Advanced Studies in Pure Mathematics 43, 2006
Singularity Theory and Its Applications
pp. 463–475

(r) does not imply (n) or (npf) for definable sets in non polynomially bounded o-minimal structures.

David Trotman and Leslie Wilson

Abstract.

It is known that if two subanalytic strata satisfy Kuo's ratio test, then the normal cone of the larger stratum Y along the smaller X satisfies two nice properties: the fiber of the normal cone at any point of X is the tangent cone to the fiber of Y over that point; the projection of the normal cone to X is open ("normal pseudo-flatness"). We present an example with X a line and Y a surface which is definable in any non polynomially bounded o-minimal structure such that the pair satisfies Kuo's ratio test, but neither of the above properties hold for the normal cone.

In **[OT2]** P. Orro and the first author defined a regularity condition (r^e) for C^2 stratifications which provides a way of quantifying Kuo's ratio test (r) **[K]**, because for subanalytic stratifications, Whitney's condition (a) and (r^e) hold, for some e, $0 < e < 1$, if and only if Kuo's ratio test (r) is satisfied. They further showed that if $0 \leq e < 1$, $(a + r^e)$ implies rather good behaviour of the normal cone along strata: the special fibre of the normal cone at a point x in a stratum X is equal to the tangent cone to the normal slice to X through x (this property is denoted by (n) in **[OT2]**), and the stratification is normally pseudo-flat (abbreviated to (npf)). Thus for subanalytic stratifications, (r) implies both (n) and (npf).

In the example below, which is not subanalytic, (r) holds, but neither (n) nor (npf) hold, and one can check that (r^e) fails for all $0 \leq e < 1$, so that in particular Verdier's condition (w) fails ((w) is equivalent to $(a + r^0)$). Example 4.2 of **[OT2]** provides a different non-subanalytic example without (n) or (npf), called a Kuo Escargot (cf. **[OT1]**), which was (b)-regular and not (r)-regular, but this example was not definable

Received March 30, 2004
Revised August 27, 2005

in any o-minimal structure, due to spiralling. The example below is log-analytic, so is definable in the o-minimal structure $R_{exp,an}$, but it is not definable in *any* polynomially bounded o-minimal structure, by Miller's dichotomy **[M]** stating that an o-minimal structure is not polynomially bounded if and only if it possesses the exponential function as a definable function. By the same dichotomy, our example is definable in *every* o-minimal structure which is not polynomially bounded.

It is straightforward to show that (r) implies (r^e) for some $e, 0 \leq e < 1$, for stratified sets whose strata are definable in a polynomially bounded o-minimal structure, as the proof of the implication in **[OT2]** uses only curve selection and the Łojasiewicz inequality (see **[DM]** or **[V]**).

One can check easily that $(r_{\mathrm{cod}\,1})$ fails for our example showing that (r) does not imply (r^*) for definable sets in non polynomially bounded o-minimal structures. The proof in **[NT]** that (r) implies (r^*) for subanalytic strata presumably works for polynomially bounded o-minimal structures (but it would be good to have a complete proof of this).

One can also check that (b) holds for the example, showing that (b) does not imply (b^*) along a stratum X for definable sets in non polynomially bounded o-minimal structures, even when $\dim X = 1$. Recall from **[NT]** that (b) implies (b^*) for subanalytic strata if $\dim X = 1$ because then (r) and (b) are equivalent, by **[K]**.

Presumably, for definable sets in polynomially bounded o-minimal structures, (r) implies (b), and (b) implies (r) if $\dim X = 1$, so that then (b) would imply (b^*) if $\dim X = 1$.

In the example below the density is actually constant along the small stratum, so in particular it is continuous. In 2000, G. Comte **[C]** has shown continuity of the density along strata of any (r)-regular subanalytic stratification (hence along 1-dimensional strata of any (b)-regular subanalytic stratification). In 2003 G. Valette found a different proof of this result **[V]** with a strengthened conclusion and has very recently (2003) announced an extension to any (b)-regular subanalytic stratification.

Are these results about the density true for definable sets in any o-minimal structure ?

Definitions. Below k will denote an integer greater than or equal to 2. Let S be a closed stratified subset of $\mathbb{R}^n$, whose strata are differentiable submanifolds of class C^k. For each stratum X of S denote by $C_X S$ the normal cone of S along X, that is the restriction to X of the closure of the set $\{(x, \mu(x\pi(x))) : x \in S - X\} \subset \mathbb{R}^n \times S^{n-1}$, where π is the local canonical projection onto X, $\mu(x)$ is the unit vector $\frac{x}{\|x\|}$, and here and throughout the paper pq denotes the vector $q-p$. In fact C_XS is the union of the normal cones C_XY_i, where $\{Y_i\}$ are the strata of S whose closures contain X.

Condition (n): The fibre $(C_XS)_x$ of C_XS at a point x of X is the tangent cone $C_x(S_x)$ to the fibre $S_x = S \cap \pi^{-1}(x)$ of S at x, for every stratum X of S.

Normal pseudo-flatness (npf): The projection $p : C_XS \to X$ is open for every stratum X of S.

When a stratification satisfies two conditions, for example Whitney (a)-regularity and (n)-regularity, we say it is $(a+n)$-regular. Subanalytic stratifications satisfying $(a+n)$ or (npf) have a normal cone with good behaviour from the point of view of the dimension of its fibres. In fact they satisfy the condition

$$\dim(C_XS)_x \leq \dim S - \dim X - 1. \qquad (*)$$

This is obvious for $(a+n)$, while for (npf) it follows from $(5.1.ii')$ of **[OT2]**. For differentiable stratifications one first needs to be able to define the dimension.

Despite this limitation, the tangent cone $C_x(S_x)$ to the fibre $S_x = S \cap \pi^{-1}(x)$ (hence the fibre $(C_XS)_x$ of the normal cone, assuming (n)) can be quite arbitrary: recent work of Ferrarotti, Fortuna and Wilson show that every closed semi-algebraic cone of codimension ≥ 1 is realised as the tangent cone at a point of a certain real algebraic variety **[FFW]**, while Kwieciński and Trotman showed that *every* closed cone is realised as the tangent cone at an isolated singularity of a certain $C^\infty(b)$-regular stratified espace **[KT]**.

Hironaka showed in **[H]** that a Whitney stratification (i.e. (b)-regular) of an analytic set (real or complex) is normally pseudo-flat along each stratum. J.-P. Henry et M. Merle **[HM2]** obtained (n) with S replaced by $X \cup Y$ when X and Y are two adjacent strata of a subanalytic Whitney stratification of $X \cup Y$.

Every C^2 (w)-regular stratification satisfies automatically (a) and (r^e), i.e. $(a+r^e)$. For subanalytic strata the combination $(a+r^e)$ is equivalent to the ratio test (r) introduced by T.-C. Kuo in 1971, which implies Whitney's condition (b) **[K]**; since **[T]** we know that (r) is strictly weaker than (w) in the semialgebraic case, and there even exist real algebraic examples **[BT]**. The equivalence of (b), (r) and (w) for complex analytic stratifications was completed by Teissier in 1982 (**[Te2]**, **[HM1]**).

In **[OT2]** it is proved that every $(a+r^e)$-regular stratification is normally pseudoflat and satisfies condition (n). Hence for (r)-regular stratifications which are definable in a polynomially bounded o-minimal structure, (n) and (npf) hold.

We recall the definitions of the conditions (a) and (b) of Whitney, (r) of Kuo **[K]**, and (w) of Kuo-Verdier **[Ve]**.

Let X and Y be two submanifolds of $\mathbb{R}^n$ such that $X \subset \overline{Y}$, and let π be the local projection onto X. Following Hironaka **[H]**, denote by $\alpha_{Y,X}(y)$ the distance of T_yY to $T_{\pi(y)}X$, which is

$$\alpha_{Y,X}(y) = \max\{< \mu(u), \mu(v) >: u \in N_yY - \{0\}, v \in T_{\pi(y)}X\},$$

and by $\beta_{Y,X}(y)$ the distance of $y\pi(y)$ to T_yY expressed as

$$\beta_{Y,X}(y) = \max\{< \mu(u), \mu(y\pi(y))\} >: u \in N_yY - \{0\}\},$$

where $<,>$ is the scalar product on $\mathbb{R}^n$.

For $v \in \mathbb{R}^n$, the distance of the vector v to a plane B is

$$\eta(v,B) = \sup\{< v, n >: n \in B^{\perp}, \|n\| = 1\}.$$

Set

$$d(A,B) = \sup\{\eta(v,B) : v \in A, \|v\| = 1\},$$

so that in particular

$$\alpha_{Y,X}(y) = d(T_{\pi(y)}X, T_yY).$$

Set also

$$R_{Y,X}(y) = \frac{\|y\|\alpha_{Y,X}(y)}{\|y\pi(y)\|} \quad \text{and} \quad W_{Y,X}(y,x) = \frac{d(T_xX, T_yY)}{\|yx\|}.$$

Definition. The pair of strata (X,Y) satisfies, at $0 \in X$:
condition (a) if, for y in Y,

$$\lim_{y \to 0} \alpha_{Y,X}(y) = 0,$$

condition (b^π) if, for y in Y,

$$\lim_{y \to 0} \beta_{Y,X}(y) = 0,$$

condition (b) if, for y in Y,

$$\lim_{y\to 0} \alpha_{Y,X}(y) = \lim_{y\to 0} \beta_{Y,X}(y) = 0,$$

condition (r) if, for y in Y,

$$\lim_{y\to 0} R_{Y,X}(y) = 0,$$

condition (w) if, for y in Y and x in X, $W_{Y,X}(y,x)$ is bounded near 0,

In **[OT2]** P. Orro and the first author introduced the following condition of Kuo-Verdier type.

Definition. Let $e \in [0,1)$. One says that (X,Y) satisfies condition (r^e) at $0 \in X$ if, for $y \in Y$, the quantity $R_e(y) = \frac{\|\pi(y)\|^e \alpha_{Y,X}(y)}{\|y\pi(y)\|}$ is bounded near 0.

This condition is a C^2 diffeomorphism invariant. It is Verdier's condition (w) when $e = 0$, hence (w) implies (r^e) for all $e \in [0,1)$. But, unlike (w), condition (r^e) when $e > 0$ does not imply condition (a) : a counter-example which is a semi-algebraic surface can be obtained by pinching a half-plane $\{z \geq 0, y = 0\}$ of $\mathbb{R}^3$, with boundary the axis $0x = X$, in a cuspidal region $\Gamma = \{x^2 + y^2 \leq z^p\}$, where p is an odd integer such that $p > \frac{2}{e}$, such that in Γ there are sequences tending to 0 for which condition (a) fails. Such an example will be (r^e)-regular.

Theorem[OT2]. *Every $(a + r^e)$-regular stratification is normally pseudo-flat and satisfies condition (n).*

Corollary. *For (r)-regular stratifications which are definable in a polynomially bounded o-minimal structure, (n) and (npf) hold.*

Now we recall the definition of E^*-regularity for E an equisingularity condition, as in **[OT1]**. This notion came from the discussion of B. Teissier in his 1974 Arcata lectures **[Te1]**. Teissier stated that one requirement for an equisingularity condition to be "good" is that it be preserved after intersection with generic linear spaces containing a given linear stratum. Various equisingularity conditions have been shown to have this property, notably Whitney (b)-regularity for complex analytic stratifications (**[Te2], [HM1]**), and Kuo's ratio test (r) and Verdier's condition (w) for subanalytic stratifications **[NT]**.

Definition. Let M be a C^2-manifold. Let X be a C^2-submanifold of M and $x \in X$. Let Y be a C^2-submanifold of M such that $x \in \overline{Y}$, and $X \cap Y = \emptyset$. Let E denote an equisingularity condition (examples: Whitney (b), (r), (w)). Then (X, Y) is said to be $E_{\mathrm{cod}\ k}$-regular at x $(0 \leq k < \mathrm{cod}\ X)$ if there is an open dense subset U^k of the Grassmann manifold of codimension k subspaces of T_xM containing T_xX such that if W is a C^2 submanifold of M with $X \subset W$ near x, and $T_xW \in U^k$, then W is transverse to Y near x, and $(X, Y \cap W)$ is E-regular at x.

Definition. (X, Y) is said to be E^*-regular at x if (X, Y) is $E_{\mathrm{cod}\ k}$-regular for all k, $0 \leq k < \mathrm{cod}\ X$.

Theorem[NT]. *For subanalytic stratifications,* (r) *implies* (r^*) *and* (w) *implies* (w^*).

Corollary. *For subanalytic* (b)*-regular stratifications,* (b^*) *holds over every* 1*-dimensional stratum.*

In the log-analytic example below, (r) and (b) hold, but (r^*) and (b^*) fail.

Example.

In $\mathbf{R}^3$ consider the graph Y of the function $f(x, z)$, for $z > 0$, and x and z small, where

$$y = f(x, z) = z - \frac{z}{\ln z} \ln(x + \sqrt{x^2 + z^2}).$$

Note that $\lim_{z \to 0} f(x, z) = 0$.

Then let X be the x-axis, so that $X \subset \overline{Y}$, and X and Y are disjoint C^∞ submanifolds of $\mathbf{R}^3$. We consider the closed stratified set S with just 2 strata (X, Y).

Remark 1. $f(x, z) = -f(-x, z)$*, i.e.* f *is an odd function of* x.

Proof.

$$\begin{aligned} f(x, z) + f(-x, z) &= 2z - \frac{z}{\ln z}[\ln(x + \sqrt{x^2 + z^2}) + \ln(-x + \sqrt{x^2 + z^2})] \\ &= 2z - \frac{z}{\ln z}[\ln(x^2 + z^2 - x^2)] = 2z - \frac{z.2 \ln z}{\ln z} = 0. \end{aligned}$$

Q.E.D.

Remark 2. $X \subset \overline{Y}$, *because* $\lim_{z\to 0} f(x,z) = 0$.

Proof. Obviously

$$\lim_{z\to 0} z = 0, \quad \text{and} \quad \lim_{z\to 0} \frac{1}{\ln z} = 0.$$

If $x > c > 0$, then $|\ln(x + \sqrt{x^2+z^2})| < |\ln(2c)|$, so that

$$\lim_{z\to 0} z \ln(x + \sqrt{x^2+z^2}) = 0.$$

By remark 1 we do not need to study the case of $x < 0$.
If both x and z tend to 0, consider the cases :
(i) $\frac{|z|}{|x|} \to 0$. Then

$$|z \ln(x + \sqrt{x^2+z^2})| < |z \ln(2x)| < |x \ln(2x)| \to 0 \quad \text{as} \quad x \to 0.$$

(ii) $\frac{|x|}{|z|}$ is bounded. Then

$$|z \ln(x + \sqrt{x^2+z^2})| \equiv |z \ln z| \to 0 \quad \text{as} \quad x \to 0.$$

Q.E.D.

We prove below that the following five properties hold :

(1) (n) **and** (npf) **fail at** $(0,0,0)$**.**
(2) (r) **holds.**
(3) (b) **holds.**
(4) (b^*) **and** (r^*) **fail at** $(0,0,0)$**.**
(5) The density of S **is constant along** X**.**

Property 1. (n) *and* (npf) *fail at* $(0,0,0)$.

Proof. We will show that the limits of secants from $(x,0,0)$ to $(x, f(x,z), z)$ as (x,z) tends to $(x_0, 0)$ are the straight lines which in the (y,z)-plane have equations

$$\begin{aligned} y = z \quad & \text{if} \quad x_0 > 0 \\ y = \sigma z \text{ for all } \sigma \in [-1,1] \quad & \text{if} \quad x_0 = 0 \\ y = -z \quad & \text{if} \quad x_0 < 0. \end{aligned} \tag{1.1}$$

However, for the secants from $(0,0,0)$ to $(0, f(0,z), z)$ as z tends to 0, the limiting secant is $y = 0$. Hence (n) fails (the tangent cone to

$C_0(S_0)$ does not equal the fibre at 0 of the normal cone). Moreover (npf) fails since for $x_0 \neq 0$ the fibre at x_0 of the normal cone is 0-dimensional, while the fibre at 0 is 1-dimensional.

Proof of (1.1). First observe that, for all $0 < z < 1$, the secant from $(0,0,0)$ to $(0, f(0,z), z)$ has slope

$$\frac{f(0,z)}{z} = 1 - \frac{\ln z}{\ln z} = 0.$$

Take $x_0 > 0$ and let (x,z) tend to $(x_0, 0)$. The slope of the secant from $(x,0,0)$ to $(x, f(x,z), z)$ is

$$\frac{f(x,z)}{z} = 1 - \frac{\ln(x + \sqrt{x^2+z^2})}{\ln z}$$

which tends to 1 as z tends to 0 and x tends to x_0.

By symmetry (Remark 1), when $x_0 < 0$ the limiting slope is -1.

Now suppose (x,z) tends to $(0,0)$.

By symmetry (Remark 1 again) it will be enough to study the case $x > 0$ and to show that all the values $\sigma \in [0, +1]$ are realised. So we must show that the limits of

$$\frac{\ln(x + \sqrt{x^2+z^2})}{\ln z}$$

take all values in $[0,1]$ as x and z tend to 0 when $x > 0$.

First notice that if $x < Cz$ for some positive constant C, then

$$\lim_{x\to 0, z\to 0} \frac{\ln(x + \sqrt{x^2+z^2})}{\ln z} = 1,$$

because

$$\ln(x + \sqrt{x^2+z^2}) = \ln z + \ln\left(\frac{x}{z} + \sqrt{\left(\frac{x}{z}\right)^2 + 1}\right),$$

and the second term is bounded and non-negative.

So it remains to check that $\ln(x + \sqrt{x^2+z^2})$ takes all values in $[0,1]$, when $z = o(x)$. Write

$$\frac{\ln(x + \sqrt{x^2+z^2})}{\ln z} = \frac{\ln x}{\ln z} + \frac{\ln\left(1 + \sqrt{1 + \left(\frac{z}{x}\right)^2}\right)}{\ln z}.$$

The second term on the right has a bounded numerator so goes to 0 as (x, z) goes to $(0, 0)$. Because $0 < z < x < 1$, the first term on the right belongs to $(0, 1)$.

Let $\alpha \in (0, 1)$. On the curve $x = z^\alpha$,

$$\frac{\ln x}{\ln z} = \alpha,$$

so that

$$\lim \frac{\ln(x + \sqrt{x^2 + z^2})}{\ln z} = \alpha.$$

On the curve $x|\ln z| = +1$,

$$\frac{\ln x}{\ln z} = x|\ln x|,$$

with limit 0 as x tends to 0.

This completes the proof of (1.1), and hence the proof of Property 1. Q.E.D.

Property 2. (r) *holds for the pair of strata* (X, Y) *at* $(0, 0)$.

Proof. Recall that Kuo's ratio test (r) holds when

$$\frac{|(x, y, z)|.d(T_{(x,0,0)}X, T_{(x,y,z)}Y)}{|(y, z)|} \to 0$$

as (x, y, z) tends to $(0, 0, 0)$ on Y.

Now,

$$d(T_{(x,0,0)}X, T_{(x,y,z)}Y) = \frac{|\frac{\partial f}{\partial x}|}{|(\frac{\partial f}{\partial x}, -1, \frac{\partial f}{\partial z})|} < |\frac{\partial f}{\partial x}|.$$

And

$$\begin{aligned}\frac{|\frac{\partial f}{\partial x}|.|(x, y, z)|}{|(y, z)|} &\approx \frac{|\frac{\partial f}{\partial x}|}{|z|}.\sqrt{x^2 + z^2} \\ &= \frac{z}{|\ln z|}.\frac{1}{x + \sqrt{x^2 + z^2}}.(1 + \frac{x}{\sqrt{x^2 + z^2}}).\frac{\sqrt{x^2 + z^2}}{z} \\ &= \frac{1}{|\ln z|}\end{aligned}$$

which tends to 0 as z tends to 0.

We check directly that (a) holds. As above, $d(T_{(x,0,0)}X, T_{(x,y,z)}Y) < |\frac{\partial f}{\partial x}|$.

But $|\frac{\partial f}{\partial x}| = \frac{|z|}{|\ln z|}.\frac{1}{\sqrt{x^2+z^2}} < \frac{1}{|\ln z|}$, which tends to 0 as z tends to 0, as required. Q.E.D.

Note that although (r) holds, this argument does not show that (r^e) (of **[OT2]**) holds. In fact we know already, by the main theorem of **[OT2]**, that (r^e) must fail, because (a) holds, while (n) and (npf) fail.

Property 3. *(b) holds for (X, Y) at $(0,0,0)$.*

Proof. We have just seen that (a) holds. Thus we need only prove that (b^π) holds.

Again by remark 1, we need only treat the case $x \geq 0$.

Suppose $0 < z < 1$, and $0 \leq x$, for x small.

Then $x + \sqrt{x^2+z^2} \geq z$, so that

$$0 < \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z} \leq 1. \qquad (*)$$

Also

$$0 < \frac{z^2}{x\sqrt{x^2+z^2}+x^2+z^2} \leq 1. \qquad (**)$$

The zy slope of the secant line from $(x,0,0)$ to $(x, f(x,z), z)$ is

$$\frac{f(x,z)}{z} = 1 - \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}.$$

The zy slope of the tangent in the z direction on Y is

$$\frac{\partial f}{\partial z}(x,z) = 1 - \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z} - z\frac{\partial}{\partial z}\left(\frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right).$$

To prove that Whitney (b^π) holds at $(0,0,0)$ we must show that

$$\lim_{(x,z)\to(0,0)} \left(z\frac{\partial}{\partial z}\left(\frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right)\right) = 0.$$

But

$$z\frac{\partial}{\partial z}\left(\frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right) = \frac{1}{\ln z}\left(\frac{z^2}{x\sqrt{x^2+z^2}+x^2+z^2} - \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right),$$

which tends to 0 as z tends to 0 for x small, by $(*)$ and $(**)$. This implies that (b^π) holds, and hence that (b) holds for (Y, X) on a neighbourhood of $(0,0,0)$ in X. Q.E.D.

Property 4. (b^*) *and* (r^*) *fail at* $(0,0,0)$.

Proof. We intersect Y with planes $\{y = az, -1 < a < 1\}$ to obtain

$$(1-a)z = \frac{z}{\ln z}\ln(x + \sqrt{x^2+z^2}),$$

which becomes

$$z^{1-a} = x + \sqrt{x^2+z^2}.$$

Squaring, we get $x^2 + z^2 = z^{2-2a} - 2xz^{1-a} + x^2$ which simplifies to

$$x = \frac{z^{1-a} - z^{1+a}}{2}.$$

This curve in the xz-plane passes through $(0,0)$ if $-1 < a < 1$, showing that (b^*) and (r^*) fail to hold, since $Y \cap \{y = az\}$ contains curves passing through $(0,0,0)$, and so $(X, Y \cap \{y = az\})$ cannot be (b)-regular, because (a) fails for $(X, Y \cap \{y = az\})$, and hence (X,Y) is not $(b_{\text{cod } 1})$-regular and (b^*) fails, implying that (r^*) fails also. Q.E.D.

Property 5. *The density of* S *is constant along* X.

Proof. We show first that the tangent cone to S at $(0,0,0)$ is the half-plane $\{y = 0, z \geq 0\}$.

Each definable curve on Y which passes through $(0,0,0)$ and which is not tangent to X has a projection to the (x,z)-plane tangent to some line $x = cz$, where c is a nonzero constant. On such a curve,

$$y = z\left(1 - \frac{\ln(cz + o(z) + \sqrt{c^2z^2 + 2cz.o(z) + o(z^2) + z^2})}{\ln z}\right)$$

$$= \frac{z\ln(c + o(1) + \sqrt{c^2 + 2c.o(1) + 1})}{\ln z} = o(z).$$

Hence such a curve on Y is tangent to $\{y = 0\}$.

Now consider a curve whose projection to the (x,z)-plane is tangent to the x-axis, so of the form $(x, y(x,z), z(x))$ where $z = o(x)$. Then

$$y = z\left(1 - \frac{\ln(x + \sqrt{x^2+z^2})}{\ln z}\right)$$

$$= O(z) = o(x),$$

so that again the curve itself is tangent to the x-axis. It follows that the tangent cone to S at $(0,0,0)$ is $\{y = 0, z \geq 0\}$.

It is easy to see that the tangent cone at $(x_0, 0, 0)$ equals $\{y = z\}$ if $x_0 > 0$ and equals $\{y = -z\}$ if $x_0 < 0$.

It follows that the density of S at points of the x-axis has the constant value $1/2$. Q.E.D.

References

[BT] H. Brodersen and D. Trotman, Whitney (b)-regularity is strictly weaker than Kuo's ratio test for real algebraic stratifications, Math. Scand., **45** (1979), 27–34.

[C] G. Comte, Equisingularité réelle : nombres de Lelong et images polaires, Ann. Sci. Ecole Norm. Sup., Paris (4), **33** (2000), 757–788.

[DM] L. van den Dries and C. Miller, Geometric categories and o-minimal structures, Duke Math. Journal, **84** (1996), 497–540.

[D] L. van den Dries, Tame Topology and 0-minimal structures, Cambridge University Press, London Mathematical Society Lecture Note Series, **248** (1998).

[FFW] M. Ferrarotti, E. Fortuna and L. Wilson, Real algebraic varieties with prescribed tangent cones, Pacific J. of Math., **194** (2000), 315–323.

[HM1] J.-P. Henry and M. Merle, Limites de normales, conditions de Whitney et éclatement d'Hironaka, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math. Amer. Math. Soc., **40** (1983), 575–584.

[HM2] J.-P. Henry and M. Merle, Stratifications de Whitney d'un ensemble sous-analytique, Comptes Rendus de l'Académie des Sciences de Paris, Série I, **308** (1989), 357–360.

[H] H. Hironaka, Normal cones in analytic Whitney stratifications, Inst. Hautes Études Sci. Publ. Math., **36** (1969), 127–138.

[K] T.-C. Kuo, The ratio test for analytic Whitney stratifications, Proceedings of the Liverpool Singularities Symposium I, (ed. C. T. C. Wall), Springer Lecture Notes in Math., **192** (1971), 141–149.

[KT] M. Kwiecinski and D. Trotman, Scribbling continua in R^n and constructing singularities with prescribed Nash fibre and tangent cone, Topology and its Applications, **64** (1995), 177–189.

[L] T. L. Loi, Verdier and strict Thom stratifications in o-minimal structures, Illinois J. Math., **42** (1998), 347–356.

[M] C. Miller, Exponentiation is hard to avoid, Proc. Amer. Math. Soc., **122** (1994), 257–259.

[NT] V. Navarro Aznar and D. Trotman, Whitney regularity and generic wings, Annales Inst. Fourier, Grenoble, **31** (1981), 87–111.

[OT1] P. Orro and D. Trotman, On the regular stratifications and conormal structure of subanalytic sets, Bull. London Math. Soc., **18** (1986), 185–191.

[OT2] P. Orro and D. Trotman, Cône normal et régularités de Kuo-Verdier, Bull. Soc. Math. France, **130** (2002), 71–85.

[Te1] B. Teissier, Introduction to equisingularity problems, Providence, Rhode Island, Proc. Sympos. Pure Math., Algebraic Geometry, Arcata, 1974, **29** (1975), 593–632.

[Te2] B. Teissier, Variétés polaires II: Multiplicités polaires, sections planes, et conditions de Whitney, Sptringer-Verlag, New York, Algebraic Geometry, Proc., La Rabida, 1981, Lecture Notes in Math., **961** (1982), 314–491.

[T] D. Trotman, Counterexamples in stratification theory: two discordant horns, Sijthoff and Noordhoff, Alphen aan den Rijn, Real and Complex Singularities, Proc. 9th Nordic Summer School, Oslo, 1976, (ed. P. Holm), 1977, 679–686.

[V] G. Valette, Détermination et stabilité du type métrique des singularités, University of Provence thesis, 2003.

[Ve] J.-L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, Inventiones Math., **36** (1976), 295–312.

David Trotman
Laboratoire d'Analyse, Topologie et Probabilites,
Centre de Mathématiques et Informatique,
Université de Provence,
39 rue Joliot-Curie,
13453 MARSEILLE Cedex 13
France

Leslie Wilson
Department of Mathematics
University of Hawaii
Honolulu, HI, 96822
USA