Lectures on Real Stratification Theory

David TROTMAN

LATP-UMR 6632
University of Provence
Marseille, France

Preliminary version : 12 October 2005.

1. Stratifications.

What is a stratification ?

The idea is to decompose a singular space into smooth manifolds with some control on how these manifolds fit together.

In 1957 Whitney [W1] showed that every algebraic variety $V = f^{-1}(0)$, where $f : \mathbf{R}^n \to \mathbf{R}^p$ has polynomial coordinates, can be partitioned into finitely many connected smooth submanifolds of $\mathbf{R}^n$. This he called a *manifold complex.* Such a partition is obtained by showing that the singular part of V is again algebraic and of dimension strictly less that that of V. One obtains thus a filtration of V by algebraic subvarieties,

$$V \supset SingV \supset Sing(SingV) \supset \dots$$

Thom proposed that a partition should exist for which transversality to strata of a map $g : \mathbf{R}^m \to \mathbf{R}^n$ is an open condition on maps in $C^\infty(\mathbf{R}^m, \mathbf{R}^n)$, and that there should be some "local triviality" in a neighbourhood of each stratum.

As a result Whitney refined his definition in 2 papers [W2, W3] which appeared in 1965, concerning stratifications of real and complex analytic varieties. Thom then developed a theory of C^∞ stratified sets, described in detail in his 1969 paper entitled "*Ensembles et morphismes stratifiés*" [Th2].

I will now describe what has become the accepted notion of Whitney stratification (due to Thom and Whitney).

Definition (C^k stratification). Let Z be a closed subset of a differentiable manifold M of class C^k. A C^k *stratification* of Z is a filtration by closed subsets

$$Z = Z_d \supset Z_{d-1} \supseteq \dots \supseteq Z_1 \supseteq Z_0$$

such that each difference $Z_i - Z_{i-1}$ is a differentiable submanifold of M of class C^k and dimension i, or is empty. Each connected component of $Z_i - Z_{i-1}$ is called a *stratum* of dimension i. Thus Z is a disjoint union of the strata, denoted $\{X_\alpha\}_{\alpha \in A}$.

Example. The filtration of a realisation of a simplicial complex defined by skeleta, where the strata are the open simplices.

We would like the stratification to "look the same" at different points on the same stratum. This turns out to be possible if "looking the same" is interpreted as "having neighbourhoods which are homeomorphic". Various equisingularity conditions have been introduced ensuring this. An obvious necessary condition is as follows:

Definition. A stratification $Z = \bigcup_{\alpha \in A} X_\alpha$ satisfies the *frontier condition* if $\forall(\alpha, \beta) \in A \times A$ such that $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, one has $X_\alpha \subseteq \overline{X_\beta}$. As the strata are disjoint this means that $X_\alpha = X_\beta$ or that $X_\alpha \subset \overline{X_\beta} \setminus X_\beta$.

One says that the stratification is *locally finite* if the number of strata is locally finite.

2. Whitney's conditions (a) and (b).

The most successful of the different regularity conditions proposed so as to provide adequate "equisingularity" are the conditions (a) and (b) of Whitney.

Take two adjacent strata X and Y, i.e. two C^1 submanifolds of M such that $Y \subset \overline{X} \setminus X$. The pair (X, Y) is said to satisfy Whitney's condition (a) at $\in Y$, or to be (a)-regular at y if : $\forall$ sequences $\{x_i\} \in X$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ in the grassmannian G_{dimX}^{dimM}, one has $T_yY \subseteq \tau$.

The pair (X, Y) is said to satisfy Whitney's condition (b) at $y \in Y$, or to be (b)-regular at y if : $\forall$ sequences $\{x_i\} \in X$ and $\{y_i\} \in Y$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ and the lines $\overline{x_i y_i}$ tend to λ, one has $\lambda \in \tau$.

When $Z = \bigcup_{\alpha \in A} X_\alpha$ is a locally finite stratification such that all pairs of adjacent strata satisfy the frontier condition and are (b)-regular at all points, we say we have a *Whitney stratification* of Z.

Remark. It will be a nontrivial consequence of the theory that the frontier condition is automatically satisfied by pairs of adjacent strata of a locally finite (b)-regular stratification.

Exercises.

Let $\pi : T_Y \to Y$ be a C^1 tubular neighbourhood of Y in M. A pair of adjacent strata (X, Y) is said to be b^π)-regular if for all sequences $\{x_i\}$ in X such that x_i tends to y and the lines $\overline{x_i \pi(x_i)}$ tend to λ and the tangent planes $T_{x_i}X$ tend to τ, then $\lambda \in \tau$.

1. $(b) \Rightarrow (a)$.
2. $(b) \Leftrightarrow (b^\pi) \quad \forall \pi$.
3. $(a) + (b^\pi)$ for some $\pi \Leftrightarrow (b)$.
4. If (X, Y) is (b)-regular at $y \in Y$, then $dimY < dimX$.

The following standard example due to Whitney shows that (a) does not imply (b).

Example. Let $Z = Z_2 = \{y^2 = t^2x^2 + x^3\} \subset \mathbf{R}^3$. Set $Z_1 = \{(0,0,t) | t \in \mathbf{R}\}$ and $Z_0 = \emptyset$. Then $Z_2 \supset Z_1 \supset Z_0 = \emptyset$ is a filtration defining a stratification with 4 strata of dimension 2 and one stratum of dimension 1. The strata are defined as follows : $X_1 = (Z_2 - Z_1) \cap \{t > 0\} \cap \{x < 0\}, X_2 = (Z_2 - Z_1) \cap \{t < 0\} \cap \{x < 0\}, X_3 = (Z_2 - Z_1) \cap \{y < 0\} \cap \{x > 0\}, X_4 = (Z_2 - Z_1) \cap \{y > 0\} \cap \{x > 0\}, Y = Z_1$.

You can check that the pairs of strata (X_3, Y) and (X_4, Y) are (b)-regular, and in fact they are each C^∞ manifolds with boundary, while (X_1, Y) and (X_2, Y) are not (b)-regular at $(0, 0, 0)$, although they are (a)-regular. Note that the frontier property does not hold for (X_1, Y) and (X_2, Y). It is possible to unite X_1 and X_2 into one connected stratum by turning Y into a circle, so that the frontier condition would hold. But (b) will still fail.

Next we give an example showing that (b^π) does not imply (a).

Example (Koike and Kucharz). Let $Z = \{x^3 + 3xy^5 + ty^6 = 0\} \subset \mathbf{R}^3$, with filtration $Z_2 \supset Z_1 = (Ot) \supset Z_0 = \emptyset$.

Theorem (Whitney 1965). Every analytic variety (in $\mathbf{R}^n$ or $\mathbf{C}^n$) admits a Whitney stratification whose strata are analytic (hence C^∞) manifolds.

Hironaka proved that the same is true of every subanalytic set (in particular every semialgebraic set). His proof uses resolution of singularities. A more elementary proof is due to Denkowska, Wachta and Stasica [DWS, DS].

One can ask why one should study Whitney's condition (a), as it is strictly weaker than condition (b). One reason is that it is both simple to understand and easy to check. A second reason is that it is both necessary and sufficient for transversality to the strata of a stratification to be an *open* condition, as we shall see in the next theorem.

We say that a map $f : N \to M$ between C^1 manifolds is *transverse* to a stratification of a closed set $Z \subset M$, if $\forall x \in N$ such that $f(x) \in Z$, then

$$(df)_x T_x N + T_{f(x)} X = T_{f(x)} M$$

where X is the stratum containing $f(x)$.

Theorem (Trotman 1979). A locally finite stratification of a closed subset Z of a C^1 manifold M is (a)-regular if and only if for every C^1 manifold N, $\{f \in C^1(N, M) | f$ is transverse to the strata of $Z\}$ is an open set in the Whitney C^1 topology.

Condition (a) for (X, Y) says that the distance between the tangent space to X at x and the tangent space to Y at y tends to zero as x tends to y. Kuo and Verdier studied what happens when the rate of vanishing of this distance is $O(|x - \pi_Y(x)|)$.

Definition. Two adjacent strata (X, Y) are (w)-regular at $y_0 \in Y$, or satisfy the Kuo-Verdier condition (w), if there exists a constant $C > 0$ and there exists a neighbourhood U of y_0 in M such that

$$d(T_y Y, T_x X) < C||x - y||$$

$\forall x \in U \cap X, y \in U \cap Y$.

Here, for vector subspaces V and W of an inner product space E,

$$d(V, W) = sup\{inf\{sin\theta(v, w) | w \in W^*\} | v \in V^*\}$$

where $\theta(v, w)$ is the angle between v and w.

Note that $d(V, W) = 0 \Leftrightarrow V \subset W$, and that $d(V, W) = 1 \Leftrightarrow \exists v \in V^*, v \perp W$.

Proposition (Kuo). For subanalytic X and Y, $(w) \Rightarrow (b)$.

So (w)-regularity is a stronger regularity condition than (b). It turns out to be generic too, as the following theorem shows.

Theorem (Verdier 1976). Every subanalytic set admits a locally finite (w)-regular stratification. This is also true for definable sets in arbitrary o-minimal structures (Loi 1998).

For complex analytic strata, $(b) \Leftrightarrow (w)$ (Teissier 1982 [Te]). Real algebraic examples showing that (b) does not imply (w) are common because (b) is a C^1 invariant while (w) is not.

Example (Brodersen-Trotman [BT]) Let $Z = \{y^4 = t^4x + x^3\} \subset \mathbf{R}^3$. Then the stratification of Z defined by $Z = Z_2 \supset Z_1 = (Ot)$ is (b)-regular but not (w)-regular. Z is actually the graph of the C^1 function $f(x,t) = (t^4x + x^3)^{1/4}$.

As we want our stratifications to "look the same" at different points of a given stratum one might hope that there is a C^1 diffeomorphism mapping neighbourhoods of a point y_1 on Y to neighbourhoods of another point y_2 on Y. This is not true in general.

Example (Whitney). Let $Z = \{(x,y,t)|xy(x-y)(x-ty), t \neq 1\} \subset \mathbf{R}^3$, stratified by $Z = Z_2 \supset Z_1 = (Ot)$. This is a family of 4 lines parametrised by t. The stratification is both (b)-regular and (w)-regular, but there is no C^1 diffeomorphism mapping Z_{t_1} to Z_{t_2} where $Z_t = Z \cap \mathbf{R}^2 \times \{t\}$, because of the crossratio obstruction. (A linear isomorphism of the plane preserving 3 lines preserves also any 4th line.)

In the next talk we will discuss the Thom-Mather isotopy theorem ensuring local topological triviality and more recent work of Mostowski and Parusinski giving generic local bilipschitz triviality of analytic varieties and subanalytic sets.

Transversal intersection of stratifications: Suppose Z and Z' are two closed stratified sets of a manifold M. Denote the set of strata by Σ and Σ' respectively. We can stratify $Z \cap Z'$ by $\Sigma \cap \Sigma' = \{X \cap X' | X \in \Sigma, X' \in \Sigma'\}$ if Σ and Σ' are transverse, i.e. $\forall X \in \Sigma, \forall X' \in \Sigma'$, X and X' are transverse.

Theorem. If (Z, Σ) and (Z', Σ') are Whitney (b)-regular (resp. (a)-regular, resp. (w)-regular), and have transverse intersections in M, then $(Z \cap Z', \Sigma \cap \Sigma')$ is (b)-regular (resp. (a)-regular, resp. (w)-regular).

This can often be useful. The case of (b)-regularity was treated in the book by Gibson, Wirthmuller, du Plessis and Looijenga (1976). A more general theorem of this kind was proved by Orro and Trotman in 2002 [OT], including (w)-regularity.

Products: If Z and Z' are Whitney stratified then so is $Z \times Z'$.

Triangulation: It is known that all Whitney stratified sets are triangulable (Goresky [G1], Johnson, Shiota).

Open question [G2]. Does a Whitney stratified set (Z, Σ) have a triangulation whose open simplexes are the strata of a Whitney stratification refining Σ ?

I will be more precise in stating what is known about local topological triviality.

Theorem (Thom-Mather). Let (Z, Σ) be a Whitney stratified subset of a C^2 manifold M. Then for each stratum $Y \in \Sigma$ and each point $y_0 \in Y$ there is a neighbourhood U of y_0 in M, a stratified set $L \subset S^{k-1}$ and a homeomorphism

$$h : (U, U \cap Z, U \cap Y) \to (U \cap Y) \times (B^k, c(L), y_0)$$

such that $p_1 \circ h = \pi_Y$, where $c(L)$ is the cone on the link L with vertex y_0, $k = codimY$, B^k is the k-ball, and π_Y is the projection onto $U \cap Y$ of a tubular neighbourhood.

This theorem applies without any hypothesis of analyticity or subanalyticity. The proof of Mather [M] uses the notion of controlled vector field, and the homeomorphism is obtained by integrating such controlled vector fields.

A (stratified) vector field v on a stratified set (Z, Σ) is defined by a collection of vector fields $\{v_X | X \in \Sigma\}$. It is *controlled* when $(\pi_Y)_* v_X(x) = v_Y(\pi_Y(x))$ and $(\rho_Y)_* v_X(x) = 0$ on a tubular neighbourhood T_Y of Y, where T_Y is part of a set of compatible tubular neighbourhoods called control data. See Mather's notes [M] for details of the theory of controlled vector fields. It was not until around 1996 that a proof was published that these stratified controlled vector fields could be assumed to be *continuous* (Shiota-du Plessis-Bekka) : given a vector field v_Y on a stratum Y, there exists a continuous controlled stratified vector field $\{v_X\}$ on M extending v_Y. This result has been used for example by Hamm [Ha] to simplify some statements in stratified Morse theory [GM], and by S. Simon to prove a stratified version of the Poincaré-Hopf theorem [Si].

One can characterise (w)-regularity using stratified vector fields.

Proposition (Brodersen-Trotman). A stratification is (w)-regular $\Leftrightarrow$ every vector field on a stratum Y extends to a rugose stratified vector field in a neighbourhood of Y.

Here a vector field is called *rugose* near y_0 when there exists a neighbourhood U of y_0 and a constant $C > 0$, such that $\forall x \in U \cap X, \forall y \in U \cap Y$,

$$\parallel v(x) - v(y) \parallel \leq C \parallel x - y \parallel .$$

This resembles an asymmetric Lipschitz condition, and poses the question of whether the extension of a Lipschitz vector field can be chosen to be Lipschitz.

3. Lipschitz stratifications.

Mostowski in 1985 introduced conditions (L) on a stratification, further strengthening (w), which imply the possibility of extending Lipschitz vector fields and are (almost) characterised by the existence of Lipschitz extensions [Pa].

Here are the definitions, which are necessarily somewhat complicated.

Let $Z = Z_d \supset \ldots \supset Z_\ell \neq \emptyset$ be a closed stratified set in $\mathbf{R}^n$. Write $\overset{\circ}{Z}_j = Z_j - Z_{j-1}$.

Definition (Mostowski). Let $\gamma > 1$ be a fixed constant. A *chain* for a point $q \in \overset{\circ}{Z}_j$ is a strictly decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = \ell$ such that each $j_s (s \geq 2)$ is the greatest integer less than j_{s-1} for which

$$dist(q, Z_{j_s - 1}) \geq 2\gamma^2 dist(q, Z_{j_s}).$$

For each $j_s, 1 \leq s \leq r$, choose $q_{j_s} \in \overset{\circ}{Z}_{j_s}$ such that $q_{j_1} = q$ and $|q - q_{j_s}| \leq \gamma dist(q, Z_{j_s})$.

If there is no confusion we call $\{q_{j_s}\}_{s=1}^r$ a chain of q.

For $q \in \overset{\circ}{Z}_j$, let $P_q : \mathbf{R}^n \to T_q(\overset{\circ}{Z}_j)$ be the orthogonal projection to the tangent space and let $P_q^\perp = I - P_q$ be the orthogonal projection to he normal space $(T_q(\overset{\circ}{Z}_j))^\perp$.

Definition (Mostowski). A stratification $\Sigma = \{Z_j\}_{j=\ell}^d$ of Z is said to be a *Lipschitz stratification*, or to satisfy the (L)-conditions, if for some constant $C > 0$ and for every chain $\{q = q_{j_1}, \ldots, q_{j_r}\}$ with $q \in \overset{\circ}{Z}_{j_1}$ and each $k, 2 \leq k \leq r$,

$$| P_q^\perp P_{q_{j_2}} \cdots P_{q_{j_k}} | \leq C \mid q - q_{j_2} \mid / d_{j_k - 1}(q) \qquad (L1)$$

and for each $q' \in \overset{\circ}{Z}_{j_1}$ such that $\mid q - q' \mid \leq (1/2\gamma)\, d_{j_1 - 1}(q)$,

$$| (P_q - P_{q'}) P_{q_{j_2}} \cdots P_{q_{j_k}} | \leq C \mid q - q' \mid / d_{j_k - 1}(q) \qquad (L2)$$

and

$$| P_q - P_{q'} | \leq C \mid q - q' \mid / d_{j_1 - 1}(q) \qquad (L3).$$

Here $dist(-, Z_{\ell - 1}) \equiv 1$, by convention.

It is not hard to show that for a given Lipschitz stratification $\exists\, C > 0$ such that $\forall x \in \overset{\circ}{Z}_j, \forall y \in \overset{\circ}{Z}_k, k < j$ then

$$|P_x^\perp P_y| \leq \frac{C|x - y|}{dist(y, Z_{k-1})},$$

and as $|P_x^\perp P_y| = d(T_y \overset{\circ}{Z}_k, \overset{\circ}{Z}_j)$, (w)-regularity follows with a precise estimation for the constant (which can tend to infinity as y approaches Z_{k-1}).

Theorem (Parusinski 1994). Every subanalytic set admits a Lipschitz stratification. Moreover such Lipschitz stratifications are locally bilipschitz trivial.

It is not true that definable sets in arbitrary o-minimal structures admit Lipschitz stratifications.

Example (Parusinski). Let $X(t)$ be the union of the x-axis and the graph $y = x^t (x > 0)$ in $\mathbf{R}^3 = (x, y, t)$. Then the Lipschitz types of $X(t)$ are distinct for all $t > 1$.

Question: Do definable sets in polynomially bounded o-minimal structures admit Lipschitz stratifications ?

It is clear that the (L)-conditions are much more of a constraint than is (w).

Example (Mostowski). In $\mathbf{C}^4$ or $\mathbf{R}^4$ let $Z = \{y = z = 0\} \cup \{y = x^3, z = tx\}$. Then (w) holds along the t-axis, but (L) fails.

Example (Koike-Juniati). In $\mathbf{R}^3$ let $Z = \{y^2 = t^2x^2 + x^3, x \geq 0\}$ stratified by $Z = Z_2 \supset Z_1 = \langle Ot \rangle$. It is easy to check that (w) holds for this semialgebraic example, while $(L2)$ fails : let $q = q_{j_1} = q_2 = (t^2, \sqrt{2}t^3, t), q' = (t^2, -\sqrt{2}t^3, t), q_{j_2} = q_1 = (0, 0, t)$, as $t \to 0$. See [JTV].

In his 1974 Arcata lectures Teissier gave criteria for a good equisingularity condition E on a stratification of a complex analytic set; E-regularity should:

1) be as strong as possible;
2) be generic, i.e. every complex analytic set should possess an E-regular stratification;
3) imply local topological triviality along strata;
4) imply equimultiplicity;
5) be preserved after intersection with generic linear spaces containing a given stratum, locally linearised ($E \Rightarrow E^*$);
6) have a Zariski equisingularity property.

Criteria 2) to 6) hold for Whitney (b)-regularity (Teissier 1982), which turns out to be equivalent to (w) in the complex case. Criterion 5) is an essential part of the proof, via the equimultiplicity of polar varieties. (Recall that (b) does not imply (w) for real algebraic varieties.)

Definition of (E^*)-regularity.
Let M be a C^2 manifold. Let Y be a C^2 submanifold of M and let $y \in Y$. Let X be a C^2 submanifold of M such that $y \in \overline{X}$ and $Y \cap X = \emptyset$. Let (E) denote an equisingularity condition (e.g. $(b), (w), (L)$). Then (X, Y) is said to be (E_{codk})-regular at $y (0 \leq k \leq codY)$ if there exists an open dense subset U^k of the grassmannian of codimension k subspaces of T_yM containing T_yY, such that if W is a C^2 submanifold of M with $Y \subset W$ near y, and $T_yW \in U^k$, then W is transverse to X near y, and $(X \cap W, Y)$ is (E)-regular at y.

One says finally that (X, Y) is (E^*)-regular at y if (X, Y) is (E_{codk})-regular for all $k, 0 \leq k < codY$.

Theorem (Navarro Aznar-Trotman). For subanalytic stratifications, $(w) \Rightarrow (w^*)$, and if $dimY = 1, (b) \Rightarrow (b^*)$.

Question: Does $(b) \Rightarrow (b^*)$ for subanalytic stratifications in general ?

Theorem [Te2]. For complex analytic stratifications, $(b) \Rightarrow (b^*)$.

Theorem [JTV]. For subanalytic stratifications, $(L) \Rightarrow (L^*)$.

We conclude that the (L)-regularity of Mostowski is possibly the best equisingularity condition. However it has disadvantages:

1) it is not generic for definable sets over non polynomially bounded o-minimal structures ((b) and (w) are generic, as proved by Ta Le Loi),

2) it has a long and complicated definition which is hard to work with ((b) and (w) have simple definitions).

4. Definable trivialisations.

We have seen that Whitney (b)-regularity ensures local topological triviality. Mostowski and Parusinski proved that a (L)-regular stratification is locally bilipschitz trivial. It is natural to ask if such trivialisations can be chosen to be definable. Or generally if Z is a semialgebraic set, is there some stratification which is locally semialgebraically trivial ? This was proved by Hardt in 1980; his method was very recently improved by G. Valette who obtained semialgebraic bilipschitz triviality.

Theorem (Hardt). Semialgebraic sets admit locally semialgebraically trivial stratifications.

Theorem (Valette). Semialgebraic sets admit locally semialgebraically bilipschitz trivial stratifications.

There are also subanalytic versions of these results. For semialgebraic (b)-regular stratifications Coste and Shiota [CS] proved a semialgebraic isotopy theorem using real spectrum methods. See the book of Shiota for further details and references.

5. Bekka's (c)-regularity.

It can be important to be more precise as to when a stratification is locally topologically trivial, for example when classifying topologically or studying topological stability (cf. work of the Liverpool School by Bruce, Giblin, Gibson, Wall, Looijenga, Wirthmuller and the book of du Plessis and Wall). Then one needs the weakest regularity condition on a stratification ensuring local topological triviality.

Definition (K. Bekka). A stratified set (Z, Σ) in a manifold M is (c)-regular if for every stratum Y of Σ there exists an open neighbourhood U_Y of Y in M and a C^1 function $\rho_Y : U_Y \to [0, \infty)$ such that ${\rho_Y}^{-1}(0) = Y$ and the restriction $\rho_Y|_{U_Y \cap Star(Y)}$ is a Thom map, where $Star(Y) = \bigcup\{X \in \Sigma | X \geq Y\}$, i.e. $\forall X \in Star(Y)$, with $\rho_{XY} = \rho_Y|_X$ and $x \in X$,

$$lim_{x \to y} T_x({\rho_{XY}}^{-1}(\rho_Y(x))) \supseteq T_y Y \quad \forall y \in Y.$$

Note that $\rho_Y : U_Y \to [0, \infty)$ is defined globally on a neighbourhood of Y. So this is not a local condition.

Theorem (Bekka). (c)-regular stratifications are locally topologically trivial along strata.

The proof is by proving the existence of an abstract stratified structure of Mather which allows the use of Mather's theory of controlled stratified vector fields. If one only requires constance of homological/cohomological data then one can weaken (c) even further - see the book of Schurmann.

We saw how (w) and (L) are characterised by the existence of appropriate lifts of vector fields. Here is the corresponding result for (c)-regularity.

Theorem (du Plessis-Bekka). A stratification is (c)-regular $\Leftrightarrow$ every C^1 vector field on a stratum Y admits a continuous controlled stratified extension to a neighbourhood of Y.

This means that there exists a family of vector fields $\{v_X | X \in Star(Y)\}$ such that $v = \bigcup v_X$ is continuous (in TM), while being controlled as defined above.

How do (c) and (b) compare ?

I proved that (b) over a stratum Y is equivalent to the property that for every C^1 tubular neighbourhood T_Y of Y the restriction to neighbouring strata of the associated map (π_Y, ρ_Y) is a submersion, where $\pi_Y : T_Y \to Y$ is the canonical retraction and $\rho_Y : T_Y \to [0, 1)$ the canonical distance function.

In comparison, (c) says that there exists some $C^1\rho$ (not necessarily associated to a tubular neighbourhood; ρ can be degenerate, e.g. weighted homogeneous, or even flat on Y) such that for every C^1 tubular neighbourhood T_Y of Y the restriction to neighbouring strata of the map (π_Y, ρ) is a submersion.

One can prove that (b) implies (c) while the converse is false.

6. Condition (t^k).

We return to the first example of Whitney, $Z = \{y^2 = t^2x^2 + x^3\}$. Slice the surface by a plane S transverse to the t-axis at 0. Then the topological type of the germ at 0 of the intersection $Z \cap S$ is constant, i.e. independent of S. Remember that Whitney $a)$ holds. Thom noticed this and mentioned it to Kuo, who proved the following theorem [K].

Theorem (Kuo 1978). If (X, Y) is (a)-regular at $y \in Y$ then (h^∞) holds, i.e. the germs at y of intersections $S \cap X$, where S is a C^∞ submanifold transverse to Y at $y \in S \cap Y$ and $dimS + dimY = dimM$, are homeomorphic.

It later turned out that one can replace (h^∞) by (h^1), meaning one considers all C^1 transversals S, and weaken (a) to (t^1), defined as follows.

Definition. A pair of strata (X, Y) is (t^k)-regular at $y \in Y$ if for every C^k submanifold S transverse to Y at $y \in Y \cap S$, there is a neighbourhood U of y such that S is transverse to X on $U \cap X$ $(1 \leq k \leq \infty)$.

Theorem (Trotman 1985). (t^1) is equivalent to (h^1).

Theorem (Trotman-Wilson 1999). For subanalytic strata, (t^k) is equivalent to the finiteness of the number of topological types of germs at y of $S \cap X$ for S a C^k transversal to Y $(k \geq 1)$.

The proofs developed with Kuo and Wilson use the "Grassmann blowup" introduced by Kuo and myself [KT]. Let

$$E^{n,d} = \{(L,x)|x \in L\} \subset G^{n,d} \times \mathbf{R}^n$$

for $d < n$, with projection to $G^{n,d}$, denote the canonical d-plane bundle. Let $\beta = \beta_{n,d}$ denote projection to $\mathbf{R}^n$. When $d = 1$ this is the usual blowup of $\mathbf{R}^n$ with centre 0.

Suppose $X, Y \subset \mathbf{R}^n$ and $0 \in Y$ with $d = codim Y$.

Let $\tilde{X} = \beta^{-1}(X)$ and let $\tilde{Y} = \{(L,0)|L \text{ is transverse to } Y \text{ at } 0\}$. the following striking theorem results from work by Kuo and myself [KT], completed by work with Wilson [TW].

Theorem. (X,Y) is (t^k)-regular at $0 \in Y$ if and only if $(\tilde{X},\tilde{Y})$ is (t^{k-1})-regular at every point of Y $(k \geq 1)$.

Meaning when $k = 1$: here (t^0) is equated with (w), the Kuo-Verdier condition. So in particular, (w)-regularity is the first in a sequence of (t^k)-regularity conditions !

Now we can see how to prove that (t^1) implies (h^1) by using the Verdier isotopy theorem for (w)-regular stratifications in the Grassmann blowup, although this was not the orginal proof.

The (t^k)- conditions were used to characterise jet sufficiency by Trotman and Wilson, generalising theorems of Bochnak, Kuo, Lu and others, and realising part of the early programme of Thom (1964). See [TW] for details. Very recent work with Gaffney and Wilson [GTW] develops an algebraic approach to the (t^k)- conditions, using integral closure of modules.

To illustrate the difference between (t^2) and (t^1), and the previous theorem, look at the Koike-Kucharz example (1979) given by $Z = \{x^3 - 3xy^5 + ty^6 = 0\} \subset \mathbf{R}^3$ stratified as usual by (X,Y) with Y the t-axis and X its complement $Z - Y$. Then (X,Y) is (t^2) but not (t^1) at 0. It is easy to check that there are 2 topological types of germs at 0 of intersections $S \cap X$ where S is a C^2 submanifold transverse to Y at 0. However the number of topological types of such germs for S of class C^1 is infinite, even uncountable.

It is easy to construct similar examples showing (t^k) does not imply (t^{k-1}).

7. Density and normal cones.

I mentioned Hironaka's theorem that complex analytic Whitney stratifications are equimultiple along strata. What is a real version of this statement ? Define the multiplicity $m(V,p)$ at a point p of a complex analytic variety V to be the number of points near p in the intersection of V with a generic plane L missing p of complementary dimension to that of V. This positive integer is equal to the Lelong number, or density $\theta(V,p)$ of V at p defined as the limit as ϵ tends to 0 of the quotient $\frac{vol(V \cap B_\epsilon(p))}{vol(P \cap B_\epsilon(p))}$. Kurdyka and Raby showed that the density is well-defined for subanalytic sets, as a positive real number. It is thus natural to conjecture (I did so in 1988) that the density of a subanalytic set is continuous along strata of a Whitney stratification, as a generalisation of Hironaka's theorem. This was partially proved by Georges Comte in his thesis (1998) for subanalytic Verdier (w)-regular stratifications [C], or more generally for subanalytic (b^*)-regular stratifications.

The general conjecture was proved for subanalytic (b)-regular stratifications by Guillaume Valette in 2003 [Va2]. Valette also showed that the density is a lipschitz function along strata of a subanalytic (w)-regular stratification.

In the paper [Hi] about equimultiplicity, Hironaka also proved results about the *normal cones* of analytic Whitney stratifications. Suppose Z is a stratified subset of $\mathbf{R}^n$ and let Y be a stratum. Let π_Y be the projection of a tubular neighbourhood of Y and let $\mu(v) = \frac{v}{\|v\|}$. The normal cone is defined to be:

$$C_Y Z = \overline{\{(x, \mu(x\pi_Y(x)))|x \in Z - Y\}}|_Y \subset \mathbf{R^n} \times S^{n-1}.$$

Let $p : C_Y Z \to Y$ be the canonical projection.

Theorem. A (b)-regular subanalytic stratification of a subanalytic set is
(npf) normally pseudo-flat, i.e. p is an open map, and
(n) for each stratum Y and each point y of Y, the fibre $(C_Y Z)_y$ of the normal cone at y is equal to the tangent cone $C_y(Z_y)$ at y to the special fibre $\pi_Y{-}1(y)$.

The proof is by integration of vector fields (cf. [Hi], also [OT]).

The result is not true for definable sets in non-polynomially bounded o-minimal structures. For an example one can take Z in $\mathbf{R^3}$ to be the graph of the function $f : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ defined by

$$z = f(x, y) = x - \frac{x}{ln(x)} ln(y + (x^2 + y^2)^{\frac{1}{2}}).$$

Stratify Z by $Z_1 = \{0y\} \subset Z$. One checks easily that $(C_Y Z)_0$ is an arc, while $C_0(Z_0)$ is a point so that the criterion (n) above fails. Moreover the example is not normally pseudoflat, nor (b^*)-regular, but it is Whitney (b)-regular.

In [OT] real algebraic (a)-regular examples are given showing that (n) does not imply (npf) and conversely. First let $(0z) = Z_1 \subset Z = \{x(x^2 + y^2)z^2 - (x^2 + y^2)^2 + xy^2 = 0\}$. Then (a) and (n) hold but (npf) fails. Finally look yet again at $\{y^2 = t^2x^2 + x^3\}$, stratified by the t-axis and its complement. Although (n) fails, because $(C_Y Z)_0$ consists of 2 points while $C_0(Z_0)$ consists of 1 point, it is normally pseudoflat.

REFERENCES

[B] K. Bekka, C-régularité et trivialité topologique, *Singularity theory and its applications, Warwick 1989, Part I*, Lecture Notes in Math. **1462**, Springer, Berlin, 1991, 42-62.

[BT] H. Brodersen, D. Trotman, Whitney (b)-regularity is weaker than Kuo's ratio test for real algebraic stratifications, *Mathematica Scandinavia* 45 (1979), 27-34.

[C] G. Comte, Équisingularité réelle : nombres de Lelong et images polaires, *Ann. Sci. École Norm. Sup.* (4) 33 (2000), no. 6, 757–788.

[CS] M. Coste, M. Shiota, Thom's first isotopy lemma: a semialgebraic version, with uniform bound, *Real analytic and algebraic geometry (Trento, 1992)*, 83–101, de Gruyter, Berlin, 1995.

[DS] Z. Denkowska, J. Stasica, *Ensembles sousanalytiques à la polonaise*, manuscript, 1985.

[DWS] Z. Denkowska, K. Wachta, J. Stasica, Stratification des ensembles sous-analytiques avec les propriétés (A) et (B) de Whitney, *Univ. Iagel. Acta Math.* No. 25, (1985), 183–188.

[GTW] T. Gaffney, D. Trotman and L. Wilson, Equisingularity of sections, (t^r) condition, and the integral closure of modules, preprint, 2005.

[GWPL] C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math. 552, Springer-Verlag, 1976.

[G1] M. Goresky, Triangulation of stratified objects, *Proc. A.M.S.* 72 (1978), no. 1, 193–200.

[G2] M. Goresky, Whitney stratified chains and cochains,. *Trans. Amer. Math. Soc.* 267 (1981), no. 1, 175–196.

[GM] M. Goresky, R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 14. Springer-Verlag, Berlin, 1988.

[Ham] H. Hamm, On stratified Morse theory, *Topology* 38 (1999), no. 2, 427–438.

[Har] R. Hardt, Semi-algebraic local-triviality in semi-algebraic mappings,*Amer. J. Math.* 102 (1980), no. 2, 291–302.

[Hi] H. Hironaka, Normal cones in analytic Whitney stratifications, *Inst. Hautes Études Sci. Publ. Math.* No. 36 1969 127–138.

[JTV] D. Juniati, D. Trotman, G. Valette, Lipschitz stratifications and generic wings, *Journal of the London Mathematical Society*, (2) 68 (2003), no. 1, 133–147.

[K] T.-C. Kuo, On Thom-Whitney stratification theory, *Math. Ann.* 234 (1978), no. 2, 97–107.

[KT] T.-C. Kuo, D. Trotman, On (w) and (t^s)-regular stratifications, *Inventiones Mathematicae* 92, 1988, 633–643.

[Lo] Ta Lê Loi, Verdier and strict Thom stratifications in o-minimal structures, *Illinois J. Math.* 42 (1998), no. 2, 347–356.

[M] J. Mather, *Notes on topological stability*, Mimeographed notes, Harvard University, 1970.

[Mo] T. Mostowski, *Lipschitz equisingularity*, Dissertationes Math. (Rozprawy Mat.) 243 (1985), 46 pp.

[NT] V. Navarro Aznar, D. Trotman, Whitney regularity and generic wings, *Annales de l'Institut Fourier, Grenoble*, 31, 1981, 87–111.

[OT] P. Orro, D. Trotman, Cône normal et régularités de Kuo-Verdier, *Bulletin de la Société Mathématique de France*, 130 (2002), 71–85.

[Pa] A. Parusinski, Lipschitz stratification of subanalytic sets, *Ann. Sci. cole Norm. Sup.* (4) 27 (1994), no. 6, 661–696.

[P] A. du Plessis, Continuous controlled vector fields, *Singularity theory (Liverpool, 1996, edited by J. W. Bruce and D. M. Q. Mond), London Math. Soc. Lecture Notes* **263**, Cambridge Univ. Press, Cambridge, (1999), 189-197.

[PW] A. A. du Plessis, C. T. C. Wall, *The Geometry of Topological Stability*, Oxford University Press, Oxford, 1995. Oxford University Press, 1995.

[Sch] J. Schurmann, *Topology of singular spaces and constructible sheaves*, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), 63. Birkhuser Verlag, Basel, 2003.

[Sh] M. Shiota, *Geometry of Subanalytic and Semialgebraic Sets*, Birkhaüser, Boston, 1997.

[Si] S. Simon, Champs totalement radiaux sur une structure de Thom-Mather, *Ann. Inst. Fourier (Grenoble)* 45 (1995), no. 5, 1423–1447.

[Te1] B. Teissier, Introduction to equisingularity problems, *Algebraic geometry (Proc. Sympos. Pure Math.*, Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 593–632. Amer. Math. Soc., Providence, R.I., 1975.

[Te2] B. Teissier, Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney, *Algebraic geometry (La Ràbida, 1981)*, 314–491, Lecture Notes in Math., 961, Springer, Berlin, 1982.

[Th1] R. Thom, Local topological properties of differentiable mappings, *Differential Analysis, Bombay Colloq.,* 1964, pp. 191–202

[Th2] R. Thom, Ensembles et morphismes stratifiés, *Bull. A.M.S.* 70, 1969, pp. 240–284.

[Tr1] D. Trotman, Stability of transversality to a stratification implies Whitney (a)-regularity, *Inventiones Mathematicae* 50, 1979, 273–277.

[Tr2] D. Trotman, Comparing regularity conditions on stratifications, *Proceedings of Symposia in Pure Mathematics, Volume 40, Arcata 1981–Singularities, Part 2,* American Mathematical Society, Providence, Rhode Island, 1983, 575–586.

[Tr3] D. Trotman, Transverse transversals and homeomorphic transversals, *Topology* 24 (1985), no. 1, 25–39.

[TW] D. Trotman, L. Wilson, Stratifications and finite determinacy, *Proceedings of the London Mathematical Society*, (3) 78, 1999, no. 2, 334–368.

[Va1] G. Valette, Lipschitz triangulations, *Illinois J. of Math.*, to appear.

[Va2] G. Valette, *Volume, density and Whitney conditions*, preprint.

[Ve] J.-L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, *Inventiones Math.* 36 (1976), 295-312.

[W1] H. Whitney, Elementary structure of real algebraic varieties, *Ann. of Math.* (2) 66 (1957), 545–556.

[W2] H. Whitney, Local properties of analytic varieties, *Differential and Combinatorial Topology*, Princeton Univ. Press, (1965), 205–244.

[W3] H. Whitney, Tangents to an analytic variety, *Annals of Math.* (2) 81 (1965), 496–549.