

# POINCARÉ-HOPF THEOREMS ON SINGULAR SPACES

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ABSTRACT. In 1991 M.-H. Schwartz proved a Poincaré-Hopf theorem for stratified radial vector fields on Whitney stratified analytic varieties with boundary. In this paper we extend her theory and results to a wide class of stratified sets with boundary, called radial manifold complexes, which include all Thom-Mather stratified sets, but also arbitrary closure orderable finite partitions of subanalytic sets into subanalytic submanifolds. In the process we give a short proof of a generalization of the Poincaré-Hopf theorem for radial vector fields of M.-H. Schwartz (our Theorem 3.2).

We then study more general stratified vector fields which need not be radial and may further be tangent to the boundary of the manifold complex, and prove the corresponding Poincaré-Hopf theorems. In particular, a notion is given of the virtual index of a vector field on a radial manifold complex and the Euler characteristic of the complex is shown to be the sum of the virtual indices.

## 1. INTRODUCTION

The classical Poincaré-Hopf theorem for a vector field  $v$  with only isolated zeros on a smooth compact manifold  $M$  without boundary says that the Euler characteristic of  $M$  equals the sum of the indices of  $v$  at its zeros. M.-H. Schwartz has developed a theory ([13], [14], [15]) giving an analogous formula for the Euler characteristic of a Whitney stratified (singular) analytic variety  $V$  of an analytic manifold  $M$  as the sum of the indices of zeros on  $V$  of a “radial” vector field  $v$  on  $M$  which is tangent to the strata of  $V$ . The index of a zero on a stratum  $X$  is defined to be the index of the restriction of  $v$  to  $X$ . The “radial” vector fields of M. H. Schwartz are defined in terms of two types of families of tubes around strata, one type called parametric tubes, given by barycentric subdivision of a triangulation, and the other called geodesic tubes, given by certain geodesic tubular neighborhoods defined using the ambient metric. Such a vector field is exiting from sufficiently small geodesic tubes around a stratum  $X$  over some closed subset of  $X$  which contains the zeros of  $v$  on  $X$ . Having two types of families of tubes without

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compatibility relations makes proving the existence of M. H. Schwartz’s “radial” fields quite complicated, and further requires delicate extension arguments and angle estimates when adapting the “radial” vector fields to yield “exiting” vector fields in a suitable form for the Poincaré-Hopf theorem (see [15]).

Now it is well-known that Whitney stratified sets (not necessarily analytic) possess compatible systems of tubes around strata, and thus they are Thom-Mather stratified sets, [20], [9]. We exploit this to define a simple notion of radial stratified vector field on radial manifold complexes. Radial manifold complexes are certain generalizations of Thom-Mather stratified sets which are not necessarily stratified sets in the classical sense, although we will continue to often refer to the “pieces” of the manifold complex as the “strata” and the complex as a whole as a stratified set. Our radial manifold complexes are much more general than Whitney stratified sets or Thom-Mather abstract stratified sets. For example, any closure orderable finite subanalytic partition of a subanalytic set into subanalytic submanifolds is a radial manifold complex in our sense. (Closure orderable just means that there is an order on the partition so that the closure of any piece only intersects lesser pieces.) In particular our radial manifold complexes are not required to have strata of increasing dimensions in successive skeleta, nor are they necessarily locally topologically trivial along strata. In general, our radial manifold complexes are not even triangulable, whereas Thom-Mather stratified sets are triangulable [5].

The proofs of existence of our radial vector fields (Lemma 3.1), and of the Poincaré-Hopf formula (Theorem 3.2), become straightforward and short. We use induction on the depth of strata, rather than on their dimension. This produces much shorter proofs. Now observe that in the abstract context of our theory, ambient continuity of stratified vector fields has no meaning. As a consequence, when our results are applied to embedded stratified sets in a manifold, they hold for stratified vector fields which are not continuous, unlike those of M. H. Schwartz whose “radial” vector fields are always required to be continuous on the ambient manifold. Natural examples of non-continuous stratified vector fields are the projection onto strata of a given vector field on  $M$ , in particular the gradient vector field on strata which is defined by a stratified Morse function [6].

In the present paper we show that the Poincaré-Hopf formula applies moreover to what we call semiradial vector fields (Corollary 6) – roughly those which are not vertically descending near strata with respect to the radial structure. Further generalization is obtained by defining a virtual index at points on a stratum  $X$  which are limits of points where  $v$  is vertically descending towards  $X$ ; we call such limit points virtual zeros. The Euler characteristic of the manifold complex is shown to equal the sum of the indices and virtual indices of a stratified vector field whose zeros and virtual zeros are isolated (Theorem 5.4). This applies to generic vector fields in many embedded situations.

Another improvement we obtain concerns behavior of  $v$  on the boundary of the manifold complex, defined to be the union of the boundaries of strata (assumed to be manifolds with boundary). We allow tangency, and deduce as a special

case the Morse-Pugh theorem (see the discussion after Theorem 3.3) which gives a Poincaré-Hopf formula for generic vector fields on a manifold with boundary [11], [12]. We note that V. Arnol'd [3], again for manifolds with boundary, gave a refined Poincaré-Hopf index formula for zeros of 1-forms. It would be interesting to understand the relation between the index of Arnol'd and our virtual index. In the case of smooth manifolds with boundary, our Poincaré-Hopf formula reduces to that of Becker and Gottlieb in [4]. We thank M. Shub, J. Dupont, and D. Cohen-Steiner for respectively indicating to us the papers of Pugh, Morse, and Becker-Gottlieb.

To sum up, we extend the theory of M.-H. Schwartz in four ways :

- Our stratified sets (radial manifold complexes) need not be analytic, nor Whitney regular, nor even possess a Thom-Mather structure. They include arbitrary closure orderable finite subanalytic decompositions of subanalytic sets into smooth manifolds.
- Our vector fields are not required to be continuous in the ambient space, even when this makes sense. Continuity was essential for the proof given by M.-H. Schwartz, where the stratified vector field is extended to a continuous vector field on a neighborhood of the stratified set so as to be able to apply the classical Poincaré-Hopf theorem.
- The vector fields we consider are not necessarily radial : we first allow semi-radiality, and then remove this condition, requiring only that both virtual zeros and zeros are isolated.
- We do not require that our vector fields be transverse to the boundary (i.e. be either exiting or entering) : tangency is allowed.

The classical Poincaré-Hopf theorem for vector fields on smooth manifolds (with boundary) can be deduced from the Lefschetz fixed-point theorem. Goresky and MacPherson (in [7]) have recently given generalized Lefschetz fixed point formulae for self-maps of stratified subanalytic sets. Our theory applies to more general sets, but even when the objects coincide, Lefschetz fixed-point formulae usually cannot be used to give Poincaré-Hopf index formulae because most stratified vector fields are not globally integrable, so will not generate self-maps.

## 2. RADIAL MANIFOLD COMPLEXES

The objects we will be considering are stratified sets of a certain general type.

**Definition 2.1.** For the purposes of this paper, a *manifold complex with boundary* is a locally compact Hausdorff topological space  $X$  and a finite nested collection of closed subsets  $X_0 \subset X_1 \subset \dots \subset X_m = X$  so that if  $S_i = X_i - X_{i-1}$  then:

- (1) Each  $S_i$  is a topological manifold with boundary  $\partial S_i$ .
- (2) The closure of each boundary satisfies  $\text{Cl}(\partial S_i) \subset \bigcup_{j=0}^i \partial S_j$ .

We call  $X_i$  the *skeleta* of  $X$  and  $S_i$  the *strata* of  $X$ . We let  $\partial X$  denote  $\bigcup_{j=0}^m \partial S_j$ . Note that  $\partial X$  has the natural structure of a manifold complex with empty boundary. We say the *depth* of  $X$  is  $m' - \ell$  where  $m'$  is the smallest index so that

$X_{m'} = X$  and  $\ell$  is the smallest index so that  $X_\ell$  is not empty. We say  $X$  is *smooth* if we put a smooth structure on each stratum. For the purposes of this paper,  $C^1$  is smooth enough (except for one place in Lemma 5.1 where Sard's Theorem is used).

Note that the strata  $S_i$  need not have dimension  $i$ . In fact the dimensions of  $S_i$  need not be monotonically increasing. For example, any finite CW complex is a smooth manifold complex (with empty boundary).

**Remark 2.2.** An open subset of a (smooth) manifold complex inherits the structure of a (smooth) manifold complex in the natural way (skeleta and strata are just intersected with the open set). Likewise there is a natural (smooth) manifold complex structure on the product of a (smooth) manifold complex with a (smooth) manifold. With an indexing convention there is even a manifold complex structure on the product of two manifold complexes.

**Definition 2.3.** If  $X$  is a smooth manifold complex with boundary then  $\mathbf{v}$  is a *vector field* on  $X$  if for each  $x \in X$  if  $x \in S_i$  then  $\mathbf{v}(x)$  is a vector in the tangent space to  $S_i$  at  $x$ . Furthermore, we ask that the restriction of  $\mathbf{v}$  to each stratum  $S_i$  be continuous.

In other words, a vector field is a collection of continuous vector fields on the strata of  $X$ . Note however, that we do not ask for global continuity of a vector field. So for example, if  $X$  happened to be embedded in some manifold, we do not ask that a vector field be the restriction of some continuous vector field.

The above notion of manifold complex is a bit too general to work with conveniently. The strata might approach each other in a very complicated way. So we add more structure. For example Thom-Mather stratified sets have the properties alluded to in Definition 2.4 below. On a first reading the reader may prefer to think only of the more familiar Thom-Mather stratified sets or Whitney stratified sets rather than the more general radial manifold complexes defined below.

**Definition 2.4.** We say that  $X$  is a radial manifold complex with boundary if it is a smooth manifold complex with boundary and for each stratum  $S_i$  there is an open neighborhood  $U_i$  of  $S_i$  in  $X - X_{i-1}$ , a continuous map  $\rho_i: U_i \rightarrow [0, 1]$  a vector field  $\xi_i$  on  $U_i - S_i$  and a homotopy  $\varphi: U_i \times [0, 1] \rightarrow X - X_{i-1}$  so that:

- (1)  $\rho_i^{-1}(0) = S_i$ .
- (2)  $\xi_i$  is tangent to  $\partial X$ .
- (3) For each  $i < j$  the restriction of  $\rho_i$  to  $U_i \cap S_j$  is smooth, with no critical points.
- (4)  $d\rho_i(\xi_i) = 1$  for all  $i$  and  $d\rho_i(\xi_j) = 0$  for all  $i < j$ .
- (5)  $\varphi_i(x, 0) = x$  and  $\varphi_i(x, 1) \in S_i$  for all  $x \in U_i$ .
- (6)  $\varphi_i(x, t) = x$  for all  $x \in S_i$  and  $t \in [0, 1]$ .
- (7)  $\rho_i \circ \varphi_i(x, t_0) \geq \rho_i \circ \varphi_i(x, t_1)$  if  $t_0 \leq t_1$  and  $\varphi_i(x, t_j) \in U_i$ .
- (8)  $\rho_i \circ \varphi_j(x, t) = \rho_i(x)$  if  $j > i$ ,  $x \in U_i \cap U_j$ , and  $\varphi_j(x, t) \in U_i$ .
- (9)  $\varphi_i((U_i \cap \partial X) \times [0, 1]) \subset \partial X$ .

So speaking loosely, a radial manifold complex with boundary is a manifold complex with boundary, together with functions  $\rho_i$  which we may think of as giving the distance to  $S_i$ , and deformations  $\varphi_i$  of neighborhoods  $U_i$  to  $S_i$ . These deformations are compatible with the distance functions  $\rho_i$  in that they never increase distance, and preserve distance to lower strata. The vector fields  $\xi_i$  are just concrete manifestations of the property that any map  $(\rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_d})$  has maximal rank  $d$  on  $(U_{i_1} - S_{i_1}) \cap \dots \cap (U_{i_d} - S_{i_d})$ . (Note that condition 4 implies that the vector fields  $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_d}$  are linearly independent on  $(U_{i_1} - S_{i_1}) \cap \dots \cap (U_{i_d} - S_{i_d})$ .)

**Example 2.5.** Any smooth manifold  $M$  with boundary is a radial manifold complex with boundary with just one stratum  $X_0 = M$ . We have  $\rho_0(x) = 0$ ,  $\varphi_0(x, t) = x$ , and  $\xi_0$  has empty domain.

**Example 2.6.** A Whitney stratified set is a radial manifold complex with boundary. The maps  $\rho_i$  are just part of the usual tubular data and  $\varphi_i$  is obtained by integrating a controlled vector field  $\xi_i$  on  $U_i - S_i$  with  $d\rho_i(\xi_i) = 1$  and  $d\pi_i(\xi_i) = 0$ .

**Remark 2.7.** By items 2 and 9, we know that  $\partial X$  inherits the structure of a radial manifold complex with boundary by just restricting each  $\rho_i$ ,  $\varphi_i$ , and  $\xi_i$  to  $U_i \cap \partial X$ . Of course  $\partial X$  has empty boundary. The boundary  $\partial X$  might not have a collar neighborhood, see Example 3.4.

**Remark 2.8.** We do not assume any smoothness of  $\varphi_i$ , at least until we get to section 5. Requiring smoothness at this point would unnecessarily complicate gluing and corner rounding. We also do not require that  $\varphi_i$  preserve strata, although in practice it usually will.

It is straightforward to prove that an arbitrary closure orderable finite subanalytic partition of a subanalytic set into smooth submanifolds has a radial manifold complex with boundary structure. (For a proof one can use a subanalytic triangulation, or a subanalytic Whitney stratification compatible with the given structure.)

Apparently, stratified sets with boundary were first considered in [2]; they were also used in [21]. Siebenmann was the first to use the useful notion of depth in [16].

### 3. POINCARÉ-HOPF THEOREMS FOR RADIAL VECTOR FIELDS

From now on  $X$  will be a radial manifold complex with boundary with the various data  $U_i$ ,  $\xi_i$ ,  $\rho_i$  and  $\varphi_i$  as above. For convenience, extend each  $\xi_i$  (noncontinuously) to be zero outside  $U_i - X_i$ .

Let  $\mathbf{v}$  be a vector field on  $X$ , i.e., for each  $x \in X$  if  $x \in S_i$  then  $\mathbf{v}(x)$  is a vector in the tangent space to  $S_i$  at  $x$ . Furthermore, the restriction of  $\mathbf{v}$  to each stratum  $S_i$  is continuous.

**Definition 3.1.** We say that a vector field  $\mathbf{v}$  is *radial* if for each  $i$  there is a neighborhood  $U'_i$  of  $S_i$  in  $U_i$  so that  $d\rho_i(\mathbf{v}(x)) > 0$  for all  $x \in U'_i - S_i$ .

**Lemma 3.1.** *Any radial manifold complex with boundary  $X$  has a radial vector field  $\mathbf{v}$  with isolated zeroes which points inward on  $\partial X$ .*

*Proof.* Pick any vector field  $\mathbf{v}'$  on  $X$  which points inward on  $\partial X$ . This is merely a matter of picking a continuous vector field on each stratum  $S_i$  which points inward on  $\partial S_i$ . We suppose by induction on  $k$  that we have constructed nonnegative functions  $\beta_i: X \rightarrow [0, \infty)$  for  $0 \leq i < k$  so that:

- a)  $\beta_i$  is continuous on each stratum.
- b)  $\beta_i$  vanishes on  $X_i$  and  $X - U_i$  and in fact, the support of the restriction of  $\beta_i$  to any stratum  $S_j$  is contained in  $U_i \cap S_j$ . Thus  $\beta_i \xi_i$  is continuous on each stratum of  $X$ .
- c) For each  $j < k$ , there is a neighborhood  $U'_j$  of  $S_j$  so that on  $U'_j - S_j$  we have  $d\rho_j(\mathbf{v}' + \sum_{i=0}^{k-1} \beta_i \xi_i) > 0$ .

It is easy to choose a suitable  $\beta_k$  to complete the induction. We only need choose  $\beta_k$  continuous on each stratum, vanishing on  $X_k$ , with support in  $U_k \cup X_k$ , thus giving a) and b), so that for some neighborhood  $U'_k$  of  $S_k$  we have

$$\beta_k(x) > -d\rho_k(\mathbf{v}'(x)) - \sum_{i=0}^{k-1} \beta_i(x) d\rho_k(\xi_i(x))$$

for all  $x \in U'_k - S_k$ . Since  $d\rho_j(\beta_k \xi_k) = 0$  for all  $j < k$ , we know c) still holds.

So we may assume  $k = m$ , the maximal index. Let  $\mathbf{v}''$  be the vector field  $\mathbf{v}' + \sum_{i=0}^{m-1} \beta_i \xi_i$ . Then  $\mathbf{v}''$  is radial because of c) and it points inward on  $\partial X$  because  $\mathbf{v}'$  did and each  $\xi_i$  is tangent to  $\partial X$ . We could almost let  $\mathbf{v}$  be  $\mathbf{v}''$ , but  $\mathbf{v}''$  might not have isolated zeroes. However, c) implies that the zeroes of  $\mathbf{v}''$  in any stratum  $S_i$  must lie in  $S_i - \bigcup_{j < i} U'_j$ . So we may, after perhaps slightly shrinking the neighborhoods  $U'_j$ , let  $\mathbf{v}$  restricted to any stratum  $S_i$  be any vector field which points inward on  $\partial S_i$ , has isolated zeroes, and equals  $\mathbf{v}''$  on  $\bigcup_{j < i} U'_j \cap S_i$ .  $\square$

A similar result for Thom-Mather stratified sets was proved independently by Simon [17].

We now give our first Poincaré-Hopf theorem in Theorem 3.2 below. As an example of its application, let  $X$  be a finite simplicial complex, the strata being the interiors of simplices. Take the vector field  $\mathbf{v}$  which on each simplex points directly to the barycenter of that simplex. Then each simplex  $\sigma$  contains exactly one zero of  $\mathbf{v}$  (at the barycenter) and its index is  $(-1)^{\dim \sigma}$ . Thus Theorem 3.2 gives the usual formula for the Euler characteristic of  $X$ .

**Definition 3.2.** Let  $\mathbf{v}$  be a vector field all of whose zeroes are isolated. Then to each point  $p \in S_i - \partial S_i$  we may assign an index  $\mathcal{I}(p, \mathbf{v})$  by restricting  $\mathbf{v}$  to  $S_i$  and taking the usual index (c.f., [10]). Note that  $\mathcal{I}(p, \mathbf{v}) = 0$  unless  $\mathbf{v}(p) = 0$ . For convenience we also set  $\mathcal{I}(p, \mathbf{v}) = 0$  for  $p \in \partial X$  where  $\mathbf{v}(p) \neq 0$  and leave  $\mathcal{I}(p, \mathbf{v})$  undefined for  $p$  a zero of  $\mathbf{v}$  on  $\partial X$ .

**Theorem 3.2.** *Let  $X$  be a compact radial manifold complex with boundary. Suppose  $\mathbf{v}$  is a radial vector field on  $X$ , the zeroes of  $\mathbf{v}$  are isolated and they do not occur on the boundary of  $X$ . Then:*

- (1) *If  $\mathbf{v}$  points inward on  $\partial X$  then  $\chi(X) - \chi(\partial X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v})$ .*
- (2) *If  $\mathbf{v}$  points outward on  $\partial X$  then  $\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v})$ .*
- (3) *In fact, if  $\mathbf{v}$  points inward on some components  $\partial X_-$  of  $\partial X$  and outward on the others, then  $\chi(X) - \chi(\partial X_-) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v})$ .*

*Proof.* We first show part 1, and then the other results will follow from that. We prove part 1 by induction on the depth of  $X$ .

Let  $k$  be the smallest index so that  $X_k$  is not empty. Then  $X_k = S_k$  is compact. Since  $\mathbf{v}$  is radial, there is a compact neighborhood  $U'_k$  of  $S_k$  in  $U_k$  so that  $d\rho_k(\mathbf{v}) > 0$  on  $U'_k - S_k$ . By compactness we may pick an  $\epsilon > 0$  which is less than  $\rho_k(x)$  for all  $x$  in the frontier  $\text{Fr}(U'_k)$ .

Let  $X' = X - \rho_k^{-1}([0, \epsilon])$  and let  $Y = \rho_k^{-1}([0, \epsilon])$ . Since  $d\rho_k(\mathbf{v}) > 0$  on  $U'_k - S_k$  we know by claim 3.2.1 that  $\mathbf{v}$  has no zeroes on  $\rho_k^{-1}([0, \epsilon])$ .

**Claim 3.2.1.**  *$Y$  is a compact subset of  $U'_k$*

To see this claim, note that if  $x \in U_k - U'_k$  and  $\rho_k(x) \leq \epsilon$ , then  $\varphi_k(x, t) \in \text{Fr}(U'_k)$  for some  $t$ , since  $\varphi_k(x, 1) \in S_k \subset U'_k$ . But then  $\rho_k \circ \varphi_k(x, t) \leq \rho_k \circ \varphi_k(x, 0) = \rho_k(x) \leq \epsilon$  which violates our choice of  $\epsilon$ . So  $Y \subset U'_k$ . But then  $Y$  is compact since  $U'_k$  is compact and  $Y$  is a closed subspace of  $U'_k$ .

**Claim 3.2.2.**  *$X'$  can be given the structure of a radial manifold complex with boundary (by rounding corners). Furthermore,  $\mathbf{v}$  points inward on  $\partial X'$  and the depth of  $X'$  is less than that of  $X$ . Also  $\partial X' = (\partial X \cap X') \cup \rho_k^{-1}(\epsilon)$ .*

The proof of claim 3.2.2 is just corner rounding. Full details are in Lemma 6.6, but a sketch follows. Set  $X'_i = X_i \cap X'$  and let  $\xi'_i$ ,  $\varphi'_i$  and  $\rho'_i$  be the restrictions of  $\xi_i$ ,  $\varphi_i$  and  $\rho_i$  for  $i > k$  and empty for  $i \leq k$ . (It may be necessary to shrink  $U_i$  a bit before restricting to make sure that  $\varphi_i((U_i \cap X') \times [0, 1]) \subset X'$ .) Since  $\epsilon$  is a regular value of  $\rho_k|_{S_i}$ , the stratum  $S'_i = S_i \cap X'$  is a smooth manifold with boundary  $(\partial S_i \cap X') \cup (S_i \cap \rho_k^{-1}(\epsilon))$  and a corner on the boundary at  $\partial S_i \cap \rho_k^{-1}(\epsilon)$ . Smooth out this corner. The restriction of  $\mathbf{v}$  to  $X'$  still points inward on  $\partial X'$  since  $\mathbf{v}$  points inward on  $\partial X$  and  $d\rho_k(\mathbf{v}) > 0$  on  $\rho_k^{-1}(\epsilon)$ . The vector fields  $\xi'_i$  are tangent to  $\partial X'$  since  $d\rho_k(\xi_i) = 0$ .

**Claim 3.2.3.**  *$\varphi_k(Y \times [0, 1]) \subset Y$ . Hence  $Y$  deformation retracts to  $X_k$  and  $Y \cap \partial X$  deformation retracts to  $\partial X_k$ .*

To see the claim, suppose that  $\varphi_k(x, t) \notin Y$  for some  $x \in Y$ . Then  $\varphi_k(x, t') \in U_k - Y$  for some  $t' \leq t$ , but then  $\epsilon \geq \rho_k \circ \varphi_k(x, 0) \geq \rho_k \circ \varphi_k(x, t') > \epsilon$ , a contradiction.

By claim 3.2.2 and induction on depth:

$$(3.1) \quad \chi(X') - \chi(\partial X') = \sum_{p \in X'} \mathcal{I}(p, \mathbf{v}) = \sum_{p \in X - X_k} \mathcal{I}(p, \mathbf{v})$$

since  $\mathcal{I}(p, \mathbf{v}) = 0$  for all  $p \in \rho_k^{-1}((0, \epsilon])$  because  $\mathbf{v}(p) \neq 0$ . Also by claim 3.2.3 and the classical Poincaré-Hopf theorem for manifolds we have

$$(3.2) \quad \sum_{p \in X_k} \mathcal{I}(p, \mathbf{v}) = \chi(X_k) - \chi(\partial X_k) = \chi(Y) - \chi(Y \cap \partial X).$$

So adding equations (3.1) and (3.2) we have

$$(3.3) \quad \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X') - \chi(\partial X') + \chi(Y) - \chi(Y \cap \partial X).$$

By Theorem 6.1 applied to  $X$ ,  $\partial X$ , and  $\partial X'$  respectively we know that  $(X', Y)$ ,  $(\partial X \cap X', \partial X \cap Y)$  and  $(\partial X \cap X', X' \cap Y)$  are excisive couples and hence the usual additive property of Euler characteristic holds. Thus:

$$(3.4) \quad \chi(X) = \chi(X') + \chi(Y) - \chi(X' \cap Y)$$

$$(3.5) \quad \chi(\partial X) = \chi(\partial X \cap X') + \chi(Y \cap \partial X) - \chi(Y \cap \partial X \cap X')$$

$$(3.6) \quad \chi(\partial X') = \chi(\partial X \cap X') + \chi(X' \cap Y) - \chi(\partial X \cap X' \cap Y).$$

Thus combining (3.3)-(3.6) we have

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X)$$

and we have proven part 1).

Note that 3) easily implies 2). So to finish the proof, it suffices to prove part 3). In fact this is done in more generality in Theorem 3.3 below, but we give here a more direct proof. Using Lemma 3.1, pick a radial vector field  $\mathbf{v}'$  on  $X$  with isolated zeroes which points inward on all of  $\partial X$ . Let  $\partial X_+ = \partial X - \partial X_-$  and let  $X''$  be the radial manifold complex with boundary  $X'' = X \times 0 \cup \partial X_+ \times [0, 1] \cup X \times 1$ . The strata of  $X''$  are  $S_i'' = S_i \times 0 \cup (\partial X_+ \cap \partial S_i) \times [0, 1] \cup S_i \times 1$ . The boundary of  $X''$  is  $\partial X'' = \partial X_- \times \{0, 1\}$ .

We have a vector field  $\mathbf{v}''$  on  $X''$  which we take to be  $\mathbf{v}$  on  $X \times 0$  and  $\mathbf{v}'$  on  $X \times 1$  and interpolate linearly between  $\mathbf{v}$  and  $\mathbf{v}'$  on  $\partial X_+ \times [0, 1]$ . Since  $\mathbf{v}$  points outward on  $\partial X_+$  and  $\mathbf{v}'$  points inward on  $\partial X_+$ , the vector field  $\mathbf{v}''$  has no zeroes on  $\partial X_+ \times [0, 1]$ . Note that  $\mathbf{v}''$  points inward on all of  $\partial X''$ . So by part 1) we have:

$$(3.7) \quad \begin{aligned} \chi(X'') - \chi(\partial X'') &= \sum_{p \in X''} \mathcal{I}(p, \mathbf{v}'') \\ &= \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{p \in X} \mathcal{I}(p, \mathbf{v}') \\ &= \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \chi(X) - \chi(\partial X). \end{aligned}$$

But  $\partial X''$  is just two disjoint copies of  $\partial X_-$  so (3.7) implies

$$\begin{aligned} \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) &= \chi(X'') - 2\chi(\partial X_-) - \chi(X) + \chi(\partial X) \\ &= \chi(X'') - \chi(\partial X_-) - \chi(X) + \chi(\partial X_+). \end{aligned}$$

However by Mayer-Vietoris we get

$$\chi(X'') = \chi(X) + \chi(X) - \chi(\partial X_+)$$

and so

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X_-).$$

□

The above Theorem 3.2 implies Theorem 6.2.2 of [15]. This theorem says that if  $W$  is a Whitney stratified analytic subset of an analytic manifold  $M$  and  $D \subset M$  is a compact codimension 0 submanifold of  $M$  whose boundary is transverse to the strata of  $W$  and if  $X^{+rad}$  (resp.  $X^{-rad}$ ) is a radial vector field on  $D$  (in the sense of [15]) which is tangent to the strata of  $W$  and points outward (resp. inward) on  $\partial D$  then we have

$$\begin{aligned} \chi(W \cap D) &= \sum_{p \in W \cap D} \mathcal{I}(p, X^{+rad}) \\ \chi(W \cap D) - \chi(W \cap \partial D) &= \sum_{p \in W \cap D} \mathcal{I}(p, X^{-rad}). \end{aligned}$$

If we choose tubular data for  $W$  so that  $\rho_i$  is the distance from  $S_i$  (which is in fact what one normally does) then the notion of radial in [15] is the same as the one we give here. So Theorem 6.2.2 in [15] follows from Theorem 3.2 above since  $W \cap D$  is a radial manifold complex with boundary  $W \cap \partial D$ .

Note however the greater generality of Theorem 3.2. It applies to other than analytic sets, also the vector fields it applies to need not have a continuous extension to an ambient manifold  $M$ .

We also have a theorem in the more general case where we do not restrict the behavior of  $\mathbf{v}$  on  $\partial X$  except to ask that it have no zeroes. But first we need some definitions.

**Definition 3.3.** Suppose that  $\mathbf{v}$  is a vector field on  $X$ . We say that a vector field  $\mathbf{v}'$  on  $\partial X$  is a *radial projection* of  $\mathbf{v}$  to  $\partial X$  if it is radial (on  $\partial X$ ), has isolated zeroes, and also  $\mathbf{v}'(x) = \mathbf{v}(x)$  whenever  $\mathbf{v}(x)$  is tangent to  $\partial X$  at  $x \in \partial X$ .

We will show in Lemma 4.1 below that any radial vector field has a radial projection to  $\partial X$ . Meanwhile we get the following theorem which easily implies Theorem 3.2.

**Theorem 3.3.** *Suppose  $\mathbf{v}$  is a radial vector field on a compact radial manifold complex with boundary  $X$ , the zeroes of  $\mathbf{v}$  are isolated and do not occur on the*

boundary of  $X$ . Let  $\mathbf{v}'$  be a radial projection of  $\mathbf{v}$  to  $\partial X$ . Then

$$(3.8) \quad \chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}')$$

where  $A$  is the set of points  $x \in \partial X$  such that  $\mathbf{v}(x)$  points into  $X$ .

*Proof.* We prove this by modifying  $X$  and  $\mathbf{v}$  until we reach the point where we may apply Theorem 3.2. Let  $X' = X \cup (\partial X \times [-1, 1])$  where we identify each point  $x \in \partial X$  with  $(x, -1) \in \partial X \times [-1, 1]$ . We take the natural stratification of  $X'$  by gluing the collar  $\partial S_i \times [-1, 1]$  onto each stratum  $S_i$  and picking a smooth structure on the result (so that  $S_i$  and  $\partial S_i \times [-1, 1]$  are smooth submanifolds). We give  $X'$  the structure of a radial manifold complex with boundary in the natural way. In particular its radial and projection functions when restricted to  $X$  are the same as for  $X$ . On  $\partial X \times [-1, 1]$  they are the obvious extensions. The submersive property 4 of the  $\rho_i$  allows us to choose a smoothing so that the new  $\rho_i$  are still smooth.

We now extend  $\mathbf{v}$  to a radial vector field  $\mathbf{v}''$  on  $X'$  as follows. Note that  $\mathbf{v}$  is defined on  $\partial X \times -1 \subset \partial X \times [-1, 1]$ , let it be  $\mathbf{v}(x, -1) = (\mathbf{v}'''(x), \alpha(x))$  where  $\mathbf{v}'''$  is a vector field on  $\partial X$  and  $\alpha$  is a function  $\alpha: \partial X \rightarrow \mathbf{R}$ . Note that  $\alpha(x) < 0$  where  $\mathbf{v}$  points into  $X$ ,  $\alpha(x) > 0$  where  $\mathbf{v}$  points out of  $X$  and  $\alpha(x) = 0$  where  $\mathbf{v}$  is tangent to  $\partial X$ .

On  $\partial X \times [-1, 1]$  we let

$$\mathbf{v}''(x, t) = \begin{cases} ((1+t)\mathbf{v}'(x) - t\mathbf{v}'''(x), \alpha(x)) & \text{if } t \leq 0 \\ (\mathbf{v}'(x), (1-t)\alpha(x) + t) & \text{if } t \geq 0. \end{cases}$$

Note that  $\mathbf{v}''(x, 1) = (\mathbf{v}'(x), 1)$  so  $\mathbf{v}$  points outward on  $\partial X' = X \times 1$ . Let us now find the zeroes of  $\mathbf{v}''$ . On  $\partial X \times [-1, 0]$  there are no zeroes of  $\mathbf{v}''$  since we would have to have  $\alpha(x) = 0$  (so  $\mathbf{v}(x)$  is tangent to  $\partial X$ ) but then  $\mathbf{v}''(x, t) = (\mathbf{v}(x), 0) \neq 0$ . On  $\partial X \times [0, 1]$ , the zeroes occur exactly when  $\mathbf{v}'(x) = 0$  and  $\alpha(x) = -t/(1-t)$  (or equivalently  $t = \alpha(x)/(\alpha(x) - 1)$ ) which means that  $\alpha(x) < 0$  so  $\mathbf{v}(x)$  points inward. In other words there is one zero  $(x_0, t_0)$  on  $\partial X \times [-1, 1]$  for each point  $x_0 \in A$  where  $\mathbf{v}'(x_0) = 0$ .

**Claim 3.3.1.** *The index of this zero  $(x_0, t_0)$  of  $\mathbf{v}''$  is exactly the index of  $\mathbf{v}'$  at  $x_0$ .*

To prove claim 3.3.1, let  $N_0$  be a neighborhood of  $x_0$  in  $\partial X$  containing no other zeroes of  $\mathbf{v}'$ . Let  $\tilde{\mathbf{v}}'$  be a perturbation of  $\mathbf{v}'$  in  $N_0$  such that  $\tilde{\mathbf{v}}'$  has only nondegenerate zeroes in  $N_0$ . Then by definition, the total index of  $\tilde{\mathbf{v}}'$  in  $N_0$  equals the index of  $\mathbf{v}'$  at  $x_0$ . The associated vector field  $\tilde{\mathbf{v}}''(x, t) = (\tilde{\mathbf{v}}'(x), (1-t)\alpha(x) + t)$  also has only nondegenerate zeroes in  $N_0 \times [0, 1]$  and the total index of  $\tilde{\mathbf{v}}''$  in  $N_0 \times [0, 1]$  equals the index of  $\mathbf{v}''$  at  $(x_0, t_0)$ . Now by Lemma 6.4 of [10] the index of each zero  $(x_1, t_1)$  of  $\tilde{\mathbf{v}}''$  equals the index of  $\tilde{\mathbf{v}}'$  at  $x_1$  and the claim follows.

Note that  $X'$  deformation retracts to  $X$  so by claim 3.3.1 and Theorem 3.2,

$$\chi(X) = \chi(X') = \sum_{p \in X'} \mathcal{I}(p, \mathbf{v}'') = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

Consequently the theorem is proven.  $\square$

In the case where  $X$  is a smooth manifold, the above Poincaré-Hopf formula (3.8) in Theorem 3.3 is that given in [4]. Pugh [12] rediscovered an alternative formula, 40 years after it first appeared in a paper of Morse ([11])! In [12], it is assumed that  $X$  is a manifold with boundary and  $\mathbf{v}$  is generic so if  $\Gamma_1$  is the set of  $x \in \partial X$  where  $\mathbf{v}$  is tangent to  $\partial X$ , then  $\Gamma_1$  is a smooth codimension one submanifold of  $\partial X$ , and if  $\Gamma_2$  is the set of  $x \in \Gamma_1$  where  $\mathbf{v}$  is tangent to  $\Gamma_1$  then  $\Gamma_2$  is a smooth codimension one submanifold of  $\Gamma_1$  and so on. Pugh then gives a Poincaré-Hopf formula with correction terms involving these  $\Gamma_i$ 's. In particular,

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X) + \sum \chi(A_i) - \chi(\Gamma_i)$$

where  $A_1$  is the set of points in  $\partial X$  at which  $\mathbf{v}$  is pointing out of  $\partial X$  and  $A_{i+1}$  is the set of points of  $\Gamma_i$  where  $\mathbf{v}$  is pointing out of  $A_i$ . Note  $\partial A_i = \Gamma_i$ . By Theorem 3.3 we have  $\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \chi(X) - \chi(\partial X) + \sum_{q \in A_1} \mathcal{I}(q, \mathbf{v}')$ , but by induction we have  $\sum_{q \in A_1} \mathcal{I}(q, \mathbf{v}') = \chi(A_1) - \chi(\Gamma_1) + \sum_{i \geq 2} \chi(A_i) - \chi(\Gamma_i)$ , so we obtain the formula in [12] and [11].

**Example 3.4.** Let  $X$  be the space in the left half of Figure 1. So  $X$  is the set of points  $(x, y)$  in the plane so that  $x^2 + y^2 \leq 1$  and if  $0 < y < \sqrt{1 - x^2}/2$  then  $x^2 + y^2 = y/3$ . We let  $X_0 = \{(x, 0) \mid -1 \leq x \leq 1\}$  and  $X_1 = X$ . Then  $\partial X$  is shown in the right half of Figure 1. Now  $X$  has the structure of a radial manifold complex with boundary so that  $\rho_0(x, y) = |y|$ , an exercise safely left to the reader. Consider the following vector field  $\mathbf{v}$  on  $X$ , drawn on the left half of Figure 1.

$$\mathbf{v}(x, y) = \begin{cases} (1, 0) & \text{if } y = 0 \\ (0, -1) & \text{if } y < 0 \\ (0, 1) & \text{if } y \geq \sqrt{1 - x^2}/2 \\ (x(1 - 6y), 6x^2) & \text{if } 0 < y < \sqrt{1 - x^2}/2 \end{cases}$$

Note that  $\mathbf{v}$  is radial since  $d\rho_0(\mathbf{v}) > 0$  near  $S_0$ , being either 1 or  $6x^2$ . The vector field  $\mathbf{v}$  has just one zero, at the point  $(0, 1/3)$  labelled  $A$  where its index is  $-1$ . The right half of Figure 1 shows a radial projection  $\mathbf{v}'$  of  $\mathbf{v}$  to  $\partial X$ . Note that  $\mathbf{v}$  is never tangent to  $\partial X$ , so any radial vector field on  $\partial X$  will be a radial projection of  $\mathbf{v}$ . For dimension reasons,  $\mathbf{v}'$  is zero at the points  $B$  and  $C$  of  $\partial S_0$  and its index is 1 at those points. In our choice of  $\mathbf{v}'$  there are three zeroes  $D$ ,  $E$ , and  $F$  in  $\partial S_1$  and the index at each is  $-1$ . The only zeroes of  $\mathbf{v}'$  where  $\mathbf{v}$  points inward are  $B$  and  $E$ . Thus Theorem 3.3 says:

$$\chi(X) = \mathcal{I}(A, \mathbf{v}) + \mathcal{I}(B, \mathbf{v}') + \mathcal{I}(E, \mathbf{v}') = -1 + 1 - 1 = -1$$

which is correct.

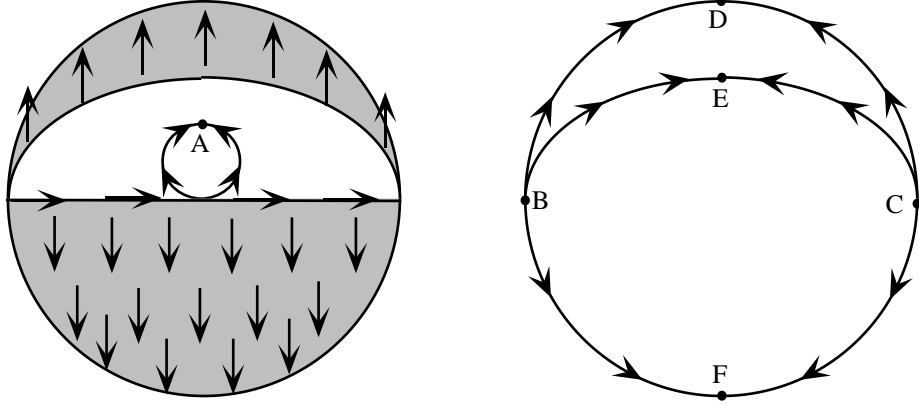


FIGURE 1. A radial vector field on a radial manifold complex and a radial projection to its boundary

#### 4. THE POINCARÉ-HOPF THEOREM FOR SEMIRADIAL VECTOR FIELDS

The next thing to do is to try to eliminate the radial condition on  $\mathbf{v}$ . We do so by introducing virtual zeroes of  $\mathbf{v}$  which are places where the behavior of  $\mathbf{v}$  in a neighborhood of a stratum requires some contribution to the index.

**Definition 4.1.** We let  $\mathbf{B}_i(\mathbf{v})$  be the set of  $x \in U_i - X_i$  so that for some numbers  $c_j \leq 0$  for  $i \leq j \leq m$  we have

$$\mathbf{v}(x) = \sum_{j=i}^m c_j \xi_j(x)$$

and  $c_i < 0$ . Then we say  $x \in S_i$  is a *virtual zero* of  $\mathbf{v}$  if  $x \in \text{Cl}(\mathbf{B}_i(\mathbf{v}))$ . (Note that by convention,  $\xi_j$  is 0 outside  $U_j$  so we may as well suppose that  $x \in U_j$  if  $c_j \neq 0$ . With this convention, the coefficients  $c_j$  are uniquely determined.)

**Remark 4.2.** A radial vector field has no virtual zeroes because near  $X_i$  we must have  $0 < d\rho_i(\mathbf{v}) = c_i$  if  $\mathbf{v}(x) = \sum_{j=i}^m c_j \xi_j(x)$ . Also, if  $X$  is a Thom stratified set and  $\mathbf{v}$  is a controlled vector field then it has no virtual zeroes since  $0 = d\rho_i(\mathbf{v}) = c_i$ .

There is a tricky point which will come up in Lemma 4.1 below. We only really care about the germ of  $\xi_j$  at  $S_j$  and as we take smaller and smaller neighborhoods  $U_j$ , the set  $\mathbf{B}_i(\mathbf{v})$  can get smaller and thus some virtual zeroes disappear. We could give a more complicated definition of virtual zero to allow this. But for the sake of simplicity in the proofs and attempting to keep technical details from obscuring the main ideas we prefer to use the above simpler definition. It just sometimes identifies a point as a virtual zero when it is not necessary to do so. Needless to say, when this happens the index contribution will be 0.

**Definition 4.3.** We say a vector field  $\mathbf{v}$  is *semiradial* if it has no virtual zeroes.

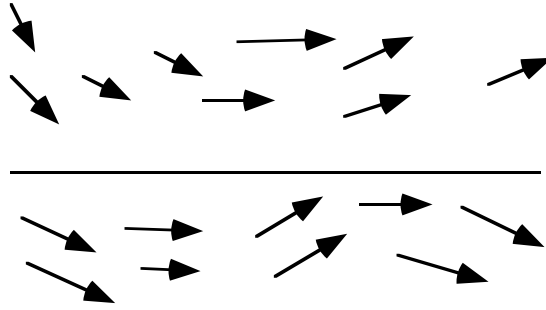


FIGURE 2. A semiradial vector field which is not radial

**Example 4.4.** For example Figure 2 shows a semiradial vector field. The center line is a stratum and  $\xi_i$  is vertical. The vector field is not radial since it sometimes slopes toward the center line. But it is semiradial since it never points straight up or down towards the center line.

**Definition 4.5.** We say that a vector field  $\mathbf{v}'$  on  $\partial X$  is a *semiradial projection* of  $\mathbf{v}$  to  $\partial X$  if it is semiradial, has isolated zeroes and also  $\mathbf{v}'(x) = \mathbf{v}(x)$  whenever  $\mathbf{v}(x)$  is tangent to  $\partial X$  at  $x \in \partial X$ .

The following lemma shows the existence of semiradial projections. There are examples which show that it is necessary to shrink the  $U_i$ 's.

**Lemma 4.1.** *Suppose  $\mathbf{v}$  is a vector field on  $X$  with no zeroes or virtual zeroes on  $\partial X$ . Then (after perhaps shrinking the  $U_i$ 's) there is a vector field  $\mathbf{v}'$  on  $\partial X$  which is a semiradial projection of  $\mathbf{v}$  to  $\partial X$ . Furthermore, if  $\mathbf{v}$  is radial we may take  $\mathbf{v}'$  to be radial.*

*Proof.* Let  $T \subset \partial X$  be the set of  $x$  at which  $\mathbf{v}(x)$  is tangent to  $\partial X$ . Suppose by induction on  $k$  that after perhaps shrinking the  $U_i$ 's, we have a vector field  $\mathbf{v}''$  on  $\partial X$  which equals  $\mathbf{v}$  on  $T$  and has no virtual zeroes in  $\partial X_{k-1}$ . Moreover, if  $\mathbf{v}$  is radial we also suppose that  $d\rho_i(\mathbf{v}''(x)) > 0$  for all  $x \in (U_i - S_i) \cap \partial X$  and all  $i < k$ .

Since  $\mathbf{v}$  has no virtual zeroes on  $\partial X$ , for every  $i \geq k$  we may choose a neighborhood  $U'_i$  of  $S_i$  in  $X$  so that

$$(4.1) \quad U'_i \subset U_i - \mathbf{B}_i(\mathbf{v}) \cap \partial X$$

and so  $U'_i \cap S_j$  is closed in  $S_j$  for all  $j > i$ . We let  $\mathbf{B}'_k(\mathbf{v}'')$  be the subset of  $\mathbf{B}_k(\mathbf{v}'') \cap U'_k$  consisting of those  $x \in \partial X$  so that for some numbers  $c_j \leq 0$  for  $k \leq j \leq m$  we have

- (1)  $\mathbf{v}''(x) = \sum_{j=k}^m c_j \xi_j(x)$ .
- (2)  $c_k < 0$ .
- (3) If  $c_j < 0$  then  $x \in U'_j$ .

In other words,  $\mathbf{B}'_k$  is just like  $\mathbf{B}_k$  except we shrink the neighborhoods  $U_j$  to  $U'_j$ .

**Claim 4.1.1.** *If  $i > k$  then  $\text{Cl}(\mathbf{B}'_k(\mathbf{v}'') \cap S_i) \cap S_i$  and  $T \cap S_i$  are disjoint closed subsets of  $\partial S_i$ .*

To see the claim, suppose  $x \in \text{Cl}(\mathbf{B}'_k(\mathbf{v}'') \cap S_i) \cap S_i$ . Then  $x \in \text{Cl}(\partial X) = \partial X$  so  $x \in \partial S_i$ . We know by condition 1 above that  $\mathbf{v}''(x) = \sum_{j=k}^m c_j \xi_j(x)$  for some  $c_j \leq 0$ . Also, since  $U'_j \cap S_i$  is closed in  $S_i$  we may suppose by condition 3 that  $x \in U'_j$  if  $c_j < 0$ . If in addition  $x \in T$ , then  $\mathbf{v}''(x) = \mathbf{v}(x)$  so  $\mathbf{v}''(x) \neq 0$  so not all  $c_j$  are 0. So  $x \in \mathbf{B}_j(\mathbf{v}) \cap U'_j$  for some  $j \geq k$ . But this violates (4.1) so the claim is shown.

Since  $\text{Cl}(\mathbf{B}'_k(\mathbf{v}'') \cap S_i) \cap S_i \subset U'_k \cap S_i \subset U_k \cap S_i$  it is also disjoint from a closed neighborhood of  $\partial S_i - U_k$ . So using the Tietze extension theorem we may then find a continuous function  $\eta_i: \partial S_i \rightarrow [0, \infty)$  which is 0 on a neighborhood of  $\partial S_i - U_k$  and on  $T \cap S_i$  and so that  $\eta_i(x) > -d\rho_k(\mathbf{v}''(x))$  for all  $x \in \mathbf{B}'_k(\mathbf{v}'') \cap S_i$ . Define  $\eta: \partial X \rightarrow [0, \infty)$  to be 0 on  $\partial X_k$  and  $\eta_i$  on  $\partial S_i$  for  $i > k$ . Now shrink each  $U_i$  until it is contained in  $U'_i$ . After shrinking, we have  $\mathbf{B}_k(\mathbf{v}'') \subset \mathbf{B}'_k(\mathbf{v}'')$ . Then the vector field  $\mathbf{v}'' + \eta \xi_k$  has no virtual zeroes in  $\partial X_k$  since  $\mathbf{B}_i(\mathbf{v}'' + \eta \xi_k) \subset \mathbf{B}_i(\mathbf{v}'')$  for  $i \leq k$  and  $d\rho_k(\mathbf{v}'' + \eta \xi_k) > 0$  on  $\mathbf{B}_k(\mathbf{v}'')$ . Thus we may by induction assume that  $\mathbf{v}''$  is semiradial.

If  $\mathbf{v}$  is radial, then we may modify the above construction. For  $i \geq k$  we choose  $U'_i$  so that  $d\rho_i(\mathbf{v}(x)) > 0$  for all  $x \in U'_i$ . Then  $d\rho_k(\mathbf{v}''(x)) = d\rho_k(\mathbf{v}(x)) > 0$  for all  $x \in T \cap U'_k$ . So for each  $i > k$  the closed sets  $(T \cap S_i) \cup (\partial S_i - U_k)$  and  $\{x \in U'_k \cap \partial S_i \mid d\rho_k(\mathbf{v}''(x)) \leq 0\}$  are disjoint. We may then choose  $\eta_i: \partial S_i \rightarrow [0, \infty)$  which is 0 on a neighborhood of  $\partial S_i - U_k$  and on  $T \cap S_i$  and so that  $\eta_i(x) > -d\rho_k(\mathbf{v}''(x))$  for all  $x \in U'_k$ . Proceed as above and the resulting  $\mathbf{v}''$  will be radial.

So we may by induction assume such a semiradial or radial  $\mathbf{v}''$  exists. Now to construct  $\mathbf{v}'$  we perturb  $\mathbf{v}''$  to make its zeroes isolated, just as we did in the proof of Lemma 3.1.  $\square$

**Lemma 4.2.** *Suppose  $\mathbf{v}$  is a semiradial vector field on  $X$  with isolated zeroes and no zeroes on  $\partial X$ . Then there is a radial vector field  $\mathbf{v}'$  on  $X$  so that:*

- 1)  $\mathbf{v}'$  has the same zeroes as  $\mathbf{v}$  and in fact  $\mathcal{I}(p, \mathbf{v}) = \mathcal{I}(p, \mathbf{v}')$  at all points  $p \in X$ .
- 2)  $\mathbf{v}'$  is tangent to  $\partial X$  at the same places as  $\mathbf{v}$  is tangent to  $\partial X$ .
- 3)  $\mathbf{v}'$  points inward on  $\partial X$  at the same places as  $\mathbf{v}$ .
- 4)  $\mathbf{v}'$  points outward on  $\partial X$  at the same places as  $\mathbf{v}$ .
- 5) If  $\mathbf{v}''$  is a semiradial projection of  $\mathbf{v}$  to  $\partial X$ , there is a radial projection  $\mathbf{v}'''$  of  $\mathbf{v}'$  to  $\partial X$  so that  $\mathbf{v}'''$  and  $\mathbf{v}''$  have the same zeroes and in fact  $\mathcal{I}(p, \mathbf{v}''') = \mathcal{I}(p, \mathbf{v}'')$  at all points  $p \in \partial X$ .

*Proof.* Pick neighborhoods  $U''_i$  of  $S_i$  in  $U_i$  so that  $U''_i \cap \mathbf{B}_i(\mathbf{v})$  and  $U''_i \cap \mathbf{B}_i(\mathbf{v}'')$  are empty, and so that  $U''_i - S_i$  contains no zeroes of  $\mathbf{v}$  or  $\mathbf{v}''$ . Let  $T \subset \partial X$  be the set of  $x$  at which  $\mathbf{v}(x)$  is tangent to  $\partial X$ .

The proof is similar to the proof of Lemma 3.1 but we must exercise more care. We will construct  $\mathbf{v}'$  and  $\mathbf{v}'''$  by letting  $\mathbf{v}' = \mathbf{v} + \sum_{i=0}^{m-1} \beta_i \xi_i$  and  $\mathbf{v}''' =$

$\mathbf{v}'' + \sum_{i=0}^{m-1} \beta_i \xi_i$  for suitably chosen nonnegative functions  $\beta_i$ . Then conditions 2), 3) and 4) are automatically satisfied, since the  $\xi_i$  are tangent to  $\partial X$ .

We suppose by induction on  $k$  that we have constructed nonnegative functions  $\beta_i: X \rightarrow [0, \infty)$  and neighborhoods  $U'_i$  of  $S_i$  in  $U''_i$  for  $0 \leq i < k$  so that:

- a)  $\beta_i$  is continuous on each stratum.
- b)  $\beta_i$  vanishes on  $X_i$  and  $X - U''_i$  and in fact for each  $j > i$  the support of the restriction  $\beta_i|_{S_j}$  is contained in  $U''_i \cap S_j$ .
- c)  $d\rho_i(\mathbf{v} + \sum_{j=0}^{k-1} \beta_j \xi_j) > 0$  on  $U'_i - S_i$ .
- d)  $d\rho_i(\mathbf{v}'' + \sum_{j=0}^{k-1} \beta_j \xi_j) > 0$  on  $U'_i \cap \partial X - S_i$ .

To complete the induction, we just choose  $\beta_k$  continuous on each stratum, vanishing on  $X_k$ , with support in  $U''_k \cup X_k$ , thus giving a) and b), so that for some neighborhood  $U'_k$  of  $S_k$  we have

$$\beta_k(x) > -d\rho_k(\mathbf{v}(x)) - \sum_{i=0}^{k-1} \beta_i(x) d\rho_k(\xi_i(x)) \quad \text{for all } x \in U'_k - S_k$$

$$\beta_k(x) > -d\rho_k(\mathbf{v}''(x)) - \sum_{i=0}^{k-1} \beta_i(x) d\rho_k(\xi_i(x)) \quad \text{for all } x \in \partial X \cap U'_k - S_k.$$

So by induction we may assume that  $k = m$ , the maximal index, and so we have  $\beta_1, \dots, \beta_{m-1}$  satisfying a)-d) above. We let  $\mathbf{v}' = \mathbf{v} + \sum_{i=0}^{m-1} \beta_i \xi_i$ . The vector field  $\mathbf{v}'$  is radial by c). Likewise,  $\mathbf{v}''' = \mathbf{v}'' + \sum_{i=0}^{m-1} \beta_i \xi_i$  is radial by d).

Let us see why 1) is true. First of all, if  $x$  is a zero of  $\mathbf{v}$  in some stratum  $S_k$  then  $x \notin U''_i$  for all  $i < k$  so  $x$  is not in the support of  $\beta_i|_{S_k}$  for all  $i$  so  $\mathbf{v}$  and  $\mathbf{v}'$  agree on a neighborhood of  $x$  in  $S_k$  and thus  $\mathcal{I}(x, \mathbf{v}) = \mathcal{I}(x, \mathbf{v}')$ . So we must only show that  $\mathbf{v}'$  has no zeroes which are not zeroes of  $\mathbf{v}$ . Suppose  $\mathbf{v}'(x) = 0$  but  $\mathbf{v}(x) \neq 0$ . Pick the smallest  $\ell$  so that  $\beta_\ell(x) \neq 0$ . Then  $x \in \mathbf{B}_\ell(\mathbf{v})$  since

$$\mathbf{v}(x) = \mathbf{v}'(x) - \sum_{j=\ell}^{m-1} \beta_j(x) \xi_j(x) = \sum_{j=\ell}^{m-1} -\beta_j(x) \xi_j(x)$$

and  $-\beta_\ell(x) < 0$ . This is a contradiction, since  $x \in \mathbf{B}_\ell(\mathbf{v}) \cap U''_\ell$  which is empty. So 1) is true. By a similar argument, 5) is true.  $\square$

We can now generalize Theorem 3.3 to semiradial vector fields.

**Corollary 4.3.** *Suppose  $\mathbf{v}$  is a semiradial vector field on a compact radial manifold complex with boundary  $X$ , the zeroes of  $\mathbf{v}$  are isolated and there are no zeroes on the boundary of  $X$ . Let  $\mathbf{v}'$  be a semiradial projection of  $\mathbf{v}$  to  $\partial X$ . Then*

$$\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}')$$

where  $A$  is the set of points  $x \in \partial X$  such that  $\mathbf{v}(x)$  points into  $X$ .

*Proof.* By Lemma 4.2, we may find a radial vector field  $\mathbf{v}''$  on  $X$  with the same zeroes as  $\mathbf{v}$  and a radial projection  $\mathbf{v}'''$  of  $\mathbf{v}''$  to  $\partial X$  so that  $\mathbf{v}'''$  and  $\mathbf{v}'$  have the same zeroes. Moreover,  $\mathcal{I}(p, \mathbf{v}) = \mathcal{I}(p, \mathbf{v}'')$  for all  $p \in X$  and  $\mathcal{I}(p, \mathbf{v}''') = \mathcal{I}(p, \mathbf{v}')$  for all  $p \in \partial X$ . Also,  $\mathbf{v}$  and  $\mathbf{v}''$  point outward on the same subset  $A \subset \partial X$ . By Theorem 3.3 we know that

$$\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}'') + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}''').$$

Hence we have

$$\chi(X) = \sum_{p \in X} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

□

## 5. A POINCARÉ-HOPF THEOREM IN THE PRESENCE OF VIRTUAL ZEROES

Now what if a vector field  $\mathbf{v}$  is not semiradial, so it has virtual zeroes. It is not hard to find examples where the conclusion of Corollary 4.3 is not true. But we show that if the index is suitably defined at a zero or virtual zero, then a Poincaré-Hopf theorem does hold.

**Example 5.1.** Let  $Y$  and  $Z$  be compact smooth manifolds without boundary and let  $f: Y \rightarrow Z$  be a smooth map. We let  $X$  be the mapping cylinder of  $f$ , the quotient space  $Y \times [0, 1] \cup Z/(y, 0) \sim f(y)$ . We stratify  $X$  by letting  $S_0 = Z$  and  $S_1 = Y \times (0, 1]$ . We have  $\partial X = Y \times 1$ . We let  $U_0 = X - \partial X$ ,  $\varphi_0((y, t), s) = (y, (1-s)t)$ ,  $\rho_0(y, t) = t$  and  $\xi_0 = (0, 1)$ . Thus  $X$  is a radial manifold complex with boundary. Take any vector field  $\mathbf{v}$  on  $Y$  with isolated zeroes. Consider any vector field  $\mathbf{v}$  on  $X$  with isolated zeroes so that on  $Y \times (0, 1]$  we have  $\mathbf{v}(y, t) = (\mathbf{v}'(y), -1)$ . Let  $K \subset Y$  be the zeroes of  $\mathbf{v}'$ . Then  $\mathbf{B}_0(\mathbf{v}) = K \times (0, 1)$  so  $f(K)$  is the set of virtual zeroes of  $\mathbf{v}$ . Note

$$\sum_{p \in X} \mathcal{I}(p, \mathbf{v}) = \sum_{p \in Z} \mathcal{I}(p, \mathbf{v}) = \chi(Z) = \chi(X).$$

But  $\mathbf{v}$  points inward on  $\partial X$  so we want this sum to be  $\chi(X) - \chi(\partial X) = \chi(X) - \chi(Y)$ . We will show below how to suitably define an index at the virtual zeroes of  $\mathbf{v}$  so that the sum does come out right. In this example, the modified index at a virtual zero  $p$  will turn out to be  $\hat{\mathcal{I}}(p, \mathbf{v}) = \mathcal{I}(p, \mathbf{v}) - \sum_{q \in f^{-1}(p) \cap K} \mathcal{I}(q, \mathbf{v}')$ .

**Definition 5.2.** A *nice contractible neighborhood* of a point  $x \in X - \partial X$  is a compact neighborhood  $W$  of  $x$  in  $X - \partial X$  so that:

- (1)  $W$  is contractible.
- (2) For each stratum  $S_i$ ,  $W \cap S_i$  is either empty or is a smooth codimension 0 submanifold of  $S_i$  whose boundary is  $\text{Fr}(W) \cap S_i$ , the intersection of the frontier of  $W$  with  $S_i$ .
- (3)  $\text{Fr}(W)$  is a closed proper radial subcomplex of  $X$  without boundary (see Definition 6.2). In other words, for each stratum  $S_i$  there is a neighborhood  $U_i''$  of  $S_i$  in  $X$  so that:

- (a)  $\xi_i$  is tangent to  $\text{Fr}(W) \cap U_i''$ .
- (b) if  $\varphi_i(x, t_0) \in U_i'' \cap \text{Fr}(W)$  then  $\varphi_i(x, t) \in \text{Fr}(W)$  for all  $t \geq t_0$ .

**Remark 5.3.** Note in Definition 5.2 that  $W$  and the complement of the interior of  $W$  each have a natural structure as a radial manifold complex with boundary since  $\rho_i$ ,  $\varphi_i$  and  $\xi_i$  will just restrict after perhaps shrinking the  $U_i$ 's, see Lemma 6.6 for details. The boundaries are just  $\text{Fr}(W)$  and  $\text{Fr}(W) \cup \partial X$ .

**Definition 5.4.** We say a radial manifold complex with boundary  $X$  is *nice* if, letting  $\pi_i: U_i \rightarrow S_i$  be the retraction  $\pi_i(x) = \varphi_i(x, 1)$ , we have:

- (1)  $\pi_i$  is smooth when restricted to each stratum. What is more, for each  $j > i$  either  $\pi_i|_{S_j \cap U_i}$  is  $k$  times differentiable where  $k \geq \dim(S_j)$  or else  $\pi_i|_{S_j \cap U_i}$  is a submersion.
- (2)  $d\pi_i(\xi_i) = 0$ .
- (3) For all  $i < j$  we have  $\pi_i \pi_j = \pi_i$  on  $U_i \cap \pi_j^{-1}(U_i \cap S_j)$ .
- (4)  $\pi_i \circ \varphi_i(x, t) = \pi_i(x)$  if  $x \in U_i$  and  $\varphi_i(x, t) \in U_i$ .

For example a Whitney stratified set is nice if it is given the radial manifold complex with boundary structure of Example 2.6.

**Lemma 5.1.** *Let  $X$  be a nice radial manifold complex with boundary. Then every point  $x \in X - \partial X$  has arbitrarily small nice contractible neighborhoods.*

*Proof.* Let  $i$  be such that  $x \in S_i$ . Pick any open neighborhood  $U$  of  $x$ . We will find a nice contractible neighborhood of  $x$  inside  $U$ . We may as well suppose that  $\text{Cl}(U)$  is a compact subset of  $U_i - \partial X$ . Pick a smooth function  $r: S_i \rightarrow [0, \infty)$  so that  $x = r^{-1}(0)$  and so that  $r^{-1}([0, \delta])$  is compact and contractible for all small enough  $\delta$ , for example the squared distance function in some Riemannian metric. By either Sard's theorem [19] or submersiveness we may pick a small  $\delta > 0$  so that  $\delta$  is a regular value of the map  $r \circ \pi_i$ , i.e., for each  $j > i$ ,  $\delta$  is a regular value of the restriction of  $r \circ \pi_i$  to  $S_j \cap U_i$ . Also we may suppose that  $r^{-1}([0, \delta]) \subset U$  and  $r^{-1}([0, \delta])$  is contractible.

Pick  $\epsilon > 0$  so that  $\rho_i^{-1}([0, \epsilon]) \cap \pi_i^{-1}r^{-1}([0, \delta]) \subset U$ . Then we may let  $W$  be  $\rho_i^{-1}([0, \epsilon]) \cap \pi_i^{-1}r^{-1}([0, \delta])$  with corners rounded. Let us be more precise about corner rounding. Pick a continuous function  $\gamma: [0, \delta] \rightarrow [\epsilon/2, \epsilon]$  so that  $\gamma$  is smooth on  $[0, \delta)$  and  $D = \{(x, y) \in [0, \delta] \times (0, \epsilon] \mid y \leq \gamma(x)\}$  is a smooth submanifold of  $[0, 2\delta] \times (0, 2\epsilon)$ . For example,  $\gamma(x) = \epsilon/2 + \epsilon/(2 - \epsilon \ln(1 - x/\delta))$ . Then let  $W = (r \circ \pi_i, \rho_i)^{-1}(D)$ . Note that  $\text{Fr}(W) = (r \circ \pi_i, \rho_i)^{-1}(C)$  where  $C = \delta \times [0, \gamma(\delta)] \cup C'$  where  $C' = \{(x, y) \in [0, \delta] \times [0, \epsilon] \mid y = \gamma(x)\}$ . Note that  $(r \circ \pi_i, \rho_i)$  is transverse to  $\delta \times [0, \gamma(\delta)]$  since  $\delta$  is a regular value of  $r \circ \pi_i$  and  $(r \circ \pi_i, \rho_i)$  is transverse to  $C'$  since  $d(r \circ \pi_i, \rho_i)(\xi_i) = (0, 1)$ . So  $(r \circ \pi_i, \rho_i)$  is transverse to  $C$  which implies condition 2 of definition 5.2. We see that  $W$  is contractible by noting that using  $\varphi_i$ , it deformation retracts to  $r^{-1}([0, \delta])$  which is contractible.

Now let us see why  $\text{Fr}(W)$  is a closed proper radial subcomplex of  $X$ . Let  $U_i'' = \rho_i^{-1}([0, \epsilon/2])$ . Then  $\text{Fr}(W) \cap U_i'' = (r \circ \pi_i)^{-1}(\delta) \cap \rho_i^{-1}([0, \epsilon/2])$ . Note that  $\xi_i$  is tangent to  $\text{Fr}(W) \cap U_i''$  since  $d\pi_i(\xi_i) = 0$ . Suppose  $\varphi_i(x, t_0) \in \text{Fr}(W) \cap U_i''$ .

Then if  $t \geq t_0$  we know  $r \circ \pi_i \circ \varphi_i(x, t) = r \circ \pi_i(x) = r \circ \pi_i \circ \varphi_i(x, t_0) = \delta$ . We also know  $\rho_i \circ \varphi_i(x, t) \leq \rho_i \circ \varphi_i(x, t_0) \leq \epsilon/2$ . Consequently,  $\varphi_i(x, t) \in \text{Fr}(W) \cap U_i''$ . So we have satisfied the condition 3 in Definition 5.2, for the  $i$  stratum. Now pick any  $k > i$  and let us see that condition 3 is satisfied for the  $k$  stratum. We have  $\text{Fr}(W) = (r \circ \pi_i, \rho_i)^{-1}(D)$ . If  $x$  is near  $S_k$ , then  $d\rho_i(\xi_k(x)) = 0$  by condition 4 of Definition 2.4 and  $d(r \circ \pi_i)(\xi_k(x)) = d(r \circ \pi_i \circ \pi_k)(\xi_k(x)) = d(r \circ \pi_i) \circ d\pi_k(\xi_k(x)) = 0$  by conditions 3 and 2 of definition 5.4. Consequently,  $\xi_i$  is tangent to  $\text{Fr}(W)$  near  $S_k$ . Suppose  $\varphi_k(x, t_0) \in \text{Fr}(W)$  near  $S_k$ . Pick any  $t \geq t_0$ . We have

$$r \circ \pi_i \circ \varphi_k(x, t) = r \circ \pi_i \circ \pi_k \circ \varphi_k(x, t) = r \circ \pi_i \circ \pi_k(x) = r \circ \pi_i \circ \varphi_k(x, t_0)$$

$$\rho_i \circ \varphi_k(x, t) = \rho_i \circ \pi_k \circ \varphi_k(x, t) = \rho_i \circ \pi_k(x) = \rho_i \circ \varphi_k(x, t_0)$$

So  $(r \circ \pi_i, \rho_i) \circ \varphi_k(x, t) = (r \circ \pi_i, \rho_i) \circ \varphi_k(x, t_0) \in D$  and thus  $\varphi_k(x, t) \in \text{Fr}(W)$ .  $\square$

**Definition 5.5.** We now define the index of a vector field  $\mathbf{v}$  at an isolated zero or virtual zero  $x$  of  $\mathbf{v}$  if  $x \notin \partial X$  and  $x$  has arbitrarily small nice contractible neighborhoods. Pick a nice contractible neighborhood  $W$  of  $x$  so that  $x$  is the only zero or virtual zero of  $\mathbf{v}$  in  $W$ . Choose any vector field  $\mathbf{v}'$  on  $\partial W$  which is a semiradial projection of  $\mathbf{v}|_W$  to  $\partial W$ . Then we define

$$\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in A} \mathcal{I}(q, \mathbf{v}')$$

where  $A$  is the set of  $q \in \partial W$  so that  $\mathbf{v}(q)$  points into  $W$ . If  $x$  is not a zero or virtual zero of  $\mathbf{v}$  then we define  $\hat{\mathcal{I}}(x, \mathbf{v}) = 0$ .

**Lemma 5.2.**  $\hat{\mathcal{I}}(x, \mathbf{v})$  is independent of the choice of  $W$  and the projection  $\mathbf{v}'$ .

*Proof.* Pick another nice contractible neighborhood  $W'$  with  $W'$  contained in the interior  $\text{Int } W$  of  $W$  and pick a semiradial projection  $\mathbf{v}''$  of  $\mathbf{v}|_{W'}$  to  $\partial W'$ . Let  $Z = W - \text{Int } W'$ . By Corollary 4.3 we know that

$$(5.1) \quad \chi(Z) = \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') + \sum_{q \in C} \mathcal{I}(q, \mathbf{v}'')$$

where  $A$  is the set of  $q \in \partial W$  so that  $\mathbf{v}(q)$  points into  $W$  and  $C$  is the set of  $q \in \partial W'$  so that  $\mathbf{v}(q)$  points out of  $W'$ , i.e., into  $Z$ . We also know by Theorem 6.1 that

$$1 = \chi(W) = \chi(Z) + \chi(W') - \chi(\partial W') = \chi(Z) + 1 - \chi(\partial W')$$

so  $\chi(Z) = \chi(\partial W')$ . But  $\chi(\partial W') = \sum_{q \in \partial W'} \mathcal{I}(q, \mathbf{v}'')$  so by (5.1) we get

$$(5.2) \quad \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') = \sum_{q \in \partial W' - C} \mathcal{I}(q, \mathbf{v}'').$$

Calculating the index using  $W$  and  $\mathbf{v}'$  we obtain

$$\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

Calculating using  $W'$  and  $\mathbf{v}''$  we obtain

$$\hat{\mathcal{I}}(x, \mathbf{v}) = 1 - \sum_{q \in A'} \mathcal{I}(q, \mathbf{v}'')$$

where  $A'$  is the set of points on  $\partial W'$  where  $\mathbf{v}$  points into  $W'$ . But  $\mathcal{I}(q, \mathbf{v}'') = 0$  if  $q \in \partial W' - (A' \cup C)$  since  $\mathbf{v}''(q) = \mathbf{v}(q) \neq 0$ . So

$$\sum_{q \in \partial W' - C} \mathcal{I}(q, \mathbf{v}'') = \sum_{q \in A'} \mathcal{I}(q, \mathbf{v}'')$$

and the result follows from (5.2).  $\square$

**Lemma 5.3.** *If  $\mathbf{v}$  is semiradial and has isolated zeroes, then  $\hat{\mathcal{I}}(x, \mathbf{v}) = \mathcal{I}(x, \mathbf{v})$  for all  $x \in X - \partial X$ .*

*Proof.* Let  $W$  be a nice contractible neighborhood of  $x$  which contains no zeroes of  $\mathbf{v}$ , except possibly  $x$ . By Corollary 4.3, we know that

$$\chi(W) = \sum_{p \in W} \mathcal{I}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

But  $W$  is contractible so  $\chi(W) = 1$ . Also  $\mathcal{I}(p, \mathbf{v}) = 0$  if  $p \in W - x$  since  $\mathbf{v}(p) \neq 0$  for  $p \in W - x$ . So we get

$$\mathcal{I}(x, \mathbf{v}) = 1 - \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') = \hat{\mathcal{I}}(x, \mathbf{v}).$$

$\square$

**Theorem 5.4.** *Let  $X$  be a nice compact radial manifold complex with boundary. Let  $\mathbf{v}$  be a vector field on  $X$  with isolated zeroes and isolated virtual zeroes and with no zeroes or virtual zeroes on  $\partial X$ . Let  $\mathbf{v}'$  be a semiradial projection of  $\mathbf{v}$  to  $\partial X$ . Let  $A$  be the set of points in  $\partial X$  where  $\mathbf{v}(x)$  points into  $X$ . Then*

$$\chi(X) = \sum_{p \in X} \hat{\mathcal{I}}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}').$$

*Proof.* We prove this by induction on the number of virtual zeroes of  $\mathbf{v}$ . If  $\mathbf{v}$  has no virtual zeroes, then this follows from Corollary 4.3 and Lemma 5.3.

Now suppose that  $\mathbf{v}$  has a virtual zero at some  $x$ . Pick a nice contractible neighborhood  $W$  of  $x$  which contains no zeroes or virtual zeroes other than  $x$ . Let  $X' = X - \text{Int}(W)$ . Pick a semiradial projection  $\mathbf{v}''$  of  $\mathbf{v}|_W$  to  $\partial W$  and let  $B$  be the set of points of  $\partial W$  at which  $\mathbf{v}$  points out of  $W$ , and hence into  $X'$ . By induction on the number of virtual zeroes we know that

$$\chi(X') = \sum_{p \in X'} \hat{\mathcal{I}}(p, \mathbf{v}) + \sum_{q \in A} \mathcal{I}(q, \mathbf{v}') + \sum_{q \in B} \mathcal{I}(q, \mathbf{v}'').$$

So we only need show that

$$\chi(X) - \chi(X') = \hat{\mathcal{I}}(x, \mathbf{v}) - \sum_{q \in B} \mathcal{I}(q, \mathbf{v}'').$$

But by Theorem 6.1,

$$\chi(X) - \chi(X') = \chi(W) - \chi(\partial W) = 1 - \chi(\partial W)$$

so the result follows since

$$\begin{aligned} \chi(\partial W) &= \sum_{q \in \partial W} \mathcal{I}(q, \mathbf{v}'') \text{ and} \\ \hat{\mathcal{I}}(x, \mathbf{v}) &= 1 - \sum_{q \in \partial W - B} \mathcal{I}(q, \mathbf{v}''). \end{aligned}$$

□

Here are two simple examples which may be helpful to the reader.

**Example 5.6.** First let  $X$  be the closed unit disc  $D^2$  in the plane, stratified by the unit circle  $S^1$  and the interior  $\text{Int}(D^2)$  of the disc. (So  $\partial X = \emptyset$ .) Let  $v$  be  $\frac{\partial}{\partial \theta}$  on  $S^1$  and  $\frac{\partial}{\partial y}$  on  $\text{Int}(D^2)$ . Then  $v$  has no zeros, is not continuous, and is not radial or semiradial, but has a single virtual zero at  $(0, 1)$  if we use the standard radial manifold complex structure near  $S^1$ , considered as a stratum. It is now an easy but instructive exercise to check that the virtual index of  $v$  at  $(0, 1)$  is 1.

**Example 5.7.** Next we give an example with a nonempty boundary : let  $S_2$  be the unit 2-sphere with the north and south poles removed, and let  $S_1$  be the closed line segment with endpoints  $(0, 0, 2)$  and  $(0, 0, -2)$ . Then set  $X = S_1 \cup S_2$ . Consider the stratified vector field  $v$  on  $X$  which equals  $\frac{\partial}{\partial z}$  on  $S_1$ , and on  $S_2$  is the orthogonal projection of  $\frac{\partial}{\partial z}$  onto the tangent plane at each point of  $S_2$ . Then  $v$  has no zeros, again is not continuous, nor radial, and has just one virtual zero at the north pole  $(0, 0, 1)$ , of virtual index 0. The formula of Theorem 5.4 works because of the boundary contribution :  $v$  is entering on  $\partial X$  at  $(0, 0, -2)$ , and the associated semiradial projection at this point, which is zero, has an index of  $+1$ , which is the Euler characteristic of  $X$ .

**Remark 5.8.** Now we indicate situations in which Theorem 5.4 applies to “generic” vector fields. Let  $X$  be a radial manifold complex with boundary embedded in a smooth manifold  $M$  such that the strata are Whitney (a)-regular. Let  $\mathbf{v}_M$  be a vector field on  $M$ , and  $\mathbf{v}$  the vector field on  $X$  obtained by projecting  $\mathbf{v}_M$  onto the tangent space of the stratum of  $X$  at each point. Then every virtual zero of  $\mathbf{v}$  is a zero. As a corollary, if  $M$  is some  $\mathbb{R}^n$ , then the set of  $\mathbf{v}_M$  such that  $\mathbf{v}$  has only isolated zeroes (and virtual zeroes) none of which lie on  $\partial X$ , is open and dense in the space of constant vector fields on  $\mathbb{R}^n$ . Hence Theorem 5.4 applies to such “generic” vector fields.

**Remark 5.9.** As another corollary, suppose  $X$  is a radial manifold complex with boundary embedded in a manifold  $M$  and  $f: M \rightarrow \mathbf{R}$  is a smooth function so that  $f|_X$  has only isolated critical points on the strata of  $X$ . (This is a generic condition.) Then Theorem 5.4 applies to the gradient vector field of  $f|_{S_i}$ .

There is another point of view of vector fields on radial manifold complexes with boundary. We can look at a resolution of singularities of a radial manifold complex with boundary to a disjoint union of manifolds  $V_i$  with boundary and gluing maps which glue parts of the boundary of some of these manifolds to others. Thom looked at a version of this in his paper [20]. Alternatively, this is similar to the resolution towers of [2] except that one splits the manifolds apart along the ticos to obtain a manifold with boundary. In other words, we obtain each skeleton  $X_i$  from  $X_{i-1}$  by attaching a manifold  $V_i$  along a reasonably nice closed subset of its boundary. It is easy to prove, by induction, the formula

$$\chi(X) - \chi(\partial X) = \sum_{i=0}^m \chi(V_i) - \chi(\partial V_i).$$

Thus Poincaré-Hopf theorems for radial manifold complexes with boundary can be reduced to the manifold case. This also explains why we only needed our vector fields to be continuous on each stratum, since the Euler characteristic only depends on the resolutions  $V_i$  of each stratum. Our virtual zeroes correspond to possible zeroes of a projection of an extension of the vector field on  $\text{Int } V_i$  to  $\partial V_i$ . Such a continuous extension might not exist, but will after perturbation.

## 6. SOME PROPERTIES OF RADIAL MANIFOLD COMPLEXES WITH BOUNDARY

In this section we discover some properties of radial manifold complexes with boundary which we used in this paper. We first define the notion of subobjects. We could define these more generally, but for convenience restrict to closed, proper subobjects which is all we need here.

**Definition 6.1.** If  $X$  is a manifold complex with boundary and  $Y \subset X$  then we say that  $Y$  is a *closed, proper subcomplex* if  $Y$  is closed and for each stratum  $S_i$  of  $X$ ,  $Y \cap S_i$  is a submanifold of  $S_i$  with boundary  $Y \cap \partial S_i$ . If  $X$  is smooth we also ask that  $Y \cap S_i$  be a smooth submanifold transverse to  $\partial S_i$ .

**Definition 6.2.** If  $X$  is a radial manifold complex with boundary and  $Y \subset X$  then we say that  $Y$  is a *closed, proper radial subcomplex* if  $Y$  is a closed, proper subcomplex of  $X$  and for each stratum  $S_i$  of  $X$ , with the usual data, there is a neighborhood  $U_i''$  of  $S_i$  in  $X$  so that

- (1)  $\xi_i$  is tangent to  $Y$  in  $U_i''$ .
- (2) If  $\varphi_i(x, t_0) \in U_i'' \cap Y$  then  $\varphi_i(x, t) \in Y$  for all  $t \geq t_0$ .

**Remark 6.3.** A closed, proper radial subcomplex inherits the structure of a radial manifold complex with boundary by shrinking  $U_i$  to  $\text{Int}(U_i'')$  and restricting  $\varphi_i$ ,  $\rho_i$ , and  $\xi_i$ .

**Theorem 6.1.** *Suppose  $X$  is a compact radial manifold complex with boundary and  $A$  and  $B$  are closed subsets of  $X$  so that  $X = A \cup B$  and so that  $A \cap B$  is a closed, proper radial subcomplex of  $X$ . Then  $\{A, B\}$  is an excisive couple. Moreover, the singular homologies of  $X$ ,  $A$ ,  $B$ , and  $A \cap B$  are all finitely generated and thus  $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$ .*

*Proof.* We will prove this by induction on the depth of  $X$ . The proof is structurally similar to the proof of Theorem 3.2. Let  $k$  be the smallest index so that  $X_k$  is not empty. Let  $U_k''$  be a compact neighborhood of  $S_k$  satisfying the conditions of Definition 6.2 for  $A \cap B$ . As in Theorem 3.2 we may pick an  $\epsilon > 0$  so that  $\rho_k^{-1}([0, 2\epsilon])$  is a compact subset of  $U_k''$ . Let  $X' = X - \rho_k^{-1}([0, \epsilon])$  and let  $X'' = \rho_k^{-1}([0, 2\epsilon])$ .

After rounding corners,  $X'$ ,  $X''$ , and  $X' \cap X''$  are all compact radial manifold complexes with boundary. (See Lemma 6.6 if you really want to see all the details.) Moreover,  $X'$  and  $X'' \cap X'$  have smaller depth than  $X$ . Furthermore,  $A \cap B \cap X'$  is a closed, proper radial subcomplex of  $X'$  and  $A \cap B \cap X' \cap X''$  is a closed, proper radial subcomplex of  $X' \cap X''$ . So by induction,  $(X' \cap A, X' \cap B)$  and  $(X' \cap X'' \cap A, X' \cap X'' \cap B)$  are both excisive couples. Also,  $X' \cap C$  and  $X' \cap X'' \cap C$  all have finitely generated homology for  $C = X, A, B, A \cap B$ .

Next note that  $\varphi_k$  gives a deformation retraction of  $X''$  to  $S_k$  which restricts to deformation retractions of  $X'' \cap A$  to  $S_k \cap A$  and  $X'' \cap B$  to  $S_k \cap B$ . So  $(X'' \cap A, X'' \cap B)$  is an excisive couple if  $(S_k \cap A, S_k \cap B)$  is. But  $(S_k \cap A, S_k \cap B)$  is an excisive couple since  $S_k \cap A \cap B$  is a bicollared submanifold of  $S_k$ . Moreover,  $X''$ ,  $X'' \cap A$ ,  $X'' \cap B$ , and  $X'' \cap A \cap B$  deformation retract to  $S_k$ ,  $S_k \cap A$ ,  $S_k \cap B$ , and  $S_k \cap A \cap B$  and hence all have finitely generated homology.

So by Lemma 6.2 below we know that  $\{A, B\}$  is an excisive couple. Using the Mayer-Vietoris sequence we then see that  $X$ ,  $A$ ,  $B$ , and  $A \cap B$  all have finitely generated homology and  $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$ .  $\square$

**Lemma 6.2.** *Let  $A$ ,  $B$ ,  $U$ , and  $V$  be closed subsets of a topological space  $X$  so that:*

- (1)  $X = A \cup B$  and  $X = \text{Int}(U) \cup \text{Int}(V)$ .
- (2)  $\{A \cap U, B \cap U\}$  is an excisive couple.
- (3)  $\{A \cap V, B \cap V\}$  is an excisive couple.
- (4)  $\{A \cap U \cap V, B \cap U \cap V\}$  is an excisive couple.

*Then  $\{A, B\}$  is an excisive couple.*

*Proof.* In the terminology of [18] we know that  $\{(A \cap U, A \cap B \cap U), (A \cap V, A \cap B \cap V)\}$  and  $\{(U, B \cap U), (V, B \cap V)\}$  are both excisive couples of pairs since  $X = \text{Int}(U) \cup \text{Int}(V)$ . Comparing the relative Mayer-Vietoris sequences of  $\{(A \cap U, A \cap B \cap U), (A \cap V, A \cap B \cap V)\}$  and  $\{(U, B \cap U), (V, B \cap V)\}$  and using the 5-Lemma we see that the inclusion  $(A, A \cap B) \subset (X, B)$  induces an isomorphism on singular homology. Hence  $\{A, B\}$  is an excisive couple.  $\square$

**Lemma 6.3.** *A radial manifold complex with boundary is locally path connected.*

*Proof.* Take any open set  $V$  in  $X$  and any  $x \in V \cap S_i$ . Let  $K$  be a compact neighborhood of  $x$  in  $U_i \cap V$ . We may as well assume that  $K \cap S_i$  is path connected. If not, let  $C \subset K \cap S_i$  be a compact path connected neighborhood of  $x$  in  $S_i$  and replace  $K$  by  $K - \bigcup_{y \in K \cap S_i - C} O_y$  where each  $O_y$  is an open neighborhood of  $y$  in  $X - C$ . Now  $K \times [0, 1] \cap \varphi_i^{-1}(X - V)$  is compact, so its projection  $L$  to  $K$  is compact. But  $x \notin L$ . Let  $W = \{y \in K - L \mid \varphi_i(y, 1) \in K\}$ . Then  $W$  is a

neighborhood of  $x$  in  $V$  and every point  $y$  of  $W$  has a path in  $V$  to  $x$ , since  $\varphi_i(y, t)$  is a path from  $y$  to  $\varphi_i(y, 1) \in K \cap S_i$ .  $\square$

**Lemma 6.4.** *A radial manifold complex with boundary is second countable, hence metrizable.*

*Proof.* Each  $S_i$  is second countable so it has a countable basis  $V_{ij}$ ,  $j = 1, 2, \dots$ . Let  $\pi_i: U_i \rightarrow S_i$  be the retraction  $\pi_i(x) = \varphi_i(x, 1)$ . A countable basis for  $X$  consists of sets of the form  $V_{ijn}$  for  $i = 0, \dots, m$ ,  $j = 1, 2, \dots$ ,  $n = 1, 2, \dots$  where  $V_{ijn}$  is the union of the path components of  $\pi_i^{-1}(V_{ij}) \cap \rho_i^{-1}([0, 1/n])$  which contain points of  $V_{ij}$ . We know  $V_{ijn}$  is open by Lemma 6.3. To see  $\{V_{ijn}\}$  forms a basis, pick any open set  $O \subset X$  and  $p \in O$ . Suppose  $p \in S_i$ . Choose  $V_{ij}$  so  $p \in V_{ij}$ ,  $\text{Cl}(V_{ij})$  is compact, and  $\text{Cl}(V_{ij}) \subset S_i \cap O$ . Let  $O'$  be a compact neighborhood of  $\text{Cl}(V_{ij})$  in  $O$ . Choose  $n$  so  $\rho_i(x) \geq 1/n$  for all  $x \in \text{Fr}(O') \cap \pi_i^{-1}(\text{Cl}(V_{ij}))$ . Then  $p \in V_{ijn} \subset O$ . To see this, suppose that  $\alpha: [0, 1] \rightarrow \pi_i^{-1}(V_{ij}) \cap \rho_i^{-1}([0, 1/n])$  is a path with  $\alpha(0) \notin O$  but  $\alpha(1) \in V_{ij}$ . Then  $\alpha(s) \in \text{Fr}(O')$  for some  $s$  but this contradicts the choice of  $n$ .  $\square$

**Lemma 6.5.** *Suppose  $X$  is a radial manifold complex with boundary and  $C$  and  $D$  are closed subsets of  $X$  so that  $X = C \cup D$  and so that  $C \cap D$  is a closed, proper radial subcomplex of  $X$ . Suppose also that  $Z \subset X$  is a closed, proper radial subcomplex of  $X$  transverse to  $C \cap D$ , i.e., so  $Z \cap S_i$  is transverse to  $C \cap D \cap S_i$  in  $S_i$  and  $Z \cap C \cap D \cap S_i$  is transverse to  $\partial S_i$  for each  $i$ . Then after perhaps shrinking the  $U_i$ s and rounding corners,  $C$  and  $D$  have a radial manifold complex with boundary structure obtained by restricting  $\varphi_i$ ,  $\rho_i$ , and  $\xi_i$ . Moreover,  $Z \cap C$  is a closed, proper radial subcomplex of  $C$  and  $Z \cap D$  is a closed, proper radial subcomplex of  $D$ .*

*Proof.* Choose neighborhoods  $U_i''$  of  $S_i$  in  $U_i$  so that:

- (1)  $\xi_i$  is tangent to  $Z \cap U_i''$  and  $C \cap D \cap U_i''$ .
- (2) If  $\varphi_i(x, t_0) \in Z \cap U_i''$  then  $\varphi_i(x, t) \in Z$  for all  $t \geq t_0$ .
- (3) If  $\varphi_i(x, t_0) \in C \cap D \cap U_i''$  then  $\varphi_i(x, t) \in C \cap D$  for all  $t \geq t_0$ .

Since  $\varphi_i(S_i \times [0, 1]) = S_i$ , we may choose neighborhoods  $U_i'$  of  $S_i$  in  $U_i''$  so that  $\varphi_i(U_i' \times [0, 1]) \subset U_i''$ .

Note now that if  $x \in C \cap U_i'$  and  $\varphi_i(x, t_0) \notin C$  for some  $t_0$ , then  $\varphi_i(x, t) \in C \cap D$  for some  $t < t_0$ , but then  $\varphi_i(x, t) \in C \cap D \cap U_i''$  so  $\varphi_i(x, t_0) \in C \cap D$ , a contradiction. So if we shrink each  $U_i$  to  $\text{Int}(U_i')$ , then  $\varphi_i$ ,  $\rho_i$ , and  $\xi_i$  will restrict to  $C$ . Likewise they will also restrict to  $D$ .

So the only thing that remains is to smooth out the corners in  $S_i \cap C$  and  $S_i \cap D$  which appear at  $\partial S_i \cap C \cap D$ . We will round the corner of  $S_i \cap C$ , the corner of  $S_i \cap D$  is done similarly. Let us be precise about what we want to do. For notational convenience, fix an  $i$ , let  $C_i = S_i \cap C$  and let  $E$  denote the corner  $E = \partial S_i \cap C \cap D$ . We must put a smooth structure  $\Sigma$  on  $C_i$  so that:

- (1)  $\Sigma$  restricts to the given smooth structure on  $C_i - E$ .
- (2)  $\Sigma$  restricts to the given smooth structure on  $E$ .

- (3)  $Z \cap C_i$  is a smooth submanifold of  $C_i$ , transverse to  $\partial C_i = (C_i \cap \partial S_i) \cup (C_i \cap D)$ .
- (4) For each  $j < i$ , the restriction of  $\rho_j$  to  $C_i \cap U_j$  is smooth, after perhaps shrinking  $U_j$ .

Following is our strategy for constructing such a  $\Sigma$ . Since  $\partial S_i$  has a collar neighborhood, there is a smooth  $\gamma: S_i \rightarrow [0, \infty)$  so that  $\gamma^{-1}(0) = \partial S_i$  and  $\gamma$  has no critical points in a neighborhood of  $\partial S_i$ . Since  $C_i \cap D$  separates  $S_i$  and is a smooth manifold, there is a smooth  $\eta: S_i \rightarrow \mathbb{R}$  so that  $\eta^{-1}(0) = C_i \cap D$  and  $\eta$  has no critical points in a neighborhood of  $C_i \cap D$ . Moreover, since  $C_i \cap D$  is transverse to  $\partial S_i$  we know that  $(\gamma, \eta)$  has no critical points in a neighborhood of  $E$ . In turn, since  $C_i \cap D$ ,  $\partial S_i$ , and  $Z \cap C_i$  are all in general position, we know that the restriction  $(\gamma, \eta)|_{Z \cap S_i}$  has no critical points in a neighborhood of  $E$ .

We claim that there is neighborhood  $W$  of  $E$  in  $S_i$  and a smooth retraction  $\pi: W \rightarrow C$  so that:

- a)  $(\gamma, \eta, \pi): W \rightarrow [0, \infty) \times \mathbb{R} \times E$  is an embedding.
- b)  $\pi^{-1}(E \cap Z) = W \cap Z$ .
- c) For each  $j < i$ , there is a neighborhood  $U_j'''$  of  $S_j$  in  $U_j'$  so that  $\rho_j \circ \pi(x) = \rho_j(x)$  for all  $x \in W \cap U_j'''$ .

Given this claim, we define  $\Sigma$  by taking charts around  $E$  to be of the form

$$\varphi': \pi^{-1}(V) \cap X' \rightarrow [0, \infty) \times \mathbb{R} \times \mathbb{R}^{n-2}$$

where

$$\varphi'(x) = \left( 2\gamma(x)\eta(x), \gamma(x)^2 - \eta(x)^2, \varphi \circ \pi(x) \right)$$

and  $\varphi: V \rightarrow \mathbb{R}^{n-2}$  is a chart in  $E$ . We then readily see that  $\Sigma$  satisfies the above conditions 1-4.

So it only remains to find the retraction  $\pi$  above. For each  $j < i$ , pick a closed neighborhood  $U_j'''$  of  $S_j$  in  $U_j'$ . By the usual partition of unity argument, there is a smooth vector field  $\mathbf{v}$  on  $S_i$  which is tangent to  $Z \cap S_i$ , so that  $d\rho_j(\mathbf{v}) = 0$  on a neighborhood of  $S_i \cap U_j'$  for all  $j < i$ , and so that  $d\gamma(\mathbf{v}) = 1$  on a neighborhood of  $\partial S_i$ . Integrate  $\mathbf{v}$  to obtain a flow  $\psi(x, t)$  and define  $\pi_1: W_1 \rightarrow \partial S_i$  by  $\pi_1(x) = \psi(x, -\gamma(x))$  for some small enough neighborhood  $W_1$  of  $\partial S_i$  in  $S_i$ . Note after perhaps shrinking  $W_1$  a bit, that  $\rho_j \circ \pi_1 = \rho_j$  on  $W_1 \cap U_j'$  for  $j < i$ . Now pick a smooth vector field  $\mathbf{v}'$  on  $\partial S_i$  which is tangent to  $Z \cap \partial S_i$ , so that  $d\rho_j(\mathbf{v}') = 0$  on a neighborhood of  $\partial S_i \cap U_j'$  for all  $j < i$ , and so that  $d\eta(\mathbf{v}') = 1$  on a neighborhood of  $E$ . Integrate  $\mathbf{v}'$  to obtain a flow  $\psi'(x, t)$  and define  $\pi_2: W_2 \rightarrow E$  by  $\pi_2(x) = \psi'(x, -\eta(x))$  for some small enough neighborhood  $W_2$  of  $E$  in  $\partial S_i$ . Note after perhaps shrinking  $W_2$  a bit, that  $\rho_j \circ \pi_2 = \rho_j$  on  $W_2 \cap U_j'$  for  $j < i$ . We now let  $\pi = \pi_2 \circ \pi_1$  and conditions b) and c) hold. To see a), note that  $(\gamma, \eta, \pi)$  has an explicit inverse  $(\gamma, \eta, \pi)^{-1}(s, t, x) = \psi(\psi'(x, t), s)$ .  $\square$

**Lemma 6.6.** *Corners can be rounded, as was claimed in Theorems 3.2 and 6.1.*

*Proof.* Apply lemma 6.5 to Theorem 3.2 by setting  $C = X'$ ,  $D = Y$ , and  $Z = \emptyset$ . Lemma 6.5 applies to Theorem 6.1 in three applications, first setting  $C = X'$ ,

$D = \rho_k^{-1}([0, \epsilon])$  and  $Z = A \cap B$ , next setting  $C = X - \rho_k^{-1}([0, 2\epsilon])$ ,  $D = X''$  and  $Z = A \cap B$ , and finally setting  $X = X''$ ,  $C = \rho_k^{-1}([0, \epsilon])$ ,  $D = X' \cap X''$  and  $Z = A \cap B \cap X''$ .  $\square$

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