COALESCENT PROCESSES ARISING IN A STUDY OF DIFFUSIVE CLUSTERING

ANDREAS GREVEN[(1)] VLADA LIMIC[(2)] ANITA WINTER[(3)]

Abstract. This paper studies the spatial coalescent on $\mathbb{Z}^2$. In our setting, partition elements are located at the sites of $\mathbb{Z}^2$ and undergo local delayed coalescence and migration. That is, pairs of partition elements located at the same site coalesce into one partition element after exponential waiting times. In addition, the partition elements perform independent random walks. The system starts in either locally finite configurations or in configurations containing countably many partition elements per site. These two situations become relevant when the coalescent is used to study the scaling limits of genealogies in interacting Moran models and interacting Fisher-Wright diffusions (or Fleming-Viot processes) on $\mathbb{Z}^2$, which is the key application of the present work.

Our goal is to determine the longtime behavior with an initial population of countably many individuals per site restricted to a box $\Lambda^{\alpha,t} := [-t^{\alpha/2}, t^{\alpha/2}]^2 \cap \mathbb{Z}^2$ and observed at time t^β with $1 \geq \beta \geq \alpha \geq 0$. We study both asymptotics, as $t \to \infty$, for a fixed value of α as the parameter $\beta \in [\alpha, 1]$ varies and for a fixed β, as the parameter $\alpha \in [0, \beta]$ varies. This exhibits the genealogical structure of the mono-type clusters arising in 2-dimensional Moran and Fisher-Wright systems.

A new random object, the so-called *coalescent with rebirth*, is constructed via a look-down procedure and shown to arise in the space-time limit of the coalescent restricted to $\Lambda^{\alpha,t}$ with $\alpha \in [0,1]$ and observed at time t, as $t \to \infty$. For the sake of completeness, and in view of future applications, we introduce the spatial coalescent with rebirth and study its longtime asymptotics as well.

Keywords: spatial coalescent, Kingman coalescent, coalescent with rebirth, look-down construction, two-dimensional random walk asymptotics, Erdös-Taylor formula, asymptotic exchangeability.

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Contents

1. Introduction

In this paper we study the spatial coalescent processes which arise in the context of neutral population models. The class of these population models contains the interacting Moran models and their diffusion limit which are the interacting Fisher-Wright diffusions, or in a multi-type setting, the interacting Fleming-Viot diffusions. They all describe populations in which individuals are associated with a type and a geographic location that evolve due to resampling and migration, where resampling means that a pair of individuals is replaced by a new pair, choosing for each new individual a parent at random whose type is inherited by both offspring (see Shiga [33] and Durrett [14]).

If one considers a fixed time population with countably many individuals per site and traces back the path leading to the ancestors, then these ancestral lines also migrate through the geographic space - now according to the time-reversed migration mechanism - and whenever two of the ancestral lines occupy the same location they merge into one ancestral line at a certain (delay) rate. The process describing this genealogy is called *the spatial (delayed) coalescent*. Notice that a time when two ancestral lines merge corresponds to an individual of the forward time population living at an earlier time that died without having a descendent due to the resampling mechanism. For many application it is necessary to consider in the backward picture also the ancestry of the pushed out individual, i.e. the ancestry of the *fossils*. This leads to the *full genealogy* whose dynamics is a process to which in the following we refer as the *spatial coalescent with rebirth*.

The spatial coalescent process plays a key rôle for the analysis of the space-time scale behavior of spatial neutral population models and also of the scaling behavior of the corresponding genealogical trees. By the duality between the forward time population models and the backward time coalescent processes, on $\mathbb{Z}^d$ our interacting neutral population models show a dichotomy in the basic long-term behavior. In high dimensions (when there is a positive probability that two ancestral lines do not merge) non-trivial equilibria exist, while in low dimension (where two ancestral lines do meet almost surely) the populations approach laws which are concentrated on local mono-type regions. Such a dichotomy is known for a huge class of spatial stochastic systems that combine migration between the sites and a stochastic mechanism acting at each site (including the voter model, branching random walks or interacting diffusions, see, for example, Liggett [29], Dawson [10], Shiga [33] and Cox and Greven [8]).

A special rôle is played by the *critical dimension* $d = 2$, which is characterized by the fact that the underlying (symmetrized) migration random walk is recurrent, while its Green's function $\sum_{k=1}^{n} \mathbf{P}\{X_k = 0\}$ grows only logarithmically in n. Therefore (as in general for the recurrent setting) the above processes converge weakly to a law concentrated on mono-type configurations as time evolves from 0 to infinity. Somewhat surprisingly, as first explored for the voter model by Cox and Griffeath in 1986 [9], by investigating the dual (instantaneously) coalescing random walks, the order of magnitude of the regions where the system looks mono-type is not asymptotically deterministic (unlike in the $d = 1$ setting where we get $\sqrt{t}$ as order of magnitude for the size of the mono-type regions). In fact, the mono-type cluster containing the origin has an area of the order t^α, as $t \to \infty$, where the *random exponent* α takes values in $[0, 1]$. This phenomenon is called the *diffusive clustering*.

There are certain functionals which can not be read off from the genealogy of the population at a fixed time. For example, consider the critical dimension $d = 2$ with diffusive clustering. Here one may be interested in the "age" of a cluster. To be more precise (for even more details, compare Remark 3.5), suppose that at some large time t, and in a fixed region of area t^α around the origin, the population contains a mono-type cluster. Then one may ask what is the amount of time that this cluster had been already present in that region. To answer such questions about the age of a cluster requires information about the full genealogy including the fossils.

From a weak approach via space-time moment formulas for the present and related models it is known that the age of such an *α-cluster* is of order of magnitude t^β with a random exponent $\beta \in (\alpha, 1)$. See, for example, Fleischmann and Greven [17, 18] for interacting diffusions with components in $[0, 1]$ on the hierarchical group with a symmetric critically recurrent migration kernel, Winter [35] for branching random walks on $\mathbb{Z}^2$, and Greven, Klenke, Wakolbinger [19], [20] and [21] for models in random medium.

Even more detailed results for the above population models can be obtained if one exploits the strong from of duality to the spatial coalescent with rebirth. This means that we represent the time-t state of the Fisher-Wright population via a spatial coalescent, compare Figure 1. This requires a formal construction of the spatial coalescent with rebirth which will replace and extend the earlier ad hoc constructions via time-space moment dualities used in previous work by various authors.

However, spatial coalescents have also possible applications outside the context of population models, and are therefore interesting in their own right. In the present paper we therefore present and prove results for the spatial coalescent processes (with and without rebirth) only. To keep things as simple as possible, in this paper the results are formulated for individuals, types and locations only without involving the structure of the associated geographical tree.

The potential of this genealogical viewpoint will become even more apparent in future applications. For example, in [23] and [26] we shall prove convergence theorems for the complete genealogical structure of the coalescent with rebirth in order to describe the genealogy in the interacting Moran models and the interacting Fisher-Wright diffusions including "fossils".

We systematically explore in this paper the long-term behavior of the spatial coalescent with and without rebirth in the geographic space $\mathbb{Z}^2$. In particular four points are important to us and different from previous work:

(1) the universality of the scaling results in the sense that the migration mechanism belongs to a *large class* of random walks,
(2) initial configuration may contain sites with a countably *infinite* number of individuals,
(3) the introduction of the coalescent with *rebirth* allowing to include fossils in the picture,
(4) a projective limit characterization of the coalescent with rebirth as Markov pure jump process as well as its construction by a random graph via the *look-down procedure*.

At the end of Subsection 3, after the outline, we comment in detail on earlier work by Cox and Griffeath ([9]) and Bramson, Cox and Griffeath ([4]) who considered the spatial *instantaneous coalescent* with a *simple random walk migration* mechanism.

2. Models

The coalescent processes considered in the present paper are describing the genealogies of neutral population models involving resampling between any two individuals to the effect that these individuals are replaced by descendants of one of them. In the time-reversed evolution these time points correspond to the times at which the ancestral lines of the two individuals *coalesce* to a common ancestral line (compare with Figure 1).

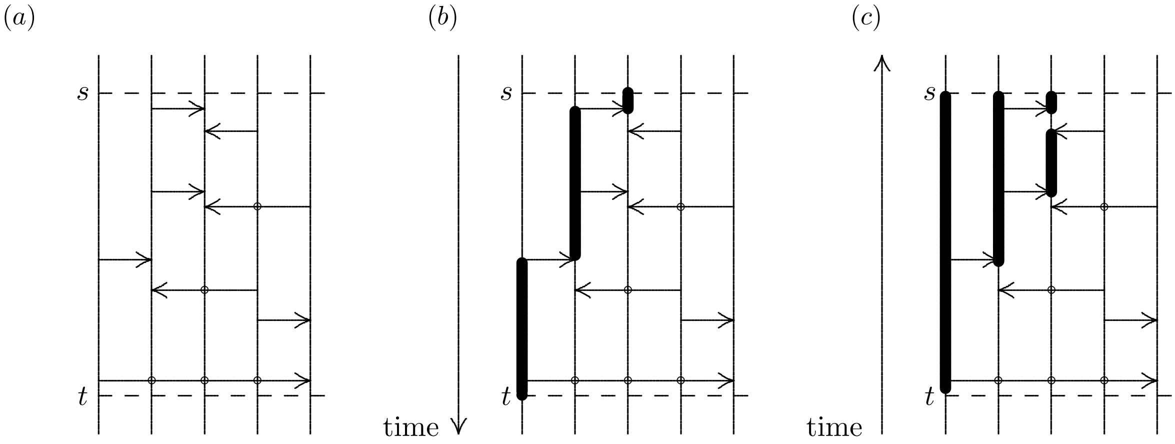

FIGURE 1. (a) illustrates a realization of the resampling. The height of an arrow indicates the times of the resampling events, while the tip of an arrow indicates that the corresponding individual gets pushed out of the population, (b) The ancestral line of the individual 3 which lives at time s is indicated in bold, and (c) the set of all descendants up to time s of the individual 1 present at time t is indicated in bold.

We shall define in this section the spatial coalescent, the spatial coalescent with rebirth and finally the so-called look-down process, which give straightforward constructions for a version of these processes.

2.1. **The spatial coalescent on $\mathbb{Z}^2$.** Processes describing the dynamics of finitely many moving and coalescing particles appeared already in the 1980'ies (see, for example, [5, 4, 9] and compare Liggett [29] for more detailed references). In order to use coalescent processes for representing genealogies of diffusion processes, it is essential to allow configurations with *countably many particles per site* on a countable geographic space. Moreover, while the above papers were only recording occupation numbers at various sites, we will provide a set-up which also exhibits the partition structure.

The *spatial coalescent* that we analyze in the current paper was introduced on a class of Abelian groups in [22]. For the benefit of the reader we briefly recall in three steps the relevant notation, appropriate topologies and its construction. We restrict the setting to $\mathbb{Z}^2$. In the following

$$\mathcal{I} := \big\{\text{set of all (labels of) individuals}\big\} \tag{2.1}$$

is a countable or finite non-empty set.

Step 1 (Migration) Let $a(\cdot,\cdot)$ be an irreducible random walk kernel which has finite exponential moments, i.e.,

$$a(x,y) = a(0, y-x), \tag{2.2}$$

for all $x, y \in \mathbb{Z}^2$, and

$$\sum_{(z_1,z_2)\in\mathbb{Z}^2} e^{\lambda_1 z_1 + \lambda_2 z_2} a(0,(z_1,z_2)) < \infty, \tag{2.3}$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$. We consider the continuous time random walk with jump rate 1 and transition probability $a(\cdot, \cdot)$.

We next present the standard way to construct particle systems with interaction between sites due to migration that start in configurations with possibly countably many particles at some or all sites. These particle systems are constructed as extensions of particle systems which start in specific *locally finite* states and the dynamics is such that they guarantees local finiteness of the particle process at all times $t > 0$ (compare also with Remark 2.2). To construct such locally finite systems we follow an approach due to Liggett and Spitzer [30].

Fix a finite measure α on $\mathbb{Z}^2$ with $\alpha(\{x\}) > 0$, for all $x \in \mathbb{Z}^2$, such that for a constant Γ

$$\sum_{y \in \mathbb{Z}^2} a(x,y)\alpha(\{y\}) \leq \Gamma \cdot \alpha(\{x\}), \tag{2.4}$$

for all $x \in \mathbb{Z}^2$. Denote by $\mathcal{N}(\mathbb{Z}^2)$ the set of all locally finite $\mathbb{N}_0$-valued measures on $\mathbb{Z}^2$. Then

$$\mathcal{E} \equiv \mathcal{E}_\alpha := \big\{\eta \in \mathcal{N}(\mathbb{Z}^2) : \sum_{x \in \mathbb{Z}^2} \eta(\{x\})\alpha(\{x\}) < \infty\big\} \tag{2.5}$$

is the *Liggett-Spitzer* space (corresponding to α).

Remark 2.1 ($\mathcal{E}$ is a state space)**.** Let $\{(X_t^i)_{t\geq 0} : i \in \mathcal{I}\}$ be a countable collection of independent random walks, and put for all $t \geq 0$, $\eta_t := \sum_{i \in \mathcal{I}} \delta_{X_t^i} \in \mathcal{N}(\mathbb{Z}^2)$. If

$$\eta_0 \in \mathcal{E}, \qquad \mathbf{P}\text{-a.s.}, \tag{2.6}$$

then an easy calculation shows that the process $(e^{-\Gamma t} \sum_{i \in \mathcal{I}} \alpha(\{X_t^i\}))_{t \geq 0}$ is a non-negative super-martingale, which by the optional sampling theorem never hits ∞. Thus under (2.6),

$$\mathbf{P}\big\{\eta \in D([0,\infty), \mathcal{E})\big\} = 1, \tag{2.7}$$

where $D([0,\infty), \mathcal{E})$ is the space of $\mathcal{E}$-valued càdlàg paths equipped with the Skorohod topology, and $\mathcal{E} \subset \mathcal{N}(\mathbb{Z}^2)$ is equipped with the vague topology. □

To build in countably many individuals per site we shall make use of the coalescence mechanism introduced next.

Step 2 (Coalescence) Recall that a *partition* of a countable set $\mathcal{I}$ is a collection $\{\pi_\lambda\}$ of pairwise non-empty disjoint subsets of $\mathcal{I}$ such that $\mathcal{I} = \cup_\lambda \pi_\lambda$. We refer to the elements of a partition as *partition elements*. Let us denote by

$$\Pi^{\mathcal{I}} := \text{collection of all partitions of } \mathcal{I}. \tag{2.8}$$

In this space we use, as usual, the discrete topology on restrictions on finite index sets turning the state space into a Polish space. We recall this construction in the Appendix A.

For all $\mathcal{I}' \subseteq \mathcal{I}$, write $\rho_{\mathcal{I}'}$ for the restriction map from $\Pi^{\mathcal{I}}$ to $\Pi^{\mathcal{I}'}$ which sends a partition $\mathcal{P}$ of $\mathcal{I}$ to a partition $\rho_{\mathcal{I}'}(\mathcal{P})$ of $\mathcal{I}'$ by first erasing in each partition element all elements of $\mathcal{I} \setminus \mathcal{I}'$ and then ignoring empty sets, i.e., we introduce for any $\mathcal{P} \in \Pi^{\mathcal{I}}$, the *induced partition*

$$\rho_{\mathcal{I}'}(\mathcal{P}) := \big\{\mathcal{I}' \cap \pi; \pi \in \mathcal{P}, \pi \cap \mathcal{I}' \neq \emptyset\big\}. \tag{2.9}$$

Definition 2.1 (The $\mathcal{I}$-Kingman coalescent)**.** *The* $\mathcal{I}$-Kingman coalescent*, or short the* Kingman-coalescent*,*

$$K := (K_t)_{t \geq 0}, \tag{2.10}$$

is the unique strong Markov process with càdlàg paths such that for all finite $\mathcal{I}' \subseteq \mathcal{I}$*, the restricted process*

$$K^{\mathcal{I}'} := \rho_{\mathcal{I}'}(K) \tag{2.11}$$

is a $\Pi^{\mathcal{I}'}$*-valued Markov chain which starts in some* $\mathcal{P} \in \Pi^{\mathcal{I}'}$*, and given* $K_t^{\mathcal{I}'}$*, independently each pair of partition elements is merging to form a single partition element after an exponential waiting time with rate* $\gamma_{\mathrm{King}} > 0$*.*

Step 3 (Migration and coalescence combined) We next combine migration and coalescence. For that purpose, fix a *site space* M. In the present paper, we consider $M := \mathbb{Z}^2$ only. However, in many applications other mark spaces are of interest, as for example the d-dimensional lattice, the d-dimensional torus or the hierarchical group. Then from any $\mathcal{P} \in \Pi^{\mathcal{I}}$ one can form a *marked* partition

$$\mathcal{P}^M := \big\{(\pi, L(\pi));\, \pi \in \mathcal{P}\big\}, \tag{2.12}$$

by assigning to each partition element $\pi \in \mathcal{P}$, its *location* $L(\pi) \in M$. Put

$$\Pi^{\mathcal{I},M} := \text{set of all marked partitions.} \tag{2.13}$$

For all $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{P} \in \Pi^{\mathcal{I},M}$, we introduce the restricted marked partition as

$$\rho^M_{\mathcal{I}'}(\mathcal{P}) := \big\{(\mathcal{I}' \cap \pi, L(\pi));\, \pi \in \mathcal{P},\, \pi \cap \mathcal{I}' \neq \emptyset\big\}. \tag{2.14}$$

We are now ready to define the spatial $\mathcal{I}$-coalescent.

Definition 2.2 (The spatial $\mathcal{I}$-coalescent)**.** *The* spatial $\mathcal{I}$-coalescent *on* $\mathbb{Z}^2$*,*

$$(C, L) := (C_t, L_t)_{t \ge 0} = \Big(\big\{(\pi, L_t(\pi));\, \pi \in C_t\big\}\Big)_{t\ge 0}, \tag{2.15}$$

is a $\Pi^{\mathcal{I},\mathbb{Z}^2}$*-valued strong Markov process with càdlàg paths such that for all subsets* $\mathcal{I}' \subseteq \mathcal{I}$ *with*

$$\sum\nolimits_{\pi \in C_0, \mathcal{I}' \cap \pi \neq \emptyset} \delta_{L_0(\pi)} \in \mathcal{E}, \tag{2.16}$$

the restricted process $\rho_{\mathcal{I}'}(C, L)$ *is a* $\Pi^{\mathcal{I}',\mathbb{Z}^2}$*-valued strong Markov particle system which undergoes the following two* independent *mechanisms:*

- Migration *The marks of the partition elements perform independent continuous time random walks with rate* 1 *and transition kernel* $a(\cdot,\cdot)$*.*
- Coalescence *Each pair of partition elements whose locations are equal merges into one partition element independently after exponential waiting times with rate* γ*.*

Remark 2.2 (Spatial coalescent is well-defined)**.**

(i) Note that for the marked $\mathcal{I}$-coalescent process above there is a natural coupling with the migration random walks $\{(X^i_t)_{t\ge 0} : i \in \mathcal{I}\}$ such that

$$\sum\nolimits_{\pi \in C_t} \delta_{L_t(\pi)}(B) \le \sum\nolimits_{i \in \mathcal{I}} \delta_{X^i_t}(B), \qquad \text{a.s.,} \tag{2.17}$$

for all $B \subseteq \mathbb{Z}^2$. Therefore, by Remark 2.1, the spatial $\mathcal{I}$-coalescent is locally finite and in particular well-defined if $\mathcal{I}$ is already such that (2.16) holds.

(ii) By Proposition 3.4 in [22], the spatial $\mathcal{I}$-coalescent is well-defined for all initial marked partitions. Specifically, it is even well-defined if started in a configuration which contains countably infinite many partition elements at each site in $\mathbb{Z}^2$. In all cases, we have that $\sum_{\pi \in C_t} \delta_{L_t(\pi)} \in \mathcal{E}$ for all $t > 0$ almost surely. □

Remark 2.3 (Consistency Property)**.** In all of our constructions of concrete realizations of coalescents below we use the following important *consistency* property: if (C, L) is the $\mathcal{I}$-coalescent and $\mathcal{I}' \subseteq \mathcal{I}$ then $\rho_{\mathcal{I}'}(C, L)$ is the $\mathcal{I}'$-coalescent started in $\rho_{\mathcal{I}'}(C_0, L_0)$. □

Remark 2.4 (Instantaneous coalescent; $\gamma = \infty$)**.** Note that if the finite parameter γ is replaced by ∞, the spatial coalescent changes into the system of *instantaneously* coalescing random walks which appear as dual for the voter model and is considered in [4, 9]. This observation will become important in Section 5. □

2.2. **The coalescent with rebirth.** In this subsection we want to introduce the (spatial) coalescent with rebirth. Recall that coalescent processes are motivated by the forward neutral population models and here in particular in the study of genealogical relationships between individuals *currently* alive by viewing them in reversed time. Each coalescent event corresponds in the forward model to a splitting of an descendant line which always goes along with a simultaneous death of another descendant line. However, if one is interested in genealogies which include also the individuals alive at earlier times, commonly referred to as *"fossils"*, then a richer object than the spatial coalescent is needed. We call this new object the *coalescent with rebirth.*

The coalescent with rebirth accounts in the forward model for the lines of descent which died before the current time. More precisely, whenever an individual dies and gets replaced by a descendent of another individual in the forward model, in the time-reversed model the coalescent dynamics with rebirth generates a new individual at the corresponding time. For finite populations this can easily be defined as a Markov pure jump process as suggested by the following example.

Example 2.1. Consider $\mathcal{I} := \{1, 2, 3\}$, the initial configuration $\{\{1\}, \{2\}, \{3\}\}$ at time s, and the transitions (without rebirth) at times t_1 and t_2, respectively,

$$\{\{1\}, \{2\}, \{3\}\} \stackrel{\text{at time } t_1}{\longrightarrow} \{\{1\}, \{2, 3\}\} \stackrel{\text{at time } t_2}{\longrightarrow} \{\{1, 2, 3\}\} \tag{2.18}$$

(following the resampling events illustrated in Figure 2). This way we code only the information about which individuals are involved in an resampling event, while the information gets lost about which of the individuals was pushed out by the other. To keep record of this as well, in the corresponding coalescent with rebirth we shall reintroduce at time t_1 a new partition element $\{(3, t_1)\}$ corresponding to the fossil individual 3 "reborn" at time t_1. That is, the coalescent with rebirth would start at time s from $\{\{(1, s)\}, \{(2, s)\}, \{(3, s)\}\}$, and at time t_1 the state would change to $\{\{(1, s)\}, \{(2, s), (3, s)\}, \{(3, t_1)\}\}$. Moreover, in addition to the transitions (2.18), at time $u \in (t_1, t_2)$ the fossil individual $(3, u)$ is born when $\{(3, t_1)\}$ merges with $\{(2, s), (3, s)\}$ at time u. This transition would be invisible in the coalescent (without rebirth) since that coalescent does not keep track of what happens to fossil individuals. Next at time t_2 the fossil individual $(2, t_2)$ is born, i.e., in the coalescent with rebirth

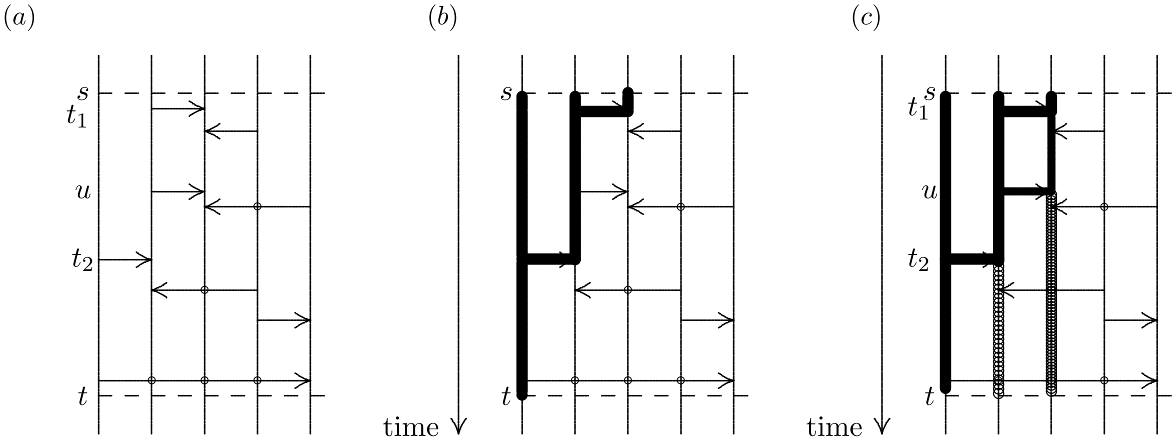

FIGURE 2.
(a) a realization of the resampling times, (b) the genealogy of the first three particles alive at time s is drawn in bold, and (c) the enriched genealogy after the fossils are included (here the different patterns correspond to different rebirth events).

we observe the following transitions

$$
\begin{aligned}
&\big\{\{(1,s)\},\{(2,s)\},\{(3,s)\}\big\}\\
\overset{\text{at time } t_1}{\longrightarrow}&\big\{\{(1,s)\},\{(2,s),(3,s)\},\{(3,t_1)\}\big\}\\
\overset{\text{at time } u}{\longrightarrow}&\big\{\{(1,s)\},\{(2,s),(3,s),(3,t_1)\},\{(3,u)\}\big\}\\
\overset{\text{at time } t_2}{\longrightarrow}&\big\{\{(1,s),(2,s),(3,s),(3,t_1)\},\{(2,t_2)\},\{(3,u)\}\big\}.
\end{aligned}
\tag{2.19}
$$

All new born partition elements also undergo coalescence, and in the spatial version of the model, also migration. □

The goal of this subsection is to introduce the coalescent with rebirth first in the non-spatial and then in the spatial setting for countably many individuals which needs some care but can be done rigorously using a graphical construction.

Step 1 (Coalescence with rebirth). Recall from Example 2.1 that in contrast to the classical coalescent which was defined in a symmetric manner, to be able to construct a dynamics for the coalescent with rebirth we always have to declare which of the partition elements is getting "lost" and simultaneously gets reborn as a singleton. However, since the functionals in which we will be interested later does not make explicit use of this information (neither would the construction of a genealogical tree), in the following we want to introduce a hierarchy of individuals and follow the rule that whenever two partition elements coalesce it is always the "higher-label" element which gets "reborn".

As before, let $\mathcal{I}$ be a countable set and distinguish an order relation $\preceq$ such that $\#\{i \preceq i_0\} < \infty$ for all $i_0 \in \mathcal{I}$. As illustrated in Example 2.1 we want to identify an individual (i,t) with its *index* $i \in \mathcal{I}$ and *birth time* $t \in \mathbb{R}$. Consider therefore the space $\mathcal{I} \times \mathbb{R}$ of all potential individuals and extend $\preceq$ on $\mathcal{I} \times \mathbb{R}$ to the *lexicographic order*, i.e., let for $(i,s),(j,t) \in \mathcal{I} \times \mathbb{R}$,

$$(i,s) \preceq (j,t) \qquad \text{if and only if} \quad i \prec j \text{ or } i = j \text{ and } s \le t. \tag{2.20}$$

Define then for each partition $\mathcal{P} \in \Pi^{\mathcal{I}}$ and each partition element $\pi \in \mathcal{P}$ the *label* $\kappa(\pi) \in \mathcal{I}$ as

$$\kappa(\pi) := \min_{\preceq} \big\{v;\, v \in \pi\big\}. \tag{2.21}$$

Recall from (2.8) the collection Π^S of all partitions of a set S. Call $\mathcal{P}$ an ordered *sub-partition* of $\mathcal{I}\times\mathbb{R}$ if $\mathcal{P}$ is a vector, $(\pi_i;\, i \in \mathcal{I})$, of pairwise disjoint non-empty subsets of $\mathcal{I}\times\mathbb{R}$. To define the state space, we call an ordered subpartition $\mathcal{P} = (\pi_i;\, i \in \mathcal{I})$ *admissible* if

(1) For all $i \in \mathcal{I}$ and all $t_0 < t_1$ in $\mathbb{R}$,

$$\#\big\{t \in [t_0, t_1] :\, (i,t) \in \cup_{j\in\mathcal{I}}\pi_j\big\}, < \infty. \tag{2.22}$$

(2) For all $i \in \mathcal{I}$ there exists an $t \in \mathbb{R}$ with

$$\kappa\big(\pi_i\big) = (i,t), \tag{2.23}$$

(3) For all $j \in \mathcal{I}$ and $i_1, i_2 \in \mathcal{I}$ with $i_1 \prec i_2$,

$$(j, s_1) \in \pi_{i_1} \text{ and } (j, s_2) \in \pi_{i_2} \text{ implies that } s_1 < s_2. \tag{2.24}$$

Denote then by

$$\Pi^{\leq,\mathcal{I}} := \text{set of all admissible ordered sub-partitions of } \mathcal{I}\times\mathbb{R}. \tag{2.25}$$

Since the coalescent with rebirth is keeping track of the birth time of an individual we need in addition (to obtain a time-homogeneous mechanism) to encode explicitly the time in the state. That is, we finally choose as the state space

$$\hat{\Pi}^{\mathcal{I}} := \big\{(t_0, \mathcal{P});\, t_0 \in \mathbb{R}, \mathcal{P} \in \Pi^{\leq,\mathcal{I}} \text{ with } \cup_{\pi\in\mathcal{P}}\, \pi \subseteq \mathcal{I}\times(-\infty, t_0]\big\}. \tag{2.26}$$

In words, we allow only those time-partition states for which all birth times are smaller than or equal to the current time. The space $\hat{\Pi}^{\mathcal{I}}$ can be equipped with a topology such that it becomes Polish (compare Appendix A).

We are now ready to define the coalescent with rebirth.

Definition 2.3 (Kingman-type coalescent with rebirth)**.** *The Kingman-type coalescent with rebirth is a $\hat{\Pi}^{\mathcal{I}}$-valued strong Markov process*

$$K^{\text{birth}} = (K^{\text{birth}}_u)_{u\geq 0}, \tag{2.27}$$

which starts in $K^{\text{birth}}_0 := (t_0, \mathcal{P}_0) \in \hat{\Pi}^{\mathcal{I}}$ such that for all finite subsets $\mathcal{I}' \subseteq \mathcal{I}$ which contain with an index also all smaller indices, the restricted process $\rho_{\mathcal{I}'}(K^{\text{birth}})$ is a $\hat{\Pi}^{\mathcal{I}'}$-valued Markov process which undergoes the following two transition mechanisms:

- Time growth *The time coordinate grows at a deterministic speed one.*
- Coalescence with rebirth *Given the current state $(t + t_0, \mathcal{P}) \in \hat{\Pi}^{\mathcal{I}'}$ at time t, for each $i_1 \prec i_2$ in $\mathcal{I}$ the partition elements $\pi_{i_1}, \pi_{i_2} \in \mathcal{P}$ merges into $\pi_{i_1} \cup \pi_{i_2}$ after an exponential waiting time with rate $\gamma_{\text{King}} > 0$, and at this time $t' > t$, simultaneously a new partition element $\{(i_2, t' + t_0)\}$ is born.*

The following will be proven in Subsection 2.3.

Proposition 2.1 (Existence and uniqueness in law)**.**

(a) *The Kingman-type coalescent with rebirth is a well-defined pure jump process whenever $\mathcal{I}$ is finite.*

(b) *For every initial state $(t_0, \mathcal{P}) \in \hat{\Pi}^{\mathcal{I}}$, there exists a unique càdlàg process satisfying the requirements of Definition 2.3.*

Step 2 (Migration and coalescence with rebirth combined) In the case of the spatial coalescent with rebirth all partition elements have in addition to an index and a birth-time also a current location that changes according to a random walk independently of all other partition elements. Recall the (at most) countable index set $\mathcal{I}$ equipped with the distinguished order $\preceq$, and fix a *countable site space* M which later will be chosen to be $\mathbb{Z}^2$. Then from any $\mathcal{P} = (\pi_i;\, i \in \mathcal{I}) \in \Pi^{\leq,\mathcal{I}}$ one can form an admissible *marked* partition

$$\mathcal{P}^M := \big\{(\pi_i, L^i);\, i \in \mathcal{I}\big\}, \tag{2.28}$$

by assigning to each index $i \in \mathcal{I}$ (and thus each partition element π_i) its *location* $L^i \in M$. Let then

$$\Pi^{\leq,\mathcal{I},M} := \text{set of all admissible } M\text{-marked subpartitions.} \tag{2.29}$$

Once more, in order to obtain a time-homogeneous mechanism we explicitly encode the time in the state. We choose as the state space

$$\begin{aligned} &\hat{\Pi}^{\mathcal{I},M} \\ &:= \big\{(t_0, \mathcal{P});\, t_0 \in \mathbb{R}, \mathcal{P}^M = ((\pi_i, L^i);\, i \in \mathcal{I}) \in \Pi^{\leq,\mathcal{I},M} \text{ with } \cup_{i\in\mathcal{I}}\, \pi_i \subseteq \mathcal{I} \times (-\infty, t_0]\big\}. \end{aligned} \tag{2.30}$$

For all $\mathcal{I}' \subseteq \mathcal{I}$, recall from (2.14) the restriction operator $\rho^M_{\mathcal{I}'}$ from $\Pi^{\leq,\mathcal{I},M}$ to $\Pi^{\leq,\mathcal{I}',M}$. With a slight misuse of notation, we also write $\rho_{\mathcal{I}'}$, $\mathcal{I}' \subseteq \mathcal{I}$, for the restriction map which sends $(t, \mathcal{P}^M) \in \hat{\Pi}^{\mathcal{I},M}$ to $(t, \rho_{\mathcal{I}'}(\mathcal{P}^M)) \in \hat{\Pi}^{\mathcal{I}',M}$.

We are now prepared to define the spatial $\mathcal{I}$-coalescent with rebirth.

Definition 2.4 (The spatial $\mathcal{I}$-coalescent with rebirth on $\mathbb{Z}^2$.)**.** *The* spatial $\mathcal{I}$-coalescent with rebirth on $\mathbb{Z}^2$ *is a $\hat{\Pi}^{\mathcal{I},\mathbb{Z}^2}$-valued strong Markov process*

$$\tilde{C}^{\text{birth}} := \big(\tilde{C}^{\text{birth}}_u\big)_{u\geq 0} = \big(t_0 + u, C^{\text{birth}}_u, L_u\big)_{u\geq 0} \tag{2.31}$$

which starts in $(t_0, ((\pi_i, L^i);\, i \in \mathcal{I})) \in \hat{\Pi}^{\mathcal{I},\mathbb{Z}^2}$ such that for all subsets $\mathcal{I}' \subseteq \mathcal{I}$ which contain with an index also all smaller indices and which satisfy

$$\sum\nolimits_{i\in\mathcal{I}'} \delta_{L^i_0} \in \mathcal{E}, \tag{2.32}$$

the restricted process $\rho_{\mathcal{I}'}(\tilde{C}^{\text{birth}})$ is a $\hat{\Pi}^{\mathcal{I}',\mathbb{Z}^2}$-valued strong Markov process which undergoes the following three independent *transition mechanisms:*

- Time growth *The time coordinate grows at deterministic rate one.*
- Migration *The marks of the partition elements perform independent rate* 1 *random walks on $\mathbb{Z}^2$ with transition probabilities satisfying (2.3).*
- Coalescence with rebirth *Given the current state $(t, ((\pi_i, L^i);\, i \in \mathcal{I}')) \in \hat{\Pi}^{\mathcal{I}',\mathbb{Z}^2}$, for each $i_1, i_2 \in \mathcal{I}'$ with $i_1 \prec i_2$ the partition elements π_{i_1} and π_{i_2} merges into $\pi_{i_1} \cup \pi_{i_2}$ after an exponentially distributed waiting time with hazard function given by the density $\gamma \cdot \mathbf{1}\{L^{i_2}_{\cdot} = L^{i_1}_{\cdot}\}$, and at this random time t', instantaneously the marked partition element $(\{(i_2, t' + t_0)\}, L^{i_2}_{t'})$ is born.*

Remark 2.5 (Consistency Property)**.** Recall the consistency property for the spatial coalescent stated in Remark 2.3. For the spatial coalescent with rebirth the analogue reads as follows: if $\tilde{C}^{\text{birth}}$ is the spatial $\mathcal{I}$-coalescent with rebirth and $\mathcal{I}' \subseteq \mathcal{I}$ such that $\mathcal{I}'$ contains which each index also all smaller ones and satisfies (2.32), then $\rho_{\mathcal{I}'}(\tilde{C}^{\text{birth}})$ is the $\mathcal{I}'$-coalescent with rebirth started in $\rho_{\mathcal{I}'}(\tilde{C}^{\text{birth}}_0)$. □

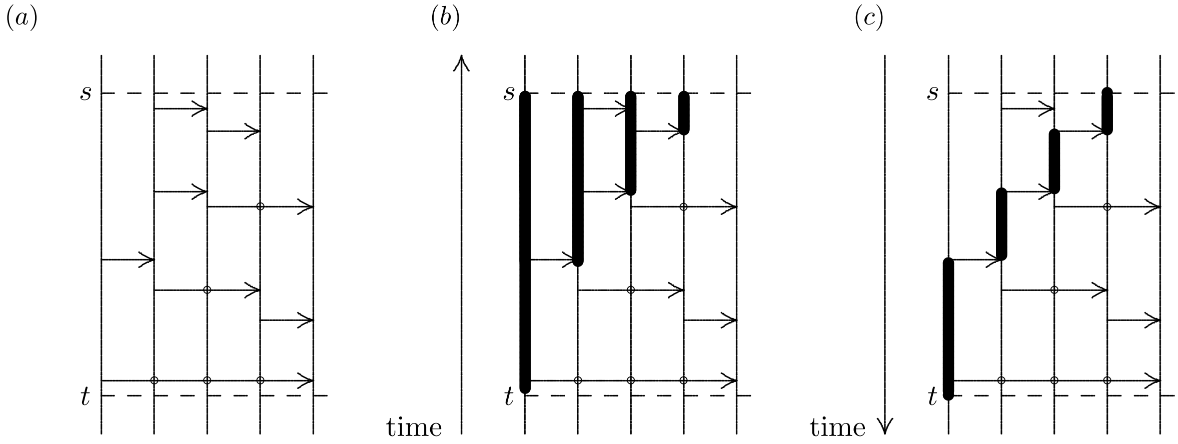

FIGURE 3. (a) illustrates the graph based on realizations of the Poisson processes Q and the family of paths L, (b) the set of all descendants up to time s of the individual labeled 1 at time t is indicated in bold, (c) the ancestral line of the individual 4 alive at time s is indicated in bold.

The following is proven in Subsection 2.3

Proposition 2.2 (The spatial $\mathcal{I}$-coalescent on $\mathbb{Z}^2$ with rebirth is well-defined)**.**

(a) *If (2.32) is satisfied for $\mathcal{I}$, then the spatial $\mathcal{I}$-coalescent with rebirth is a well-defined particle system.*

(b) *For every initial state $(t_0, (\pi_i, L^i); i \in \mathcal{I}) \in \hat{\Pi}^{\mathcal{I},\mathbb{Z}^2}$ there exists a unique càdlàg process satisfying the requirements of Definition 2.4.*

2.3. **The look-down construction (Proof of Propositions 2.1 and 2.2).** In this subsection we give an explicit construction of a version of the coalescent and the coalescent with rebirth. For that purpose we will rely on the look-down process introduced first by Donnelly and Kurtz in [13] and generalized to the spatial setting in [22]. The look-down process provides a graphical representation which allows to read off the forward time population model (after random permutations of individuals) and therefore also the backward time coalescent processes with and without rebirth.

In order to give the explicit construction for the backward model based on a random graph we proceed as follows. Fix a rate $\gamma > 0$, and a non-empty countable set $\mathcal{I}$ referred to as the *set of all individuals.* Assume that we are given a total order $\preceq$ on $\mathcal{I}$, and let

$$Q := \{Q^{i,j} : i,j \in \mathcal{I};\ i \prec j\} \tag{2.33}$$

be a family of independent Poisson point processes on $\mathbb{R}$ with intensity measure $\gamma\,\mathrm{d}t$. Furthermore consider a collection

$$L := \{(L^i_t)_{t\geq 0}, i \in \mathcal{I}\} \tag{2.34}$$

of independent continuous time irreducible random walks on an Abelian group M. The random collections in (2.33) and (2.34) are independent. This specifies our probability space. Starting at time 0 for each index $i \in \mathcal{I}$, we follow the random walk $(L^i_t)_{t\geq 0}$ and draw an arrow from i to j at time t if t is a point of $Q^{i,j}$ and $L^i_t = L^j_t$. This defines a random graph embedded in $\mathcal{I} \times \mathbb{R}$ with (random) marks in M, which is defined on our probability space.

The following key result follows then immediately (compare with Figure 3).

Lemma 2.1 (Ancestors are well-defined.)**.** *For each $i \in \mathcal{I}$ and $s \geq 0$, there exists a unique function $(A^i_{s,t})_{t\leq s}$ from $(-\infty, s]$ into $\mathcal{I}$ with càdlàg paths such that*

(1) $A^i_{s,s} = i$,

(2) *if* $t \in Q^{j, A^i_{s,t-}}$ *and* $L^{A^i_{s,t-}}_{(s-t)} = L^j_{(s-t)}$ *for some* $j \in \mathcal{I}$ *then* $A^i_{s,t} = j$, *and*

(3) $A^i_{s,t-} \neq A^i_{s,t}$ *if and only if* $t \in Q^{A^i_{s,t}, A^i_{s,t-}}$ *and* $L^{A^i_{s,t-}}_{(s-t)} = L^{A^i_{s,t}}_{(s-t)}$.

For each $(j,s) \in \mathcal{I} \times \mathbb{R}$, $(A^j_{s,t})_{t\leq s}$ is referred to as the *ancestral line* of individual j which lives at time s. Reversing the direction of time, for each $i \in \mathcal{I}$, $s \geq 0$, and $t \leq s$, denote by

$$\pi^s_i(t) := \big\{ j \in \mathcal{I} :\ A^j_{s,t} = i \big\}, \tag{2.35}$$

the set of all *descendants* at time s of the individual i which lived at time t in the past.

Remark 2.6 (The look-down process and the spatial $\mathcal{I}$-coalescent)**.** The spatial $\mathcal{I}$-coalescent was introduced in [22]. It can be easily read off from the graphical representation by putting $s = 0$, and letting for all $t \geq 0$,

$$C_t := \big\{ \pi^0_j(-t) :\ j \in \mathcal{I}, \pi^0_j(-t) \neq \emptyset \big\} \in \Pi^{\mathcal{I}}. \tag{2.36}$$

Notice that if $\pi \in C_t$ for some $t \geq 0$, then $A^i_{0,-t} = A^{i'}_{0,-t}$ for all $i, i' \in \pi$. Write therefore $A^\pi_{0,-t}$ for the common ancestor of all individuals in π at time $-t$ in the past, and put

$$L_t(\pi) := L_t^{A^\pi_{0,-t}}. \tag{2.37}$$

Then the process $(C, L) := (\{(\pi, L_t(\pi));\ \pi \in C_t\})_{t\geq 0}$ is a version of the spatial $\mathcal{I}$-coalescent and the càdlàg path property follows immediately from the choice of topology (compare the appendix). □

We next turn to the coalescent processes with rebirth.

Proof of Proposition 2.1. Proposition 2.1 is a special case of Proposition 2.2. □

Proof of Proposition 2.2. Assertion (a) follows by standard arguments.

(b) *Uniqueness* of the process is a direct consequence of Assertion (a). For *existence* we will use the look-down construction from above.

Fix the initial time t_0. W.l.o.g. we can assume that the initial state $\tilde{C}^{\text{birth}}_0$ is of the form $(t_0, ((\{(i, t_0)\}, L^i_0);\ i \in \mathcal{I}))$, for some $t_0 \in \mathbb{R}$. Consider in analogy to (2.35) for each $t \geq 0$ and $i \in \mathcal{I}$,

$$\hat{\pi}_{(i,t)} := \big\{ j \in \mathcal{I};\ \exists\, s \in [0, t] \text{ such that } A^j_{s, t_0 - t} = i \big\}, \tag{2.38}$$

the set of all descendants at any time $t_0 - s \in [t_0 - t, t_0]$ of the individual i back at time $t_0 - t$. For each $j \in \hat{\pi}_{(i,t)}$ denote by

$$u^i_t(j) := \inf\{ t_0 - s \in [t_0 - t, t_0] :\ A^j_{t_0 - s, t_0 - t} = i \} \tag{2.39}$$

its *birth time*.

Then set for all $t \geq 0$,

$$\tilde{C}^{\text{birth}}_t := \Big(t_0 + t, \big(\big(\big\{ (j, u^j_t(i));\ j \in \hat{\pi}_{(i,t)} \big\}, L^i_t \big);\ i \in \mathcal{I} \big) \Big) \tag{2.40}$$

The process $(\tilde{C}_t)_{t\geq 0}$ is the spatial coalescent with rebirth, provided we can show the *càdlàg* path property.

Recall first that the topology on $\hat{\Pi}^{\mathcal{I}}$ can be metrized, for example, by the metric defined in (A.4).

Fix an enumeration $\mathcal{I} := \{i_1, i_2, ...\}$. For all $n \in \mathbb{N}$, the restricted processes $K^{n,\text{birth}} := \rho_{\{i_1,...,i_n\}}(K^{\text{birth}})$ is a jump process with càdlàg paths. We will show that

$$(K^{n,\text{birth}}_s)_{s\geq 0} \underset{n\to\infty}{\longrightarrow} (K^{\text{birth}}_s)_{s\geq 0}, \tag{2.41}$$

in Skorohod topology, almost surely.

For that fix $T > 0$. We will show that for all $(t_n)_{n\in\mathbb{N}}$ with $(t_n) \downarrow t$, $K^{n,\text{birth}}_{t_n} \underset{n\to\infty}{\longrightarrow} K^{\text{birth}}_t$ and for all $(t_n)_{n\in\mathbb{N}}$ with $(t_n) \uparrow t$, $K^{n,\text{birth}}_{t_n} \underset{n\to\infty}{\longrightarrow} K^{\text{birth}}_{t-}$, almost surely. Indeed, if $(t_n)_{n\in\mathbb{N}}$ with $(t_n) \downarrow t$ and $\varepsilon > 0$ are given then there exists a random $N = N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, the set $\cup_{1\leq i\leq j\leq \lfloor 1-\log_2\varepsilon\rfloor} Q^{i,j}[t, t_n]$ is empty and therefore $d(K^{n,\text{birth}}_{t_n}, K^{\text{birth}}_t) < \varepsilon$ (with the metric d from (A.4)). The other convergence relation follows by a similar argument and the càdlàg path property follows by the choice of the topology. □

3. Main results

We will be interested in the asymptotic behavior of the spatial coalescent with or without rebirth which initially start in the region $\Lambda^{\alpha,t}$, where for $\alpha \in (0, 1]$ and $t > 0$,

$$\Lambda^{\alpha,t} := \big[- t^{\frac{\alpha}{2}}, t^{\frac{\alpha}{2}}\big]^2 \cap \mathbb{Z}^2. \tag{3.1}$$

Our parameter tending to infinity will be t. The size in geographic space will be measured on the scale $t^{\alpha/2}$ and the time at which we observe the process is on the scale t^β with $\beta \in [0, \infty)$. We therefore refer to α and β as the corresponding macroscopic *space* and *time parameter*, respectively.

We consider three settings, each in a separate subsection,

(1) the spatial coalescent (without rebirth) as a process in the macroscopic time parameter β for a fixed space parameter α,
(2) the spatial coalescent (without rebirth) as a process in the macroscopic space parameter α for a fixed time parameter β, and
(3) the spatial coalescent with rebirth.

In all settings we state that certain functionals of the spatial coalescent started from a configuration which contains particles at each site of $\Lambda^{\alpha,t}$, and which is observed at times t^β, for a $\beta \geq \alpha$, converge to the corresponding functionals of the Kingman coalescent with or without rebirth.

3.1. **The spatial coalescent in the macroscopic time parameter.**

Throughout this subsection we are in our setting (1), i.e., we fix $\alpha \in (0, 1]$. Recall from Definition 2.2 the spatial coalescent (C, L) on $\mathbb{Z}^2$, and let K be the Kingman coalescent. Denote for all $t > 0$ and each $\rho \in (0, \infty) \cup \{\infty\}$ by

$$C^{\alpha,t,\rho} \tag{3.2}$$

the spatial coalescent that starts in a Poisson configuration with either intensity $\rho \in (0, \infty)$ or with intensity "$\rho = \infty$", i.e. with initially countable infinitely many particles at each site of $\Lambda^{\alpha,t}$. We refer to these processes as *α-coalescents.*

Remark 3.1. The case $\rho < \infty$ is used in the study of the so-called interacting Moran models, while the case $\rho = \infty$ is needed to analyze its diffusion limit, the so-called Fisher-Wright diffusions, or in the more general setting of infinitely many types, the so-called interacting Fleming-Viot processes. See [10] and [11] for more on these processes. □

In each of the following subsubsections we state a theorem which considers the cases $\rho < \infty$, $\rho = \infty$ and refinements of the statements separately.

3.1.1. *The number of partition elements as a process indexed by the time parameter; finite intensity (Theorem 1).* The following result states the convergence of the number of partition elements of the α-coalescent observed at time t^β in the space $D([\alpha', \infty), \mathbb{N})$ of $\mathbb{N}$-valued càdlàg paths, equipped with the Skorokhod topology, where $\alpha' > \alpha$. Here and in the remainder of the paper, for any (marked) partition $\mathcal{P}$, we denote by $\#\mathcal{P}$ the number of partition elements in $\mathcal{P}$.

Theorem 1 (Number of partition elements as processes in β; $\rho < \infty$)**.** *Fix $0 < \rho < \infty$ and consider the spatial coalescent with rate $\gamma > 0$ and the Kingman coalescent with rate* 1*. Then for all $\alpha' > \alpha > 0$,*

$$\mathcal{L}\big[(\#C^{\alpha,t,\rho}_{t^\beta})_{\beta\in[\alpha',\infty)}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_{\log(\beta/\alpha)})_{\beta\in[\alpha',\infty)}\big]. \tag{3.3}$$

More generally, we can drop the assumption of a initial Poisson configuration. That is, the statement also holds if $(C_0, L_0) \in \Pi^{\mathcal{I},\mathbb{Z}^2}$ such that if in addition to (2.16) the following properties are satisfied:

$$\sup_{z\in\mathbb{Z}^2} \mathbf{E}\Big[\#\big\{\pi \in C_0,\ L_0(\pi) = z\big\}\Big] < \infty, \tag{3.4}$$

and

$$\#\big\{\pi \in C_0, L_0(\pi) \in \Lambda^{1,t}\big\} \underset{t\to\infty}{\longrightarrow} \infty, \quad \textit{in probability.} \tag{3.5}$$

Note that the scaling limit on the right hand side in (3.3) does not depend on the parameter $\rho \in (0, \infty)$. Indeed the more general statement given in Subsection 3.1.2 shows that there is very little dependence between the initial state and the scaling limit.

3.1.2. *The number of partition elements as a process indexed by the time parameter; infinite intensity.* We next turn to $\rho = \infty$. This case arises as dual process if one studies the genealogies in a model corresponding to interacting measure-valued Fleming-Viot diffusions. These models are limits of the spatial Moran model as the number of individual per site tends to ∞. The genealogy of the limiting model can be represented by the spatial coalescent starting with countable many particles at each site, see [22].

In this situation the total number of initial individuals (partition elements) does not come down from infinity in positive time (compare [3]) since partition elements can escape into empty space. Therefore we cannot expect Theorem 1 to hold in this case. However we will show that the fraction of the (original) partition elements which do avoid coalescing with the ones sharing their sites is very small and that in fact its relative frequency in the total population is 0. The reason is that avoiding coalescence can only happen to those partition elements which jump immediately.

To make this precise we *fix an enumeration* $(i_1, i_2, ...)$ of $\mathcal{I}$. Then the following result is the analogue of Theorem 1 for $\rho = \infty$.

Theorem 2 (Number of partition elements as processes in β; $\rho = \infty$)**.** *Let $\rho = \infty$. There exists a random subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ such that*

(a) *with probability 1,*

$$\tfrac{1}{n}\#(\tilde{I} \cap \{i_1, \cdots, i_n\}) \underset{n\to\infty}{\longrightarrow} 1, \tag{3.6}$$

and

(b) *for all $\alpha' > \alpha > 0$,*

$$\mathcal{L}\big[(\#\rho^{\mathbb{Z}^2}_{\tilde{\mathcal{I}}}(C^{\alpha,t,\infty}_{t^\beta})_{\beta\in[\alpha',\infty)}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_{\log(\beta/\alpha)})_{\beta\in[\alpha',\infty)}\big]. \tag{3.7}$$

Remark 3.2. Proving results for the system with the exception of a set of frequency 0 of initial individuals becomes useful if one describes the genealogy of the Fleming-Viot process by a metric measure tree since it implies convergence in the canonical Gromov-weak topology (compare, for example, [25]). □

3.1.3. *The number of partition elements as a process indexed by the time parameter; refinement.* The next goal is to extend the results in Theorems 1 and 2 to the case where $\alpha' = \alpha$. The number of partition elements then diverges in the limit. Let therefore $\mathbb{N}$ be equipped with the discrete topology and denote by

$$\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\} \tag{3.8}$$

its one point compactification. This means that a sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\bar{\mathbb{N}}$ converges in $\bar{\mathbb{N}}$ if either $(n_k)_{k\in\mathbb{N}}$ is a convergent sequence in $\mathbb{N}$, or $(n_k)_{k\to\infty}$ diverges to infinity.

Now we can consider the processes

$$\big(\#C^{\alpha,t,\rho}_{t^\beta}\big)_{\beta\geq\alpha}, \text{ and } \big(\#K_{\log\beta/\alpha}\big)_{\beta\geq\alpha} \tag{3.9}$$

in the Skorokhod space $D([\alpha,\infty),\bar{\mathbb{N}})$. For brevity, and in mind of future applications (see [23, 24]), we will consider only particular initial configurations.

Theorem 3 (Convergence to the entrance law)**.** *Fix $\alpha > 0$.*

(i) *Assume that the initial configuration is either a Poisson process with intensity $\rho < \infty$ or a Bernoulli field with success probability $p \in (0,1]$. For the corresponding coalescent processes, $(C^{\alpha,t}_s, L^{\alpha,t}_s)_{s\geq 0}$, the following hold:*

$$\mathcal{L}\big[(\#C^{\alpha,t}_{t^\beta})_{\beta\in[\alpha,\infty)}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_{\log(\beta/\alpha)})_{\beta\in[\alpha,\infty)}\big]. \tag{3.10}$$

(ii) *Assume $\rho = \infty$. Then there exists a random subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ with (3.6) and such that*

$$\mathcal{L}\big[(\#\rho^{\mathbb{Z}^2}_{\tilde{\mathcal{I}}}(C^{\alpha,t,\infty}_{t^\beta})_{\beta\in[\alpha,\infty)}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_{\log(\beta/\alpha)})_{\beta\in[\alpha,\infty)}\big]. \tag{3.11}$$

3.1.4. *The partition-valued process indexed by the time parameter.* Recall the space of partitions $\Pi^{\mathcal{I}}$ from (2.8). So far we have shown that the number of partition elements in the suitably re-scaled spatial $\mathcal{I}$-coalescent converges to those of a Kingman coalescent. Notice that in the spatial coalescent dynamics the local geographic structure is important at small times. For example, individuals at neighboring sites will coalesce much quicker than the ones further away. However asymptotically, as $t \to \infty$, the evolution at much later times treats all individuals exchangeably which allows a description with Kingman's coalescent. This suggests that the statement holds also for the partition-valued objects provided we abstract from the individuals names and only keep the partition structure which reflects the concept of isomorphy in this context. That is, we view the state of the coalescent not as a partition of a specific set $\mathcal{I}$ but rather allow for *bijective relabelings* of the individuals.

Formally we proceed as follows. Assume we are given two non-empty countable sets $\mathcal{I}$ and $\tilde{\mathcal{I}}$, as well a bijection φ from $\mathcal{I}$ to $\hat{\mathcal{I}}$. Let then φ^* be the induced map which sends a partition $C \in \Pi^{\mathcal{I}}$ to $\varphi^*(C) \in \Pi^{\hat{\mathcal{I}}}$ which is obtained by replacing each $i \in \mathcal{I}$ by $\varphi(i) \in \hat{\mathcal{I}}$.

We can then strengthen Theorem 3 as follows:

Theorem 4 (Convergence as partition-valued process). *Fix $\rho \in (0, \infty]$. For all $t \in [0, \infty)$ we can find a bijection $\varphi_t : \mathcal{I} \to \mathcal{I}$ such that*

$$\mathcal{L}\Big[\big(\varphi_t^*(C_{t^\beta}^{\alpha,t})\big)_{\beta\in[\alpha,\infty)}\Big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(K_{\log(\beta/\alpha)})_{\beta\in[\alpha,\infty)}\big]. \tag{3.12}$$

3.2. **Spatial coalescent as a function of macroscopic spatial parameter α.** For all $\rho \in [0, \infty]$, recall $C^{1,t,\rho}$ from (3.2). We are now in our Setting 2. That is, we fix the time parameter $\beta = 1$ and let the space parameter α vary. To be more precise, we first fix

$$0 \le \alpha_l < \alpha_u < 1, \tag{3.13}$$

. We are then interested in the limit behavior, as $t \to \infty$, of the process

$$\big(\#\rho^{\mathbf{Z}^2}_{\mathcal{I}^{\alpha,t}}(C_t^{1,t,\rho})\big)_{\alpha\in[\alpha_l,\alpha_u]}\big)_{\alpha\in[\alpha_l,\alpha_u]}, \tag{3.14}$$

where $\mathcal{I}^{\alpha,t} := \{i \in \mathcal{I} :\, L_0^{\{i\}} \in \Lambda^{\alpha,t}\}$.

To be in a position to state the limiting process, denote for all $\alpha \ge \alpha_l$ by

$$K^{\text{birth}}[\log\alpha_l, \log\alpha] := \big(K_t^{\text{birth}}[\log\alpha_l, \log\alpha]\big)_{t\ge\log\alpha_l} \tag{3.15}$$

the *Kingman coalescent process with time-restricted rebirth* which starts at time $\log\alpha_l$ in

$$K^{\text{birth}}_{\log\alpha_l}[\log\alpha_l, \log\alpha] := \Big(0, \big((\{i\}, 0);\, i \in \mathcal{I}\big)\Big) \tag{3.16}$$

and follows a Kingman coalescent with rebirth during the time interval $[\log\alpha_l, \log\alpha]$, while rebirth is suppresed after $\log\alpha$.

Let pr_{index} and pr_{time} be the projection maps of individuals to their *indices* and *birth times*, i.e.,

$$(i, t) = \big(\text{pr}_{\text{index}}(i, t), \text{pr}_{\text{time}}(i, t)\big), \tag{3.17}$$

for all $(i, t) \in \mathcal{I} \times \mathbb{R}$.

We want to think of $(K_t^{\text{birth}}[\log\alpha_l, \log\alpha])_{\alpha\ge\alpha_u}$ as coupled sub-coalescents obtained from the Kingman coalescent with rebirth which starts in (3.16) by putting

$$K_t^{\text{birth}}[\log\alpha_l, \log\alpha] := \big\{\pi \in K_t^{\text{birth}} :\, \text{pr}_{\text{time}}(\kappa(\pi)) \le \log\alpha\big\}. \tag{3.18}$$

The following result describes the asymptotic law of the process of sizes of the spatially defined sub-coalescents.

Theorem 5 (Convergence as processes in α). *Fix $0 \le \alpha_l < \alpha_u < 1$.*

(i) *For all $\rho \in [0, \infty)$,*

$$\mathcal{L}\big[\big(\#\rho_{\mathcal{I}^{\alpha,t}}(C_t^{1,t,\rho})\big)_{\alpha\in[\alpha_l,\alpha_u]}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_0^{\text{birth}}[\log\alpha_l, \log\alpha])_{\alpha\in[\alpha_l,\alpha_u]}\big]. \tag{3.19}$$

(ii) *Assume $\rho = \infty$. Then there exists a random subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ with (3.6) and such that*

$$\mathcal{L}\big[\big(\#\rho_{\mathcal{I}^{\alpha,t}\cap\tilde{\mathcal{I}}}(C_t^{1,t,\infty})\big)_{\alpha\in[\alpha_l,\alpha_u]}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_0^{\text{birth}}[\log\alpha_l, \log\alpha])_{\alpha\in[\alpha_l,\alpha_u]}\big]. \tag{3.20}$$

Remark 3.3. Notice that since $\#K_0^{\text{birth}}[\log\alpha_l, \log\alpha] \to \infty$, as $\alpha \to 1$, and $\#C_t^{1,t,\rho} \to \infty$, as $t \to \infty$, the result holds also for $\alpha_u = 1$. However, in order to rigorously include $\alpha_u = 1$ in the statement formally we would again have to consider the one point compactification of $\bar{\mathbb{N}}$ and apply similar techniques as in the proof of Theorem 3.

We also refrain here from writing down the corresponding partition-valued statement. □

3.3. **Rescaling the spatial coalescent with rebirth.** We are now in the Setting 3. Recall from Definition 2.4 the spatial coalescent $\left(C^{\text{birth}}, L^{\text{birth}}\right)$ with rebirth. At a time t we observe the spatial coalescent with rebirth which started in a Poisson configuration on $\mathbb{Z}^2$ with intensity $\rho \in (0,\infty]$ at time 0. Since all the partition elements which "die" due to coalescence get replaced, the configuration of locations on $\mathbb{Z}^2$ of partition elements remains Poisson for times in $[0,t]$.

Fix $\alpha \in (0,1)$, $m \in \mathrm{N}$ and $\alpha < u_1 < ... < u_m < 1$. We are interested in the behavior of the coalescent in a growing time-space window and want to describe this behavior by a suitable limit process. To be more precise, we observe the spatial coalescent with rebirth at a late time t and focus on those partition elements which were located in a box $\Lambda^{\alpha,t}$ at the earlier times t^{u_1}, t^{u_2}, ..., t^{u_m}. We are particularly interested in the number of these partition elements in the limit as $t \to \infty$. This quantity plays an important role in the study of interacting Fleming-Viot processes observed at certain space-time windows since the corresponding sub-coalescent arises as a dual object in resampling models. See Remark 3.5 for a more detail explanation.

Recall the label $\kappa(\pi)$ of a partition element π from (2.21). For $j = 1, ..., m$ and $\rho \in (0,\infty]$, we therefore define

$$C_j^{\alpha,t,\vec{u},\rho} := \left\{\pi \in C_t^{\text{birth}} : \exists i \in \{1,...,j\} \text{ s.t. } \mathrm{pr}_{\text{time}}(\kappa(\pi)) \le t^{u_i},\ L_{t^{u_i}}^{\text{birth}}(\pi) \in \Lambda^{\alpha,t}\right\}. \tag{3.21}$$

To be in a position to state our asymptotics we next introduce the limit. Consider the time-inhomogeneous $\Pi^{\mathbb{N}}$-valued coalescent process

$$K^{\text{merge}} := \left(K_s^{\text{merge}}\right)_{s\ge 0} \tag{3.22}$$

which starts in $\{\{n\};\, n \in \mathbb{N}\}$ at time 0 and has the following dynamics:

- If $\pi_1, \pi_2 \in K_t^{\text{merge}}$ with $[\kappa(\pi_1)]_{\text{mod}m} = [\kappa(\pi_2)]_{\text{mod}m}$ then they coalesce with rate 1, where $[n]_{\text{mod}m}$ is the unique element in $\{0,1,2,...,m-1\}$ such that $n - [n]_{\text{mod}m}$ is a multiple of m.
- If $\pi_1, \pi_2 \in K_t^{\text{merge}}$ with $[\kappa(\pi_1)]_{\text{mod}m} \ne [\kappa(\pi_2)]_{\text{mod}m}$ then they coalesce at rate $\mathbf{1}_{t \ge \log\left(\frac{u_{[\kappa(\pi_1)]_{\text{mod}m} \vee [\kappa(\pi_2)]_{\text{mod}m}}}{\alpha}\right)}$.

That is, the process K^{merge} is given by a *family of m merging coalescents* where each coalescent starts in the partition $\{\{k \cdot m + i - 1\};\, k \in \mathbb{N}\}$ with $i = 1, ..., m$ and evolves like a Kingman coalescent and at certain specified times each of the m coalescents starts to coalescence also with the other $(m-1)$ members of the family of merging coalescents. The times of enabling the mutual coalescence are given in terms of the u_i.

Our result reads then as follows:

Theorem 6 (Asymptotics of coalescent with rebirth)**.** *Fix $0 < \alpha < \beta < 1$ and $\alpha \le u_1 < .. < u_m \le \beta$.*

(a) *If $\rho < \infty$ then*

$$\mathcal{L}\Big[\big(\#C_1^{\alpha,t,\vec{u},\rho},...,\#C_m^{\alpha,t,\vec{u},\rho}\big)\Big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\Big[\big(\#\rho_{\{km;\,k\in\mathbb{N}\}}(K^{\text{merge}}_{-\log\alpha}),\#\rho_{\{km+j-1;\,k\in\mathbb{N},j=1,2\}}(K^{\text{merge}}_{-\log\alpha}),...,\#K^{\text{merge}}_{-\log\alpha}\big)\Big]. \tag{3.23}$$

(b) *If $\rho = \infty$, then there exists a random set $\tilde{\mathcal{I}} \subset \mathcal{I}$ with (3.6) and such that*

$$\mathcal{L}\Big[\big(\#\rho_{\tilde{\mathcal{I}}}(C_1^{\alpha,t,\vec{u},\rho}),...,\#\rho_{\tilde{\mathcal{I}}}(C_m^{\alpha,t,\vec{u},\rho})\big)\Big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\Big[\big(\#\rho_{\{km;\,k\in\mathbb{N}\}}(K^{\text{merge}}_{-\log\alpha}),\#\rho_{\{km+j-1;\,k\in\mathbb{N},j=1,2\}}K^{\text{merge}}_{-\log\alpha},...,\#K^{\text{merge}}_{-\log\alpha}\big)\Big]. \tag{3.24}$$

Remark 3.4 (No path-wise convergence)**.** Since the times t^u and $t^{u'}$ separate for all $u' \neq u$ in this limit, we cannot use a continuous macroscopic time parameter in our analysis but rather have to discretize. □

Remark 3.5 (Space-time cluster formation)**.** Theorem 6 is the key to the study of the space-time cluster formation of interacting Moran models or interacting Fleming-Viot diffusions on $\mathbb{Z}^2$. As already indicated, the space-time genealogy of interacting Moran models is described by the spatial coalescent with rebirth. To make this more precise, let us fix some large t and introduce the reversed time $s^{\leftarrow}(s) \equiv s^{\leftarrow}_t(s) := t - s$. Then, provided that the original configuration of particles is Poisson, and that the particles evolve according to the enriched interacting Moran models in forward time (where the types that die due to resampling are kept as fossils), then their paths observed in reversed time evolve according to the spatial coalescent with rebirth. Moreover, a resampling event that occurs at time $t - s$ corresponds to a unique rebirth event occurring at time $s^{\leftarrow}_t(s) = s$.

Assume that at the initial time 0 *each individual (particle) carries its own type.* The following questions arise naturally in this context: if we fix a time $t > 0$ and a large window W of observation, how far back in time do we have to look so that most of the population present in W at time t has a single ancestor and hence carries a single type (color)? How does this information changes if the population is sampled at several time instances from the same window W?

Theorems 1 through 3 suggest that the right space scale for the window of observation is the *α-box* $\Lambda^{t,\alpha}$, while the right time scale for answering these questions is the logarithmic time scale. Fix therefore $\alpha \in (0,1]$ and $u \in (0,1]$, and and assume we observe two sub-populations of individuals in the Moran model which are at time t respectively at time $t - t^u$ in $\Lambda^{\alpha,t}$. We can then ask whether or not these two sub-populations have one common ancestor. That is, whether or not we see in the window one color or several colors at both times. In the first case we say that the age of the cluster in logarithmic scale is at least u.

Another question arises when the subpopulation observed at time t in $\Lambda^{\alpha,t}$ has one common ancestor in $\Lambda^{\alpha,t}$ at time $t - t^u$ but the other individuals at that time in $\Lambda^{\alpha,t}$ have more than one common ancestor by time 0 or their common ancestor is different from the one of the previous subpopulation. In this latter case $\Lambda^{\alpha,t}$ has at time t a color which does not cover this block at the earlier time $t - t^u$. We then say that the age of the α-cluster is at most u.

To be able to say something on the *age* of the cluster, we want to know for which u both these scenarios have positive probability. Asymptotically, as $t \to \infty$, we can read this off from the system of merging Kingman coalescents appearing in the right hand side of (3.23). An immediate observation is that $u > \alpha$ is needed to see these scenarios.

More generally, fix $m \geq 1$ and $u_1, ..., u_m$ such that $0 < u_1 < ... < u_m \leq 1$. During the time interval $[t - t^{u_m}, t]$, consider the joint evolution of m different sub-populations of the Moran model where the 0^{th} subpopulation consists of particles present in the α-box at time t, and for $i = 1, ..., m-1$, the i^{th} subpopulation consists of particles present in the α-box at time $t - t^{u_i}$. By reversing time all the interesting information about their joint genealogy is expressed precisely in terms of the quantities $(\#C_1^{\alpha,t,(u_1,...,u_m),\rho}, ..., \#C_m^{\alpha,t,(u_1,...,u_m),\rho})$ as defined in (3.21). For example, the event on which the latter vector takes value $(1, ..., 1)$ is precisely the event that all the individuals (in m sub-populations combined) have a common ancestor at time $t - t^{u_m}$. □

Outline. The rest of the paper is organized as follows: In Section 4 we recall and extend some basic facts on coalescents on $\mathbb{Z}^2$, and in Sections 5 and 6 we provide the asymptotic analysis of coalescents which allows us to prove Theorems 1 -Theorem 4 in Section 7, and Theorems 5 and 6 in Section 9. Section 10 contains the proof of a moment estimate on the number of partition elements. □

Result and Problem History. Here we give some information concerning the history of the problems treated in this paper. In the setting of instantaneous coalescence for simple random walks on $\mathbb{Z}^2$, i.e., two partition elements coalesce immediately when the hit the same site, Lemma 5.2 was proved in [9], and Proposition 6.1 in [4]. Propositions 5.1 and 6.2 and Lemma 7.1, are to the best of our knowledge novel in the setting of any spatial coalescent model on $\mathbb{Z}^2$. Due to the applications we have in mind (using duality with the IMM and IFWD) in the subsequent papers, we are primarily interested in the spatial (and delayed) coalescents, and therefore the results are phrased and proved in the current setting. However, it is important to note that the arguments, and therefore statements, in Section 3 remain to hold in the setting of [4] and [9]. □

4. Preliminaries

In this section we present several basic techniques on coalescents and present the key properties of random walks which we will need for our subsequent arguments. We first state in Subsection 4.1 some notational conventions which will be used throughout the rest of the paper. In Subsection 4.2 we recall a famous result by Erdös-Taylor which gives the asymptotics of the hitting time of a planar random walk. In Subsection 4.3 we state the asymptotic exchangeability for the spatial coalescent on $\mathbb{Z}^2$. In Subsection 4.4 we recall some consequences of monotonicity properties.

4.1. **Notational conventions.**

In the rest of the paper we often use the following convention concerning notation.

- For functions $g, h : [0, \infty) \to \mathbb{R}$, we write $g(t) = O(h(t))$ or $g(t) = o(h(t))$ if and only if $\limsup_{t\to\infty} \frac{g(t)}{h(t)} < \infty$ or $\lim_{t\to\infty} \frac{g(t)}{h(t)} = 0$, respectively.
- For a set A, we denote by A^c its complement (with respect to the natural superset, determined by the context).
- Recall $\bar{\mathbb{N}}$ from (3.8), and denote for a finite or countable set A by $\#A \in \bar{\mathbb{N}}$ the number of elements in A.
- If $a, b \in \mathbb{R}$, let $a \wedge b$ denote the minimal, and $a \vee b$ the maximal element of $\{a, b\}$.
- Poisson(ρ) random variable (or distribution) has intensity (rate, expectation) ρ.

- For a partition $\mathcal{P}$, recall that $\#\mathcal{P}$ denotes the number of partition elements of $\mathcal{P}$.
- If $\mathcal{P}$ is a partition then we write $i \sim_{\mathcal{P}} j$ if i and j belong to the same partition element of $\mathcal{P}$. If $(\mathcal{P}_t,\, t \geq 0)$ is a partition-valued process then $i \sim_{\mathcal{P}_t} j$ will be sometimes abbreviated as $i \sim^t j$.

4.2. **Erdös-Taylor formula.** Recall a well-known result by Erdös and Taylor [15] for planar random walks with finite variance: if τ is the first hitting time of the origin of a two-dimensional random walk, then

$$\lim_{t\to\infty} \mathbf{P}^{xt^{\alpha/2}}\{\tau > t^{\beta}\} = \frac{\alpha}{\beta} \wedge 1, \tag{4.1}$$

for all $\alpha, \beta \in [0,1]$, and all $x \in \mathbb{R}^2 \setminus \{(0,0)\}$ (see, for example, Proposition 1 in [9]). In particular, the right hand side of (4.1) does not depend on $x \in \mathbb{R}^2 \setminus \{(0,0)\}$. Due to this peculiar (specific to $d = 2$) property, the behavior of the spatial coalescent started in $\Lambda^{\alpha,t}$ and observed at time t^{β}, asymptotically as $t \to \infty$, depends only on the logarithmic scales α and β, while all the finer distinctions are washed out.

For $c \in (0,\infty)$, define

$$I_\alpha(c,t) := \big[A_\alpha^-(t), A_\alpha^+(t)\big] := \big[(c\log t)^{-1} \cdot t^{\frac{\alpha}{2}}, (c\log t)\cdot t^{\frac{\alpha}{2}}\big]. \tag{4.2}$$

Say that a set of locations (marks) $\{x_1,\dots,x_n\}$ *is contained in* $I_\alpha(c,t)$ if and only if $\| x_i - x_j \| \in I_\alpha(c,t)$, for all $1 \leq i < j \leq n$.

¿From (4.1) one sees immediately that if $\{x_1,\dots,x_n\}$ is contained in $I_\alpha(c,t)$, then for the corresponding random walks $\{(X_t^j)_{t\geq 0},\, j = 1,\dots,n\}$ with $X_0^j := x_j$,

$$\mathbf{P}\big\{X_s^i \neq X_s^j,\ \forall i \neq j, \forall s \in [0, g(t)]\big\} \underset{t\to\infty}{\longrightarrow} 1, \tag{4.3}$$

whenever $g(t)$ is a function satisfying $g(t) = O(t^{\alpha+\varepsilon})$ for all $\varepsilon > 0$.

4.3. **Asymptotic exchangeability.** In this subsection we perform some preliminary calculations implying "asymptotic exchangeability" that will be useful in the sequel. The main result is Proposition 4.1 below.

Let $\alpha \in (0,1]$, and set

$$g_\alpha(t) := t^\alpha \log^3 t. \tag{4.4}$$

Remark 4.1. *In fact, any function $g_\alpha(t)$ with $t^\alpha \log^2 t = o(g_\alpha(t))$ could be used instead of $t^\alpha \log^3 t$.* □

For $k \in \mathbb{N}$, let ζ be a permutation on $\{1,\dots,k\}$. Given $\{x_1(t),\dots,x_k(t)\} \subset \mathbb{Z}^2$, we denote by $(C_s^{\alpha,t}, L_s^{\alpha,t})_{s\geq 0}$ the spatial coalescent that starts from $C_0^{\alpha,t} = \{\{1\},\dots,\{k\}\}$, $L_0^{\alpha,t}(\{i\}) = x_i(t)$, $i = 1,\dots,k$, and by $(C_s^{\alpha,t,\zeta}, L_s^{\alpha,t,\zeta})_{s\geq 0}$ the spatial coalescent that starts from $C_0^{\alpha,t,\zeta} = C_0^{\alpha,t}$, $L_0^{\alpha,t}(\{i\}) = x_{\zeta_i}(t)$, $i = 1,\dots,k$.

Proposition 4.1 (Asymptotic exchangeability for the spatial coalescent)**.** *Fix $\alpha \in (0,1]$ and $k \in \mathbb{N}$, and assume that $\{x_1(t),\dots,x_k(t)\} \subset \mathbb{Z}^2$ is contained in $I_\alpha(c,t)$. If the spatial coalescent $(C_s^{\alpha,t}, L_s^{\alpha,t})_{s\geq 0}$ starts in the marked partition $\{(\{1\},x_1(t)),\dots,(\{k\},x_k(t))\}$, then for all $M \in \mathcal{B}(D([0,\infty),\Pi^{\mathcal{I}}))$,*

$$\lim_{t\to\infty} \Big|\mathbf{P}\big\{(C_s^{\alpha,t})_{s\geq g_\alpha(t)} \in M\big\} - \mathbf{P}\big\{(C_s^{\alpha,t,\zeta})_{s\geq g_\alpha(t)} \in M\big\}\Big| = 0. \tag{4.5}$$

We prepare the proof by stating the corresponding result for the underlying random walks.

Lemma 4.1 (Asymptotic exchangeability for random walks)**.** *Fix $\alpha \in (0,1]$, $k \in \mathbb{N}$, and let ζ be a permutation on $\{1,\ldots,k\}$. Let $(Y_s)_{s\geq 0}$ be the $2k = k\times 2$ dimensional random walk*

$$Y_s := \big(X^1_{1,s}, X^1_{2,s}, X^2_{1,s}, X^2_{2,s}, \ldots, X^k_{1,s}, X^k_{2,s}\big), \tag{4.6}$$

where, for each $i \in \{1,\ldots,k\}$, $(X^i_s)_{s\geq 0} = (X^i_{1,s}, X^i_{2,s})_{s\geq 0}$ is the two dimensional random walk with transition kernel $a(x,y)$, and the k random walks are taken to be independent. Moreover, let

$$(Y^\zeta_s)_{s\geq 0} := \big(X^{\zeta_1}_{1,s}, X^{\zeta_1}_{2,s}, X^{\zeta_2}_{1,s}, X^{\zeta_2}_{2,s}, \ldots, X^{\zeta_k}_{1,s}, X^{\zeta_k}_{2,s}\big). \tag{4.7}$$

Then for all $M \in \mathcal{B}(D([0,\infty),\mathbb{Z}^{2k}))$,

$$\Big|\mathbf{P}\big\{(Y_s)_{s\geq g_\alpha(t)} \in M\big\} - \mathbf{P}\big\{(Y^\zeta_s)_{s\geq g_\alpha(t)} \in M\big\}\Big| \underset{t\to\infty}{\longrightarrow} 0. \tag{4.8}$$

Proof. The proof relies on a consequence of the local central limit theorem for continuous time random walks that we recall next: if $(Z_s)_{s\geq 0}$ is a random walk in $\mathbb{Z}^d$ (here no moment assumption is needed), then there exists a finite constant c (see, for example for our setting [31]) that depends on the dimension and the transition mechanism only, such that for all $y \in \mathbb{Z}^2$,

$$\sum_{z\in\mathbb{Z}^2} \big|\mathbf{P}(Z_s = z|Z_0 = 0) - \mathbf{P}(Z_s = z|Z_0 = y)\big| \leq \frac{c \parallel y \parallel}{s^{1/2}}. \tag{4.9}$$

We will apply the above difference estimate (4.9) to $(Y_s)_{s\geq 0}$ and $(Y^\zeta_s)_{s\geq 0}$.

Let $M \in \mathcal{B}(D([0,\infty),\mathbb{Z}^{2k}))$. For each $2k$-tuple $(z^1_1, z^1_2, z^2_1, z^2_2, \ldots, z^k_1, z^k_2) \in \mathbb{Z}^{2k}$, set

$$q\big(z^1_1, z^1_2, z^2_1, z^2_2, \ldots, z^k_1, z^k_2\big) := \mathbf{P}\big((Y_s)_{s\geq 0} \in M | Y_0 = (z^1_1, z^1_2, z^2_1, z^2_2, \ldots, z^k_1, z^k_2)\big). \tag{4.10}$$

Denote by $B(r)$ the ball in $\mathbb{R}^2$ of radius r centered at 0. Suppose $x^1,\ldots,x^k \in \mathbb{Z}^2 \cap B(c\log(t)\,t^{\alpha/2})$, and let $X^1,\ldots,X^k$ be k independent random walks with transition kernel $a(\cdot,\cdot)$ started at locations $x^1, x^2, \ldots, x^k$, respectively. Let Y be the walk formed as in (4.6) but using the walks $X^1, X^2, \ldots, X^k$ as input. For a permutation ζ of $\{1,2,\ldots,k\}$, let Y^ζ be the walk formed as in (4.7) using $X^{\zeta_1}, X^{\zeta_2}, \ldots, X^{\zeta_k}$ as input, instead. Then clearly Y and Y^ζ have the same transition mechanism, and the difference $Y_0 - Y^\zeta_0$ of their starting locations is a vector with norm bounded by $O(t^{\alpha/2}\log t)$. Therefore, by (4.9), for all $u \in [0,1]$,

$$\Big|\mathbf{P}\big\{q(Y_{g_\alpha(t)}) \geq u\big\} - \mathbf{P}\big\{q(Y^\zeta_{g_\alpha(t)}) \geq u\big\}\Big| \leq \frac{O(t^{\alpha/2}\log t)}{t^{\alpha/2}\log t^{3/2}} \underset{t\to\infty}{\longrightarrow} 0. \tag{4.11}$$

That is, the $[0,1]$-valued random variables $q(Y_{g_\alpha(t)})$ and $q(Y^\zeta_{g_\alpha(t)})$ are asymptotically equal in distribution. In particular,

$$\begin{aligned} &\Big|\mathbf{E}\big[q(Y_{g_\alpha(t)})\big] - \mathbf{E}\big[q(Y^\zeta_{g_\alpha(t)})\big]\Big| \\ &= \Big|\mathbf{P}\big\{(Y_s)_{s\geq g_\alpha(t)} \in M\big\} - \mathbf{P}\big\{(Y^\zeta_s)_{s\geq g_\alpha(t)} \in M\big\}\Big| \\ &\underset{t\to\infty}{\longrightarrow} 0, \end{aligned} \tag{4.12}$$

and we are done. □

Proof of Proposition 4.1. Let the $2k$-dimensional processes Y^{sc} and $Y^{sc,\zeta}$ ("sc" stands for semi-coalescent) be formed as in (4.6) and (4.7), however the input random processes $X^1,\ldots,X^k$ are changed so that X^is are independent continuous-time random walks with kernel $a(\cdot,\cdot)$ until time $g_\alpha(t)$, and after time $g_\alpha(t)$ their joint evolution is the evolution of the location process of the spatial coalescent with initial configuration $(X^1_{g_\alpha(t)},\ldots,X^k_{g_\alpha(t)})$. Moreover, let Y^c and $Y^{c,\zeta}$ ("c" stands for coalescent) be the $2k$-dimensional processes whose joint evolution is the evolution of the location process of the spatial coalescent with initial configuration $(X^1_{g_\alpha(t)},\ldots,X^k_{g_\alpha(t)})$.

It is obvious how to construct couplings (Y^{sc},Y^c) and $(Y^{sc,\zeta},Y^{c,\zeta})$, so that on the event $\{$no coalescence up to time $g_\alpha(t)\}$ the two processes, the coalescent and the corresponding semi-coalescent, in both couplings above agree for all times. Hence,

$$\begin{aligned}
&\Big|\mathbf{P}\big\{(Y^c_s)_{s\ge g_\alpha(t)}\in M\big\}-\mathbf{P}\big\{(Y^\zeta_s)_{s\ge g_\alpha(t)}\in M\big\}\Big|\\
&\quad\le 2\mathbf{P}\big\{\text{coalescence occurs before time } g_\alpha(t)\big\}\\
&\qquad+\Big|\mathbf{P}\big\{(Y^{sc}_s)_{s\ge g_\alpha(t)}\in M\big\}-\mathbf{P}\big\{(Y^{sc,\zeta}_s)_{s\ge g_\alpha(t)}\in M\big\}\Big|.
\end{aligned}\tag{4.13}$$

The claim follows immediately from the previous observations and from the fact

$$\mathbf{P}\big\{\text{coalescence occurs before time } g_\alpha(t)\big\}\underset{t\to\infty}{\longrightarrow}0,\tag{4.14}$$

which is a direct consequence of (4.3). □

4.4. **Monotonicity and consequences.** Recall the set of marked partitions $\Pi^{\mathcal{I},\mathbb{Z}^2}$ from (2.13). It is convenient to introduce a partial order "$\le$" on $\Pi^{\mathcal{I},\mathbb{Z}^2}$. Let for $\mathcal{P}_1,\mathcal{P}_2\in\Pi^{\mathcal{I},\mathbb{Z}^2}$,

$$\mathcal{P}_1\le\mathcal{P}_2\tag{4.15}$$

iff for each $g\in\mathbb{Z}^2$ the number of partition elements in $\mathcal{P}_1$ with mark g is bounded above by the number of partition elements in $\mathcal{P}_2$ with mark g. For brevity reasons, we will often omit from (4.15) the dependence on the location processes when evident from the context, so we will write

$$C_1\le C_2\tag{4.16}$$

to mean $(C_1,L_1)\le(C_2,L_2)$.

Remark 4.2. Note that if $\mathcal{P}_1\le\mathcal{P}_2$, one can easily construct a coupling $((C^1_s,L^1_s),(C^2_s,L^2_s))_{s\ge0}$ of the spatial coalescents where $(C^j_0,L^j_0)=\mathcal{P}_j$, $j=1,2$, such that $C^1_s\le C^2_s$, for all $s\ge0$, almost surely. □

Suppose that $f:\Pi^{\mathcal{I},\mathbb{Z}^2}\to\mathbb{R}$ is non-decreasing, and let $g:[0,\infty)\to(0,\infty)$. For $a,b\in[-\infty,\infty]$ consider asymptotic behavior(s) of the type

$$\limsup(\liminf)_{t\to\infty}\frac{f(C_t,L_t)}{g(t)}=a,\quad \limsup(\liminf)_{t\to\infty}\frac{E(f(C_t,L_t))}{g(t)}=b.\tag{4.17}$$

An important observation is the next easy consequence of monotonicity and Remark 4.2. Namely, if any of the four types of asymptotic behavior (4.17) holds for both spatial coalescents $(C^j_t,t\ge0)$, $j=1,3$, and if

$$\mathbf{P}\big\{C^1_0\le C^2_0\le C^3_0\big\}=1,\tag{4.18}$$

then the same asymptotic behavior holds for the spatial coalescent $(C^2_t,t\ge0)$.

Moreover, let $\mathcal{A} \subseteq \mathbb{R}$, and suppose we are given three coalescent families

$$\big\{(C_s^{j,\alpha})_{s\ge 0};\, \alpha \in \mathcal{A}, j \in \{1,2,3\}\big\}, \tag{4.19}$$

with initial states such that

$$\mathbf{P}\big\{C_0^{1,\alpha} \le C_0^{2,\alpha} \le C_0^{3,\alpha},\, \forall \alpha \in \mathcal{A}\big\} = 1. \tag{4.20}$$

In addition, assume that càdlàg path such that

$$\lim_{\alpha\to\alpha_0} (C_s^{1,\alpha})_{s\ge 0} = \lim_{\alpha\to\alpha_0} (C_s^{3,\alpha})_{s\ge 0}, \tag{4.21}$$

where the above convergence is weak convergence on $D([0,\infty),\Pi^{\mathcal{I}})$equipped with the Skorokhod topology.

Lemma 4.2. *If (4.20) and (4.21) hold, then $(C_s^{2,\alpha})_{s\ge 0}$ also converges in law as $\alpha \to \alpha_0$, and*

$$\lim_{\alpha\to\alpha_0} (C_s^{2,\alpha})_{s\ge 0} = \lim_{\alpha\to\alpha_0} (C_s^{1,\alpha})_{s\ge 0}. \tag{4.22}$$

The next result will, together with the above consequences of monotonicity, eventually be used for deducing various asymptotics for the spatial coalescent started from infinite configurations, given the results for the spatial coalescents started from finite configurations.

Let $(K_s)_{s\ge 0}$ be the Kingman coalescent.

Lemma 4.3. *For each $\delta > 0$ there exists $\rho = \rho(\delta) \in (0,\infty)$ such that*

$$\mathbf{P}\big\{\#K_\delta \ge n\big\} \le \mathbf{P}\big\{X_\rho \ge n\big\}, \tag{4.23}$$

where $X_\rho \stackrel{\mathrm{d}}{=} 1 + \mathrm{Poisson}(\rho)$. That is, $\mathbf{P}\{X_\rho = k\} = e^{-\rho}\rho^{(k-1)}/(k-1)!$, for all $k \ge 1$.

Remark 4.3. The shift by one unit is necessary here since $\mathbf{P}(K_\delta \ge 1) = 1$. □

Proof. Let $\{\Upsilon_n;\, n \ge 1\}$ be the family of independent exponential random variables where Υ_n has rate $n(n+1)/2$. Then by construction of Kingman's coalescent (see, for example, [27, 2]),

$$\mathbf{P}\{\#K_\delta > n\} = \mathbf{P}\{\sum_{k\ge n} \Upsilon_k > \delta\} \le e^{-\theta\delta}\mathbf{E}\big[e^{\theta\sum_{k\ge n}\Upsilon_k}\big], \tag{4.24}$$

for all $\theta \in \mathbb{R}$. Assume that $\theta < n(n+1)/2$, and consequently that $\mathbf{E}\big[e^{\theta\sum_{k\ge n}\Upsilon_k}\big] < \infty$.

Since

$$\begin{aligned}\mathbf{E}\big[e^{\theta\sum_{k\ge n}\Upsilon_k}\big] &= \prod_{k=n}^{\infty} \frac{\frac{(k+1)k}{2}}{\frac{(k+1)k}{2} - \theta} \\ &= \exp\Big[\sum_{k=n}^{\infty} \log\big(1 + \tfrac{\theta}{\frac{(k+1)k}{2} - \theta}\big)\Big] \\ &\le \exp\Big[\sum_{k=n}^{\infty}\Big(\frac{\theta}{\frac{(k+1)k}{2} - \theta} + O\big(\frac{\theta^2}{(\frac{(k+1)k}{2} - \theta)^2}\big)\Big)\Big],\end{aligned} \tag{4.25}$$

by (4.24)

$$\mathbf{P}\big\{\#K_\delta > n\big\} \le \exp\big[-\delta\theta + \sum_{k=n}^{\infty} \frac{\theta}{\frac{(k+1)k}{2} - \theta}\big]. \tag{4.26}$$

Plugging in, for example, $\theta = n\log^2 n$ gives

$$\mathbf{P}\big\{\#K_\delta > n\big\} = O(e^{-\delta n(\log n)^2/2}), \tag{4.27}$$

which is of a smaller order than

$$\mathbf{P}\big\{\text{Poisson}(\rho) + 1 > n\big\} \asymp C(\rho)e^{-n(\log n + O(1))}, \tag{4.28}$$

for all large n, where $O(1)$ indicates a term that stays bounded as $n \to \infty$. Since the sum of independent Poisson random variables is another Poisson random variable, we can choose ρ appropriately large so that $\mathbf{P}\big\{\#K_\delta > n\big\} \le \mathbf{P}\big\{\text{Poisson}(\rho) > n-1\big\}$, for all $n \ge 1$. □

5. Asymptotics for sparse particles

Fix throughout this section $\alpha \in (0, 1]$. Our goal in this section is to analyze the behavior of a finite coalescent with particles spaced at distance $t^{\alpha/2}$ and observed at time t^β, $\beta > \alpha$, as $t \to \infty$.

Recall the instantaneous coalescent that corresponds to the spatial coalescent with resampling rate $\gamma = \infty$. In our setting $\gamma \in (0, \infty)$ is fixed. Nevertheless, we still can rely on the "loss of the spatial structure" property of the coalescent on time scales t^β for the instantaneous coalescent with partition elements situated initially at mutual distances of order $t^{\alpha/2}$ that was exploited in [9].

Recall $\Lambda^{\alpha,t}$ from (3.1). We denote by

$$(C^\alpha_s)_{s\ge 0}, \qquad \text{and} \quad (IC^\alpha_s)_{s\ge 0} \tag{5.1}$$

the spatial coalescent and the instantaneous coalescent starting from initial configuration C^α_0 with marks contained in $\Lambda^{\alpha,t}$. Notice that t is suppressed from the notation, but this should not cause confusion.

There are classical results on $(IC^\alpha_s)_{s\ge0}$ with initially N individuals spread out in $\Lambda^{\alpha,t}$, and observed at time t^β, where $\beta > \alpha$, which we wish to recall first. Let $c > 0$ and recall $I_\alpha(c, t)$ from (4.2). The following result was proved in a beautiful paper by Cox and Griffeath [9] under the additional assumption that the underlying random walks are simple random walks: for fixed $N \in \mathbb{N}$, the initial locations $\{x_1(t), \dots, x_N(t)\}$ contained in $I_\alpha(c, t)$ and for each $\beta > \alpha$,

$$\mathcal{L}\big[\#IC^\alpha_{t^\beta}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[\#K_{\log\frac{\beta}{\alpha}}\big]. \tag{5.2}$$

We next consider the spatial (delayed) coalescent, and show the stronger form of weak convergence in two ways: (i) in the sense of path-valued random variables where β is the "time"-parameter, and (ii) accounting for the partition structure. Note that the weak convergence is done in the sense of the discrete topology.

Proposition 5.1 (Finite sparse coalescents: large time scales)**.** *Fix $N \in \mathbb{N}$, $\alpha \in (0, 1]$ and $c > 0$, and assume that $\{x_1(t), \dots, x_N(t)\} \subset \mathbb{Z}^2$ is contained in $I_\alpha(c, t)$. Let the spatial coalescent $(C^\alpha_s)_{s\ge0}$ start in $\{(\{1\}, x_1(t)), \dots, (\{N\}, x_N(t))\}$. Then*

$$\mathcal{L}\big[(C^\alpha_{t^\beta})_{\beta\in[\alpha,\infty)}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(K^N_{\log(\frac{\beta}{\alpha})})_{\beta\in[\alpha,\infty)}\big], \tag{5.3}$$

where $(K^N_t)_{t\ge0}$ is the Kingman coalescent started in $\{\{1\}, \dots, \{N\}\}$.

The proof of this result is given in the next two subsections.

5.1. **Convergence of marginal distributions.** A key element of the proof is the following fact which we state for future reference.

Lemma 5.1 (Lemma 1 from [9]). *Fix $\alpha_0 > 0$, and $c > 0$. Let $\{(X^i_s)_{s\geq 0}; i = 1,\dots,4\}$ be a family of independent random walks with $X^i_0 = x_i$, for $i = 1,\dots,4$. Then uniformly in $\alpha \in [\alpha_0,\infty)$ and $\{x_1,\dots,x_4\} \subset \mathbb{Z}^2$ contained in $I_\alpha(c,t)$, we have*

$$\int_{t^\alpha}^{\infty} \mathrm{d}s\, \mathbf{P}\big(\{X^1_s = X^2_s\} \cap \{X^1_s, X^3_s, X^4_s \text{ not contained in } I_1(4c,s)\}\big) \underset{t\to\infty}{\longrightarrow} 0. \tag{5.4}$$

Remark 5.1. In the setting of [9] the walks are simple symmetric walks, but the proof of the corresponding lemma is more general, depending solely on the uniform bound

$$\mathbf{P}\{X_s = y\} \leq \frac{c}{s}, \tag{5.5}$$

for all $y \in \mathbb{Z}^2$, and $s \geq 0$. It therefore applies to our setting (see [34]). □

The first major step to prove Proposition 5.1 is to show:

Lemma 5.2 (Finite sparse coalescents: convergence of marginals). *Fix $N \in \mathbb{N}$, $\alpha \in (0,1]$ and $c > 0$, and assume that $\{x_1(t),\dots,x_N(t)\} \subset \mathbb{Z}^2$ is contained in $I_\alpha(c,t)$. Let the spatial coalescent $(C^\alpha_s)_{s\geq 0}$ start in $\{(\{1\},x_1(t)),\dots,(\{N\},x_N(t))\}$. Then for all $\beta > \alpha$,*

$$\mathcal{L}\big[\#C^\alpha_{t^\beta}\big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[\#K^N_{\log(\frac{\beta}{\alpha})}\big], \tag{5.6}$$

where $(K^N_s)_{s\geq 0}$ is the Kingman coalescent started in $\{\{1\},\dots,\{N\}\}$.

Proof. The argument makes use of an obvious coupling of $(C^\alpha_\cdot, L^\alpha_\cdot)$ and $(IC^\alpha_\cdot, IL^\alpha_\cdot)$ where $IC^\alpha_0 := C^\alpha_0$. We proceed by induction on $N \in \mathbb{N}$.

We start with $N = 2$. Put

$$\tau'_1(t) := \inf\big\{s > 0 : \#IC^\alpha_s = 1\big\}, \tag{5.7}$$

and set $C^\alpha_s := IC^\alpha_s$, for all $s \in [0,\tau'_1(t)]$. Define $(C^\alpha_s)_{s>\tau'_1(t)}$ in a standard way, using additional (independent) randomness. Let then

$$\tau_1(t) := \tau^{\alpha,t}_1 := \inf\big\{s > 0 : \#C^\alpha_s = 1\big\}, \tag{5.8}$$

so that $\tau'_1(t)$ and $\tau_1(t)$ are the coalescence times of the two particles in IC^α, and C^α, respectively. Then clearly

$$\tau'_1(t) \leq \tau_1(t) \leq \tau'_1(t) + \sum_{i=0}^{G} \tau^0_i \tag{5.9}$$

where G has shifted geometric distribution with success probability $\gamma/(2+\gamma)$, i.e., $\mathbf{P}\{G \geq m\} = (2/(2+\gamma))^{m-1}$, for all $m \geq 1$, τ^0_i, $i \geq 1$, is distributed as the length of the (almost surely finite) excursion away from 0 for the underlying migration walk, and where the family $\{\tau'_1(t), G, \{\tau^0_i, i \geq 0\}\}$ is an independent family of random variables.

The result of Cox and Griffeath discussed above is based on the Erdös-Taylor asymptotics (4.1) and stronger estimates of a similar type. In particular, we rewrite (4.1) in the current setting, where $\beta > \alpha$ and the random walk is twice as fast as the simple one, as

$$\mathbf{P}\big\{\tau'_1(t) > t^\beta/2\big\} \underset{t\to\infty}{\longrightarrow} \frac{\alpha}{\beta}. \tag{5.10}$$

Note that (5.10) can be restated as the following convergence in distribution: for all $u \geq 0$,

$$\lim_{t\to\infty} \mathbf{P}\Big\{ \log\big(\frac{\log \tau_1'(t)}{\alpha \log t}\big) < u \Big\} = 1 - e^{-u}. \tag{5.11}$$

We would like to show the same convergence holds with $\tau_1(t)$ in place of $\tau_1'(t)$. Due to (5.9) it suffices to show that, as $t \to \infty$, with overwhelming probability,

$$\sum_{i=0}^{G} \tau_i^0 \leq \tau_1'(t), \tag{5.12}$$

since then $\log(\tau_1'(t) + \sum_{i=0}^{G} \tau_i^0) \leq \log \tau_1'(t) + \log 2$, and $\log 2/\log t$ becomes negligible in the limit. Since $\sum_{i=0}^{G} \tau_i^0 < \infty$, almost surely, and $\tau_1'(t) \to \infty$, as $t \to \infty$, in probability, (5.12) trivially follows, and we have

$$\lim_{t\to\infty} \mathbf{P}\Big\{ \log\big(\frac{\log \tau_1(t)}{\alpha \log t}\big) < u \Big\} = 1 - e^{-u}, \tag{5.13}$$

for all $u \geq 0$.

Now note that for $N > 2$ and for $\beta \geq \alpha$, using analogous coupling of $(C^\alpha_\cdot, L^\alpha_\cdot)$ and $(IC^\alpha_\cdot, IL^\alpha_\cdot)$ up to the first coalescence time $\tau_{N-1}'(t)$ in IC^α,

$$\lim_{t\to\infty} \mathbf{P}\big\{\#C^\alpha_{t^\beta} = N\big\} = \lim_{t\to\infty} \mathbf{P}\big\{\#IC^\alpha_{t^\beta} = N\big\} = \Big(\frac{\alpha}{\beta}\Big)^{\binom{N}{2}}. \tag{5.14}$$

where the second limit above was evaluated in Proposition 2 of [9]. Moreover, if

$$\tau_{N-1}(t) := \inf\big\{s > 0 : \#C^\alpha_s = N-1\big\}, \tag{5.15}$$

due to the fact that $|\log \tau_{N-1}'(t) - \log \tau_{N-1}(t)| \to 0$, as $t \to \infty$, almost surely (argue as for (5.12) above), the induction step in the proof of [9] Theorem 3 can be carried out verbatim. The details are tedious, so we omit them, and state instead that

$$p_{N,k}(\alpha/\beta) := \lim_{t\to\infty} \mathbf{P}\big\{\#C^\alpha_{t^\beta} = k\big\} \tag{5.16}$$

satisfies the recursion of [9] Theorem 3,

$$p_{N+1,k}(\frac{1}{s}) = \binom{N+1}{2} \int_1^s \mathrm{d}y\, y^{-\binom{N+1}{2}-1} p_{N,k}(y/s), \tag{5.17}$$

for all $s \geq 1$ and $1 \leq k \leq N+1$.

Since the initial conditions (5.13), (5.14) to the recursion are identical to those in Theorem 3 in [9], as argued above, the solution is the same, and so we have verified that for each $\beta > \alpha$ and each $k \in \{1, \ldots, N\}$,

$$\lim_{t\to\infty} \mathbf{P}\big\{\#C^\alpha_{t^\beta} = k\big\} = \lim_{t\to\infty} \mathbf{P}\big\{\#IC^\alpha_{t^\beta} = k\big\} = \mathbf{P}\big\{\#K^N_{\log(\frac{\beta}{\alpha})} = k\big\}, \tag{5.18}$$

where the last identity was again obtained in [9]. □

5.2. **Convergence in path space.** In order to show path convergence of $(\#C^{\alpha}_{t^{\beta}})_{\beta\geq\alpha}$ to $(\#K_{\log\beta/\alpha})_{\beta\geq\alpha}$ one defines a sequence of random times $\{\tau^{\alpha}_{k}(t);\, 1\leq k\leq N\}$, where for each $k\geq 1$,

$$\tau^{\alpha}_{k}(t):=\inf\big\{s\geq 0:\ \#C^{\alpha}_{s}\leq k\big\}, \tag{5.19}$$

where as usual $\tau^{\alpha}_{k}(t)=\infty$ if $\inf_{s\geq 0}\#C^{\alpha}_{s}>k$. That is, $\tau^{\alpha}_{N}=0$, and $\tau^{\alpha}_{N-1}(t)$ is the first coalescence time, (also denoted by $\tau_{N-1}(t)$ in the proof of Lemma 5.2), $\tau^{\alpha}_{N-2}(t)$ is the second coalescence time, etc. It is not difficult to see that the arguments of the proof of Theorem 3 in [9] extend to showing that, with probability one $\#C_{\tau^{\alpha}_{k}(t)}=k$, for all $k=N-1,\dots,1$ (see also Lemma 5.1), and that with respect to convergence in probability,

$$\lim_{t\to\infty}\frac{\tau^{\alpha}_{k}(t)}{\tau^{\alpha}_{k-1}(t)}=0, \tag{5.20}$$

for each $k\geq 2$. (Note here that the remaining k partition elements are spread out). Moreover, the following joint convergence in distribution holds

$$\begin{aligned}&\Big(\log\big(\tfrac{\log(\tau^{\alpha}_{N-1}(t))}{\alpha\log t}\big),\log\big(\tfrac{\log(\tau^{\alpha}_{N-2}(t)-\tau^{\alpha}_{N-1}(t))}{\log(\tau^{\alpha}_{N-1}(t))}\big),\dots,\log\big(\tfrac{\log(\tau^{\alpha}_{1}(t)-\tau^{\alpha}_{2}(t))}{\log(\tau^{\alpha}_{2}(t))}\big)\Big)\\&\underset{t\to\infty}{\Longrightarrow}\big(U_{N-1},U_{N-2},..,U_{1}\big),\end{aligned} \tag{5.21}$$

where $\{U_i;\, i=1,\dots,N-1\}$ is a family of independent random variables such that for all $i=1,\dots,N-1$, U_i has the rate $\binom{i+1}{2}$ exponential distribution. Now (5.20) and (5.21) imply the convergence of random vectors

$$\begin{aligned}&\Big(\log\big(\tfrac{\log(\tau^{\alpha}_{N-1}(t))}{\alpha\log t}\big),\log\big(\tfrac{\log(\tau^{\alpha}_{N-2}(t))}{\alpha\log t}\big)-\log\big(\tfrac{\log(\tau^{\alpha}_{N-1}(t))}{\alpha\log t}\big),\dots,\log\big(\tfrac{\log(\tau^{\alpha}_{1}(t))}{\alpha\log t}\big)-\log\big(\tfrac{\log(\tau^{\alpha}_{2}(t))}{\alpha\log t}\big)\Big)\\&\underset{t\to\infty}{\Longrightarrow}\big(U_{N-1},U_{N-2},\dots,U_{1}\big).\end{aligned} \tag{5.22}$$

Since

$$\#C^{\alpha}_{t^{\beta}}=1_{\big[\frac{\log(\tau^{\alpha}_{1}(t))}{\alpha\log t},\infty\big)}\big(\tfrac{\beta}{\alpha}\big)+\sum_{k=2}^{N}k1_{\big[\frac{\log(\tau^{\alpha}_{k}(t))}{\alpha\log t},\frac{\log(\tau^{\alpha}_{k-1}(t))}{\alpha\log t}\big)}\big(\tfrac{\beta}{\alpha}\big) \tag{5.23}$$

and with $\bar{U}_k=exp(U_N+\dots+U_k)$,

$$\#K^{N}_{\log(\beta/\alpha)}=1_{\big[\bar{U}_1,\infty\big)}\big(\tfrac{\beta}{\alpha}\big)+\sum_{k=2}^{N}k1_{\big[\bar{U}_k,\bar{U}_{k-1}\big)}\big(\tfrac{\beta}{\alpha}\big), \tag{5.24}$$

it immediately follows that the process $(\#C^{\alpha}_{t^{\beta}})_{\beta\geq\alpha}$ converges in the sense of Skorokhod topology to the process $(\#K^{N}_{\log\frac{\beta}{\alpha}})_{\beta\geq\alpha}$, as $t\to\infty$.

In order to upgrade the above convergence to the one on the level of partitions, as stated in Proposition 5.1, we need to make sure that for any fixed N and any choice of initial locations $x_1,\dots,x_N$ contained in $I_{\alpha}(c,t)$, asymptotically as $t\to\infty$, any two current partitions elements coalesce equally likely and independently of the coalescent time. That is,

$$\mathbf{P}\big(i,j\text{ coalesce at time }\tau^{\alpha}_{N-1}(t)|\tau^{\alpha}_{N-1}(t)\big)\underset{t\to\infty}{\longrightarrow}\binom{N}{2}^{-1}. \tag{5.25}$$

Assume without loss of generality that $i < j$. Fix $\beta > \alpha$, and let (with $C_0^\alpha := \{\{1\},\dots,\{N\}\}$ and g_α as in (4.4))

$$M_{i,j}^{\beta}(t) := \bigcup_{s\in[0,t^\beta - g_\alpha(t)]} \big\{C_{s-}^\alpha = C_0^\alpha,\, C_s^\alpha = \{i,j\} \cup C_{s-}^\alpha \setminus \{\{i\},\{j\}\}\big\}, \tag{5.26}$$

and put

$$N^\beta(t) := \bigcup_{1\le i<j\le N} M_{i,j}^\beta(t). \tag{5.27}$$

Note that the events $\{M_{i,j}^\beta(t);\, 1 \le i < j < \infty\}$ are disjoint.

Recall from the proof of Proposition 4.1 that $(Y_s^c)_{s\ge 0}$ denotes the $2N$-dimensional process (i.e. $\mathbb{Z}^{2N}$-valued), whose joint evolution is the evolution of the location processes of $(C_s^\alpha, L_s^\alpha)_{s\ge0}$ but started at time 0 in $(X^1_{g_\alpha(t)},\cdots,X^N_{g_\alpha(t)})$ where the latter are N-independent $a(\cdot,\cdot)$-random walks on $\mathbb{Z}^2$. We consider the path of Y^c after time $g_\alpha(t)$ up to time t and ask whether the coalescent with these paths in the time interval $[0, t^\beta - g_\alpha(t)]$ would have a first coalescence event, we write $(Y_s^c)_{s\ge g_\alpha(t)} \in M^\beta(t)$ for this event.

Then

$$\Big|\mathbf{P}\big\{(Y_s^c)_{s\ge g_\alpha(t)} \in N^\beta(t)\big\} - \mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le t^\beta\big\}\Big| \le \mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le g_\alpha(t)\big\}, \tag{5.28}$$

and similarly for each $i < j$,

$$\Big|\mathbf{P}\big\{(Y_s^c)_{s\ge g_\alpha(t)} \in M_{i,j}^\beta(t)\big\} - \mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le t^\beta, i \sim^{\tau_{N-1}^\alpha(t)} j\big\}\Big| \le \mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le g_\alpha(t)\big\}. \tag{5.29}$$

Proposition 4.1 together with Lemma 5.2 and (5.29) imply

$$\Big|\mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le t^\beta, 1 \sim^{\tau_{N-1}^\alpha(t)} 2\big\} - \mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le t^\beta, i \sim^{\tau_{N-1}^\alpha(t)} j\big\}\Big| \underset{t\to\infty}{\longrightarrow} 0, \tag{5.30}$$

and again due to (5.26), (5.27), and (5.28),

$$\Big|\mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le t^\beta, 1 \sim^{\tau_{N-1}^\alpha(t)} 2\big\} - \binom{N}{2}^{-1} \mathbf{P}\big\{\tau_{N-1}^\alpha(t) \le t^\beta\big\}\Big| \underset{t\to\infty}{\longrightarrow} 0, \tag{5.31}$$

which proves (5.25).

Due to the asymptotic exchangeability given by Proposition 4.1 and uniform estimates (5.4) on locations of partition elements at each coalescence time, it is easy to extend (for example by induction) (5.25) to an analogous statement at any future coalescence time. This indeed confirms that the limiting object K^N is the Kingman coalescent, since the right hand sides of (5.22) and (5.25) characterizes its law completely.

6. Asymptotics for dense particles at small times

This section concentrates on the behavior of the system for fixed $\alpha \in [0,\infty)$ at times of order only slightly larger than the area of the rectangle on which the initial configuration is supported. More precisely, we set

$$\Lambda(r) := [-r, r]^2 \cap \mathbb{Z}^2. \tag{6.1}$$

and study the corresponding restricted spatial coalescent.

6.1. **Coupled spatial coalescents and moment bound.** Here and at many other occasions it is useful to couple coalescents starting in different but comparable initial configurations. We next describe a formal setting that will be used in Sections 6, 7 and 9.

Let

$$F := \{F_z,\, z \in \mathbb{Z}^2\} \tag{6.2}$$

be a family of $\bar{N}$-valued valued random variables. We think of F_z as the number of partition elements (particles) present at site $z \in \mathbb{Z}^2$ in the coalescent at time 0. In symbols,

$$F_z := \#\{\pi \in C_0 :\, L_0(\pi) = z\}. \tag{6.3}$$

Typically we will choose the collection F such that $\sum_{z\in\mathbb{Z}^2} F_z\delta_z \in \mathcal{E}$, almost surely. In addition, for the applications we have, we often assume F to be a family of independent random variables with the same Poisson (rate $\rho \in (0,\infty)$) distribution.

Assume we are given the coupled spatial coalescents from above and recall $\{F_z;\, z \in \mathbb{Z}^2\}$ from (6.2). Assume that

$$\mathbf{E}\big[F_z\big] > 0, \quad \text{and } \mathbf{Var}\big[F_z\big] < \infty, \tag{6.4}$$

for all $z \in \mathbb{Z}^2$.

Our goal is to show next that the sparse initial configurations necessary for the results of the previous section arise if the coalescent is started in the torus $\Lambda^{\alpha,t} = \Lambda(t^{\alpha/2})$, and observed at time $t^{\alpha'}$ for $\alpha' > \alpha$ and α' approaching α.

We will rely on the following tightness result for $C^{\alpha,t}_{t^\beta}$ started in $\Lambda^{\alpha,t}$, whose somewhat technical proof is given in Section 10.

Proposition 6.1 (Uniformly bounded expectation on logarithmic scale)**.** *There are finite constants M and t_0 such that for all $t \ge t_0$, satisfying $\alpha \in (0,\infty)$, and $\beta \in (\alpha, 3\alpha/2)$,*

$$\mathbf{E}\big[\#C^{\alpha,t}_{t^\beta}\big] \le M\,\Big\{\frac{\alpha}{2(\beta-\alpha)} \vee \frac{\mathbf{E}\big[\#C^{\alpha,t}_{2}\big]}{t^\alpha} \vee 1\Big\}. \tag{6.5}$$

Remark 6.1. The $C^{\alpha,t}_2$ in (6.5) denotes the coalescent partition evaluated at time 2, any finite positive time could be taken instead of 2 here, and the two constants t_0 and M would change accordingly. Our special choice of the time point 2 is convenient from the perspective of the time discretization used in the proof of Proposition 6.1 (compare with (10.3)). □

6.2. **Consequences of the expectation bound: Tightness.** Assume that

$$\rho := \limsup_{t\to\infty} \sup_{z\in\Lambda^{1,t}} \mathbf{E}\big[F_z\big] < \infty. \tag{6.6}$$

The next result states that as $t \to \infty$ the coalescents in remain finite and localized in certain boxes.

Proposition 6.2 (The asymptotically infinite spatial case: small time scales)**.** *Consider the coalescent restricted to $\Lambda^{\alpha,t}$. Let t_0 be as specified in Proposition 6.1. Then the following holds.*

(a) *For each fixed $\alpha' > \alpha$, there exists a sequence $(a_N)_{N\in\mathbb{N}} \uparrow 1$ such that for all $N \in \mathbb{N}$,*

$$\inf_{t\geq t_0} \mathbf{P}\big\{\#C^{\alpha,t}_{t^{\alpha'}} \leq N\big\} \geq a_N, \tag{6.7}$$

and ($\sim t$ denoting the equivalence relation w.r.t. time t partition)

$$\liminf_{t\to\infty} \mathbf{P}\big\{ \max_i \parallel L^{\alpha,t}_{t^{\alpha'}}(\{i \sim^{t^{\alpha'}}\}) \parallel\leq t^{\alpha'/2}\log t\big\} \geq a_N. \tag{6.8}$$

(b) *For each fixed $\alpha' > \alpha$ and each $N \in \mathbb{N}, L^{\alpha,t}_{t^{\alpha'}}$, the set of all marks at time $t^{\alpha'}$ and $I_\alpha(1,t)$ as in (4.2) we have:*

$$\mathbf{P}\big(\{L^{\alpha,t}_{t^{\alpha'}} \text{ is contained in } I_{\alpha'}(1,t)\}\big|\#C^{\alpha,t}_{t^{\alpha'}} \leq N\big) \underset{t\to\infty}{\longrightarrow} 1. \tag{6.9}$$

(c) *For each $N \in \mathbb{N}$,*

$$\lim_{\alpha'\downarrow\alpha} \liminf_{t\to\infty} \mathbf{P}\big\{\#C^{\alpha,t}_{t^{\alpha'}} \geq N\big\} = 1. \tag{6.10}$$

Proof. Assertion (6.7) is now an immediate consequence of Proposition 6.1 and the Markov inequality.

Assertion (6.8) follows from a large deviation estimate. It will be convenient here and below to set

$$\alpha^* = \alpha^*(\alpha,\alpha') := (\alpha+\alpha')/2. \tag{6.11}$$

Let $\{(X^i_s)_{s\geq 0};\ i \geq 1\}$ be an infinite collection of independent random walks with kernel $a(\cdot,\cdot)$ such that the initial locations $\{X^i_0;\ i\geq 1\}$ are distributed as the location process $L^{\alpha,t}_0$ of the coalescent restricted to the box $\Lambda^{1,t}$. Take $\varepsilon < (\alpha'-\alpha)/2$ so that $\alpha^*+\varepsilon < \alpha'$. Since (6.6) holds, we have that $\#C^{\alpha,t}_0$ is bounded by $2\rho t^\alpha$ with overwhelming probability. Due to a large deviation estimate (for example, (10.8) is more than needed here)

$$\lim_{t\to\infty} \mathbf{P}\big\{ \max \parallel X^i_{t^{\alpha^*}} \parallel> t^{(\alpha^*+\varepsilon)/2} : i \in \{1,\cdots,\lfloor 2\rho t^\alpha\rfloor\}\big\} = 0, \tag{6.12}$$

and hence

$$\lim_{t\to\infty} \mathbf{P}\big\{ \max \parallel \{X^i_{t^{\alpha^*}} \parallel> t^{(\alpha^*+\varepsilon)/2} : i \in \{1,\cdots,\#C^\alpha, t_0\}\big\} = 0. \tag{6.13}$$

Therefore

$$\lim_{t\to\infty} \mathbf{P}\big\{ \max_i \parallel L^{\alpha,t}_{t^{\alpha'}}(\{i\sim^{t^{\alpha'}}\}) \parallel> t^{(\alpha'+\varepsilon)/2}\big\} = 0. \tag{6.14}$$

In order to get (6.8) from (6.14) we use (6.7) together with the fact that during the remaining time $t^{\alpha'}-t^{\alpha^*}$ none of the finitely many partition classes reaches distance larger than $t^{\alpha'/2}\log t$, with overwhelming probability.

In order to prove (6.9) fix $\alpha' > \alpha$.

Fix $N \geq 1$, and note that (6.7) implies the uniform lower bound $\bar{p}$ on the probability of $\{\#C^{\alpha,t}_{t^{\alpha'}} \leq N\}$. So (6.9) will follow provided we show that for any $\varepsilon > 0$ we have

$$\mathbf{P}\big\{L^{\alpha,t}_{t^{\alpha'}} \text{ is contained in } I_{\alpha'}(1,t)\big\} \geq 1-\varepsilon\bar{p}. \tag{6.15}$$

Again due to part (a), it is possible to pick M_ε so that $C^{\alpha,t}_{t^{\alpha^*}}$ contains at most M_ε equivalence classes, with probability higher than $1-\bar{p}\varepsilon/3$, and such that any pair of them is at mutual distance smaller than $2t^{\alpha^*/2}\log t$ with probability higher than $1-\bar{p}\varepsilon/3$. During the remaining time interval $(t^{\alpha^*}, t^{\alpha'}]$ of length $t^{\alpha'}-t^{\alpha^*}$, which is of order $t^{\alpha'}$, each pair of non-coalescing walks (out of at most $\binom{M_\varepsilon}{2}$ many pairs) achieves, with overwhelming probability, a mutual distance of order $N(0,1)\times\, 2\sigma^2 t^{\alpha'/2}$, which is with overwhelming probability in the interval $I_\alpha(1,t)$. The set of distances between pairs of elements of $C^{\alpha,t}_{t^{\alpha'}}$ is a subset of the set of distances between the pairs of above random walks. Therefore, one can choose t large enough so that

$$\begin{aligned}(6.16)\qquad &\mathbf{P}\big\{L^{\alpha,t}_{t^{\alpha'}} \text{ is contained in } I_\alpha(1,t)\}\big|\#C^{\alpha,t}_{t^{\alpha^*}} \le M_\varepsilon, \max_i \parallel L^{\alpha,t}_{t^{\alpha'}}(\{i\sim^{t^{\alpha'}}\}) \parallel \le 2t^{\alpha^*/2}/logt\big\}\\ &\ge 1-\bar{p}\varepsilon/3,\end{aligned}$$

so (6.15), and therefore (6.9) holds.

It still remains to prove (6.10). Fix $\alpha' > \alpha > 0$. Note that for any N particles started at locations $x_1,\dots,x_N$ contained in $I_\alpha(1,t)$ we have by convergence of the first component in (5.14) that

$$\mathbf{P}\big\{\text{no coalescence by time } t^{\alpha'}\big\} \underset{t\to\infty}{\longrightarrow} (\alpha/\alpha')^{\binom{N}{2}}. \tag{6.17}$$

For fixed N first choose large t so that it is possible to find N particles from the initial configuration at time 0 with locations contained in $I_\alpha(1,t)$, and then note that as $\alpha'\downarrow\alpha$ the right hand side above converges to 1. □

7. Large time-space scale asymptotics of coalescent

In this section we combine the results of Sections 4, 5 and 6 to prove Theorems 1 through 3.

7.1. **Proof of Theorem 1.** Fix $\alpha > 0$ and a marked partition $(C_0, L_0) \in \hat{\Pi}^{\mathcal{I},\mathbb{Z}^2}$ such that (2.16), (3.4) and (3.5) hold. Denote by $C^{\alpha,t}$ the spatial coalescent which starts in (C_0, L_0) restricted to those partition elements which are initially in $\Lambda^{\alpha,t}$ as defined in (3.1), where $t > 0$.

Next fix $1 \ge \alpha' > \alpha > 0$. By (6.10), for all $N \in \mathbb{N}$ and $\varepsilon \in (0, \alpha' - \alpha)$ there exists an $\alpha^* = \alpha^*(\varepsilon) \in (\alpha, \alpha+\varepsilon) \subset (\alpha, \alpha')$ and $t_1 = t_1(N,\varepsilon)$ such that for all $t \ge t_1$,

$$\mathbf{P}\big\{\#C^{\alpha,t}_{t^{\alpha^*}} \ge N\big\} \ge 1-\varepsilon. \tag{7.1}$$

From now on assume that $t \ge t_0$ where t_0 is specified as in Proposition 6.1. Proposition 6.2 implies that with probability tending to 1, as $t\to\infty$, the configuration $C^{\alpha,t}_{t^{\alpha^*}}$ has finitely many particles in locations contained in $I_{\alpha^*}(1,t)$.

Put

$$n^{\alpha^*,t} := \#C^{\alpha,t}_{t^{\alpha^*}}. \tag{7.2}$$

Then Proposition 5.1 joint with Proposition 6.2 (a) and (b), yield

$$d_{\mathrm{Pr}}\Big(\mathcal{L}\big[(\#C^{\alpha,t}_{t^\beta})_{\beta\in[\alpha^*,\infty)}\big], \mathcal{L}\big[(\#K^{n^{\alpha^*,t}}_{\log(\beta/\alpha^*)})_{\beta\in[\alpha^*,\infty)}\big]\Big) \underset{t\to\infty}{\longrightarrow} 0, \tag{7.3}$$

where d_{Pr} is the Prohorov metric which is known to metrize the weak topology (see, for example, [16]). Moreover, for a random variable n and $s \ge 0$, $\#K^n_s$ is a random variable which, given $n = k$, is distributed as the Kingman coalescent started in $\{\{1\},\{2\},\dots,\{k\}\}$ and evaluated at time s.

Recall that we denote by $K.$ the Kingman coalescent started from the trivial infinite partition $\{\{i\} : i \in \mathbb{N}\}$. Easy properties of the Kingman coalescent guarantee that for all $\delta > 0$,

$$(\#K_s^n)_{s \geq \delta} \underset{n\to\infty}{\Longrightarrow} (\#K_s)_{s \geq \delta}, \tag{7.4}$$

and

$$(\#K_{s+u})_{s \geq \delta} \underset{u\to 0}{\Longrightarrow} (\#K_s)_{s \geq \delta}. \tag{7.5}$$

Note that Proposition 6.2(a) insures that the family $\{n^{\alpha^*,t}; t \geq t_0\}$ is tight. Choose $(t_m) \to \infty$ and n^{α^*} such that $n^{\alpha^*,t_m} \to n^{\alpha^*}$, as $m \to \infty$. Then n^{α^*} is a finite random variable and

$$(\#C^{\alpha,t_m}_{t_m^\beta})_{\beta\in[\alpha',\infty)} \underset{m\to\infty}{\Longrightarrow} (\#K^{n_{\alpha^*}}_{\log(\beta/\alpha^*)})_{\beta\in[\alpha',\infty)}. \tag{7.6}$$

The left hand side of (7.6) does not depend on ε. By (7.1), (7.4) and (7.5) we have, after letting $\varepsilon \to 0$,

$$(\#C^{\alpha,t_m}_{t_m^\beta})_{\beta\in[\alpha',\infty)} \underset{m\to\infty}{\Longrightarrow} (\#K_{\log(\beta/\alpha)})_{\beta\in[\alpha',\infty)}. \tag{7.7}$$

Since one obtains the same limit regardless of the choice of the subsequence (t_m), the statement of the theorem follows.

7.2. **Proof of Theorem 2.** In this subsection we assume that $\{F_z^\rho; z \in \Lambda^{1,t}, \rho \geq 1\}$ is for fixed ρ a family of independent identically distributed random variables with Poisson(ρ) distribution. In fact, due to thinning and superposition properties of the Poisson process on the line we can consider a coupling such of the families for different ρ that if $\rho_1 \leq \rho_2$ then

$$F_z^{\rho_1} \leq F_z^{\rho_2}, \tag{7.8}$$

for all $z \in \Lambda^{1,t}$.

Due to this coupling and the monotonicity properties collected in Subsection 4.4,

$$(C_s^{\alpha,t,\rho}, L_s^{\alpha,t,\rho})_{s\geq 0} \underset{\rho\to\infty}{\longrightarrow} (C_s^{\alpha,t,\infty}, L_s^{\alpha,t,\infty})_{s\geq 0}, \tag{7.9}$$

here convergence is meant in the sense of convergence defined (2.14).

The goal of this subsection is to show that the results obtained in Subsection 6.2 hold in the limit $\rho \to \infty$.

Fix $\delta > 0$ and define the spatial coalescent thinned out by those particles which were attempted to jump in the time period $[0, \delta]$ as follows: Assume we are given a realization of $(C_s, L_s)_{s\geq 0}$ with $C_0 := \{\{i\}; i \in \mathcal{I}\}$ which is constructed from collections of independent random walks for migration $\{(L_s^i)_{s\geq 0}, i \in \mathcal{I}\}$ and Poisson point processes $\{Q^{i,j}; i < j\}$. We construct now for each $\delta > 0$ a sub-coalescent of this spatial coalescent by first removing every individual in the original configuration for which L^i jumps before time $\delta > 0$, and then removing all empty partition elements. The sub-caolescent

$$(C_s^{\alpha,t,\infty,\delta})_{s\geq 0}. \tag{7.10}$$

is defined as the sub-coalescent of $(C_s^{\alpha,t,\infty})_{s\geq 0}$ which starts in the thinned out partition.

Notice that for each $s \geq 0$, $C_s^{\alpha,t,\infty,\delta} \longrightarrow C_s^{\alpha,t,\infty}$, as $\delta \downarrow 0$, almost surely, (for the topology used see the appendix).

Lemma 7.1 (The limit of infinite density)**.** *For each $\delta > 0$ fixed,*

$$\lim_{N\to\infty}\liminf_{t\to\infty}\mathbf{P}\big\{\#C^{\alpha,t,\infty,\delta}_{t^{\alpha'}} \le N\big\} = 1. \tag{7.11}$$

Proof. Recall from Lemma 4.3 that the number of partition elements of a Kingman coalescent can be dominated by a Poisson variable with suitably large parameter ρ_0. By monotonicity we can construct a coupling

$$\Big(\big(C^{\alpha,t,\infty,\delta}_s, L^{\alpha,t,\infty,\delta}_s\big), \big(C^{\text{Poisson}(\rho_0)+1}_s, L^{\text{Poisson}(\rho_0)+1}_s\big)\Big)_{s\ge\delta}, \tag{7.12}$$

where $(C^{\text{Poisson}(\rho_0)+1}_s, L^{\text{Poisson}(\rho_0)+1}_s)_{s\ge 0}$ is started from the initial configuration where $\{F_z;\, z \in \Lambda^{1,t}\}$ is a family of independent random variables which equal in distribution one plus a rate ρ_0 Poisson distributed random variable such that $C^{\alpha,t,\infty,\delta}_s \le C^{\text{Poisson}(\rho_0)+1}_s$, for all $s \ge \delta$, almost surely. The statement now follows from Proposition 6.2(a) applied to $(C^{\text{Poisson}(\rho_0)+1}_s)_{s\ge 0}$. □

In addition, notice that $\mathbf{P}\{\#C^{\alpha,t,\infty,\delta}_\delta \ge 1\} = 1$, so if $(C^1_s, L^1_s)_{s\ge\delta}$ is the family of spatial coalescents started with 1 particle at each site of $\Lambda(t^{1/2})$, we have

$$C^1_\delta \le C^{\alpha,t,\infty,\delta}_\delta \le C^{\text{Poisson}(\rho)+1}_\delta. \tag{7.13}$$

The extension of Theorem 1 (in Proposition 2.2) proved in Subsection 7.1 clearly applies to both the left-most and the right-most family of coalescents. Therefore, by Lemma 4.2, for fixed $\alpha' > \alpha > 0$,

$$\big(\#C^{\alpha,t,\infty,\delta}_{t^\beta}\big)_{\beta\in[\alpha',\infty)} \underset{t\to\infty}{\Longrightarrow} \big(\#K_{\log(\beta/\alpha)}\big)_{\beta\in[\alpha',\infty)}. \tag{7.14}$$

Thus Theorem 2 holds with

$$\tilde{\mathcal{I}} := \bigcap_{\delta>0}\big\{i \in \mathcal{I} :\, L^i_s \ne L^i_0 \text{ for some } s \in [0,\delta]\big\}. \tag{7.15}$$

7.3. **Proof of Theorem 3.** The proof of Theorem 3 makes use of a convergence result stated in Theorem 1 in [12], which applies in a much more general setting than ours. For the benefit of the reader, we will rephrase it in our setting.

Lemma 7.2 (Donnelly, 1991)**.** *Suppose $\{(B^N_s)_{s\ge 0};\, N \ge 1\}$ is a family of $D([0,\infty),\mathbb{N})$-valued random variables which satisfy the following three assumptions:*

(A1) *For all $N \in \mathbb{N}$, $l \ge n \in \mathbb{N}$, $s \ge \alpha$ and $y \ge 1$,*

$$\mathbf{P}\big(\inf_{u\in[\alpha,s]} B^N_u \le y | B^N_\alpha = l\big) \le \mathbf{P}\big(\inf_{u\in[\alpha,s]} B^N_u \le y | B^N_\alpha = n\big). \tag{7.16}$$

(A2) *For all $n \in \mathbb{N}$,*

$$\mathcal{L}\big[(B^N_u)_{u\ge\alpha} | B^N_\alpha = n\big] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K^n_{\log(u/\alpha)})_{u\ge\alpha}\big], \tag{7.17}$$

(A3) *Suppose we have a sequence $(n_M) \to \infty$, such that for each $u > \alpha$,*

$$\lim_{M\to\infty}\lim_{N\to\infty}\mathbf{P}\big(B^N_u \le M | B^N_\alpha = n_M\big) = 1. \tag{7.18}$$

Then

$$\mathcal{L}\big[(B^N_u)_{u\ge\alpha} | B^N_\alpha = n_N\big] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}\big[(\#K_{\log(u/\alpha)})_{u\ge\alpha}\big]. \tag{7.19}$$

Proof of Theorem 3. (i) Take a subsequence $(t_N)\uparrow\infty$, and let

$$s_N := \#\Lambda^{\alpha,t_N}. \tag{7.20}$$

We consider first a *special* case. Draw Bin_N according to the Binomial distribution with parameters s_N and $p\in(0,1]$ or the Poisson distribution with parameter $s_N\cdot\rho$. Given $\mathrm{Bin}_N=k$, place k particles uniformly without and with replacement at k positions in Λ^{α,t_N}. Notice that the random configurations obtained this way will be equal in law to C_0^{α,t_N} under the assumption that $\{F_z;\, z\in\Lambda^{\alpha,t_N}\}$ are independent and identically distributed random variables with the Bernoulli (parameter p) or with the Poisson(ρ) distribution, respectively.

Put for all $u\ge\alpha$,

$$B_u^N := \#C^{\alpha,t_N}_{(t_N)^u}. \tag{7.21}$$

By Lemma 5.2, given that $\mathrm{Bin}_N=k$, with probability tending to 1, $B_\alpha^N=k$. The advantage of the above construction(s) is that the assumption (A1) is automatically satisfied provided we keep the same algorithm for "positioning the k particles in Λ^{α,t_N}", for all $k\in\mathbb{N}$, i.e., provided that for each $k\in\mathbb{N}$ and all $l<k$, the first l points in C_0^{α,t_N} given $\mathrm{Bin}_N=l$ match those in C_0^{α,t_N} given $\mathrm{Bin}_N=k$. The assumptions (A2) and (A3) (provided that $n_N=O(s_N)$) are implied by Lemma 5.2 and (6.7), respectively. Therefore, (7.19) holds in the (special) Binomial case for any $p\in(0,1]$ and any sequence $n_N\le s_N$ going to ∞. Similarly, (7.19) holds in the (special) Poisson case for any $\rho\in(0,\infty)$ and any sequence $n_N=O(s_N)$.
In particular, if $p=1$ and $n_N=s_N$ (almost surely) then

$$\big(\#C^{\alpha,t_N}_{{t_N}^\beta}\big)_{\beta\ge\alpha} \underset{N\to\infty}{\Longrightarrow} \big(\#K_{\log(\beta/\alpha)}\big)_{\beta\ge\alpha}. \tag{7.22}$$

Since the limit is uniform in the choice of the subsequence $t_N\to\infty$, we conclude the statement of the theorem in this case.

The general Bernoulli(p) case can be dealt with similarly as the general Poisson(ρ) case, as we explain next. Fix $\rho\in(0,\infty)$ and note that (7.19) holds both with $n_N:=\lfloor\rho s_N/2\rfloor$ and with $n_N:=\lfloor 2\rho s_N\rfloor$. Since with probability tending to 1, the Poisson (ρs_N) distributed random variable Bin_N satisfies

$$\lfloor\rho s_N/2\rfloor \le \mathrm{Bin}_N \le \lfloor 2\rho s_N\rfloor, \tag{7.23}$$

we can apply Lemma 4.2 to conclude the needed statement as done before.

(ii) Note that due to part (i), the family of processes (t_N is a sequence with $t_N\uparrow\infty$ as $N\to\infty$)

$$\big(\#C^{\alpha,t_N}_{t_N^\beta}\big)_{\beta\in[\alpha,\infty)}, \tag{7.24}$$

where the family $\{F_z;\, z\in\Lambda^{1,t}\}$ is drawn from the "Poisson(ρ)+1" distribution is tight in $D([\alpha,\infty),\bar{\mathbb{N}})$ since we can sandwich it between from below the case where we start with exactly 1 particle per site (Bernoulli with $p=1$) and from above with the independent sum of two spatial coalescent processes one started in Poisson(ρ) and the other one with exactly 1 particle per site (Bernoulli with $p=1$). Here we use monotonicity in β for every N. Moreover, the process $(\#K_{\log(\beta/\alpha)})_{\beta\in[\alpha,\infty)}$ is the only possible (subsequential) limit due to Theorem 1. Therefore, applying monotonicity and using (7.13) as in the proof of Theorem 2 implies the statement. □

8. Proof of Theorem 4

Fix an enumeration $\mathcal{I} := \{i_1, i_2, ...\}$ and $\alpha > 0$. Recall from (A.3) the topology on the state space of partitions of $\mathcal{I}$.

Assume that $\rho > 0$. For all $t > 0$, sample $\#\{i \in \mathcal{I} :\, L_0^i \in \Lambda^{t,\alpha}\}$-many individuals which are initially located in $\Lambda^{t,\alpha}$ independently and without repetition. Denote the random outcome by $(\tilde{i}_1, \tilde{i}_2, ..., \tilde{i}_{\#\Lambda^{t,\alpha}})$. Concatenate this vector by an arbitrary enumeration of the individuals in $\mathcal{I}$ which are initially located outside $\Lambda^{t,\alpha}$. Put then for all $n \in \mathbb{N}$, $\varphi_t(i_n) := \tilde{i}_n$.

Fix a finite non-empty subset $\mathcal{I}' \subset \mathcal{I}$. Then for all $t > 0$ suitably large, $\{L_0^{\varphi_t(i)};\, i \in \mathcal{I}'\} \in I_\alpha(c, t)$, almost surely. Hence by Proposition 5.1,

$$\mathcal{L}\Big[\big(\rho^{\mathbb{Z}^2}_{\mathcal{I}'}(\varphi_t^*(C^{\alpha,t}_{t^\beta}))\big)_{\beta\in[\alpha,\infty)}\Big] \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big[(\rho_{\mathcal{I}'}(K_{\log(\beta/\alpha)})_{\beta\in[\alpha,\infty)})\big]. \tag{8.1}$$

Since $\mathcal{I}'$ was chosen arbitrarily, (3.12) follows (recall the topology from Appendix A.1).

Assume next, that $\rho = \infty$, and sample now $t \cdot \#\Lambda^{t,\alpha}$-many individuals which are initially located in $\Lambda^{t,\alpha}$ independently and without repetition, and then concatenate the random outcome $(\tilde{i}_1, \tilde{i}_2, ..., \tilde{i}_{t\#\Lambda^{t,\alpha}})$ by an arbitrary enumeration of all remaining individuals in $\mathcal{I}$. The rest of the argument follows similar lines as in the proof of convergence for $\rho < \infty$.

9. Convergence on the spatial scale (Proof of Theorems 5 and 6)

In this section we prove the two results which involve the coalescent with rebirth using the results established in Sections 5 and 6.

9.1. Proof of Theorem 5.

Let $\rho \in [0, \infty]$, and consider the family $\{\rho_{\mathcal{I}^{\alpha,t}}(C^{1,t,\rho});\, \alpha \in [0, 1)\}$ from (3.14).

Since $\#K_0^{\text{birth}}[\log \alpha_l, \log \alpha]$ equals in distribution $\#K_{-\log\alpha}$, the claims hold for a fixed $\alpha \in (0, 1]$ by Theorem 1 and 2 (with $\beta = 1$). It therefore remains to first extend this convergence of one-dimensional marginal distributions to f.d.d. converge and to then establish the convergence in path space.

We begin by f.d.d.-convergence and tightness properties.

Proposition 9.1 (Partition number f.d.d. convergence)**.**

(a) (i) *If $\rho \in [0, \infty)$, then*

$$\#\rho_{\mathcal{I}^{\alpha,t}}(C_t^{1,t,\rho}) \underset{t\to\infty}{\overset{\text{f.d.d.}}{\longrightarrow}} \#K_0^{\text{birth}}[\log \alpha_l, \log \alpha]. \tag{9.1}$$

(ii) *If $\rho = \infty$, then there exists a random set $\tilde{\mathcal{I}} \subset \mathcal{I}$ with (3.6) and such that*

$$\#\rho_{\mathcal{I}^{\alpha,t}\cap\tilde{\mathcal{I}}}(C_t^{1,t,\infty}) \underset{t\to\infty}{\overset{\text{f.d.d.}}{\longrightarrow}} \#K_0^{\text{birth}}[\log \alpha_l, \log \alpha]. \tag{9.2}$$

(b) *For any $\varepsilon > 0$ and $\rho \in (0, \infty]$, the family (indexed by α and t)*

$$\Gamma := \Big\{\mathcal{L}\big[\#\rho_{\mathcal{I}^{\alpha,t}}(C_t^{1,t,\rho})\big];\, \alpha \in [0, 1-\varepsilon], t > 0\Big\} \tag{9.3}$$

is tight.

Remark 9.1. *Note that the generalization of the proposition in terms of the corresponding convergence of the partition structure could be formulated in the setting of finite (and sparse) initial configurations considered in the proof of (a.i) below, and proved by applying the technique of Subsection 5.2 (see also Lemma 7.3 in* [22] *or Proposition 14 in* [32]*).* □

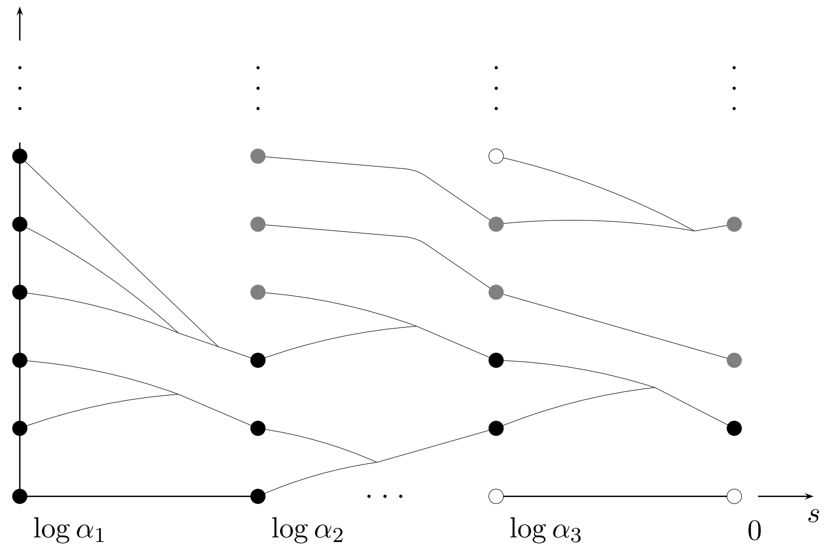

FIGURE 4. illustrates the evolution of the process $K_\cdot^{\vec{\alpha}}[\{\log \alpha_1, \dots, \log \alpha_3\}]$ to the *left* of time 0, where partition elements with birth time $\log \alpha_1$ are colored black, those with birth time $\log \alpha_2$ are colored gray, etc. For this realization we see that $N_1^{\vec{\alpha}} \geq 1$, $N_2^{\vec{\alpha}} \geq 3$, and $N_3^{\vec{\alpha}} \geq 4$.

Proof of Proposition 9.1. (a) Fix $m \in \mathbb{N}$ and $\alpha_l \leq \alpha_1 < \dots < \alpha_m \leq \alpha_u$. Consider the process $(K_0^{\text{birth}}[\log \alpha_u, \log \alpha])_{\alpha \geq \alpha_l}$ and replace in every possible partition element each element of the form (i, s) by $(i, \log \alpha_k)$ whenever $\log \alpha_{k-1} < s \leq \log \alpha_k$, $k = 1, \dots, m$. Notice that the resulting process

$$\left(K_s\big[\{\log \alpha_1, \dots, \log \alpha_m\}\big]\right)_{s \geq \log \alpha_\ell}. \tag{9.4}$$

behaves like a Kingman coalescent without rebirth during an interval of the form $[\log \alpha_{k-1}, \log \alpha_k)$, while the partition elements that were "lost" during that interval, get "reborn" at time α_k. Compare Figure 4 for an illustration.

For $k = 1, \dots, m$, denote by $\tilde{N}_k^{\vec{\alpha}}$ the total number of partition elements of $K_0^{\text{birth}}[\log \alpha_{u_k}, \log \alpha]$ with birth time *equal to or smaller than* $\log \alpha_k$, i.e.,

$$\tilde{N}_k^{\vec{\alpha}} := \# K_0\big[\{\log \alpha_1, \dots, \log \alpha_k\}\big] \tag{9.5}$$

Note that $1 \leq \tilde{N}_1^{\vec{\alpha}} \leq \tilde{N}_2^{\vec{\alpha}} \leq \dots \leq \tilde{N}_m^{\vec{\alpha}}$, almost surely. Obviously,

$$\tilde{N}^{\vec{\alpha}} = \big(\# K_0^{\text{birth}}[\log \alpha_u, \log \alpha_1], \dots, \# K_0^{\text{birth}}[\log \alpha_u, \log \alpha_m]\big). \tag{9.6}$$

(a.i) Assume that $\rho < \infty$. We want to show that

$$\big(\#\rho_{\mathcal{I}^{\alpha_1,t} \cap \tilde{\mathcal{I}}}(C_t^{1,t,\infty}), \dots, \#\rho_{\mathcal{I}^{\alpha_m,t} \cap \tilde{\mathcal{I}}}(C_t^{1,t,\infty})\big) \underset{t \to \infty}{\Longrightarrow} \big(\tilde{N}_1^{\vec{\alpha}}, \dots, \tilde{N}_m^{\vec{\alpha}}\big). \tag{9.7}$$

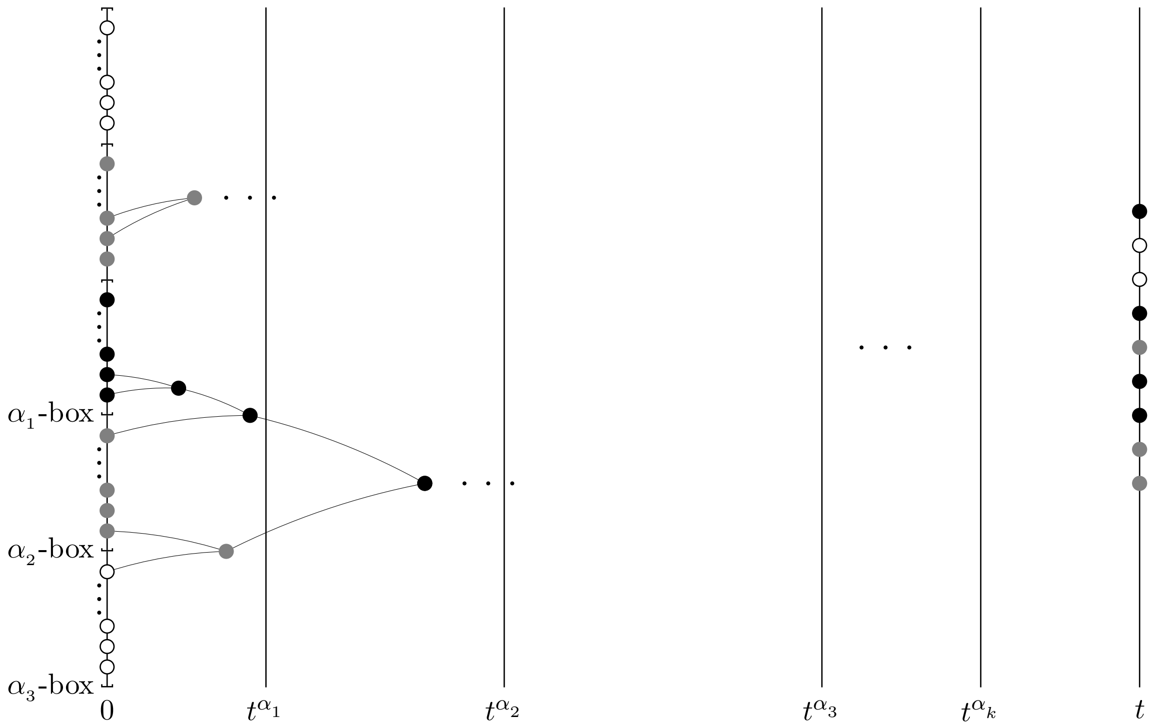

FIGURE 5. illustrates the occurrence of $\tilde{N}^{\vec{\alpha}} = (\tilde{N}_1^{\vec{\alpha}}, ..., \tilde{N}_m^{\vec{\alpha}})$ in the limit of the spatial coalescent. Notice that the colors of the particles in Figures 4 and 5 match on purpose to emphasize the correspondence between space (for the spatial coalescent) and time (for the Kingman-type coalescent with re-birth).

We will rely on Theorem 6 of [9] which gives some information for the case of the *sparse* particles and for *instantaneous* coalescence in terms of the convergence in the sense of finite dimensional distributions.

The gap between the instantaneous coalescent f.d.d. convergence case of [9] and our delayed coalescent path space convergence case is bridged as in Section 5. It would be tedious to write out (again) all the details, yet we encourage the reader to verify the steps of the argument outlined below.

Step 1 (Sparse individuals). We first treat finitely many sparse particles as initial state, where we can use some techniques from [9]. That is, consider initially *finitely many particles* (independent of t) in each of the boxes $\Lambda^{\alpha_i,t}$, $i = 1, ..., m$, such that, in analogy to the statement of Proposition 5.1, the initial positions of particles in the box $\Lambda^{\alpha_1,t}$ are contained in $I_{\alpha_1}(c,t)$ and moreover that, for each $i = 2, ..., m$, the positions of all particles initially in $\Lambda^{\alpha_i,t} \setminus \Lambda^{\alpha_{i-1},t}$, is in $I_{\alpha_i}(c,t)$. To be more concrete, assume that there are initially

$$n_i \text{ particles in } \Lambda^{\alpha_i,t} \setminus \Lambda^{\alpha_{i-1},t}, \tag{9.8}$$

for $i = 1, ..., m$ (and with $\Lambda^{\alpha_0,t} := \emptyset$). We write for the spatial coalescent with such a particular initial state

$$(C_t^{\vec{n},t}, L_t^{\vec{n},t})_{t\geq 0}. \tag{9.9}$$

In Theorem 6 in [9] it is proved for the instantaneous coalescent that

$$\begin{aligned} &\mathbf{P}\big(\#C_t^{\vec{n}} = k\big) \\ &\underset{t\to\infty}{\longrightarrow} \sum_{i_1,\cdots,i_{m-1}} \mathbf{P}(\#K^{n_1}_{\log(\alpha_2/\alpha_1)} = i_1)\cdot \mathbf{P}(\#K^{n_2+i_1}_{\log(\alpha_3/\alpha_2)} = i_2)\cdot \ldots \cdot \mathbf{P}(\#K^{n_m+i_{m-1}}_{\log(1/\alpha_m)} = k). \end{aligned} \tag{9.10}$$

It is straightforward to see that the proof of Theorem 6 in [9] actually implies a stronger statement (with a reinterpretation of formula (5.3) and (3.10) in [9]), namely that the following convergence in distribution for instantaneous coalescence:

$$\big(\#C_t^{n_1,t}, \#C_t^{(n_1,n_2)}, ..., \#C_t^{\vec{n},t}\big) \underset{t\to\infty}{\Longrightarrow} (N_1^{\vec{n}}, ..., N_m^{\vec{n}}), \tag{9.11}$$

where $N_1^{\vec{n}}$ equals in distribution $K^{n_1}_{-\log\alpha_1}$, and given $(N_1^{\vec{n}}, ..., N_l^{\vec{n}})$ for some $l = 1, ..., m-1$, $N_{l+1}^{\vec{n}}$ equals in distribution $K^{n_{l+1}+N_l^{\vec{n}}}_{-\log\alpha_{l+1}}$. The latter determines the joint distribution of a "Kingman coalescent with immigration" evaluated at time 0: we start with n_1-individuals at time $\log\alpha_1$ and evolve until time $\log\alpha_2$ where n_2 new individuals are added, then continue evolving until time $\log\alpha_3$ where n_3 new individuals are added $\cdots$, and continue until time $\log\alpha_m$ where the last immigration takes place. Then the coalescent runs until time 0, without further immigration.

The point here is that the above assumptions ensure that with overwhelming probability, for each $i = 2, ..., m$, none of the particles initially in $\Lambda(t^{\alpha_u}) \setminus \Lambda(t^{\alpha_{i-1}})$ coalesce with any other particle during the time interval $[0, g_{\alpha_i}(t)]$ (see (4.4) for the definition of $g_\alpha(t)$), while during the same time interval, on the appropriate time scale, the evolution of the partitions containing particles with initial positions in $\Lambda(t^{\alpha_{i-1}})$ is approximately that of the "Kingman coalescent with immigration", where at time $\log\alpha_j$, $j < i$, a population of size n_j is adjoined to the existing configuration. By Lemma 5.1 the partitions stay sparse with overwhelming probability, so that the asymptotic exchangeability applies, and an easy inductive argument yields the convergence in this finite setting, where the limit is the described coalescent with immigration (which is different from the limit on the r.h.s. of (9.1), since here we only have sparse individuals). Our arguments give hence a convergence statement for *instantaneous* coalescence in the sparse case. The structure of the arising coalescents is depicted in Figure 5.

As in Lemma 5.2, the convergence of Theorem 6 in [9] Theorem 6, extends to the convergence in the delayed coalescent setting. Moreover, using the asymptotic exchangeability as in Subsection 5.2, this can be extended to the convergence in path space.

Step 2 (Initially a Poisson field). In the previous step we started with a bounded number of sparse particles. If we now start in a Poisson field with intensity $\rho \in (0,\infty)$ (and restrict to particles in $\Lambda^{\alpha_i,t} \setminus \Lambda^{\alpha_{i-1},t}$), as $t\to\infty$, the number of initial particles tends to infinity. This will lead to the actual limit in (9.7).

The above mentioned "immigration" becomes infinite in the limit as $t\to\infty$, $i = 2, ..., m$. However, the reasoning of Section 7, in particular that of the proof of Theorem 1 which was based on the estimates of Proposition 6.1 and Proposition 6.2 will extend to the current setting and yield (9.1). The proof is by induction on m.

We start with $m = 2$. Consider the asymptotics of the joint distribution of

$$(\#\rho_{\mathcal{I}^{\alpha_1,t}}(C_t^{1,t,\rho}), \#\rho_{\mathcal{I}^{\alpha_2,t}}(C_t^{1,t,\rho})). \tag{9.12}$$

We know that $\#\rho_{\mathcal{I}^{\alpha_1,t}}(C_{t^{\alpha_2}}^{1,t,\rho})$ follows approximately the law of $\#K_{\log(\alpha_2/\alpha_1)}$, where K is the Kingman coalescent started with infinitely many particles. In particular, $\{\#\rho_{\mathcal{I}^{\alpha_1,t}}(C_{t^{\alpha_2}}^{1,t,\rho}),\ t \geq t_0\}$ is a tight family of random variables. Moreover, for any $\varepsilon > 0$, $\{\#\rho_{\mathcal{I}^{\alpha_2}}(C_{t^{\alpha_2+\varepsilon}}^{1,t,\rho}),\ t \geq t_0\}$ is a tight family as well.

Due to (6.9) for each $\varepsilon > 0$, the total collection of partition elements $\rho_{\mathcal{I}^{\alpha_2,t}}(C_{t^{\alpha_2+\varepsilon}}^{1,t,\rho})$ has positions in $I_{\alpha_2+\varepsilon}(1,t)$ with overwhelming probability, as $t \to \infty$. Hence the sparse particle convergence stated in Proposition 5.1 applies. By letting $\varepsilon \to 0$, and using (6.8) and (6.10) as in the proof of Theorem 1, we obtain the statement (a.i) in the case $m = 2$. The induction step uses the same line of arguments.

(a.ii) Again the statements can be easily extended to $\rho = \infty$ with $\tilde{\mathcal{I}}$ as in (7.15) and following the lines of argument from Subsection 7.2.

(b) To prove (9.3) note that by the construction in Subsection 6.1, $\#\rho_{\mathcal{I}^{\alpha,t}}(C_t^{1,t})$ has monotone non-decreasing and càdlàg (or càglàd) paths in α, for all $t > 0$, almost surely. Furthermore by Theorems 1 and 2 we know that the family $\{\#\rho_{\mathcal{I}^{\alpha,t}}(C_t^{1,t});\ t \geq 0\}$ is tight, for each $\alpha < 1$. Therefore we obtain (9.3). □

So far we have shown with Proposition 9.1 the f.d.d. convergence. It remains to show the tightness in path space. However, this is a direct consequence of the monotonicity of the process $(\#K_0[\log \alpha_l, \log \alpha])_{\alpha \in [\alpha_l, \alpha_u]}$, as well as of all the processes $\{(\#\rho_{\mathcal{I}^{\alpha,t}}(C_t^{1,t}))_{\alpha \in [\alpha_\ell, \alpha_u]};\ t > 0\}$ in α. More precisely, it is a consequence of the fact that their paths are non-decreasing and bounded from below (by identity 0), almost surely and from above by (9.3).

9.2. **Proof of Theorem 6.** Fix $\alpha \in (0,1)$. For $m \in \mathbb{N}$, consider the parameters $\alpha < u_1 < u_2 < ... < u_m < 1$, and put $\vec{u} := (u_1, ..., u_m)$. Recall $\{C_j^{\alpha,t,\vec{u},\rho};\ j = 1, ..., m\}$ from (3.21) and K^{merge} from (3.22).

Proof of Theorem 6. Assume first that $m = 1$. By construction of K^{merge}, the first component $\rho_{\{km;\, k\in\mathbb{N}\}}(K^{\text{merge}})$ is Kingman coalescent. Moreover, $\#C_1^{\alpha,t,u_1,\rho}$ equals the number of partition elements in a spatial coalescent (without rebirth) which starts at time t^{u_1} in $\rho^{\mathbb{Z}^2}_{\mathcal{I}^{\alpha,t}}(C_0^{1,t,\rho})$ and is evaluated at time t. It follows therefore from Theorem 1 that

$$\mathcal{L}\big(\#C_1^{\alpha,t,u_1,\rho}\big) \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big(\#K_{-\log \alpha}\big). \tag{9.13}$$

Thus the case $m = 1$ is covered by Theorem 1.

We next assume that $m \geq 2$. The key is to understand the case $m = 2$, since the same line of argument will easily allow also for the induction step from m to $m+1$. We will concentrate on (3.23), and we comment at the very end on the extension (3.24).

Fix a finite $\rho \in (0, \infty)$ and $t \geq t_0$, where as usual t_0 is taken from Proposition 6.1. For $i = 1, ..., m$, define

$$\bar{C}^i := \big\{\pi \in C_{t^{u_i}}^{\text{birth}} : L_{t^{u_i}}^{\text{birth}}(\pi) \in \Lambda^{\alpha,t}\big\}. \tag{9.14}$$

We consider the joint evolution of partition elements $\bar{C}^i$, $i = 1, ..., m$. There are Poisson(ρ) many partition elements present at each site of the α-box, at all times $s \geq 0$, almost surely. In particular, $\#\bar{C}^i$ has Poisson($\rho \cdot \#\Lambda^{\alpha,t}$) distribution.

Note that, for t large, due to (6.9) we will have that, with overwhelming probability,

$$\pi^1 \not\subseteq \pi^2, \quad \forall\, i < j \in \{1, ..., m\}, \pi^1 \in \bar{C}^i, \pi^2 \in \bar{C}^j. \tag{9.15}$$

In words, it is highly unlikely to have any partition element of $\bar{C}^i$ reappear (as a subset of an partition element) in the α-box at any of the later times t^{u_l}, $l \in \{i+1, ..., m\}$. For the rest of the argument, we will henceforth consider our realization on the event (9.15).

Fix a small $\delta \in (0,1)$ which will be sent to 0 eventually. For each $i \in \{1, ..., m\}$ and $s \geq t^{u_i}$, denote by $\bar{N}^i_s$ the number of partition elements of C^{birth}_s containing at least one element of $\bar{C}^i$. Once more, $\bar{N}^1_{t^{u_2}}$ follows approximately the law of $\#K_{\log(u_2/\alpha)}$ by Theorem 1. By (6.9), these partition elements have locations in $I_{u_2}(1,t)$ at time t^{u_2}, and stay in $I_{u_2}(2+\delta, t)$ during the time interval $[t^{u_2}, 2t^{u_2}]$, with overwhelming probability. Moreover, $\bar{N}^1_{2t^{u_2}} = \bar{N}^1_{t^{u_2}}$ with overwhelming probability, as $t \to \infty$.

We also know form Theorem 1 that

$$\mathcal{L}\big(\#\rho_{\bar{C}^2}(C^{\text{birth}}_s)\big)_{s\in[t^{u_2}, 2t^{u_2}]} \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\big(\#(K_s)_{s\in[0,\log(u_2/\alpha)]}\big). \tag{9.16}$$

Moreover, during the time interval $[t^{u_2}, t^{u_2} + t^{u_2-\delta}]$, with overwhelming probability, there is no coalescence interaction between the equivalence classes of C^{birth} containing at least one element of $\bar{C}^1$ and those containing at least one element of $\bar{C}^2$. In other words, for any $\delta > 0$, the two restricted coalescents evolve separately and independently during $[t^{u_2}, t^{u_2} + t^{u_2-\delta}]$, asymptotically as $t \to \infty$. For δ very small it is very likely that the event $A^{1,2}_\delta := \{\bar{N}^2_{2t^{u_2}} = \bar{N}^2_{t^{u_2}+t^{u_2-\delta}}\}$ happens.

Also note that on $A^{1,2}_\delta$ the positions of the

$$\bar{N}^1_{2t^{u_2}} + \bar{N}^2_{2t^{u_2}} = \bar{N}^1_{t^{u_2}} + \bar{N}^2_{t^{u_2}+t^{u_2-\delta}} \tag{9.17}$$

partition elements in $C^{\text{birth}}_{2t^{u_2}}$ that contain at least one element either of $\bar{C}^1$ or of $\bar{C}^2$ are contained in $I_{u_2}(2+\delta, t)$. Therefore the joint evolution of these equivalence classes during a time interval $[2t^{u_2}, t^{u_3}]$ (by Lemma 5.2 and Section 5.2) is again well approximated by that of the $(\#K_s^{\bar{N}^1_{2t^{u_2}}+\bar{N}^2_{2t^{u_2}}}, s \in [0, \log(u_3/u_2)])$, where the last coalescent process depends on $\bar{N}^1_{2t^{u_2}}$ and $\bar{N}^2_{2t^{u_2}}$ solely through its initial configuration.

Denote by $\vec{u}^{\,i}/\alpha$ the vector $(u_1/\alpha, ..., u_i/\alpha) \in \mathbb{R}^i$. It is now clear by the above argument that

$$\mathcal{L}\big(\bar{N}^1_{t^{u_3}}, \bar{N}^1_{t^{u_3}} + \bar{N}^2_{t^{u_3}}\big) \underset{t\to\infty}{\Longrightarrow} \mathcal{L}\Big[\big(\#\rho_{\{2k;\, k\in\mathbb{N}\}}(K^{\text{merge}}_{\log(\frac{u_3}{\alpha})}), \#K^{\text{merge}}_{\log(\frac{u_3}{\alpha})}\big)\Big]. \tag{9.18}$$

By setting $u_3 = 1$, one obtains the result for $m = 2$. By now it should be obvious that the induction step from m to $m+1$ follows indeed the same line of argument.

Part (b) will follow as usual from (3.23) by using the same kind of arguments as in the proof of Theorem 2. □

10. Proof of the moment bound

In this section we present the proof of Proposition 6.1 which follows the proof of a similar statement for the instantaneous coalescent stated in the proposition on page 615 in [4]. In [4] the particles move according to the nearest neighbor random walks, while here the partition elements move according to more general random walks. Moreover, coalescence happens with a rate γ delay, and it is therefore possible (often likely) to have more than 1 (up to countable many) partition elements per site.

Proof of Proposition 6.1. Recall the box $\Lambda(r)$ from (6.1), and let for $A, B \subseteq \mathbb{Z}^2$,

$$(C_s^A, L_s^A)_{s\geq 0}, \tag{10.1}$$

be the coalescent started from the configuration (6.4) restricted to locations in A. This coalescent was denoted by $C^{\mathcal{I}_A}$ in Subsection 6.1. If $A = \Lambda(t)$ we will in most cases omit the superscript from the notation. For $A, B \subseteq \mathbb{Z}^2$ and $s \geq 0$, let

$$\#C_s^A(B) := \#\{\pi \in C_s^A : L_s(\pi) \in B\}. \tag{10.2}$$

As done before, if $B = \mathbb{Z}^2$ we simply write $\#C_s^A := \#C_s^A(\mathbb{Z}^2)$.

Following the lines of Section 3 in [4], we introduce an auxiliary spatial coalescing system $(\widetilde{C}, \widetilde{L})$ which follows the spatial coalescent dynamics over the time interval $[0, 2]$, then keeps coalescing as long as the number of partition elements is not decreasing too quickly, while otherwise the "coalescence is switched off for a while". More precisely, we discretize the time on a logarithmic scale, i.e., set for $T \geq 0$,

$$m(T) := \begin{cases} 0, & \text{if } T \leq 1, \\ 2^{\lfloor \log_2 T \rfloor}, & \text{if } T > 1. \end{cases} \tag{10.3}$$

In this way we have $T \in [m(T), m(2T) \vee 1]$, $T \geq 0$.

Now, let $(\widetilde{C}_0, \widetilde{L}_0) := (C_0^{\Lambda(t)}, L_0^{\Lambda(t)})$, and run the coalescent until time $T = 2$. To define $(\widetilde{C}_t, \widetilde{L}_t)$, we proceed by induction. Put

$$\tau^{\lfloor \log_2 T \rfloor} := m(2T) \wedge \inf\Big\{s \in [m(T), m(2T)] : \mathbf{E}\big[\#\widetilde{C}_s\big] \leq \tfrac{1}{2}\mathbf{E}\big[\#\widetilde{C}_{m(T)}\big]\Big\}, \tag{10.4}$$

and start $\widetilde{C}$ at time $m(T)$ in the spatial configuration given by $\widetilde{C}_{m(T)}$. The coalescent $(\widetilde{C}, \widetilde{L})$ follows the same dynamics as the spatial coalescent on $[m(T), \tau^{\lfloor \log_2 T \rfloor}]$, while its partition elements perform independent random walks with kernel $a(x, y)$ on $[\tau^{\lfloor \log_2 T \rfloor}, m(2T)]$ yielding the random configuration $(\widetilde{C}_{m(2T)}, \widetilde{L}_{m(2T)})$. Now reset $T := 2T$ and repeat the induction step starting at (10.4). Obviously, $\mathbf{E}\big[\#\widetilde{C}_t\big] \geq \mathbf{E}\big[\#C_t\big]$, for all $t \geq 0$. In fact, one can easily construct a coupling in such a way that the corresponding inequality for processes holds for all times, almost surely. Hence it suffices to prove Proposition 6.1 with (C, L) replaced by $(\widetilde{C}, \widetilde{L})$.

Set

$$Y_T := \mathbf{E}\big[\#\widetilde{C}_T\big] = \mathbf{E}\big[\#\widetilde{C}_T^{\Lambda(t)}(\mathbb{Z}^2)\big], \tag{10.5}$$

and note that Y_T also depends on t through the initial configuration (6.4), although this is suppressed from the notation.

We will need a few preliminary lemmas. We start with a basic fact estimating the "speed" of escape from large balls centered at the origin for a zero mean random walk with finite exponential moments.

Lemma 10.1. *Let $(\xi_t)_{t\geq 0}$ be the unit rate continuous time random walk on $\mathbb{Z}$ with transition kernel $b_t(x, y)$. If $\sum_{x\in\mathbb{Z}} x b_1(0, x) = 0$ and $\varphi(\lambda) := \sum_{x\in\mathbb{Z}} e^{\lambda x} b_1(0, x) < \infty$, for all $\lambda > 0$, then there exists a finite constant $c_0 = c_0(\xi)$ such that*

$$\mathbf{P}\big\{\xi_t > u\sqrt{t}\big\} \leq e^{-c_0 u} \tag{10.6}$$

for all $u, t \geq 1$.

Proof. The argument is based on standard large deviation techniques. For all $s, t, \lambda > 0$,

$$\mathbf{P}\big\{\xi_t > st\big\} = \mathbf{P}\big\{e^{\lambda\xi_t} > e^{\lambda st}\big\} \le e^{-\lambda st} e^{t(\varphi(\lambda)-1)}. \tag{10.7}$$

In particular, if $I(s) := \sup_{\lambda>0}\big\{s\lambda - (\varphi(\lambda)-1)\big\}$, then

$$\mathbf{P}\big\{|\xi_t| > st\big\} \le e^{-I(s)t}. \tag{10.8}$$

Note that $I(s) : [0,\infty) \to [0,\infty)$ is a convex function, such that $I(s) = 0$ if and only if $s = 0$. Therefore, there exists a positive constant c_0^1 such that

$$I(s) \ge c_0^1 s, \qquad \text{if } s \ge 1. \tag{10.9}$$

Moreover, under our assumptions on exponential moments, there exists a finite constant c_0^2 (without loss of generality can assume that $c_0^2 \ge 1$) such that $\varphi(\lambda) \le 1 + c_0^2\lambda^2$, for all $\lambda \in [0,1]$. Thus, for all $s \le 1$,

$$I(s) \ge \sup_{\lambda\in[0,1]} \big\{s\lambda - c_0^2\lambda^2\big\} \ge \frac{1}{4c_0^2} s^2, \tag{10.10}$$

where we have used the fact that if $s \le 1$ then $\lambda^* := \frac{s}{2c_0^2} \le 1$.

Now set $c_0 := \min\{c_0^1, (4\tilde{c}_0^2)^{-1}\}$, and take $u, t \ge 1$. If $u \ge \sqrt{t}$ we obtain (10.6) from (10.8) by substituting $s = u/\sqrt{t}$ into (10.9). Similarly, if $1 \le u \le \sqrt{t}$ we obtain (10.6) by substituting $s = u/\sqrt{t}$ into (10.10). □

The next result states that if the spatial coalescent starts in $\Lambda(t)$, then at time T the fraction of partition elements which lie outside of $\Lambda(t+u\sqrt{T})$ decreases at least exponentially fast, as $u \to \infty$.

Lemma 10.2. *Fix $t > 0$. Let $\bar{R} := (\bar{R}^1, \bar{R}^2)$ be the random walk on $\mathbb{Z}^2$ with kernel $a(x,y)$. Fix $c_0 = c_0(\bar{R})$ such that (10.6) holds. Put $c_1 := 2 \cdot (2^{\overline{\frac{5}{(\sqrt{2}-1)}}} \wedge e^{c_2})$ where $c_2 = c_2(\bar{R}) := \sqrt{\frac{2}{7}}(c_0(\bar{R}^1) \wedge c_0(\bar{R}^2))$. Then*

$$\mathbf{E}\Big[\#\widetilde{C}_T\big(\Lambda^c(t+u\sqrt{T})\big)\Big] \le c_1\, e^{-c_2 u} Y_T, \tag{10.11}$$

for all $u \ge 0$ and $T \ge 1$.

Choosing a large enough so that $c_1 e^{-c_2 a} \le 1/3$ we obtain the following:

Corollary 10.1. *For sufficiently large $a \ge 1$,*

$$\mathbf{E}\Big[\#\widetilde{C}_T\big(\Lambda^c(t+a\sqrt{T})\big)\Big] \le \tfrac{1}{3} Y_T, \tag{10.12}$$

for all $T \ge 1$.

Proof of Lemma 10.2. The proof is by induction over $\lfloor \log_2 T \rfloor$. First, suppose that $2 \le T \le 2^4$ and $u \ge 1$. By comparison with the independent random walks equal in law to $\bar{R} := (\bar{R}^1, \bar{R}^2)$ on $\mathbb{Z}^2$, we obtain (with $\|\cdot\|$ the maximum norm)

$$\begin{aligned}\mathbf{E}\Big[\#\widetilde{C}_T\big(\Lambda^c(t+u\sqrt{T})\big)\Big] &\le \mathbf{E}\big[\#C_0^{\Lambda(t)}\big]\, \mathbf{P}^{(0,0)}\big\{\|\bar{R}_T\| \ge u\sqrt{T}\big\} \\ &\le \mathbf{E}\big[\#C_0^{\Lambda(t)}\big]\Big(\mathbf{P}^0\big\{|\bar{R}_T^1| \ge u\sqrt{T}\big\} + \mathbf{P}^0\big\{|\bar{R}_T^2| \ge u\sqrt{T}\big\}\Big) \\ &\le 4 \cdot \mathbf{E}\big[\#C_0^{\Lambda(t)}\big]\, e^{-(c_0(\bar{R}^1)\wedge c_0(\bar{R}^2))u}.\end{aligned} \tag{10.13}$$

By definition, $Y_T \ge Y_{2^4} \ge \frac{1}{2}Y_{2^3} \ge \ldots \ge 2^{-4}\#C_0$. Moreover the map $s \mapsto \mathbf{E}\big[\#C_s\big]$ is continuous, and therefore

$$\mathbf{E}\Big[\#\widetilde{C}_T\big(\Lambda^c(t+u\sqrt{T})\big)\Big] \le 2^6 \cdot e^{-(c_0(\bar{R}^1)\wedge c_0(\bar{R}^2))u} \cdot Y_T, \tag{10.14}$$

as required. So (10.11) holds in the case $2 \le T \le 2^4$, for all $u \ge 1$, and for $u \in [0,1]$, (10.11) holds trivially due to the fact that $c_1 e^{-c_2} \ge 1$.

Suppose now that for some $m \ge 1$, (10.11) holds for all $2 \le T \le 2^{m+3}$. Then for $T \in (2^{m+3}, 2^{m+4}]$,

$$\begin{aligned}&\mathbf{E}\Big[\#\widetilde{C}_T\big(\Lambda^c(t+u\sqrt{T})\big)\Big]\\ &\le \mathbf{E}\Big[\#\widetilde{C}_{2^m}\big(\Lambda^c(t+\frac{u}{2}\sqrt{T})\big)\Big] + Y_{2^m}\mathbf{P}^{(0,0)}\big\{\|\bar{R}_{T-2^m}\| \ge \frac{u}{2}\sqrt{T}\big\}\\ &\le \mathbf{E}\Big[\#\widetilde{C}_{2^m}\big(\Lambda^c(t+(\sqrt{2}u)2^{\frac{m}{2}})\big)\Big] + Y_{2^m}\mathbf{P}^{(0,0)}\big\{\|\bar{R}_{T-2^m}\| \ge \frac{u}{2}\sqrt{\tfrac{T}{(T-2^m)}}\sqrt{(T-2^m)}\big\}.\end{aligned} \tag{10.15}$$

The first inequality above is obtained by the following observation: each partition element in $\Lambda^c(t+u\sqrt{T})$ at time T corresponds to some partition element, located either in $\Lambda(t+\frac{u}{2}\sqrt{T})$ or its complement, at time 2^m. Applying the induction hypotheses to the first term, and Lemma 10.1 to the second term on the right hand side of (10.15), we obtain that

$$\begin{aligned}\mathbf{E}\Big[\#\widetilde{C}_T\big(\Lambda^c(t+u\sqrt{T})\big)\Big] &\le Y_{2^m}\big(c_1 e^{-c_2\sqrt{2}u} + 2e^{-c_2 u}\big)\\ &\le Y_T c_1 e^{-c_2 u}\big(2^4 e^{-c_2(\sqrt{2}-1)u} + \frac{2^6}{c_1}\big),\end{aligned} \tag{10.16}$$

where we have used the facts that $\sqrt{T/(T-2^m)} \ge \sqrt{8/7}$, for all $T \in (2^{m+3}, 2^{m+4}]$, and $Y_T \ge 2^{-4}Y_{2^m}$.

Define $u_0 := \frac{5\log 2}{c_2(\sqrt{2}-1)}$. Then an elementary calculation shows that for all $u \ge u_0$,

$$2^4 e^{-c_2(\sqrt{2}-1)u} + \frac{2^6}{c_1} \le 2^4 e^{-c_2(\sqrt{2}-1)u_0} + \frac{1}{2} \le 2^4 2^{-5} + \frac{1}{2} \le 1, \tag{10.17}$$

while for all $u \in [0, u_0]$, $c_1 e^{-c_2 u} \ge c_1 2^{-\frac{5}{\sqrt{2}-1}} \ge 1$, so (10.11) trivially holds for all $u \in [0, u_0]$. This completes the induction step and the proof. □

We next provide an estimate of the rate of decrease for the number of partition elements during an interval of time, provided that the coalescence dynamics is switched on.

For two partition elements $\{i\}, \{j\} \in C_0$, put

$$\sigma^{\{i,j\}} := \min\big\{u \ge 0 : L_u(\{i\}) = L_u(\{j\})\big\} \tag{10.18}$$

as the waiting time until these particles share the same location, and set

$$h_s^\gamma(A) := \inf_{i,j \in \mathcal{I}^A} \mathbf{P}\big\{\sigma^{\{i,j\}} \le s\big\}. \tag{10.19}$$

One can verify using a last-exit-time decomposition and the assumption (2.3) (compare Lemma 5 in [5]) that for fixed $b > 0$,

$$h_{r^2}\big(\Lambda(br)\big) \ge M(b)\frac{1}{\log(r)}, \tag{10.20}$$

for some $M(b) > 0$.

Similarly, define $\tau^{\{i,j\}} := \min\{u \geq 0 : i \sim^u j\}$, and set for $A \subseteq \mathbb{Z}^2$,

$$H_s^\gamma(A) := \inf_{i,j \in \mathcal{I}^A} \mathbf{P}\{\tau^{\{i,j\}} \leq s\}. \tag{10.21}$$

We are particularly interested in bounding from below the quantity

$$H^\gamma_{4R_T^2}(\Lambda(\sqrt{2}R_T)), \tag{10.22}$$

where

$$R_T = R_T^{a,t} := 7(1+a)\sqrt{\tfrac{t^2+aT}{Y_T}}, \tag{10.23}$$

with $a \geq 1$ chosen according to Corollary 10.1 such that (10.12) holds. We will henceforth assume that $T \leq t^3$ (as in (10.39) below). Then, if

$$s_T := 4R_T^2, \tag{10.24}$$

inequality (10.20) implies that

$$h_{s_T/2}\big(\Lambda(\sqrt{2}R_T)\big) \geq \frac{M(1)}{\log R_T} \geq \frac{M_1}{\log t}, \tag{10.25}$$

where $M_1 \in (0, 2M(1)/3) \subset (0,\infty)$ is chosen depending on a. Recalling inequality (7.48) from [22], we obtain that

$$H^\gamma_{s_T}\big(\Lambda(\sqrt{2}R_T)\big) \geq \frac{\gamma}{2+\gamma}\Big(1 - \exp\big(-\frac{2+\gamma}{2}s_T\big)\Big)h_{s_T/2}\big(\Lambda(\sqrt{2}R_T)\big) \geq \frac{M_2}{\log t}, \tag{10.26}$$

for some $M_2 \in (0,\infty)$, for all $t \geq 2$, where we use $s_T \geq 4 \cdot 49 \cdot (1+a)^2 \cdot \frac{t^2}{Y_0} > 0$, since $t \geq 2$.

Lemma 10.3 (Rate of decay for the auxiliary coalescent)**.** *Let $2 \leq T \leq r < r+s \leq 2T$. Suppose that $Y_T \geq 49$, and that $\widetilde{C}$ is coalescing during the entire time interval $[T, r+s)$. Then*

$$Y_{r+s} \leq Y_r \exp\Big[-\frac{1}{3}H_s^\gamma\big(\Lambda(\sqrt{2}R_T)\big)\Big]. \tag{10.27}$$

Proof. Write C_s^C for the spatial coalescent started in the random partition C at time 0, and evaluated at time s. For all $T \leq r < r+s \leq 2T$,

$$Y_{r+s} \leq \mathbf{E}\big[\#C_s^{\widetilde{C}_r(\Lambda(t+a\sqrt{r}))}\big] + \mathbf{E}\big[\#\widetilde{C}_r(\Lambda^c(t+a\sqrt{r}))\big]. \tag{10.28}$$

Choose a covering of $\Lambda(t+a\sqrt{r})$ by

$$n_T := \Big\lfloor 1 + [\mathrm{Area}(\Lambda(t+a\sqrt{T}))]^{1/2}/R_T \Big\rfloor^2 \tag{10.29}$$

disjoint boxes $\{\Lambda_{i,r},\, i = 1,\dots,n_T\}$ of side length

$$l_T := \Big(\frac{\mathrm{Area}(\Lambda(t+a\sqrt{r}))}{n_T}\Big)^{1/2} \leq \sqrt{2}R_T. \tag{10.30}$$

The last inequality holds since $r \in [T, 2T]$.

After ignoring coalescing events between partition elements that are located in different sub-boxes $\Lambda_{r,i} \cap \Lambda_{r,j} = \emptyset$ at time r, one can bound from above the first term on the right hand side of (10.28) by

$$\sum_{i=1}^{n_T} \sum_{C:C(\Lambda^c_{i,r})=\emptyset} \mathbf{P}\big\{\widetilde{C}_r(\Lambda(t+a\sqrt{r}) \cap \Lambda_{i,r}) = C(\Lambda(t+a\sqrt{r}) \cap \Lambda_{i,r})\big\}\mathbf{E}\big[\#C^C_s\big]. \tag{10.31}$$

It is straightforward to conclude, as in (7.44)–(7.46) in [22], that for C as above

$$\mathbf{E}\big[\#C^C_s\big] \le \#C - \big(\#C - 1\big)H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big). \tag{10.32}$$

Insert (10.32) into (10.31) to get

$$\begin{aligned}
&\mathbf{E}\big[\#C_s^{\widetilde{C}_r(\Lambda(t+a\sqrt{r}))}\big] \\
&\le \sum_{i=1}^{n_T} \sum_{C:C(\Lambda^c_{i,r})=\emptyset} \mathbf{P}\big\{\widetilde{C}_r(\Lambda(t+a\sqrt{r}) \cap \Lambda_{i,r}) = C(\Lambda(t+a\sqrt{r}) \cap \Lambda_{i,r})\big\}\cdot \\
&\qquad \cdot \Big(\#C(\Lambda(t+a\sqrt{r}) \cap \Lambda_{i,r}) - \big(\#C(\Lambda(t+a\sqrt{r}) \cap \Lambda_{i,r}) - 1\big)H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big) \\
&= \mathbf{E}\Big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big] - \Big(\mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big] - n_T\Big)H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big) \\
&\le \mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big]\Big(1 - \frac{1}{2}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big).
\end{aligned} \tag{10.33}$$

For the last inequality in (10.33) we use (10.29) and the following observations

(a) $Y_u \ge Y_T/2$, for all $u \in [T, r+s)$, and therefore in particular, $Y_r \ge Y_T/2$, since otherwise the coalescing would not last during the entire interval $[T, r+s)$,

(b) for any $r \ge 1$,

$$\begin{aligned}
Y_r &= \mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big] + \mathbf{E}\big[\#\widetilde{C}_r(\Lambda^c(t+a\sqrt{r}))\big] \\
&\le \mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big] + \frac{Y_r}{3},
\end{aligned} \tag{10.34}$$

by Corollary 10.1, and

$$\begin{aligned}
n_T &\le \big(\frac{2}{7}\sqrt{Y_T}\big)^2 \le \frac{4}{49}\cdot 4Y_r \\
&\le \frac{4\cdot 4}{49}\cdot\frac{3}{2}\mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big] < \frac{1}{2}\mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big].
\end{aligned} \tag{10.35}$$

Now by (10.28), (10.33), (10.34) and (10.12), we have

$$\begin{aligned}
Y_{r+s} &\le \mathbf{E}\big[\#\widetilde{C}_r(\Lambda(t+a\sqrt{r}))\big]\Big(1 - \frac{1}{2}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big) + \mathbf{E}\big[\#\widetilde{C}_r(\Lambda^c(t+a\sqrt{r}))\big] \\
&= Y_r\Big(1 - \frac{1}{2}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big) + \frac{1}{2}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\mathbf{E}\big[\#\widetilde{C}_r(\Lambda^c(t+a\sqrt{r}))\big] \\
&\le Y_r\Big(1 - \frac{1}{2}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big) + \frac{1}{6}Y_r H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big) \\
&= Y_r\Big(1 - \frac{1}{3}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big) \\
&\le Y_r \exp\Big[-\frac{1}{3}H^\gamma_s\big(\Lambda(\sqrt{2}R_T)\big)\Big],
\end{aligned} \tag{10.36}$$

as required. □

Lemma 10.4 (Upper bound for the decay rate of partition elements)**.** *Fix $t \geq 2$, and let for $T \geq 2$,*

$$g(T) := \frac{\log(1+\frac{T}{t^2})}{\log t} \cdot Y_T \cdot \Big(1 \vee \frac{\mathbf{E}\big[\# C_2^{\Lambda(t)}\big]}{t^2}\Big)^{-1}, \qquad T \geq 2. \tag{10.37}$$

Then there exists a finite constant M such that

$$g(T) \leq M, \qquad 2 \leq T \leq 4, \tag{10.38}$$

and

$$g(2T) \leq g(T) \vee M, \qquad 2 \leq T \leq t^3. \tag{10.39}$$

Proof. Recall M_2 from (10.26), and fix $a \geq 1 \vee \frac{M_2 \log_2 5}{48}$ suitably large such that (10.12) holds. Put

$$M := \frac{3 \cdot 16 \cdot 49 \cdot a(1+a)^2}{M_2}, \tag{10.40}$$

and notice that $M \geq 49 \cdot \log_2 5$.

Assume first that $2 \leq T \leq 4$. In this case, since $Y_T/t^2 \leq Y_2/t^2 \leq 1 \vee \mathbf{E}[C_2^{\Lambda(t)}]/t^2$, and since $\log(1+x) \leq x$, for all $x > -1$,

$$g(T) \leq \frac{Tt^2}{t^2 \log t} \leq \frac{4}{\log 2} \leq M. \tag{10.41}$$

Next assume that $2 \leq T \leq t^3$ and $Y_T \leq 49$. Then since $Y_{2T} \leq Y_T \leq 49$, we get

$$g(2T) \leq 49 \frac{\log(1+2t)}{\log t} \leq 49 \log_2 5 \leq M. \tag{10.42}$$

It therefore remains to prove (10.39) for $Y_T > 49$. Without loss of generality we may assume that

$$\widetilde{C} \text{ is coalescing during the entire interval } (T, \tfrac{3}{2}T). \tag{10.43}$$

Indeed otherwise we could find an $m \in \mathbb{N}$ such that $\tau^m \in (T, \frac{3}{2}T)$ (recall 10.4) and therefore since $Y_{2T} \leq Y_{\tau^m} \leq Y_T/2$, we get $\frac{g(2T)}{g(T)} \leq \frac{1}{2} \cdot \frac{\log(1+\frac{2T}{t^2})}{\log(1+\frac{T}{t^2})} \leq 1$.

However, under (10.43), Lemma 10.3 applies with any $(r, r+s] \subset (T, \frac{3}{2}T]$, so that $\lfloor \frac{T}{2s_T} \rfloor$ iterations of (10.27) yield that

$$Y_{2T} \leq Y_T \exp\big[-\frac{1}{3}\Big\lfloor \frac{T}{2s_T} \Big\rfloor H^{\gamma}_{s_T}\big(\Lambda(\sqrt{2}R_T)\big)\big]. \tag{10.44}$$

By (10.24),

$$\begin{aligned} \Big\lfloor \frac{T}{2s_T} \Big\rfloor \geq \frac{T}{4s_T} &= \frac{Y_T T}{16 \cdot 49 \cdot (1+a)^2(t^2+aT)} \\ &\geq g(T) \frac{T \log t}{16 \cdot 49 \cdot (1+a)^2 (t^2+aT) \log(1+\frac{T}{t^2})}. \end{aligned} \tag{10.45}$$

Finally, inserting (10.45) and (10.44) into (10.37), and recalling (10.26), yields

$$\begin{aligned}\frac{g(2T)}{g(T)} &\le \frac{\log\,(1+\frac{2T}{t^2})}{\log\,(1+\frac{T}{t^2})}\exp\Big[-\frac{1}{3}\Big\lfloor\frac{T}{2s_T}\Big\rfloor H^{\gamma}_{s_T}\big(\Lambda(\sqrt{2}R_T)\big)\Big]\\ &\le \exp\Big[\frac{T}{(t^2+T)\log\,(1+\frac{T}{t^2})}-\frac{1}{3}\Big\lfloor\frac{T}{2s_T}\Big\rfloor H^{\gamma}_{s_T}\big(\Lambda(\sqrt{2}R_T)\big)\Big]\\ &\le \exp\Big[\frac{T}{(t^2+T)\log\,(1+\frac{T}{t^2})}\big(1-M^{-1}g(T)\big)\Big].\end{aligned} \tag{10.46}$$

We therefore find that either $g(T)\le M$ or if $g(T)>M$ then $g(2T)\le g(T)$, which proves (10.39). □

To finish off the proof of the proposition, note that Lemma 10.4 readily implies $g(T)\le M$, for all $t\ge 2$, $0\le T\le t^3$. Therefore,

$$Y_T \le M\frac{\log t}{\log(1+\frac{T}{t^2})}\Big(1\vee\frac{\mathbf{E}\big[\#C_2^{\Lambda(t)}\big]}{t}\Big), \qquad 2\le T\le t^3, \tag{10.47}$$

and after replacing t with $t^{\alpha/2}$ and T with t^{β} where $\beta\in(\alpha,3\alpha/2]$,

$$Y_{t^\beta} \le M\frac{\log t^{\alpha/2}}{\log(1+\frac{t^\beta}{t^\alpha})}\Big(1\vee\frac{\mathbf{E}\big[\#C_2^{\Lambda(t^{\alpha/2})}\big]}{t^\alpha}\Big) \le M\Big(1\vee\frac{\alpha}{2(\beta-\alpha)}\vee\frac{\mathbf{E}\big[\#C_2^{\Lambda(t^{\alpha/2})}\big]}{t^\alpha}\Big). \tag{10.48}$$

□

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Appendix A. Topologies on spaces of partitions

In this appendix we want to present all topologies we are using on our state spaces of partitions. We will recall in Subsection A.1 the canonical topologies used for marked partitions of a given countable set which were introduced for example in [22]. In Subsection A.2 we then generalize the topology to the space of admissible sub-partitions as the appear in the context of coalescent processes with rebirth.

A.1. **The state space topology of the spatial coalescent.** As before consider a non-empty and at most countable set $\mathcal{I}$, and fix an enumeration $\mathcal{I}:=\{i_1,i_2,...\}$. We begin by recalling the topology on the state space of a non-spatial $\mathcal{I}$-coalescent. For that purpose, recall from (2.8) the space $\Pi^{\mathcal{I}}$ of all partitions of $\mathcal{I}$ and from (2.9) the partitions $\rho_{\mathcal{I}'}(\mathcal{P})$ induced by the restriction map $\rho_{\mathcal{I}'}:\Pi^{\mathcal{I}}\to\Pi^{\mathcal{I}'}$.

Each $\mathcal{P}\in\Pi^{\mathcal{I}}$ can be identified with the sequence

$$\big(\rho_{\{i_1\}}(\mathcal{P}),\rho_{\{i_1,i_2\}}(\mathcal{P}),...\big)\in\Pi^{\{i_1\}}\times\Pi^{\{i_1,i_2\}}\times... \tag{A.1}$$

Give $\Pi^{\mathcal{I}}$ the topology it inherits as a subset of $\Pi^{\{i_1\}}\times\Pi^{\{i_1,i_2\}}\times...$ with the product of discrete topologies. So $\Pi^{\mathcal{I}}$ is compact and metrizable, and hence Polish. In particular, a sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ *converges in* $\Pi^{\mathcal{I}}$ if for all finite subsets $\mathcal{I}'\subseteq\mathcal{I}$, the sequence $(\rho_{\mathcal{I}'}(\mathcal{P}_n))_{n\in\mathbb{N}}$ converges in $\Pi^{\mathcal{I}'}$ equipped with the discrete topology. Note that a function $f:\Pi^{\mathcal{I}}\to\mathbb{R}$ is continuous if it depends on $\Pi^{\mathcal{I}}$ only through $\Pi^{\mathcal{I}'}$, for some finite subset $\mathcal{I}'\subseteq\mathcal{I}$.

We next turn to marked partitions. As before, let the mark space M be an non-empty, countable set. Recall from (2.13) the space $\Pi^{\mathcal{I},M}$ of all marked partitions and from (2.14) the partitions $\rho^M_{\mathcal{I}'}(\mathcal{P})$ induced by the restriction map $\rho^M_{\mathcal{I}'} : \Pi^{\mathcal{I},M} \to \Pi^{\mathcal{I}',M}$. Once more we can identify each $\mathcal{P} \in \Pi^{\mathcal{I},M}$ with the sequence

$$(\rho^M_{\{i_1\}}(\mathcal{P}), \rho^M_{\{i_1,i_2\}}(\mathcal{P}), ...) \in \Pi^{\{i_1\},M} \times \Pi^{\{i_1,i_2\},M} \times ... \tag{A.2}$$

For each $n \in \mathbb{N}$, consider on $\Pi^{\{i_1,...,i_n\},M}$ the topology induced by the discrete topology on $\Pi^{\{i_1,...,i_n\}}$ and the product topology on M. Give then $\Pi^{\mathcal{I},M}$ the topology it inherits as a subset of $\Pi^{\{i_1\},M} \times \Pi^{\{i_1,i_2\},M} \times ....$ So once more, $\Pi^{\mathcal{I},M}$ is compact and metrizable, and hence Polish.

A.2. **The state space topology of the spatial coalescent with rebirth.** As before consider an at most countable non-empty set $\mathcal{I}$ with a fixed enumeration $\mathcal{I} := \{i_1, i_2, ...\}$.

We first consider the non-spatial state space $\Pi^{\leq,\mathcal{I}}$ as defined in (2.25). We shall introduce a topology which accounts for the differences in both the indices and the birth times. We therefore say that a sequence

$$\big(\mathcal{P}_n\big)_{n\in\mathbb{N}} \text{ converges in } \Pi^{\leq,\mathcal{I}}, \tag{A.3}$$

if and only if for each finite subset $\mathcal{I}' \subseteq \mathcal{I}$ such that $\mathcal{I}'$ contains with an element $i \in \mathcal{I}$ all elements smaller than i the projections to the index component of the restricted partitions $\rho_{\mathcal{I}'}(\mathcal{P}_n)$ converge in the discrete topology and the corresponding birth times converge with respect to the Euclidian distance.

More precisely, we consider the topology generated by the following metric d. We let for each $N \in \mathbb{N}$ and $\mathcal{P}, \mathcal{P}' \in \Pi^{\leq,\mathcal{I}}$,

$$d\big(\mathcal{P}, \mathcal{P}'\big) \geq 2^{-N} \tag{A.4}$$

if and only if for all $\mathcal{I}_N \subseteq \mathcal{I}$ with $\#\mathcal{I}_N = N$, and for all one-to-one maps ι_N from $\cup_{\pi\in\rho_{\mathcal{I}_N}(\mathcal{P})}\pi$ to $\cup_{\pi\in\rho_{\mathcal{I}_N}(\mathcal{P}')}\pi$ there exist $v, v' \in \pi$ for some $\pi \in \rho_{\mathcal{I}_N}\mathcal{P}$ such that

- $\iota_N(v)$ and $\iota_N(v')$ do not belong to the same partition element in $\rho_{\mathcal{I}_N}(\mathcal{P}')$, or
- $|\text{pr}_{\text{time}}(v) - \text{pr}_{\text{time}}(\iota(v))| \geq 2^{-N}$.

We next consider a countable mark space $M \neq \emptyset$ and a extend the latter to $\Pi^{\leq,\mathcal{I},M}$ from (2.13). We say that a sequence

$$(\mathcal{P}^M_n)_{n\in\mathbb{N}} \quad \text{converges in } \Pi^{\leq,\mathcal{I},M} \tag{A.5}$$

if and only if the projections on the index component of the restricted partitions $\rho_{\mathcal{I}',M}(\mathcal{P}_n)$ converge in the discrete topology and their corresponding birth times and locations converge in the topology on $\mathbb{R}^m$ respectively M^m if m is the number of partition elements.

Again a distance can be defined by letting for each $N \in \mathbb{N}$ and $\mathcal{P}^M, (\mathcal{P}')^M \in \Pi^{\leq,\mathcal{I},M}$,

$$d\big(\mathcal{P}^M, (\mathcal{P}')^M\big) \geq 2^{-N} \tag{A.6}$$

if for all $\mathcal{I}_N \subseteq \mathcal{I}$ with $\#\mathcal{I}_N = N$, and for all one-to-one maps ι_N from $\cup_{\pi\in\rho_{\mathcal{I}_N}(\mathcal{P}^M)}\pi$ to $\cup_{\pi\in\rho_{\mathcal{I}_N}((\mathcal{P}')^M)}\pi$ there exist a $v, v' \in \pi$ for some $\pi \in \rho_{\mathcal{I}_N}\mathcal{P}_M$ such that

- $\iota_N(v)$ and $\iota_N(v')$ do not belong to the same partition element in $\rho_{\mathcal{I}_N}((\mathcal{P}')^M)$, or
- $|\text{pr}_{\text{time}}(v) - \text{pr}_{\text{time}}(\iota(v))| \geq 2^{-N}$,

or if for all $v, v' \in \pi$ for some $\pi \in \rho_{\mathcal{I}_N}\mathcal{P}^M$ we have that

- $\iota_N(v)$ and $\iota_N(v') \in \pi'$ for some $\pi' \in \rho_{\mathcal{I}_N}((\mathcal{P}')^M)$ but $d_M(L(\iota(\pi)), L(\pi)) \geq 2^{-N}$.

It is then easy to check that the topology defined this way on $\Pi^{\leq,\mathcal{I},M}$ is Polish whenever the topology on (M, d_M) is Polish. Moreover, d is complete whenever d_M is complete.

Consider finally the state spaces $\hat{\Pi}^{\mathcal{I}}$ and $\hat{\Pi}^{\mathcal{I},M}$ which can be considered as subspaces of $\mathbb{R} \times \Pi^{\leq,\mathcal{I}}$ and $\mathbb{R} \times \Pi^{\leq,\mathcal{I},M}$, respectively, and equip them with the product topology. Once more the topologies are Polish whenever the topology on $\mathbb{R}$ is.

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