\beta - y_1^\beta| + |x_2^\beta - y_2^\beta|.$$ 

Proposition

*For any $\beta \in [0, 1)$ and $\epsilon > 0$,*

$$\lim_{T \to 0} \sup_{T \geq 0 \ \ x_0 \in D_\epsilon} \mathbb{P}_{x_0} \left( \sup_{t \leq T \land T_{D_\epsilon}(x_0)} d_\beta(X_t, x_t) \geq \epsilon \right) = 0$$

The proof consists in gluing a collections of domains (cones) where

$$F_{\beta, \gamma}(x) = (x_1^\beta, \gamma x_2^\beta)$$

is non-expansive and apply the previous result.
Approximation by the flow coming down from infinity

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The proof consists in gluing a collection of domains (cones) where

$$F_{\beta, \gamma}(x) = (x_1^\beta, \gamma x_2^\beta)$$

is non-expansive and apply the previous result.
(i-intraspecific) If $b > c$ and $a > d$, then there exists $x_\infty \in (0, \infty)^2$ such that for any $x_0 \in (0, \infty)^2$ and $\eta > 0$,

$$
\lim_{T \to 0} \lim_{r \to \infty} \mathbb{P}_{rx_0} \left( \sup_{\eta T \leq t \leq T} \| tX_t - x_\infty \|_2 \geq \epsilon \right) = 0.
$$

(ii-interspecific) If $c > b$ and $d > a$, then for any $\epsilon > 0$ and $\beta \in (0, 1)$,

$$
\lim_{T \to 0} \lim_{r \to \infty} \mathbb{P}_{rx_0} \left( \sup_{t \leq T} d_\beta(X_t, \phi(rx_0, t)) \geq \epsilon \right) = 0.
$$

(iii-unbalanced) If $b > c$ and $d > a$, then for any $T > 0$,

$$
\lim_{r \to \infty} \mathbb{P}_{rx_0} \left( \inf\{ t \geq 0 : X_t^2 = 0 \} \leq T \right) = 1
$$
Simulations: $a = b = 1$, $c = 0.3$, $d = 0.5$ (i-intra)

2 simulations and the dynamic system for 2 initial large values ($10^5$).
Simulations: $a = b = 1, c = 1.3, d = 1.4$ (ii-inter)

2 simulations and the dynamic system for 2 initial large values ($10^5$).
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Simulations: $a = b = 1$, $c = 1/3$, $d = 3$

2 simulations and the dynamic system for 2 initial large values ($10^5$).
Two dimensional competition Lotka Volterra diffusion

Thanks for your attention!

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