

EQUISINGULARITY OF SECTIONS, (t^r) CONDITION, AND THE INTEGRAL CLOSURE OF MODULES

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The theory of the integral closure of modules provides a powerful tool for studying stratification conditions which are connected with limits of linear spaces. It gives an expression which is both algebraic and geometric for these conditions. This connection allows one to control many geometric phenomena through associated numerical invariants. This paper will illustrate these points by examining the (t^r) conditions which were introduced by Thom and the second author. We will show how to apply these conditions to the study of certain families of sections of an analytic space.

The (t^r) conditions deal with the C^r sections of some stratified set; they were introduced initially by Thom in 1964 and developed by the second author from 1976 on, more recently in collaboration with Kuo and the third author ([Th], [Tr1], [Tr2], [Tr3], [Tr4], [Ku-Tr], [T-W]), and were applied to prove various equisingularity results. For real and complex analytic sets, we show that the (t^r) conditions have algebraic formulations in terms of integral closure of modules. Our formulation gives a new simple proof, for analytic sets, of the change in the conditions under Grassmann modification proved by Kuo and the second author [Ku-Tr] for subanalytic sets; this is used in conjunction with the principle of specialization of integral dependence to give numerical criteria for families of plane sections of complex complete intersections to be Whitney equisingular. Some of the results in this paper were announced by the first author in the 1994 Sao Carlos proceedings [G4].

In Section 1, we review the notions of integral closure, reduction, and strict dependence for submodules of $\mathcal{O}_{X,x}^p$, where X, x is the germ of a complex analytic set. We describe the analogues which are needed for the case of real analytic sets. We will apply these concepts to submodules of the Jacobian module $JM(F)$, where $X = F^{-1}(0)$. We review results from [G4] which use these tools to analyse the limits of tangent hyperplanes to X, x and to characterize the Whitney (a) and Verdier (w) regularity conditions.

In Section 2, we define jets of transversals and the (t^r) conditions. The main result of this section is Theorem 2.7, which is the characterization of condition (t^r) in terms of integral closure of modules. We note that condition (t^0) is in fact (w) . The Grassmann modification is a generalization of the blow-up, in which the projective space is replaced by the Grassmann space of planes of the dimension of the transversals we are using. The Grassmann modification theorem (2.13) says that the modification improves condition (t^r) to condition (t^{r-1}) . This is used at the end of Section 2 and several places in Section 4 for

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inductive proofs. Another useful tool is the characterization in 2.19 of which families of transversals are Verdier equisingular. This is applied (in 2.28) to families of transversals sharing a common r -jet for which (t^r) holds, and (in 2.23) to families of transversals sharing a common $(r-1)$ -jet for which (t^r) holds for generic parameter values. Then we describe the (t^{r^-}) condition, which strengthens (t^r) and was introduced in [T-W], and we show that (t^{r^-}) holds for an $(r-1)$ -jet iff (t^r) holds for all r -jets lying over that $(r-1)$ -jet (assuming $r \geq 1$; (t^{0^-}) is the condition known as strong Verdier regularity). We conclude by briefly discussing the ambient versions of the (t^r) conditions.

In Section 3, we prove theorems relating conditions (a) and (t^1) to the conormal modification and the aureole of X , and use these to give new examples showing that (t^1) is strictly weaker than (a) .

In Section 4, wholly devoted to complex analytic sets, we state and apply the principle of specialisation of integral dependence for coherent submodules of \mathcal{O}_X^p where X is equidimensional, due to the first author and S. Kleiman [G-K] (extending the work of Teissier, who introduced this principle for ideals). We characterise (t^r) in terms of the genericity of the multiplicity of a certain submodule of the jacobian module, then use the principle of specialisation of integral dependence to give an equimultiplicity criterion for (t^r) . As a consequence we obtain numerical criteria for Verdier equisingularity of families of plane sections in various situations.

§1. BACKGROUND ON THE THEORY OF INTEGRAL CLOSURE OF MODULES

In this section we review some basic facts about the integral closure of a module. We begin with a definition.

Definition 1.1. Suppose X, x is a complex analytic set germ, M a submodule of $\mathcal{O}_{X,x}^p$. Then $h \in \mathcal{O}_{X,x}^p$ is in the integral closure of M , denoted \overline{M} , iff for all analytic $\phi : (\mathbf{C}, 0) \rightarrow (X, x)$, $h \circ \phi \in (\phi^*M)\mathcal{O}_1$.

An algebraic definition of the integral closure of a module has been given by David Rees in [R], cf. [H-M2]. If we are working in the real analytic case, we simply use real analytic curves instead of complex analytic curves. The set of germs satisfying the condition of Definition 1.1 in this case is called the *real integral closure* of M and is sometimes denoted by $\overline{M}_{\mathbf{R}}$, but for simplicity we will use \overline{M} for real integral closure in this paper. To see that these notions are different, it is easy to check that the integral closure in $\mathcal{O}_{\mathbf{C}^2}$ of the ideal $(x^2 + y^2)$ is $(x^2 + y^2)$, while the integral closure in $\mathcal{O}_{\mathbf{R}^2}$ of the ideal $(x^2 + y^2)$ is (x^2, y^2, xy) . In the first case, it is clear that h must vanish on the curve defined by $(x^2 + y^2)$, hence must be divisible by $(x^2 + y^2)$. In the second case, it is clear that $\phi^*(x^2 + y^2)\mathcal{O}_1$ is just (s^{2k}) where k is the minimum of the orders of the first non-vanishing term in the components of ϕ . It is clear that the same equality holds for $\phi^*(x^2, y^2, xy)\mathcal{O}_1$. An algebraic definition of the real integral closure of an ideal has been given by Brumfiel [B]. Some of the properties of the real integral closure can be found in [G1]. Our results will hold in both the real and complex cases unless we say otherwise. Let \mathbf{k} denote either \mathbf{R} or \mathbf{C} .

The connection between the integral closures of ideals and modules is given by the next proposition, which in fact proves that our definition agrees with that of Rees. Let A be the $p \times q$ -matrix representing the left-most homomorphism in any exact sequence

$\mathcal{O}_{X,x}^q \rightarrow \mathcal{O}_{X,x}^p \rightarrow \mathcal{O}_{X,x}^p/M \rightarrow 0$. We denote by $J_k(M)$ the $(p-k)$ -th Fitting ideal of $\mathcal{O}_{X,x}^p/M$, i.e. the ideal in $\mathcal{O}_{X,x}$ generated by the $k \times k$ -minors of A ; if h is an element of $\mathcal{O}_{X,x}^p$, we denote by (h, M) the module generated by M and h .

Proposition 1.2. *Suppose that M is a submodule of $\mathcal{O}_{X,x}^p$, $h \in \mathcal{O}_{X,x}^p$ and the rank of (h, M) is k on each component of X, x . Then $h \in \overline{M}$ iff $J_k(h, M) \subseteq \overline{J_k(M)}$.*

Proof. Cf. [1.7] and [1.8] of [G1]. \square

Roughly speaking an element h is in the integral closure of an ideal I if the order of vanishing of h on the zero set of I is the same as the order of vanishing of I . Sometimes we require the order of vanishing of h to be greater. The notion of strict dependence makes this precise. Suppose M is a submodule of $\mathcal{O}_{X,x}^p$, $h \in \mathcal{O}_{X,x}^p$. Then h is *strictly dependent* on M if for all $\phi : (\mathbf{k}, 0) \rightarrow (X, x)$ we have $h \circ \phi \in \mathfrak{m}_1 \phi^* M$, where \mathfrak{m}_1 is the maximal ideal in \mathcal{O}_1 . We denote by M^\dagger the set of elements strictly dependent on M .

Note that M in general neither contains nor is contained in M^\dagger . (For example if $M = (x^3, y^3)$ in \mathcal{O}_2 , then $M^\dagger = \mathfrak{m}_2^4$.) If M is an \mathcal{O}_n -module then M^\dagger clearly contains $\mathfrak{m}_n \overline{M}$ but this inclusion may be strict.

The precise analogue of 1.2 is:

Corollary 1.3. *Suppose M is a submodule of $\mathcal{O}_{X,x}^p$, $h \in \mathcal{O}_{X,x}^p$ and the rank of (h, M) is k on each component of X, x . Then $h \in M^\dagger$ iff each minor in $J_k(h, M)$ which depends on h lies in $J_k(M)^\dagger$.*

Proposition 1.4. *Suppose X, x is a real or complex analytic set germ with irreducible components X_i , $i = 1, \dots, l$, M a submodule of $\mathcal{O}_{X,x}^p$. In the real case, assume for each i that the regular points of X_i are dense in X_i in the metric topology. Let U be a Zariski open, dense subset of $X - S(X)$, where $S(X)$ denotes the singular set of X (hence $U_i = U \cap X_i$ is dense in X_i in the metric topology). Then $h \in \mathcal{O}_{X,x}^p$ is in \overline{M} (respectively, in M^\dagger), iff for all analytic $\phi : (\mathbf{k}, 0) \rightarrow (X, x)$ with $\phi(t) \in U$ for all $t \neq 0$, $h \circ \phi \in (\phi^* M) \mathcal{O}_1$ (respectively, $h \circ \phi \in (\phi^* M) \mathfrak{m}_1$).*

Proof. We follow the proofs of [1.7], [1.8] and [4.2] of [G1]. Assume $h \in \mathcal{O}_{X,x}^p$ and for all analytic $\phi : (\mathbf{k}, 0) \rightarrow (X, x)$ with $h(t) \in U$ for all $t \neq 0$, $h \circ \phi \in (\phi^* M) \mathcal{O}_1$. Let $S = X - U$ and $S_i = X_i - U$. For the case of integral closure, it suffices to show that if ϕ is a curve in S , then $h \circ \phi \in (\phi^* M) \mathcal{O}_1$. Let ϕ be such a curve; necessarily ϕ lies in some S_i . If X_i is singular, let (\tilde{X}_i, π) be a smooth resolution of X_i ; then we may as well assume \tilde{X}_i is the germ at 0 of \mathbf{k}^m , $m = \dim X_i$. Since the regular points of X_i are dense in X_i , $\pi(\tilde{X}_i) = X_i$. Hence ϕ lifts to a curve $\tilde{\phi}$ to \tilde{X}_i .

For any k there is a curve $\tilde{\phi}_1$ so that $\tilde{\phi} - \tilde{\phi}_1 \in \mathfrak{m}_1^{k+1} \mathcal{O}_1^p$, and so that $\phi_1(t) = \pi \circ \tilde{\phi}_1(t) \in U$ for all $t \neq 0$. Note that $\phi - \phi_1 \in \mathfrak{m}_1^{k+1} \mathcal{O}_1^p$.

Suppose $\phi^*(M) \subsetneq \phi^*((h, M))$. Then by Artin-Rees and Nakayama's Lemma, there exists ν_0 such that for all $k > \nu_0$

$$\phi^*(M) \mathcal{O}_1 \subsetneq \phi^*((h, M)) \mathcal{O}_1 \text{ mod } \mathfrak{m}_1^k \mathcal{O}_1^p.$$

Assume $k > \nu_0$. Then $\phi_1^*((h, M)) = \phi_1^*(M)$ by assumption, hence $\phi^*((h, M)) = \phi^*(M) \text{ mod } \mathfrak{m}_1^k \mathcal{O}_1^p$, which is a contradiction.

For the case of strict integral closure, the same argument works, replacing $\phi^*((h, M))$ by $(\phi^*h, \mathfrak{m}_1\phi^*M)$. \square

In the applications we make, we are interested in an analytic space $X = F^{-1}(0)$ defined by $F : (\mathbf{k}^N, 0) \rightarrow (\mathbf{k}^p, 0)$. The modules we use are the jacobian module, denoted by $JM(F)$, which is the submodule of $\mathcal{O}_{X,x}^p$ generated by the first order partial derivatives of F , various submodules of it, and their reductions. (A submodule M' of M is a *reduction* of M if $\overline{M'} = \overline{M}$.)

To see the connection between integral closure and geometry, suppose that the function $F : \mathbf{C}^{n+1}, 0 \rightarrow \mathbf{C}, 0$ defines the germ of a hypersurface X at 0 in \mathbf{C}^{n+1} . Suppose X contains the y -axis, $\{0\} \times \mathbf{C}$. When does every sequence of limiting tangent planes to X at the origin contain the y -axis? In [G3] it is shown that for each limiting tangent plane H to a hypersurface X one can find a curve $\phi(t)$, such that the image of ϕ lies in the smooth set of X except for $\phi(0)$, and the limit as t tends to 0 of the tangent planes to X at $\phi(t)$ is H . This means that

$$\lim_{t \rightarrow 0} (1/t^k)(DF(\phi(t))) = (a_1, \dots, a_{n+1})$$

where $a_1x_1 + \dots + a_nx_n + a_{n+1}y = 0$ defines H , and k is the minimum of the orders of the first non-vanishing terms in $\frac{\partial F}{\partial x_1} \circ \phi, \dots, \frac{\partial F}{\partial x_n} \circ \phi, \frac{\partial F}{\partial y} \circ \phi$. The condition that H contains the y -axis is just that $a_{n+1} = 0$. This is equivalent to asking that $\frac{\partial F}{\partial y} \circ \phi \in \mathfrak{m}_1\phi^*((\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})\mathcal{O}_{n+1})$. This means that the condition that every limit of tangent planes to X at the origin contains the y -axis is equivalent to $\frac{\partial F}{\partial y} \in ((\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})\mathcal{O}_{n+1})^\dagger$.

We can also describe the limiting tangent planes in terms of integral closure. If H is a limiting tangent plane, and $v = (v_1, \dots, v_{n+1})$ is any vector in H , then $\sum a_iv_i = 0$, so the order in t of $DF(\phi(t))(v)$ must be greater than k . Let $JM(F)_H$ denote the submodule of $JM(F)$ generated by $\frac{\partial F}{\partial v}$ for all $v \in H$. Then we see that H is a limiting tangent plane iff $JM(F)_H$ is not a reduction of $JM(F)$. Both of these results hold more generally than for hypersurfaces, with an important change. In the hypersurface case, the Jacobian matrix of F had only one row, and this determined the tangent plane to X at a smooth point of X . In general, the rows of the Jacobian matrix of F will each determine a tangent hyperplane to X at a smooth point, and the row space of the Jacobian matrix of F at a smooth point will determine all of the tangent hyperplanes. Thus our results in the general case will be stated in terms of tangent hyperplanes. Here are the corresponding results.

Proposition 1.5. (Theorem 2.2 of [G3]) *Suppose X is an equidimensional complex analytic set, defined by a map germ F . A hyperplane H is a limiting tangent hyperplane to X at 0 iff $JM(F)_H$ is not a reduction of $JM(F)$.*

Proposition 1.6. (Theorem 2.3 of [G3]) *Suppose F and X are as above. Then every limiting tangent hyperplane contains a given vector v iff $DF(v)$ is in $JM(F)^\dagger$.*

Integral closure can also be used to describe the tangent cones of curves. Given $\phi : \mathbf{C} \rightarrow \mathbf{C}^n$, then the tangent cone to the image of ϕ at the origin will be a line, and the next proposition gives a test for a plane to contain this line.

Proposition 1.7. *Suppose ϕ is as above, and P is a plane. Then the tangent line to ϕ at the origin lies in P iff $\phi^*(I(P)) \subseteq \mathfrak{m}_1\phi^*\mathfrak{m}_n$.*

Proof. Let k be the minimum of the orders of the first non-vanishing term in the components of ϕ . Then $\phi^* \mathfrak{m}_n = (t^k)$. We have that

$$\lim_{t \rightarrow 0} (1/t^k)(\phi(t)) = (v_1, \dots, v_n).$$

If $h = a_1 x_1 + \dots + a_n x_n$ is a generator of $I(P)$, then the tangent direction (v_1, \dots, v_n) at 0 lies in P iff

$$\lim_{s \rightarrow 0} (1/s^k)(h(\phi(s))) = a_1 v_1 + \dots + a_n v_n = 0$$

iff the order of vanishing of every element of $\phi^*(I(P))$ is greater than k . \square

Two important conditions in the study of stratifications are Whitney's conditions (a) and (b). Whitney's condition (a) holds for a pair of strata (X, Y) at a point y if every limit of tangent hyperplanes to X at y contains the tangent space to Y at y . This condition seems to be necessary for any reasonable condition of equisingularity along a stratum Y (as opposed to the weaker (t^r) conditions which give equisingularity of families of sections through a fixed point y). For example, if $\{X_s\}$ is any topologically trivial family of complex analytic hypersurfaces with isolated singularities, and X is the total space of the family, with S the singular set, then $(X - S, S)$ satisfies Whitney (a) [L-S]. It is an open question, posed in particular by Thom, whether topologically trivial families of complex analytic sets are always Whitney (a)-regular over the smooth parameter space.

Whitney's condition (b) holds at y if for any sequence of pairs of points (x_i, y_i) which converge to (y, y) , such that the secant line joining the pair converges to a line l , and the tangent plane to X at x_i converges to a plane T , then T contains l . This condition implies local topological triviality but it also preserves some important local infinitesimal structure, the aureole [L-T]. For further discussion of these conditions see [GWPL] and [T3]. We are interested in a third regularity condition, Verdier's condition (w) [V]. Roughly speaking, this condition says that as you approach y from X , the distance between the tangent space to X at x and the tangent space to Y at y goes to zero at least as fast as the distance between x and Y goes to zero. In the complex analytic case (w) has been proved to be equivalent to Whitney's condition (b) ([T3], [H-M1]), while in the real analytic case it implies (b), but there are real algebraic examples found by Brodersen and the second author [B-T] which show that (b) can hold even though (w) fails. (See [Tr3] for a discussion of the implications between the various conditions.) We are interested in (w) because integral closure methods connect more directly with (w) than with (b) in the real analytic case. The precise definition is as follows:

Definition 1.8. Suppose A, B are linear subspaces at the origin in \mathbf{C}^N , then let

$$\text{dist}(A, B) = \sup_{\substack{u \in B^\perp - \{0\} \\ v \in A - \{0\}}} \tan \arcsin \frac{|(u, v)|}{\|u\| \|v\|}.$$

In the applications A is the "small" space and B the "big" space. Note that $\text{dist}(A, B) = 0$ iff $A \subseteq B$ and $\text{dist}(A, B) = \infty$ iff A has a nonzero vector perpendicular to B . Note also that if A and B are lines meeting in an angle θ , then $\text{dist}(A, B) = \tan \theta$. This distance allows us to talk about the Whitney condition (a) holding with a certain exponent.

Definition 1.9. Suppose $y \in Y \cap \bar{X}$, where X and Y are strata in a stratification of an analytic space such that $\text{dist}(T_y Y, T_x X) \leq C \text{dist}(x, Y)^\epsilon$ for some constant $C > 0$. Then (X, Y) is said to satisfy Whitney (a) with exponent ϵ at $y \in Y$. Verdier's condition (w) is Whitney (a) with exponent 1.

Verdier proved in [V] that (X, Y) is locally topologically trivial if (w) holds and $X \cup Y$ is locally closed.

The usual definition of $\text{dist}(A, B)$ omits the term $\tan \arcsin$; the definition above was introduced in [T-W]; it is needed to make (2.4) below, which represents condition (t^r), work correctly in the case $r = 0$, where we want (2.4) to represent condition (w).

In this paper we let X_0 denote the smooth points of X (if X is an analytic space given as $X = F^{-1}(0)$, then we mean that X_0 is the set of points at which X is smooth and the component functions of F define the reduced structure of X). The theory of the integral closure of modules allows us to show:

Theorem 1.10. *Suppose $X, 0 \subseteq \mathbf{C}^N$ is an equidimensional complex analytic set, $X = F^{-1}(0)$, Y a smooth subset of X . Then Whitney (a) holds for the pair (X_0, Y) at the origin iff $\frac{\partial F}{\partial y}$ is strictly dependent on $JM(F)$ for all tangent vectors $\frac{\partial}{\partial y}$ to Y at the origin.*

Proof. Cf. [G3] Cor. 2.4. \square

Theorem 1.11. *Let X, Y be as above with coordinates chosen so that $0 \times \mathbf{C}^k = Y$, $\mathfrak{m}_n = (x_1, \dots, x_n)$ denoting the ideal defining Y , and let $F : \mathbf{C}^N \rightarrow \mathbf{C}^p$ define X with reduced structure. Then $\frac{\partial F}{\partial y} \in \overline{\mathfrak{m}_n(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})} \mathcal{O}_X$ for all tangent vectors $\frac{\partial}{\partial y}$ to Y iff (w) holds for (X_0, Y) .*

Proof. Cf. [G1] Theorem 2.5. \square

The analogous results hold in the real analytic case using the real integral closure instead.

In Definition 1.8, suppose that B is a hyperplane, the kernel of $\omega \in \text{Hom}(\mathbf{C}^N, \mathbf{C})$. Then

$$\text{dist}(A, B) = \sup_{v \in A - \{0\}} \tan \arcsin \frac{|\omega(v)|}{\|\omega\| \|v\|}.$$

If $A(t)$ and $B(t) = \ker \omega(t)$ are analytic in t , and $v_1(t), \dots, v_N(t)$ are analytic and form a basis of \mathbf{C}^N for all t small such that $v_1(t), \dots, v_n(t)$ is a basis of $A(t)$ for all t small, then the order of vanishing

$$\text{ord}(\text{dist}(A(t), B(t))) = \text{ord} \left(\frac{\max_{1 \leq i \leq n} \omega(t)(v_i)}{\max_{1 \leq j \leq N} \omega(t)(v_j)} \right),$$

which is non-negative and, if positive,

$$(1.12) \quad \text{ord}(\text{dist}(A(t), B(t))) = \min_{1 \leq i \leq n} \text{ord}(\omega(t)(v_i)) - \min_{n+1 \leq j \leq N} \text{ord}(\omega(t)(v_j)).$$

We will use this formula in the proof of the main theorem of the next section.

§2. THE (t^r) CONDITIONS AND THE GRASSMANN MODIFICATION

Definition 2.1. Suppose X and Y are disjoint smooth submanifolds of \mathbf{R}^N in a neighborhood of $0 \in Y$. For r a positive integer, we say X is (t^r) regular over Y at 0 when every C^r submanifold Z transverse to Y at 0 , and of complementary dimension to Y , is transverse to X near 0 .

This concept was first introduced by Thom [Th]; Thom allowed Z to be of dimension greater than the complementary dimension of Y . Transversals Z of dimension complementary to Y are called *direct transversals*. The second author realized that the degree of smoothness of the transversal was important: (t^r) implies (t^{r+1}) , but the converse does not hold in general.

Further the second author showed that if you work with direct transversals, although Whitney (a) implies (t^1) , the converse is only known to hold when X and Y are subanalytic and the dimension of Y is 1 ([Tr1], [Tr2], [Tr3], [Tr4]). In [Tr4] the second author gave an example of a semialgebraic X and Y , with $\dim Y = 2$, satisfying (t^1) but not (a). We will give algebraic examples of this phenomenon in Section 3, where we relate conditions (a) and (t^1) to the conormal modification and the aureole of an analytic set X .

A little thought reveals the connection between these ideas and the family of sections of X by planes which are direct transversals to Y . If, as we move through the family of slices, one of the planes has a higher order of contact with X than the others, than that slice should not be equisingular with the other slices. The failure of (t^1) for a plane is a way of detecting this higher contact.

It was shown in [T-W] that it is convenient to refine the concept of (t^r) to one in which we only require that those transversals to Y which share a common r -jet be transverse to X ; we give the appropriate definition of r -jet of a transversal next.

Definition 2.2. First suppose $r > 0$. If $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^k, 0)$ are C^r map-germs with $j^r f(0) = j^r g(0)$, we say that $P = \Gamma(f)$ and $Q = \Gamma(g)$ are r -equivalent. The equivalence class is called the r -jet of P , denoted $j^r P$. Note that this condition is equivalent to $|f(x) - g(x)| = o(|x|^r)$. Now suppose $r = 0$. We want our transversals to have well-defined tangent planes away from the origin, so we don't want to consider graphs of continuous functions when forming a 0-jet. Instead we look at graphs of map-germs f of class $C^{0,-1}$, that is, f is C^0, C^1 except possibly at 0 , and $|f'(x)| = o(|x|^{-1})$. For two such f and g , we will say that $j^0 f(0) = j^0 g(0)$ if $f(0) = g(0)$; as above we say that $P = \Gamma(f)$ and $Q = \Gamma(g)$ are 0-equivalent. The graph of the r -th Taylor polynomial of f is called the degree r polynomial representative of $j^r P$, denoted P_r . If $f : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^k, 0)$, then the r -jet of $P = \Gamma(f)$ is the equivalence class of f as a real map $f : (\mathbf{R}^{2n}, 0) \rightarrow (\mathbf{R}^{2k}, 0)$.

We say f is C^{r-} if it is C^0, C^1 except possibly at 0 , and $|f(x)| = O(|x|^r)$, $|f'(x)| = O(|x|^{r-1})$. For example, for r odd, $f(x) = |x|^r$ is C^{r-} , but not C^r . The r^- jet of the graph of f is the set of graphs of all C^{r-} g such that $|f(x) - g(x)| = O(|x|^r)$ and $|f'(x) - g'(x)| = O(|x|^{r-1})$. For example $|x|^r$ and $2|x|^r$ have the same r^- -jet. These notions are discussed in more detail, and generalized, in [TW]. They will only be used in Theorems 2.30–2.32.

Remark. Assume that f and g are C^1 map-germs, and $|f(x) - g(x)| = o(|x|^r)$, $r > 0$. Then $\text{dist}(z, \Gamma(g)) = o(|z|^r)$ for all $z \in \Gamma(f)$. Conversely, if $\text{dist}(z, \Gamma(g)) = o(|z|^r)$ for all

$z \in \Gamma(f)$, then there exists a $\phi : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ such that

$$|(x, f(x)) - (\phi(x), g(\phi(x)))| = o(|(x, f(x))|^r) \text{ and hence } = o(|x|^r),$$

since the derivative of f is bounded near 0. Hence,

$$|x - \phi(x)| \text{ and } |f(x) - g(\phi(x))| = o(|x|^r).$$

But since the derivative of g is bounded near 0,

$$|g(x) - g(\phi(x))| \leq C|x - \phi(x)| = o(|x|^r) \text{ for some } C > 0.$$

Thus, $|f(x) - g(x)| = o(|x|^r)$.

One can define r -equivalence of germs P and Q of n -manifolds by requiring that both $\text{dist}(z, Q) = o(|z|^r)$ for all $z \in P$ and $\text{dist}(z, P) = o(|z|^r)$ for all $z \in Q$ hold. This notion of r -equivalence is a C^1 invariant. So r -equivalence as in (2.2) is the restriction of this notion to graphs of C^r map-germs. While we won't use this more general notion of r -equivalence in this paper, for completeness we will sketch the proof of C^1 invariance. Assume P and Q are r -equivalent. Suppose ϕ is a C^1 diffeomorphism-germ such that $\phi(P) = Q$. Without changing notation, we can replace ϕ , P and Q by representatives such that $\phi(P) = Q$ and $\frac{1}{C} \leq |d\phi(x)| \leq C$ holds for some $C \geq 1$ for all x in the domain of ϕ , which can be assumed convex. By the Mean Value Theorem, $|x - y| \frac{1}{C} \leq |\phi(x) - \phi(y)| \leq C|x - y|$ on the domain of ϕ . For each $x \in P$ let $y \in Q$ be such that $d(x, Q) = \|x - y\|$. Thus $d(\phi(x), \phi(Q)) \leq |\phi(x) - \phi(y)| \leq C|x - y| = o(|x|^r) = o(|\phi(x)|)$. Now repeating the argument with the role of P and Q reversed, we see that $\phi(P)$ and $\phi(Q)$ are r -equivalent.

Definition 2.3. Suppose $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^k, 0)$ is C^r , $r > 0$, or $C^{0,-1}$ (in the latter case let $r = 0$), $P = \Gamma(f)$, and W is a submanifold of \mathbf{R}^{n+k} containing 0 in its closure. Then W is (t^r) for P if for all Q such that $j^r Q = j^r P$ (i.e. they are r -equivalent) then $Q \pitchfork W$ near (but not necessarily at) 0.

If W is a complex analytic submanifold of \mathbf{C}^{n+k} , then we can take $f : (\mathbf{R}^{2n}, 0) \rightarrow (\mathbf{R}^{2k}, 0)$, so $P \subset \mathbf{R}^{2n+2k}$ and the above definition applies to W as before.

Later on we will give another approach to the (t^r) conditions in the analytic case.

Remark. Suppose $Y = \{0\} \times \mathbf{k}^k$ and P is the graph of the constant function $f(x) = 0$. It is shown in [T-W] that the pair (W, Y) satisfies Verdier's condition (w) at $(0, 0)$ iff W is (t^0) for P . In this case, we will also say that W is (t^0) regular over Y at 0.

By [T-W], for $r \geq 0$, a manifold W disjoint from $Y = \{0\} \times \mathbf{k}^k$ fails to be (t^r) for $P = \Gamma(f)$ if and only if

(2.4): there exist $a_i = (x_i, y_i) \in W \subseteq \mathbf{k}^{n+k}$, such that $x_i \rightarrow 0$ and there exist n -dimensional planes $T_i \not\pitchfork T_{a_i} W$ such that

$$(2.4.1) \quad |y_i - f(x_i)| = o(|x_i|^r)$$

$$(2.4.2) \quad \text{dist}(T_b P, T_i) = o(|x_i|^{r-1}),$$

where $b_i = (x_i, f(x_i))$.

Clearly $T \not\subseteq T_a W$ iff there exists a hyperplane $H \supseteq T + T_a W$. If $T \subseteq H$, $\text{dist}(A, T) \geq \text{dist}(A, H)$. Given A and H , there exists $T \subseteq H$ with $\dim T = \dim A$ such that $\text{dist}(A, T) = \text{dist}(A, H)$. From this, it follows that (2.4) is equivalent to:

$x_i \rightarrow 0$ and hyperplanes $H_i \supseteq T_{a_i} W$ such that

$$\begin{aligned} |y_i - f(x_i)| &= o(|x_i|^r) \\ \text{dist}(T_{b_i} P, H_i) &= o(|x_i|^{r-1}) \end{aligned}$$

where $b_i = (x_i, f(x_i))$.

If, in addition, W is subanalytic and P is analytic, an application of the Curve Selection Lemma (applied as in Theorem 5.3 of [T-W]) shows that (2.4) is equivalent to:

(2.5): there exists an analytic curve $\phi(t) = (x(t), y(t))$ such that $\phi(t) \in W$ for $t \neq 0$ and $\phi(0) = 0$, and there exists an analytic curve of hyperplanes $H(t) \supseteq T_{\phi(t)} W$ such that

$$(2.5.1) \quad |y(t) - f(x(t))| = o(|x(t)|^r)$$

$$(2.5.2) \quad \text{dist}(T_{b(t)} P, H(t)) = o(|x(t)|^{r-1})$$

where $b(t) = (x(t), f(x(t)))$.

Definition 2.6. Let $Y = \{0\} \times \mathbf{k}^k$. Suppose $P = \Gamma(f)$ is a C^r (or $C^{0,-1}$) direct transversal to Y , $r > 0$ (or $r = 0$). Suppose X is an analytic subset of \mathbf{k}^{n+k} containing 0 with singular set $S(X)$, and S is a closed subset such that $S(X) \cup (Y \cap X) \subseteq S \subseteq X$. Let $W = X - S$; note that $W \cap Y = \emptyset$. Then we say (X, S) is (t^r) for P if W is (t^r) for P and

$$(2.6.1) \quad \text{for all } Q \text{ such that } j^r Q = j^r P, \text{ then } Q \cap S = \{0\}$$

(Often the S will be understood and we will simply say that X is (t^r) for P , or even (t^r) holds for P ; S is assumed to be $S(X) \cup (X \cap Y)$ if we don't say otherwise.)

Remarks. Suppose that P is the graph of the constant function $f(x) = 0$. Then X is (t^0) for P at 0 iff the germ of S at 0 is contained in Y , and the pair (W, Y) satisfies Verdier's condition (w) at $(0, 0)$. In this case, we will also say that X is (t^0) (or (w)) regular over Y at 0.

It is important to note that Definition 2.6 has two arguments—in applications we can vary both P and X (or F). Here is an example on \mathbf{R}^4 with coordinates (x_1, x_2, y_1, y_2) . Let $F(x_1, x_2, y_1, y_2) = (x_1 - y_1^2 - y_2^2)^3 - x_2^2$, let $X = F^{-1}(0)$ and let $W = X_0$. Then W is (t^0) regular over Y at 0, but X is not (t^0) regular over Y at 0 (S is the surface $x_1 = y_1^2 + y_2^2$, $x_2 = 0$). Note W being (w) regular over Y at 0 does not imply $Y \subseteq X$, although it can be shown that X being (w) regular over Y at 0 does imply $Y \subseteq X$. In some of our applications we will be interested in proving results about the family of sets $\{X \cap P_a\}$ where P_a is a family of direct transversals, so we want to control the intersection of P_a with all of X , not just $X - S$. One way to accomplish this is to stratify X (W being the

top stratum) such that each stratum is (t^r) for P . We discuss this briefly after Cor 2.29. A special case of this is to require that (X, S) be (t^r) for P (as defined above); then the transversals will miss all the strata in S (except at 0) and hence satisfy the (t^r) condition. In the complex case, this is exactly the case where the dimension of S is less than or equal to the codimension of P . In the real case, we can have S of larger dimension than that but still have transversals P missing S away from 0.

It is obvious that X being (t^r) for P depends only on the r -jet of P . Thus we may talk about X being (t^r) for z , where z is an r -jet of some P . Further observe that disjoint manifolds W and Y are (t^r) regular over Y at 0 (as in Definition 2.1) if, and only if, $X = W \cup Y$ is (t^r) for all C^r direct transversals to Y through 0; in the complex case we say X is (t^r) regular over Y at 0 if X is (t^r) for all r -jets of graphs of complex polynomials vanishing at 0. (In this complex analytic setting, the curve of 2.5 can be taken to be complex analytic.) We say X is (t^r) regular over Y at y if $\{x - y \mid x \in X\}$ is (t^r) regular over Y at 0.

If $\mathbf{k} = \mathbf{C}$, then X can be (t^r) over Y in the real or in the complex sense: the distinction is whether one requires (t^r) over all r -jets, or just over r -jets of graphs of complex polynomial mappings from \mathbf{k}^n to \mathbf{k}^k of degree $\leq r$. However, we do not know of any example of an analytic $X \cup Y$ so that X is (t^r) over Y in the complex but not in the real sense.

The next theorem provides our algebraic criterion for condition (t^r) . To motivate this criterion, consider analytic map-germs f and g , with $f(0) = 0$ and the components of g in m_n^r . Let $f_t = f + tg$; the family of transversals $P_t = \Gamma(f_t)$ are $(r-1)$ -equivalent. By the proof of (4.6.1) of [T-W], if $X = F^{-1}(0)$ and $Y = \{0\} \times \mathbf{k}^k$ contains the points of X at which F is singular, and if $(X, Y \cap X)$ is (t^r) for $P = \Gamma(f)$, then there is a family of homeomorphism-germs $h_t : P \rightarrow P_t$, $h_t(0) = 0$, preserving X , for t sufficiently small. Temporarily we ask instead for diffeomorphism-germs, and require them to preserve all fibers of F . In fact let us suppose somewhat more: assume that the h_t are diffeomorphism germs on $(\mathbf{k}^{n+k}, 0)$ (not merely defined on P), preserve Y and all the fibers of F (not merely $F^{-1}(0)$), and $h_t(P) = P_t$ for all t . In this case we say that the family P_t is “strong analytically \mathcal{R}_F -equisingular”. Let η be the vector field $\frac{\partial h_t}{\partial t}|_{t=0}$ (note h_t is not necessarily the flow of η). Then $DF(\eta) = 0$.

Let $\Theta(\mathbf{k}^{n+k}, P, Y)$ denote the \mathcal{O}_{n+k} -module of germs of vector fields which are tangent to P and to Y . Since P and Y are direct transversals, this consists of all vector fields tangent to both P and Y (necessarily 0 at 0). Consider any differentiable family $k_t(x, y)$ of diffeomorphism-germs preserving Y and 0 and such that $k_t(P) = P_t$. Let $\kappa = \frac{\partial k_t}{\partial t}|_{t=0}$. Let $G_t(x, y) = f(x) + tg(x) - y$; so $G_t \circ k_t(P) = 0$ for all t . Then $\frac{\partial G_t \circ k_t}{\partial t}|_{t=0} = g(x) + DG_0(x, y)\kappa(x, y)$, and equals 0 when restricted to P . Consequently, the difference of two such vector fields κ_1 and κ_2 is tangent to P , hence is in $\Theta(\mathbf{k}^{n+k}, P, Y)$. One such k_t is h_t , giving rise to $\kappa_t = \eta$, and another is $k_t(x, y) = y + tg(x)$, giving rise to the vector field κ_2 assigning to each (x, y) the vector $(0, g(x))$ (we also use $g(x)$ to denote this vector field). Thus $\eta = g \pmod{\Theta(\mathbf{k}^{n+k}, P, Y)}$. Since $DF(\eta) = 0$, $\frac{\partial F}{\partial y}g = DF(g) \in DF(\Theta(\mathbf{k}^{n+k}, P, Y))$.

Let $I(P)$ denote the ideal of analytic function-germs vanishing on P (if $P = \Gamma(f)$, then $I(P) = \{y_1 - f_1, \dots, y_k - f_k\}\mathcal{O}_{n+k}$). We let $JM(F)_P$ denote the submodule of $JM(F)$ generated by $\frac{\partial F}{\partial x_i} + \sum_j \frac{\partial f_j}{\partial x_i} \frac{\partial F}{\partial y_j}$, $i = 1, \dots, n$; this submodule is obtained by applying to F vector fields tangent to the fibers of the submersion $y - f(x)$ defining P . (Note

that if P is a plane this module is the module we denoted by $JM(F)_P$ previously.) If $(x_1, \dots, x_n, y_1, \dots, y_k)$ are coordinates on \mathbf{k}^{n+k} , let $JM_y(F)$ denote the submodule of $JM(F)$ generated by the partials of F with respect to y . Hence for strong analytically \mathcal{R}_F -equisingularity to hold for all g , we need that

$$\mathfrak{m}_n^r JM_y(F) \subset DF(\Theta(\mathbf{k}^{n+k}, P, Y)) = m_n JM(F)_P + I(P)JM_y(F)$$

(equality of these two modules is an easy exercise). If one only requires that the diffeomorphisms preserve X (rather than all fibers of F), then one replaces these \mathcal{O}_{n+k} -modules by the corresponding \mathcal{O}_X -modules. To formulate our algebraic criterion for (t^r) , we replace the module on the right-hand side of this inclusion by its integral closure, as in the next Theorem.

Definition 2.6 (continued). We say that a smooth submanifold $W \subset X \subset \mathbf{k}^{n+k}$ is defined by $F : (\mathbf{k}^{n+k}, 0) \rightarrow (\mathbf{k}^p, 0)$ and $Y = 0 \times \mathbf{k}^k$ if $X = F^{-1}(0)$ and W is the Zariski open subset of $X - Y$ at which F has rank $c = \text{cod}(X - Y)$ (possibly less than p , to allow for non-complete intersections).

In the complex analytic case, if the Zariski closure of $X - Y$ is equidimensional, and F has generic rank $c = \text{cod}(X - Y)$ on each component of the Zariski closure of $X - Y$, then the set of points W at which F has rank $c = \text{cod}(X - Y)$ is a metric dense subset of $X - Y$ (in fact of X). The real analytic case is different. Consider $F(x, z, y) = z^2 - y^2x$. The zero set of F , X is the Whitney umbrella with handle the x axis. Let Y be the y axis, so the metric closure of $X - Y$ is X . Then F and Y do not define a metric dense W , as the metric closure doesn't contain the handle. (Of course, if we choose Y to be the handle, then W is metric dense in $X - Y$.)

In relating the (t^r) conditions to the Grassmann modification in the real analytic case, we will assume W is metric dense in $X - Y$; and this condition will be preserved by Grassmann modification, whereas the condition that the smooth points of X be metric dense in general won't.

Note that W consists of regular points, but we don't insist that it be all regular points. Let $\mathcal{O}_{X,0}$ be the ring defined by quotienting by the ideal generated by the components of F . Let $S = X - W$; so this contains all singular points of X as well as regular points at which F doesn't define the reduced structure, as well as all points of $X \cap Y$. (By construction, W contains no points of Y , hence no points of $X \cap Y$.) We will usually assume that W is metric dense in $X - Y$. The germs of X , F , etc. induce germs of sheaves.

We summarize some of the standard assumptions we will make:

Standard Setup: Suppose $X \subseteq \mathbf{k}^{n+k}$ is the germ of an analytic space at the origin defined by F , W is defined by F and Y as in (2.6), $S = X - W$ and W is metric dense in $X - Y$.

Theorem 2.7. *Suppose F , X , Y , W and S are as in the Standard Setup. Suppose P is an analytic direct transversal to Y . For the (t^0) case, assume in addition that $Y \subset X$. For $r \geq 0$, W is (t^r) for the transversal P iff*

$$(2.7.1) \quad \mathfrak{m}_n^r JM_y(F) \subseteq \overline{\mathfrak{m}_n JM(F)_P + I(P)JM_y(F)}$$

(take the integral closure inside $\mathcal{O}_{X,0}^p$, where p is the number of components of F ; in the real analytic situation we take real integral closure).

Furthermore, X is (t^r) for the transversal P iff W is (t^r) for the transversal P and

$$(2.7.2) \quad \mathfrak{m}_n^r \mathcal{O}_{S,0} \subseteq \overline{I(P)\mathcal{O}_{S,0}}.$$

First we prove several preliminary results. We always assume $r \geq 0$ unless we state otherwise.

Lemma 2.8. *A curve $\phi(t) = (x(t), y(t))$ satisfies (2.5.1) iff $\phi^*I(P) \subseteq \mathfrak{m}_1\phi^*\mathfrak{m}_n^r$.*

Proof. Let $\text{ord}(u(t))$ denote the order of the power series $u(t)$; if $u(t)$ is vector valued, use the minimum order of the components, equivalently the order of $|u(t)|$; if I is an ideal, $\text{ord}(I)$ denotes the minimum order of all members of I .

Suppose $\text{ord}(x(t)) = l$. Then $\text{ord}(|x(t)|^r) = lr = \text{ord}(\phi^*\mathfrak{m}_n^r)$ and $\text{ord}(|y(t) - f(x(t))|) = \text{ord}(\phi^*I(P))$. The result is immediate. \square

Lemma 2.9. *Let $\phi(t)$ be a curve satisfying (2.5.1), with $\phi(t)$ in W when $t \neq 0$. Then the following are equivalent:*

$$(2.9.1) \quad \begin{aligned} \phi^*(\mathfrak{m}_n^{r-1}JM_y(F)) &\subseteq \phi^*(JM(F)_P) \text{ (if } r \geq 1) \text{ or} \\ \phi^*(JM_y(F)) &\subseteq \phi^*(\mathfrak{m}_n JM(F)_P) \text{ (if } r = 0) \end{aligned}$$

and

$$(2.9.2) \quad \phi^*(\mathfrak{m}_n^r JM_y(F)) \subseteq \phi^*(\mathfrak{m}_n JM(F)_P + I(P)JM_y(F)).$$

Proof. Clearly (2.9.1) implies (2.9.2). Assume (2.9.2) holds. By Lemma (2.8),

$$\phi^*(I(P)JM_y(F)) \subseteq \mathfrak{m}_1\phi^*(\mathfrak{m}_n^r JM_y(F)).$$

Therefore,

$$\begin{aligned} &\phi^*(\mathfrak{m}_n JM(F)_P + I(P)JM_y(F)) \subseteq \\ &\phi^*(\mathfrak{m}_n JM(F)_P) + \mathfrak{m}_1\phi^*(\mathfrak{m}_n^r JM_y(F)) \subseteq \\ &\phi^*(\mathfrak{m}_n JM(F)_P) + \mathfrak{m}_1\phi^*(\mathfrak{m}_n JM(F)_P + I(P)JM_y(F)) \text{ by (2.9.2)}. \end{aligned}$$

By Nakayama's Lemma,

$$(2.9.3) \quad \phi^*(\mathfrak{m}_n^r JM_y(F)) \subseteq \phi^*(\mathfrak{m}_n\phi^* JM(F)_P).$$

If $r = 0$, we are done. Assume $r \geq 1$. For $l = \text{ord}(\phi(t))$, $\phi^*(\mathfrak{m}_n^r JM_y(F)) = t^{rl}\phi^*(JM_y(F))$ and $\phi^*(\mathfrak{m}_n JM(F)_P) = t^l\phi^*(JM(F)_P)$. Thus (2.9.1) holds. \square

Proposition 2.10. *Condition (2.9.1) is equivalent to requiring that there does not exist a curve of hyperplanes $H(t) \supseteq T_{\phi(t)}W$ ($t \neq 0$) satisfying (2.5.2).*

Proof. Assume $r > 0$; a slight modification of the argument works for $r = 0$.

By Proposition 1.11 of [G1] and the remark after Proposition 4.2 of [G1], condition (2.9.1) is equivalent to

$$\phi^*(\mathfrak{m}_n^{r-1}\psi JM_y(F)) \subseteq \phi^*\psi JM(F)_P \text{ for all analytic } \psi(t) \in \text{Hom}(\mathbf{k}^p, \mathbf{k}).$$

Letting $v_i(t) = \frac{\partial}{\partial x_i} + \sum_j \frac{\partial f_j(x(t))}{\partial x_i} \frac{\partial}{\partial y_j}$, $i = 1, \dots, n$ and $w_j = \frac{\partial}{\partial y_j}$, $j = 1, \dots, k$, the above is equivalent to

$$\forall \psi(t) \in \text{Hom}(\mathbf{k}^p, \mathbf{k}), \min_{j=1, \dots, k} \text{ord}(\phi^* \mathfrak{m}_n^{r-1} \psi DF \circ \phi \cdot w_j) \geq \min_{i=1, \dots, n} \text{ord}(\psi DF \circ \phi \cdot v_i).$$

This fails to hold iff

$$\exists \psi(t) \in \text{Hom}(\mathbf{k}^p, \mathbf{k}), \min_{j=1, \dots, k} \text{ord}(\phi^* \mathfrak{m}_n^{r-1} \psi DF \circ \phi \cdot w_j) < \min_{i=1, \dots, n} \text{ord}(\psi DF \circ \phi \cdot v_i).$$

If this latter happens, then $\psi DF \circ \phi$ is not identically 0. Then $H(t) := \ker(\psi DF \circ \phi)$ is a hyperplane containing $T_{\phi(t)}W$ and, applying 1.12 to $\omega(t) = \psi DF \circ \phi$, we have that the above inequality is equivalent to

$$\text{dist}(T_{b(t)}P, H(t)) = o(|x(t)|^{r-1}),$$

since $b(t) = (x(t), f(x(t)))$ and $T_{b(t)}P$ is spanned by $\{v_1(t), \dots, v_n(t)\}$. \square

Proof of the Theorem. Recall that the failure of $W = X - S$ to be (t^r) for P is equivalent to (2.5) holding for some curve (call it ϕ) in W .

If (2.5.1) fails for curve ϕ , then $\phi^* \mathfrak{m}_n^r \subseteq \phi^* I(P)$, which implies that (2.9.2) holds for ϕ . If (2.5.1) holds but (2.5.2) fails, then Proposition 2.10 and Lemma 2.9 imply that (2.9.2) holds for that ϕ .

Thus W being (t^r) for P implies that (2.5) does not hold for any analytic curve ϕ in $W, 0$, which implies that (2.9.2) holds for all analytic curves ϕ in $W, 0$. If ϕ is a curve in $Y \cap X, 0$ and $r > 0$, then (2.9.2) holds trivially. If we let $r = 0$, then (2.9.2) holds for such a curve since $Y \subseteq X$ implies $\phi^*(JM_y(F)) = 0$. Hence by Proposition 1.4 the integral closure condition of the Theorem holds.

Now assume (2.5) holds. Then Proposition 2.10 and Lemma 2.9 imply that (2.9.2) fails to hold for the curve ϕ of (2.5). Thus the integral closure condition (2.7.1) fails to hold.

The proof of the statement in the second paragraph of (2.7) is much easier. Recall from Definition 2.6 that X being (t^r) over P means W is (t^r) over P and the r -jet of P misses $X - W$ except at 0. The condition (2.7.2) means: given any curve $\phi(t) = (x(t), y(t))$ in S , there exists a $C > 0$ such that $|x(t)|^r \leq C|y(t) - f(x(t))|$ for small enough t . This inequality implies that the same inequality holds for any g such that $j^r g(0) = j^r f(0)$, and hence the r -jet of P misses S except at 0 (if $r = 0$, the inequality implies that $S \subseteq Y$ near 0). To prove the converse, assume there exists $t_i \rightarrow 0$ such that the inequality above fails: i.e. $|y(t_i) - f(x(t_i))| = o(|x(t_i)|^r)$. It follows from the Interpolation Lemma (2.6) of

[T-W] that there exists a subsequence t_{i_j} of t_i and a C^r g ($C^{0,-1}$ in case $r = 0$) such that $j^r g(0) = j^r f(0)$ and $g(x(t_{i_j})) = y(t_{i_j}) \in S$. Thus the r -jet of P fails to miss $S - \{0\}$, establishing the converse. \square

If X is t^1 for P , then this implies that no line in the tangent cone to S is a tangent line to P .

In case $r = 0$, letting $P = \mathbf{k}^n \times \{0\}$, Theorem 2.7 says that (w) is equivalent to

$$\frac{\partial F}{\partial y_i} \in \overline{(\mathfrak{m}_n JM_x(F) + \mathfrak{m}_k JM_y(F))}, \quad i = 1, \dots, k.$$

By Nakayama's Lemma, this is equivalent to

$$\frac{\partial F}{\partial y_i} \in \overline{\mathfrak{m}_n JM_x(F)}, \quad i = 1, \dots, k.$$

Thus Theorem 1.11 is the special case of Theorem 2.7 in which $r = 0$.

In the real analytic, (t^0) case it is possible for the integral closure condition of the theorem to hold even if $Y \cap X$ is a point. This is true of the example that appears after definition 2.6. There, if $\phi(t)$ is a curve on X , then the order in t of $\phi^* J(F)_y$ is greater than or equal to the order of $\phi^*(x_1 x_2) \in \phi^* \mathfrak{m}_Y J(F)_z$. It seems likely that this cannot happen in the complex analytic case.

The flexibility in choice of P and F allows the (t^r) condition to be used to study the stratification of a complex analytic set (take F to be the defining equations for X , P a "probing set"), or the order of V -determinacy of a map-germ $f : \mathbf{C}^n \rightarrow \mathbf{C}^p$. To do this, work on the ambient space of the graph of f , take P to be the graph of f , and let $F = (y_1, \dots, y_p)$; X then becomes \mathbf{C}^n , while Y is still given by $(x_1 = \dots = x_n = 0)$. If we restrict $I(P)$ to X we get $f^*(\mathfrak{m}_p) \mathcal{O}_X$, while $JM_y(F)$ becomes \mathcal{O}_n^p . The condition of Definition 2.3, via Theorem 2.7, then becomes

$$\mathfrak{m}_n^r \mathcal{O}_n^p \subseteq \overline{\mathfrak{m}_n JM(f) + f^*(\mathfrak{m}_p) \mathcal{O}_n^p}.$$

In [G1], Corollary 4.6, it is shown that if this condition holds for f , then any deformation of f which fixes the r -jet of f gives a Verdier equisingular deformation of $f^{-1}(0)$, hence f is r V -determined.

Next we will discuss the Grassmann modification of X , and how it affects the (t^r) condition.

Definition 2.11. Let $G_{n+k,n}$ denote the Grassmannian of n -planes containing the origin in \mathbf{k}^{n+k} . The total space $E_{n+k,n}$ of the canonical bundle associated to $G_{n+k,n}$ is a submanifold of $G_{n+k,n} \times \mathbf{k}^{n+k}$ with projection maps induced by the maps to the components. The fiber of $E_{n+k,n}$ over a point P is just the set of points of \mathbf{k}^{n+k} in P . Denote the projection of $E_{n+k,n}$ to \mathbf{k}^{n+k} by $\beta_{n+k,n}$. If X is a subset of \mathbf{k}^{n+k} , we call $\tilde{X} = \beta_{n+k,n}^{-1}(X)$, the $G_{n+k,n}$ modification of X . When n and k are clear from the context, we will simply refer to the Grassmann modification of X .

This notion was introduced in connection with the (t^r) conditions by Kuo and the second author [Ku-Tr]. Note that $G_{n+k,n}$ is embedded in $E_{n+k,n}$ as the zero section of

$E_{n+k,n}$. This means that, if X is an analytic set and $0 \in X$, we can think of $0 \times G_{n+k,n}$ as a stratum of \tilde{X} ; note that the projection to $0 \times G_{n+k,n}$ makes \tilde{X} a family of analytic sets with $0 \times G_{n+k,n}$ as the parameter space. The members of this family are just $\{X \cap P\}$ as P varies through the points of $G_{n+k,n}$. This means that if we want to study the equisingularity of the family $\{X \cap P\}$, then we should study the regularity conditions that the pair $(\tilde{X}_0, 0 \times G_{n+k,n})$ satisfy. In this paper, we will be interested only in the behavior of this pair at points of $G_{n+k,n}$ which correspond to direct transversals; we denote this set by \tilde{Y} . Kuo and the second author proved a remarkable result in [Ku-Tr] about the behavior of regularity conditions under Grassmann modification. In the following result, $(t^0) = (w)$.

Theorem 2.12. (*Kuo-Trotman*) *Suppose X and Y are disjoint smooth submanifolds of \mathbf{R}^N in a neighborhood of the origin, Y is analytic, and X is subanalytic. Then, for each $r \geq 0$, X is (t^{r+1}) regular over Y at 0 iff \tilde{X} is (t^r) regular over \tilde{Y} at every point of \tilde{Y} .*

Proof. Cf. Theorem 1 in [Ku-Tr] and [T-W] (there a more general statement is proved, without the hypotheses that Y be analytic and X subanalytic). \square

This result shows in particular that if X is (t^1) regular over Y at the origin, $X \cup Y$ is locally closed, and Y contains the singular set of $X \cup Y$, then the family $\{X \cap P\}$ is Verdier equisingular (i.e. (w) holds), hence is topologically trivial.

We will prove a version of Theorem 2.12 that holds in both the real and complex analytic cases using our integral closure criterion. In the Grassmann modification of \mathbf{k}^{n+k} we will always be working in a neighborhood of a direct transversal P . (Recall we regard P as a point of the zero section of $E_{n+k,n}$.) Since we are always working in $E_{n+k,n}$, we denote the projection $\beta_{n+k,n}$ by β . Since all planes near P are also direct transversals, they are also graphs over the x -plane, and they have equations $\{y_1 = \sum_j a_{1,j}x_j, \dots, y_k = \sum_j a_{k,j}x_j\}$.

This means we have local coordinates on $E_{n+k,n}$ given by $(x_1, \dots, x_n, a_{1,1}, \dots, a_{k,n})$. In these coordinates we have

$$\beta(x_1, \dots, x_n, a_{1,1}, \dots, a_{k,n}) = (x_1, \dots, x_n, \sum_j a_{1,j}x_j, \dots, \sum_j a_{k,j}x_j).$$

Given $\phi : (\mathbf{k}, 0) \rightarrow (\tilde{X}, \{0\} \times P)$, then $\beta \circ \phi$ is tangent to P at the origin.

Theorem 2.13. *Suppose F, X, Y, W and S are as in the Standard Setup (before (2.7)). Then, for each $r \geq 0$, W (respectively X) is (t^{r+1}) regular over Y at 0 iff \tilde{W} (respectively \tilde{X}) is (t^r) regular over \tilde{Y} at every point of \tilde{Y} . Moreover, every $(r+1)$ -jet z of direct transversals to Y at 0 lifts to a non-unique r -jet \tilde{z} of a direct transversal to \tilde{Y} , and W (respectively X) is (t^{r+1}) regular for z iff \tilde{W} (respectively \tilde{X}) is (t^r) for \tilde{z} .*

Proof. First note that, since $0 \in X$, $\tilde{Y} = \beta^{-1}(0) \subseteq \tilde{X}$.

Since F defines X , $G := F \circ \beta$ defines \tilde{X} . From the chain rule we note that

$$\frac{\partial G}{\partial a_{i,j}} = x_j \frac{\partial F}{\partial y_i} \circ \beta, \text{ and } \frac{\partial G}{\partial x_j} = \frac{\partial F}{\partial x_j} \circ \beta + \sum_i a_{i,j} \frac{\partial F}{\partial y_i} \circ \beta.$$

Let $p : \mathbf{k}^n \rightarrow \mathbf{k}^{nk}$ be a polynomial map of degree $\leq r$; $\Gamma(p)$ is a direct transversal to \tilde{Y} . Then $\beta(\Gamma(p)) = \Gamma(q)$ for a polynomial mapping q of degree $\leq r + 1$, $q(0) = 0$,

$$q = \left(\sum_j p_{1,j} x_j, \dots, \sum_j p_{k,j} x_j \right).$$

Conversely every such q has $\Gamma(q) = \beta(\Gamma(p))$ for some (not unique) p of degree $\leq r$.

We need to show that \tilde{W} is (t^r) for $\Gamma(p)$ iff W is (t^{r+1}) for $\Gamma(q)$. Let $\phi : (\mathbf{k}, 0) \rightarrow (\tilde{W}, p(0))$ be an analytic curve which has order $r + 1$ contact with $\Gamma(p)$:

$$\phi(t) = (x(t), a(t)), \quad |a(t) - p(x(t))| = o(|x(t)|^r).$$

Then $\phi_1 = \beta \circ \phi$ has order $r + 2$ contact with $\Gamma(q)$:

$$\phi_1(t) = (x(t), y(t)), \quad y(t) = \left(\sum_j a_{1,j}(t)x_j(t), \dots, \sum_j a_{k,j}(t)x_j(t) \right),$$

so

$$y(t) - q(x(t)) = \left(\sum_j (a_{1,j}(t) - p_{1,j}(x(t)))x_j(t), \dots \right) = o(|x(t)|^{r+1}).$$

By the proof of Theorem 2.7, we know that the inclusion

$$(2.14) \quad \phi_1^*(\mathfrak{m}_n^{r+1}(JM_y(F))) \subseteq \phi_1^*(\mathfrak{m}_n(JM(F)_{\Gamma(q)}) + I(\Gamma(q))(JM_y(F)))$$

(which is the module condition defining (t^{r+1}) along the curve ϕ_1) is equivalent to

$$(2.15) \quad \phi_1^*(\mathfrak{m}_n^r JM_y(F)) \subseteq \phi_1^*(JM(F)_{\Gamma(q)}).$$

The vector fields tangent to $\Gamma(q)$ are generated by $\frac{\partial}{\partial x_j} + \sum_i \frac{\partial q_i}{\partial x_j} \frac{\partial}{\partial t_i}$, $j = 1, \dots, n$, so (2.15) is equivalent to

$$(2.16) \quad \phi^*(\beta^* \mathfrak{m}_n^r \frac{\partial F}{\partial y_i} \circ \beta) \subseteq \phi^* \left(\frac{\partial F}{\partial x_j} \circ \beta + \sum_i \frac{\partial q_i}{\partial x_j} \frac{\partial F}{\partial y_i} \circ \beta \right) \mathcal{O}_1 = \\ \phi^* \left(\frac{\partial F}{\partial x_j} \circ \beta + \sum_i \left(\sum_l \frac{\partial p_{i,l}}{\partial x_j} x_l + p_{i,j} \right) \frac{\partial F}{\partial y_i} \circ \beta \right) \mathcal{O}_1$$

(and note that $\beta^* \mathfrak{m}_n^r = \mathfrak{m}_n^r$).

Similarly the module condition for (t^r) along ϕ reduces to

$$(2.17) \quad \phi^*(\mathfrak{m}_n^{r-1} JM_a(G)) \subseteq \phi^*(JM(G)_{\Gamma(p)}).$$

Now $JM(G)_{\Gamma(p)}$ is generated by

$$\left(\frac{\partial}{\partial x_j} + \sum_{i,l} \frac{\partial p_{i,l}}{\partial x_j} \frac{\partial}{\partial a_{i,l}} \right) (F \circ \beta) = \frac{\partial F}{\partial x_j} \circ \beta + \sum_i a_{i,j} \frac{\partial F}{\partial y_i} \circ \beta + \sum_{i,l} \frac{\partial p_{i,l}}{\partial x_j} x_l \frac{\partial F}{\partial y_i} \circ \beta$$

and $JM_a(G)$ is generated by $\frac{\partial G}{\partial a_{i,j}} = x_j \frac{\partial F}{\partial y_i} \circ \beta$.

Thus the left hand sides of (2.16) and (2.17) are identical. The difference of the right hand sides of (2.16) and (2.17) is generated by

$$\phi^* \left(\sum_i (a_{i,j} - p_{i,j}) \frac{\partial F}{\partial y_i} \circ \beta \right) = \sum_i (a_{i,j}(t) - p_{i,j}(x(t))) \phi^* \beta^* \frac{\partial F}{\partial y_i} \in \mathfrak{m}_1 \phi^* (\mathfrak{m}_n^r \beta^* JM_y(F))$$

for $j = 1, \dots, n$. So if either inclusion (2.16) or (2.17) holds, by Nakayama's Lemma both must hold and the two modules are in fact the same. So if we assume (t^{r+1}) for $\Gamma(q)$, then (2.16) holds for all curves ϕ_1 which have order $r+2$ contact with $\Gamma(q)$, which implies (2.17) holds for all curves ϕ having order $r+1$ contact with $\Gamma(p)$, and hence we have (t^r) for $\Gamma(p)$.

The converse is similar.

The proof that X is (t^{r+1}) iff \tilde{X} is (t^r) is similar, but much simpler, and we omit it. \square

It should be noted that, even if F defines the reduced structure on X , the Grassmann modification G of F may not define the reduced structure on \tilde{X} . For example, suppose X is the curve $F(x, y, z) = (y, x^3 + z^2) = 0$. Let $\beta(x, y, u, v) = (x, y, ux + vy)$. Then the ideal generated by $F \circ \beta$ is $(y, x^2(x + u^2))$, which gives a non-reduced structure on the component \tilde{Y} of \tilde{X} .

The Grassmann modification will be a major tool in section 4. Next we will treat an important tool for studying equisingularity of families of transversals.

Suppose F, X, Y, W and S are as in the Standard Setup (before (2.7)). Let $f(x, u) : (\mathbf{k}^{n+c}, 0) \rightarrow (\mathbf{k}^k, 0)$ determine a germ of a family of direct transversals $\Gamma(f_u)$, where $f_u(x) = f(x, u)$. Assume $f(0, u) = 0$ for all u . Let $\beta(x, u) = (x, f(x, u))$, $G = F \circ \beta$ and $\tilde{X} = \beta^{-1}X = G^{-1}(0)$; let $\tilde{S} = \beta^{-1}S$ and $\tilde{W} = \beta^{-1}W$; and $\tilde{Y} = \beta^{-1}Y$ is the germ at $(0,0)$ of $\{0\} \times \mathbf{k}^c$. (Note that $\tilde{Y} \subseteq \tilde{X}$; note that the germ of the map to the base at a point of \tilde{Y} of the Grassmann modification is a special case of such a β).

Definition 2.18. The family $\Gamma(f_u)$ has *Verdier equisingular intersection with W* (respectively, *with X*) if \tilde{W} is (w) regular over \tilde{Y} at each point of \tilde{Y} (respectively, \tilde{X} is (w) regular over \tilde{Y} at each point of \tilde{Y}).

Recall that \tilde{X} being (w) regular over \tilde{Y} at each point of \tilde{Y} implies, in particular, that $\tilde{S} - \tilde{Y}$ misses a neighborhood of \tilde{Y} (this follows from (2.6.1)); since $\tilde{W} \cup \tilde{Y}$ is then locally closed in this neighborhood of \tilde{Y} , it further implies that there is a family of rugose (with respect to 0) homeomorphisms h_t from $\Gamma(f_0)$ to $\Gamma(f_u)$ preserving X .

Theorem 2.19 (of Thom-Levine type). *The family $\Gamma(f_u)$ has Verdier equisingular intersection with W iff, for all $i = 1, \dots, c$,*

$$(2.20) \quad \left(\frac{\partial F}{\partial y} \circ \beta \right) \cdot \frac{\partial f}{\partial u_i} \in \overline{\mathfrak{m}_n(dF \circ \beta)_* JM_x(\beta)},$$

where $\frac{\partial F}{\partial y} \circ \beta$ takes its values in $\text{Hom}(\mathbf{k}^k, \mathbf{k}^p)$, and $(dF \circ \beta)_* JM_x(\beta)$ is the $\mathcal{O}_{\tilde{X}}$ -module generated by composing $dF \circ \beta$ with elements of $JM_x(\beta)$.

In addition, $\Gamma(f_u)$ has Verdier equisingular intersection with X iff it has Verdier equisingular intersection with W and $\beta^{-1}S - \tilde{Y}$ misses a neighborhood of \tilde{Y} .

Proof. Pick any analytic curve $\phi(t) = (x(t), u(t))$ in \tilde{X} such that $\phi(0) = 0$. Then (2.20) says that, for all $i = 1, \dots, c$ and for all such ϕ ,

$$(2.21) \quad \phi^* \left(\frac{\partial F}{\partial y} \circ \beta \cdot \frac{\partial f}{\partial u_i} \right) \in \phi^* (\mathfrak{m}_n (dF \circ \beta)_* JM_x(\beta) \mathcal{O}_{\tilde{X}}).$$

The Verdier equisingularity of the intersections of $\Gamma(f_u)$ with W is equivalent to

$$(2.22) \quad \phi^* \frac{\partial G}{\partial u_i} = \phi^* \left(\frac{\partial F}{\partial y} \circ \beta \cdot \frac{\partial f}{\partial u_i} \right) \in \phi^* (\mathfrak{m}_n JM_x(G)) = \phi^* (\mathfrak{m}_n (dF \circ \beta)_* JM_x(\beta) \mathcal{O}_{\tilde{X}}),$$

for all $i = 1, \dots, c$. Clearly (2.21) and (2.22) are the same.

The last statement of the Theorem follows immediately from the definitions. \square

Let $f_0 : (\mathbf{k}^n, 0) \rightarrow (\mathbf{k}^k, 0)$ be some analytic mapping. Perturb f_0 by all homogeneous terms of degree $r > 0$:

$$f_u(x) = f_0(x) + \sum_{\substack{|\omega|=r \\ 1 \leq i \leq k}} u_i^\omega x^\omega e_i,$$

where $e_i = (0, \dots, 1, \dots, 0)$ is the i th standard basis vector in \mathbf{k}^k . So $f(x, u) = f_u(x) : (\mathbf{k}^n \times \mathbf{k}^c, 0 \times \mathbf{k}^c) \rightarrow (\mathbf{k}^k, 0)$ is a family of representatives of all r -jets lying over the $(r-1)$ -jet determined by f_0 . Let f^i denote the i -th component function of f .

Corollary 2.23. *For any $u_0 \in \mathbf{k}^c$, W (respectively X) is (t^r) for $\Gamma(f_{u_0})$ iff the germ at u_0 of the family $\Gamma(f_u)$ has Verdier equisingular intersection with W (respectively X).*

Proof. The (t^r) condition for $\Gamma(f_{u_0})$ is:

$$(2.24) \quad \mathfrak{m}_n^r \mathcal{O}_X \frac{\partial F}{\partial y} \subseteq \overline{(\mathfrak{m}_n JM(F)_{\Gamma(f_{u_0})} + I(\Gamma(f_{u_0})) \frac{\partial F}{\partial y})} \mathcal{O}_X$$

(here for convenience we write $\mathcal{O}_X \frac{\partial F}{\partial y}$ for $JM_y(F)$). By Theorem 2.19, Verdier equisingularity of the intersection of $\Gamma(f_u)$ with X is equivalent to

$$(2.25) \quad \mathfrak{m}_n^r \mathcal{O}_{\tilde{X}} \frac{\partial F}{\partial y} \circ \beta \subseteq \overline{\mathfrak{m}_n \mathcal{O}_{\tilde{X}} (dF \circ \beta) \frac{\partial \beta}{\partial x}},$$

where \tilde{X} will be taken to mean the germ of \tilde{X} at $(0, u_0)$.

Fix a curve ϕ in \tilde{X} , $\phi(0) = u_0$, and let $\phi_1 = \beta \circ \phi$ in X . On ϕ_1 , (2.24) becomes

$$(2.26) \quad \phi^* (\mathfrak{m}_n^r \mathcal{O}_{\tilde{X}} \frac{\partial F}{\partial y} \circ \beta) \subseteq \phi^* \left(\mathfrak{m}_n \beta^* JM(F)_{\Gamma(f_{u_0})} + (f^i - f_{u_0}^i) \mathcal{O}_{\tilde{X}} \frac{\partial F}{\partial y} \circ \beta \right),$$

and (2.25) is

$$(2.27) \quad \phi^* (\mathfrak{m}_n^r \mathcal{O}_{\tilde{X}} \frac{\partial F}{\partial y} \circ \beta) \subseteq \phi^* \left(\mathfrak{m}_n \mathcal{O}_{\tilde{X}} \left(\frac{\partial F}{\partial x} \circ \beta + \left(\frac{\partial F}{\partial y} \circ \beta \right) \frac{\partial f}{\partial x} \right) \right).$$

Since

$$(f^i - f_{u_0}^i) \circ \phi(t) = \sum (u_\omega^i \circ \phi(t) - u_0^i) x^\omega \circ \phi(t) \in \mathfrak{m}_1 \phi^* \mathfrak{m}_n^r,$$

$$\left(\frac{\partial f^i - f_{u_0}^i}{\partial x_j} \right) \circ \phi(t) = \sum (u_\omega^i \circ \phi(t) - u_0^i) \frac{\partial x^\omega}{\partial x_j} \circ \phi(t) \in \mathfrak{m}_1 \phi^* \mathfrak{m}_n^{r-1},$$

and since

$$\beta^* JM(F)_{\Gamma(f_{u_0})} = \left(\frac{\partial F}{\partial x} \circ \beta + \left(\frac{\partial F}{\partial y} \circ \beta \right) \frac{\partial f_{u_0}}{\partial x} \right) \mathcal{O}_{\tilde{X}},$$

the difference of the right hand sides of (2.26) and (2.27) is

$$\phi^* \left(\mathfrak{m}_n \mathcal{O}_{\tilde{X}} \left(\frac{\partial F}{\partial y} \circ \beta \right) \frac{\partial}{\partial x} (f - f_{u_0}) + (f^i - f_{u_0}^i) \mathcal{O}_{\tilde{X}} \frac{\partial F}{\partial y} \circ \beta \right) \in \mathfrak{m}_1 \phi^* (\mathfrak{m}_n^r \mathcal{O}_{\tilde{X}} \frac{\partial F}{\partial y} \circ \beta),$$

which is \mathfrak{m}_1 times the left hand side of (2.26) (or (2.27)). Therefore (2.26) and (2.27) are equivalent. \square

The next Corollary states that if X is (t^r) for some r -jet, then all representatives of the jet have Verdier equisingular intersections with X .

Corollary 2.28. *Let $h : (\mathbf{k}^n, 0) \rightarrow (\mathbf{k}^k, 0)$ be some analytic mapping, and let g be its Taylor polynomial of degree r . Let $f_u = (1 - u)h + ug$. If W (respectively X) is (t^r) for $\Gamma(h)$, then $\Gamma(f_u)$ has Verdier equisingular intersection with W (respectively X).*

Proof. The proof is a minor variation of that of the previous theorem. It suffices to prove the result for the germ of the family at u_0 , for each $u_0 \in [0, 1]$. Since $\Gamma(f_{u_0})$ has the same r -jet as $\Gamma(h)$, X is (t^r) for $\Gamma(f_{u_0})$, the proof goes through with minor notational changes. \square

If $F(x, y) = y$, then $X = \mathbf{k}^n \times \{0\}$. Let $S = \{0\}$ and so $W = X - \{0\}$. Since $\Gamma(f_u) \cap X = f_u^{-1}(0)$, $\Gamma(f_u)$ having Verdier equisingular intersection with X is the same as saying “the functions f_u are Verdier V equisingular” as defined in [G1].

In this case Theorem 2.19 yields the following result:

Corollary 2.29. *(see [G1] for related results). The family f_u are Verdier V equisingular iff*

$$\frac{\partial f}{\partial u_i} \in \overline{\mathfrak{m}_n JM_x(f)}.$$

In this paper we have always assumed that our sections miss $S(X)$ (or more generally S) away from 0. It is possible to avoid this restriction. We will describe this briefly now, but won't pursue it further in this paper. Suppose there are mappings F_i , $i = 0, \dots, l$ defining $X = X^0 \supset X^1 \supset \dots \supset X^l$. Assume $W_i = X^i - X^{i+1}$ is a manifold containing X^{i+1} in its closure, and the collection of W_i 's is a (w) -regular stratification (strata are not necessarily connected). Then say this stratification \mathcal{S} is (t^r) for P if each W_i is. We can apply the results we have proved with F , X , S and W replaced by F_i , X^i , $S_i = X^{i+1}$ and W_i . The assumption that \mathcal{S} be (w) -regular allows one to apply the results of Section 4 of [T-W] to extend various theorems of our paper to this stratified case. For example, any finite dimension family Q_t of representatives of the r -jet of P will have Verdier equisingular

intersection with X . The pullbacks of the W_i by this family will be a family \tilde{W}_i which is (w) -regular among themselves by the original (w) -regularity and its invariance under transverse intersection, and they are (w) -regular over the \tilde{Y} by the results of our paper.

A condition (t^{r^-}) was introduced in [T-W]. It can be defined as for (t^r) using the notions of C^{r^-} functions and r^- -jets given in Definition 2.2. Let X, Y, P, W and S be as in (2.6). The failure of W to be (t^{r^-}) for P is equivalent to (2.4) or (2.5) holding with $o(\)$ replaced by $O(\)$. If $f, g : (\mathbf{k}^n, 0) \rightarrow (\mathbf{k}^k, 0)$ are C^{r^-} , so in particular if they are C^r , with $j^{r-1}f(0) = j^{r-1}g(0)$, then $|f(x) - g(x)| = O(|x|^r)$ and $\text{dist}(T_x(\Gamma(f)), T_x(\Gamma(g))) = O(|x|^{r-1})$ (in case $r = 0$, one cannot use functions f which are merely C^0 —one must also assume they are C^1 except at 0 and that $\|df(x)\| = O(|x|^{-1})$; the assumption $j^{-1}f(0) = j^{-1}g(0)$ is vacuous). Hence, the analogue of (2.4) implies that W is (t^{r^-}) for $\Gamma(f)$ iff it is (t^{r^-}) for $\Gamma(g)$. Furthermore, X is (t^{r^-}) for P iff W is and, for all C^{r^-} representatives Q of $j^{r-1}P$, $Q \cap S = \{0\}$ (if we restrict to C^r representatives, the “if” of this “iff” fails). Then the analogue of Theorem 2.7 is:

Theorem 2.30. *Suppose F, X, Y, W and S are as in the Standard Setup (before (2.7)). Suppose P is an analytic direct transversal to Y . For the (t^{0^-}) case, assume in addition that $Y \subset X$. For $r \geq 0$, W is (t^{r^-}) for the transversal P iff*

$$(2.30.1) \quad \mathfrak{m}_n^r JM_y(F) \subseteq (\mathfrak{m}_n JM(F)_P + I(P)JM_y(F))^\dagger$$

(take the integral closure inside $\mathcal{O}_{X,0}^p$, where p is the number of components of F ; in the real analytic situation we take real integral closure).

Furthermore, X is (t^{r^-}) for the transversal P iff W is (t^r) for the transversal P and

$$(2.30.2) \quad \mathfrak{m}_n^r \mathcal{O}_{S,0} \subseteq (I(P)\mathcal{O}_{S,0})^\dagger.$$

We omit the proof.

Theorem 2.31. *Suppose F, X, Y, W and S are as in the Standard Setup (before (2.7)). Then, for each $r \geq 0$, W (respectively X) is $(t^{(r+1)^-})$ regular over Y at 0 iff \tilde{W} (respectively \tilde{X}) is (t^{r^-}) regular over \tilde{Y} at every point of \tilde{Y} . Moreover, every r -jet z of direct transversals to Y at 0 lifts to a non-unique $(r-1)$ -jet \tilde{z} of a direct transversal to \tilde{Y} , and W (respectively \tilde{X}) is $(t^{(r+1)^-})$ for z iff \tilde{W} (respectively \tilde{X}) is (t^{r^-}) for \tilde{z} (saying that a condition holds for a (-1) -jet is taken to mean that it holds for all 0-jets).*

Proof. The proof is a minor variation of the proof of (2.13). Reduce the degrees of p and q by 1, and replace the Landau symbol o by O . In (2.14)–(2.17), multiply the right hand sides of the \subseteq by \mathfrak{m}_1 . The rest of the proof is the same. \square

Theorem 2.32. *If $r \geq 1$, W (respectively X) is (t^{r^-}) for $\Gamma(f)$ iff W (respectively X) is (t^r) for all graphs of degree $\leq r$ polynomial representatives of the $(r-1)$ -jet of f .*

Proof. Using the preceding theorem, we can reduce to the case $r = 1$. This was proved in [G4], Corollary 2.15. \square

Condition (t^{0^-}) turns out to be the so-called “strong Verdier regularity”, where X is strong Verdier over Y at 0 if for every $\epsilon > 0$ there is an ambient neighbourhood of 0 on which $\text{dist}(T_x X, T_y Y) < \epsilon \text{dist}(x, Y)$. This condition was called “differentiably regular” in [K-T-L]. It is a non-generic condition and implies a rather strong local topological triviality—the trivialising homeomorphisms, C^1 off Y , have a differential on Y which is the identity in the normal direction to Y (shown in [T-W]), so that for example the density of the singular variety X is *constant* along Y [Va1]. Verdier equisingularity merely ensures that the density is lipschitz along Y , as shown by G. Valette [Va2] (G. Comte [C] previously proved continuity).

There is also the notion of the ambient or relative t^r conditions.

Set-up for ambient (t^r) . Let $X \subset \mathbf{k}^{n+k}$ be an analytic space defined by a function G , and let f be a function mapping X to \mathbf{k} . Pick an extension of f to the ambient space, which we also denote by f . Denote the mapping with components (f, G) by F . Let $X(f)$ denote $F^{-1}(0)$. Let $S = S(f, G) \cup \{0\} \times \mathbf{k}^k$.

In what follows the results are independent of the chosen extension of f .

We use the same definition of the r -jet of a C^r manifold as before.

Definition 2.33. In the ambient (t^r) set up, f is (t^r) for P if for all Q such that $j^r Q = j^r P$ (i.e. they are r -equivalent) then Q misses S near (but not at) 0 and is transverse to the fibers of $f : X - S \rightarrow \mathbf{k}$ near (but not necessarily at) 0.

It is possible to define a notion of ambient (t^r) for $f : X \rightarrow \mathbf{k}^p$, but in the complex case it rarely holds.

To see this consider a map-germ $f : \mathbf{C}^3 \rightarrow \mathbf{C}^2$, whose singular set $S(f)$ is a non-empty curve. Let P be the germ of any smooth complex analytic surface at the origin which intersects $S(f)$ only at the origin. Then the restriction of f to P is not a submersion, hence $S(f|P)$ is at least a curve. At each point of $S(f|P)$ different from the origin, P is tangent to the fiber of f at that point. So (t^r) fails for such P for all r .

If Y is a submanifold passing through the origin, then we say that f is (t^r) regular over Y at 0 if and only if f is (t^r) for all C^r direct transversals to Y through 0. In the complex case we say that f is (t^r) regular over Y at 0 if f is (t^r) for all r -jets of graphs of complex polynomials vanishing at 0.

Let X_f denote the subset of X consisting of the non-singular points of X where f is a submersion. Suppose Y is embedded in X , $f(Y) = 0$. To say that f is (t^0) regular over Y means that f satisfies the (w_f) condition for the pair (X_f, Y) ((w_f) for (X, Y) means that $\text{dist}(T_0 Y, \ker d(f|_X)) \leq C \text{dist}(x, Y)$ for some constant $C > 0$, cf. Definition 1.9). If $X_f = X - Y$, then this condition ensures that the family of functions on X parametrised by Y induced from f has a rugose trivialization ([V], [T-W]).

We again have an integral closure version of this condition. The analogue of 2.7 is:

Theorem 2.34. *In the ambient (t^r) set-up, suppose $X \subseteq \mathbf{k}^{n+k}$ is the germ of an analytic space at the origin, Y a submanifold embedded as $\{0\} \times \mathbf{k}^k$, P an analytic direct transversal to Y . For $r \geq 0$, f is (t^r) for the transversal P iff*

$$\mathfrak{m}_n^r JM_y(F) \subseteq \overline{\mathfrak{m}_n JM(F)_P + I(P)JM_y(F)}$$

(take the integral closure inside $\mathcal{O}_{X,0}^p$, where p is the number of components of F ; in the real analytic situation we take real integral closure).

The analogue of 2.13 is similar; we use the same Grassmann modification of X and replace f by $f := f \circ \beta$.

Remarkably, although ambient (t^r) seems much stronger than asking that X and $X(f)$ are (t^r) , in the complex analytic case they are equivalent. The precise statement in the two stratum case is:

Theorem 2.35. *Suppose $X, 0 \subseteq \mathbf{C}^{n+k}$ is the germ of a complex analytic space, which contains $Y = \{0\} \times \mathbf{C}^k$ as a stratum, $f : (X, Y) \rightarrow (\mathbf{C}, 0)$, $X_f = X - Y$, and P a direct transversal to Y . Then f satisfies (t^r) for P iff X_0 and $X(f)_0$ do.*

Proof. It is clear that if f is ambient (t^r) for P , then $X - Y$ and $X(f)_0$ (the regular points of $X(f)$) are (t^r) for P , because the limiting tangent hyperplanes to the fibers of f include the limiting tangent hyperplanes to X and to $X(f)$.

The other direction of the proof is by induction on r . Suppose $r = 1$. Then we can assume $X - Y$ and $X(f)_0$ are (t^1) for P . Further, the assumption that $X_f = X - Y$ implies by definition that the points of $X - Y$ are smooth, the restriction of f to $X - Y$ is a submersion, hence $X(f) - Y$ is smooth. Therefore, $X(f)_0$ is $X(f) - Y$. Then, if we consider the Grassmann modification of X , $\widetilde{X - Y}$ and $\widetilde{X(f)_0}$ are the complements of $G_{n+k,n}$ in \widetilde{X} and $\widetilde{X(f)}$ respectively. Hence, by 2.13, the pairs $(\widetilde{X - Y}, G_{n+k,n})$ and $(\widetilde{X(f)}, G_{n+k,n})$ both satisfy the Whitney conditions at P . (Recall we regard P as a point of the zero section of $E_{n+k,n}$.) Since \widetilde{X} is smooth at points of $\widetilde{X(f)_0}$, the Whitney conditions hold trivially for this pair.

Thus $\{\widetilde{X - Y - X(f)}, \widetilde{X(f)_0}, G_{n+k,n}\}$ form a Whitney stratification of \widetilde{X} at $(j^1P, 0)$. By the result of Briangon, Maisonobe and Merle [BMM], this implies that $(w_{\tilde{f}})$ holds at $(j^1P, 0)$ for $\{\widetilde{X - Y}, G_{n+k,n}\}$. Then by the proof of 2.13, altered for the ambient condition, (t^1) holds for f and P .

Assuming (t^r) holds for $X - Y$ and $X(f)_0$ for P , again construct the Grassmann modification. Applying 2.13 to $X - Y$ and to $X(f)_0$ we have that (t^{r-1}) holds for the pairs $(\widetilde{X - Y}, G_{n+k,n})$ and $(\widetilde{X(f)_0}, G_{n+k,n})$ at P . Now using the induction hypothesis, we have ambient (t^{r-1}) holds for \tilde{f} , hence by the proof of 2.13, altered for the ambient condition, (t^r) holds for f and P . \square

We will be using these results at the end of section 4.

§3. CONDITIONS (a) , (t^1) AND THE AUREOLE

Let $Y \subseteq X$ be analytic subsets of \mathbf{C}^n , Y smooth, $S(X) \subseteq Y$. The relation between the (t^1) condition and Whitney's condition (a) for (X_0, Y) depends on the relation between the stratum Y and the *aureole* ([H-L], [L-T]). Recall that the conormal modification of X is the closure in $X \times \mathbf{P}^{n-1}$ of the set of hyperplanes containing $T_x X$ at a regular point x . Assume $0 \in X$. The fiber of the conormal modification over 0 will be denoted $\mathcal{C}(X)$. The aureole of X at 0 is constructed in the complex analytic case by blowing up the conormal modification of X by the pull-back of the maximal ideal of X

at 0, then projecting the irreducible components of the exceptional divisor E of the blow-up to the projectivised tangent cone $\mathbf{P}C_T(X)$ of X . We will identify subsets of $\mathbf{P}C_T(X)$ with the corresponding subcones of $C_T(X)$. The images of these components consist of a set of subcones; this set \mathcal{A} is the aureole. (The components of the tangent cone are the largest members of the aureole; any members which are of lower dimension than the cone are called *exceptional subvarieties* of $C_T(X)$; exceptional subvarieties cannot have dimension 0.)

As proved in [L-T], the aureole has the property that the limiting tangent hyperplanes to X at 0 are exactly the planes tangent to one of the elements of the aureole, i.e. $\mathcal{C}(X) = \cup_{A \in \mathcal{A}} \mathcal{C}(A)$. Thus the aureole tracks limiting tangent hyperplanes and the limiting tangents of the curves along which the limits are achieved. Without loss of generality, assume Y is linear. If Y contains an element of the aureole properly then Whitney (a) will fail: at a regular point of the contained cone, the tangent space of the contained cone will be a proper subspace of Y , hence there will be a tangent hyperplane H tangent to the contained cone but not containing Y ; by the property of aureoles mentioned above, $H \in \mathcal{C}(X)$, hence (a) fails. If there is an $\ell \in C_T(X)$, $\ell \not\subseteq Y$, a curve $\phi(t)$ in X with $\phi(0) = 0$, $\phi(t) \in X_0$ for $t \neq 0$ and ϕ is tangent to ℓ at 0, and a family of hyperplanes $H(t) \subseteq T_{\phi(t)}X_0$ such that $H = \lim H(t)$ does not contain Y , then (t^1) fails, and conversely (see [G4] Cor 2.8). This allows us to characterize both (a) and (t^1) in terms of the aureole.

Let \mathcal{E} denote the set of irreducible components of the exceptional divisor E . Let π_1 denote the projection from E to the tangent cone; so π_1 of the elements of \mathcal{E} gives the members of \mathcal{A} ; let π_2 denote the projection from E to $\mathcal{C}(X)$ (i.e. we ignore which line in the tangent cone the hyperplanes come from). Let \mathcal{E}_1 denote those elements C of \mathcal{E} for which $\pi_2(C) \subseteq \mathcal{C}(Y)$; let $\mathcal{E}_2 = \mathcal{E} - \mathcal{E}_1$.

Proposition 3.1. *X satisfies (a) over Y at 0 iff $\mathcal{E}_2 = \emptyset$.*

Proof. Note that $\mathcal{C}(X) = \cup_{C \in \mathcal{E}} \pi_2(C)$. Thus (a) holds iff $\mathcal{C}(X) \subseteq \mathcal{C}(Y)$ iff $\pi_2(C) \subseteq \mathcal{C}(Y)$ for all $C \in \mathcal{E}$.

Proposition 3.2. *X satisfies (t^1) over Y at 0 iff $\{C \in \mathcal{E}_2 : \pi_1(C) \not\subseteq Y\} = \emptyset$.*

Proof. By Cor 2.8 of [G4], (t^1) fails iff there is an $(\ell, H) \in E$ such that $\pi_1(\ell, H) = \ell \not\subseteq Y$ and $\pi_2(\ell, H) = H \notin \mathcal{C}(Y)$ iff there exists $C \in \mathcal{E}_2$ such that $\pi_1(C) \not\subseteq Y$.

The next result links Whitney (a), the aureole and (t^1) .

Theorem 3.3. *Let $Y \subseteq X$ be analytic subsets of \mathbf{C}^n , Y smooth. Assume X satisfies (t^1) over Y at $y \in Y$; then X satisfies (a) over Y at $y \in Y$ iff there are no exceptional subvarieties of $C_T(X)_y$ properly contained in $T_y Y$.*

Proof. Without loss of generality we reduce to the case $y = 0$ and Y is linear. Choose $C \in \mathcal{E}$. Since (t^1) holds, either (i) $\pi_2(C) \subseteq \mathcal{C}(Y)$, or (ii) $\pi_2(C) \not\subseteq \mathcal{C}(Y)$ and $\pi_1(C) \subseteq Y$. (Note that in (ii) the inclusion must be proper; if not Y is an element of the aureole, and by [L-T], $\pi_2(C) \subseteq \mathcal{C}(Y)$.)

Assume case (i); then this C presents no obstruction to (a) holding. Furthermore, if $A = \pi_1(C)$ is a proper analytic subset of Y , then there will be a tangent hyperplane tangent to the A , but not containing Y , thus contradicting the assumption. Assume case (ii); then (a) fails, and $A = \pi_1(C)$ is a proper subvariety of Y . \square

Observe that if $\dim Y = 1$, no member of the aureole can be a proper analytic subset of Y , so (t^1) implies (a) (which was proved in the subanalytic case by the second author ([Tr1], [Tr4])). If $\dim Y = 2$, Theorem 3.3 implies that (t^1) is equivalent to (a) iff there are no exceptional lines in Y .

We wish to give examples for which (t^1) holds but (a) fails. Theorem 3.3 shows that it suffices to find X and Y for which the aureole \mathcal{A} contains proper subvarieties of Y , and for which all other members A of \mathcal{A} have $\mathcal{C}(A) \subseteq \mathcal{C}(Y)$.

Example 3.4. Let $I = (xyf(x, y, z, w))$, $f = x^a + y^b + z^c + w^d$ with $2 \leq a < b < c < d$; let $X = V(I)$, $Y = S(X)$ the wz plane. The purpose of the xy term is to have $Y \subseteq X$. The aureole of X is the union of the aureole of the three components. The aureole consists of $x = 0$, $y = 0$ and the members of the aureole of $f = 0$. It is shown in [G3] that the aureole of $f = 0$ consists of a flag—the linear spaces defined by $x = 0$, $x = y = 0$ and $x = y = z = 0$; call these A_1 , A_2 and A_3 , respectively. Then A_3 is a proper algebraic subset of Y , and A_1 and A_2 satisfy $\mathcal{C}(A) \subseteq \mathcal{C}(Y)$. Thus (t^1) holds but (a) fails.

Without going into the details, we consider what happens in this example if some of the strict inequalities are replaced with equalities. If $2 \leq a = b < c < d$, $\mathcal{A}(f = 0) = \{\{x^a + y^a = 0\}, \{x = y = 0\}, \{x = y = z = 0\}\}$, where $x^a + y^a = 0$ is the union of a hyperplanes intersecting in $x = y = 0$; (t^1) holds but (a) fails. If $2 \leq a < b = c < d$, $\mathcal{A}(f = 0) = \{\{x = 0\}, \{x = 0, y^b + z^b = 0\}, \{x = y = z = 0\}\}$, where $x = 0, y^b + z^b = 0$ is the union of b planes intersecting in $x = y = z = 0$; (t^1) fails. If $2 \leq a = b = c < d$, $\mathcal{A}(f = 0) = \{\{x^a + y^a + z^a = 0\}, \{x = y = z = 0\}\}$; (t^1) fails.

In the real case, the structure of the conormal modification is much more complicated (see [O-W] for the case of surfaces in \mathbf{R}^3). However we state one collection of examples whose properties are not too hard to check.

Example 3.5. Now we work in \mathbf{R}^4 . Let $I = (xyf(x, y, z, w))$, $f = x^a + y^b - z^c + w^d$ with $2 \leq a = b < c < d$, a, b, c, d even; let $X = V(I)$, Y the wz plane. The geometric tangent cone of $f = 0$ is Y , so direct transversals to Y intersect $f = 0$ only at 0. $\mathcal{C}(X)$ consists of all hyperplanes containing $x = y = z = 0$. Thus (t^1) holds but (a) fails. In [Tr4] the second author previously gave semialgebraic examples in \mathbf{R}^4 showing that (t^1) can hold without (a) .

§4. THE PRINCIPLE OF SPECIALIZATION OF INTEGRAL DEPENDENCE AND THE (t^r) CONDITIONS

So far we have not used very strongly the algebraic character of the theory of integral closure. However, if we restrict ourselves to complex analytic sets, there is a remarkable result which comes from this side of the theory. In many applications we have a family of sets, and a sheaf of modules on each set. These modules often come from specializing a sheaf of modules contained in a locally free sheaf on the total space of the family to the members of the family. Often we know that some element of the free sheaf is in the integral closure of the modules on the members of the family generically. Is it in the integral closure of the member-module for all members? Is it in the integral closure of the sheaf of modules which is defined on the total space? This type of question often arises in mathematics, and the answer is usually no.

The theory of integral closure in some cases associates an invariant to the member-modules called their multiplicity. If these multiplicities are constant then the answer to both questions is yes! This is called the Principle of Specialization of Integral Dependence, first stated for ideals by Bernard Teissier (see [T1], 3.2, p.330 and [T2], App. I).

For all statements and results of this section, we will assume $\mathbf{k} = \mathbf{C}$. If one considers the maximal ideal \mathfrak{m}_x of a local ring $\mathcal{O}_{X,x}$, where X is equidimensional at x , then the multiplicity of \mathfrak{m}_x is just the multiplicity of X at x —the number of points near x in which a generic linear space of dimension complementary to X will intersect X . If one considers an ideal I of finite colength (the colength of a module $M \subseteq \mathcal{O}_X^p$ is the dimension of the quotient module $\frac{\mathcal{O}_X^p}{M}$ as a complex vector space), one can pick a set of generators of I , and use these to define a map-germ f with image $f(X)$. The multiplicity of I is just the product of the multiplicity of $f(X)$ and the degree of f .

The multiplicity of a module is defined for all modules of finite colength ([B-R] and [K-Th]). If we are working over a ring which is Cohen-Macaulay then the multiplicity $e(M; X)$ of $M \subseteq \mathcal{O}_X^p$ can be obtained by taking the colength of the submodule generated by $d+p-1$ generic elements of M , where d is the dimension of X ([G2], proposition 2.3, and the fact that any submodule generated by these generic elements has the same multiplicity as M [K-Th]). Before stating the Principle of Specialization of Integral Dependence precisely, we give an example to make these ideas concrete.

Example 4.1. Consider the ideal $I = (x^2, y^2, txy)$ in \mathcal{O}_3 ; we think of \mathbf{C}^3 as the total space of a family of planes parametrised by \mathbf{C} embedded as the t -axis. The ideal I induces a family of ideals of finite colength I_t . Clearly the term $xy \in I_t$ for $t \neq 0$. The multiplicity of I_t is 4 for all values of t , so we know that $xy \in \bar{I}$.

If we have a mapping between two sets X and Y , and the restriction of a sheaf of modules to the fiber over Y is supported at a finite number of points, then we denote by $e(y)$ the sum over the points in the fiber of y of the multiplicities of the stalks of the sheaf.

Theorem 4.2. (*Specialization of integral dependence*)

Assume that X is equidimensional, and that $F: (X, x_0) \rightarrow (Y, y_0)$ is a map of germs of analytic spaces. Assume that the fibers $X(y)$ are equidimensional of the same dimension d , d at least 1, and that Y has dimension at least 1. Let M be a coherent submodule of \mathcal{O}_X^p .

Set $S := \text{Supp}(\frac{\mathcal{O}_X^p}{M})$, and assume that S is finite over Y . Assume $y \mapsto e(y)$ is constant on Y . Let h be a section of \mathcal{O}_X^p whose image in $\mathcal{O}_{X_y}^p$ is integrally dependent on the image of M for y in a Zariski open set. Then (after X and Y are replaced by neighborhoods of x_0 and y_0 if necessary) h is integrally dependent on M .

Proof. The proof appears in [G-K]. \square

Now suppose $X = F^{-1}(0)$; we use the multiplicity of the module $JM(F)_{P_r}$ (P_r is defined in (2.2)) thought of as a submodule of a free $\mathcal{O}_{X \cap P_r, 0}$ module to control the Whitney equisingularity of a family of generic intersections with X . It is not surprising that the family of generic intersections is Whitney equisingular; what is surprising is that only one invariant is needed. In general, $d+1$ invariants of the Jacobian module are needed where d is the dimension of the elements of the family ([G2]).

Theorem 4.3. (*Genericity Theorem*) Suppose $X \subseteq \mathbf{C}^n \times \mathbf{C}^k$ is a d -dimensional complete intersection, and suppose P is a direct transversal to $Y = \{0\} \times \mathbf{C}^k$, $P \cap S(X) = \{0\}$. Then $\epsilon(JM(F)_{P_r}; P_r \cap X)$ is the generic (finite) value of this invariant for the r -jet of P among all transversals with the same $(r-1)$ -jet iff (t^r) holds for P .

Proof. The proof is by induction on r . Note that (t^r) holds for P iff (t^r) holds for P_r , so we can reduce to the r -jet of P . Further, since $P \cap S(X) = \{0\}$, it follows that the dimension of $S(X)$ is less than or equal to that of Y , hence the generic r -jet intersects $S(X)$ only at 0. (This claim is obvious for the generic 1-jet; if we fix the $r-1$ jet and apply the Grassmann modification $r-1$ times, varying the r -jet amounts to varying the family of planes in the $r-1$ modification.)

Before proceeding to the induction we prove that that the corresponding rings and modules at the level of $X \cap P_r$ and at the level of $\tilde{X} \cap \tilde{P}_r$ are isomorphic. This will show that the multiplicities of the corresponding modules are the same. The induction hypothesis for (t^{r-1}) at the level of \tilde{X} will then imply the result for (t^r) at the level of X . The hypotheses on P and $S(X)$ are to ensure that $\epsilon(JM(F)_{P_r}; P_r \cap X)$ is generically finite. Suppose P_r is given by $\{y_i - p_i(x) = 0\}$, p_i a polynomial of degree r . Suppose $p_i(x) = \sum p_{i,j}(x)x_j$, giving a lift \tilde{P} of P_r with equations $\{a_{i,j} - p_{i,j}(x) = 0\}$. We can define maps $p = (p_1, \dots, p_k)$ and $\tilde{p} = (p_{i,j})$ such that P_r and \tilde{P} are isomorphic to \mathbf{C}^n by the graph maps p_Γ and \tilde{p}_Γ , and we have that $\beta \circ \tilde{p}_\Gamma = p_\Gamma$. Then the rings $\mathcal{O}_{\tilde{X} \cap \tilde{P}, P_1}$ and $\mathcal{O}_{X \cap P_r, 0}$ are isomorphic, since

$$\tilde{p}_\Gamma^*((a_{i,j} - p_{i,j}), \beta^*(I(X))) = \tilde{p}_\Gamma^* \beta^*(I(X)) = p_\Gamma^* I(X) = p_\Gamma^*(I(X), (y_i - p_i)).$$

Now we consider the modules. If we restrict the module used in the formulation of the (t^r) condition to $X \cap P_r$ we see that it is generated by \mathfrak{m}_n times

$$\left\{ \frac{\partial F}{\partial x_j} + \sum_{i=1}^k \frac{\partial p_i}{\partial x_j} \frac{\partial F}{\partial y_i} : 1 \leq j \leq n \right\},$$

where $X = F^{-1}(0)$.

The corresponding module which controls the (t^{r-1}) condition on \tilde{X} when restricted to $\tilde{X} \cap \tilde{P}$ is generated by \mathfrak{m}_n times

$$\left\{ \frac{\partial G}{\partial x_j} + \sum_{i,l} \frac{\partial p_{i,l}}{\partial x_j} \frac{\partial G}{\partial a_{i,l}} : 1 \leq j \leq n \right\},$$

where $G = F \circ \beta$.

Now, $\frac{\partial G}{\partial a_{i,l}} = x_l \frac{\partial F}{\partial y_i} \circ \beta$, while $\frac{\partial G}{\partial x_j} = \frac{\partial F}{\partial x_j} \circ \beta + \sum_i a_{i,j} \frac{\partial F}{\partial y_i} \circ \beta$. So, the generators of the module on \tilde{X} become \mathfrak{m}_n times

$$\left\{ \frac{\partial F}{\partial x_j} \circ \beta + \sum_i a_{i,j} \frac{\partial F}{\partial y_i} \circ \beta + \sum_{i,l} \frac{\partial p_{i,l}}{\partial x_j} x_l \frac{\partial F}{\partial y_i} \circ \beta \right\}.$$

Now, $\frac{\partial p_i}{\partial x_j} = \frac{\partial \sum_l p_{i,l} x_l}{\partial x_j} = (\sum_l \frac{\partial p_{i,l}}{\partial x_j} x_l) + p_{i,j}$. By summing over l , we can rewrite the generators of the module on \tilde{X} restricted to $\tilde{X} \cap \tilde{P}$ as \mathfrak{m}_n times

$$\left\{ \frac{\partial F}{\partial x_j} \circ \beta + \sum_i a_{i,j} \frac{\partial F}{\partial y_i} \circ \beta + \sum_{i=1}^k \frac{\partial p_i}{\partial x_j} \frac{\partial F}{\partial y_i} \circ \beta - \sum_i p_{i,j} \frac{\partial F}{\partial y_i} \circ \beta \right\} = \left\{ \frac{\partial F}{\partial x_j} \circ \beta + \sum_{i=1}^k \frac{\partial p_i}{\partial x_j} \frac{\partial F}{\partial y_i} \circ \beta + \sum_i (a_{i,j} - p_{i,j}) \frac{\partial F}{\partial y_i} \circ \beta \right\}.$$

Since these generators pull back to the same elements when composed with p_Γ and \tilde{p}_Γ , the corresponding quotient modules are isomorphic, hence have the same multiplicity.

Now we start the induction. Let $r = 1$. We can view the modification \tilde{X} as the family of intersections $\{X \cap P\}$ parameterised by the Grassmannian where P is a plane which is a direct transversal.

In the proof of Theorem 3.3 of [G4], it is shown that the planes for which the Jacobian module of $\{X \cap P\}$ has the generic value of the multiplicity are the same as the planes in which $JM(F)_P$ has the generic value. This is done by noticing the isomorphism of modules

$$\frac{\mathcal{O}^{n+2k-d}_{X \cap P}}{JM(F, f_P)} \approx \frac{\mathcal{O}^{n+k-d}_{X \cap P}}{JM(F)_P}$$

induced by the inclusion of the module on the right into the module on the left. (Here f_P defines P , and is a submersion of $\mathbf{C}^{n+k} \rightarrow \mathbf{C}^k$; the inclusion is induced from the inclusion of $\mathcal{O}^{n+k-d}_{X \cap P}$ into the first $n+k-d$ factors of $\mathcal{O}^{n+2k-d}_{X \cap P}$.) This isomorphism implies that the multiplicities of the modules $JM(F, f_P)$ and $JM(F)_P$ are the same.

Call the Z -open subset of the Grassmannian which parameterises these generic planes U . Since the multiplicity of the Jacobian module is constant along U , it follows that there exists some open neighborhood W' of U in \tilde{X} which consists only of points in U and in $\beta^{-1}(X - S)$. Since the multiplicity is defined, for each P there is a neighborhood of the origin such that except at the origin, P misses the singular set and intersects the smooth part of X transversely. Since the multiplicity of the Jacobian module is constant along U , these neighborhoods cannot shrink to 0, as, if some point z of U was in the closure of the points of $\beta^{-1}(S)$ the corresponding family of planes would cause the multiplicity to jump at z . The existence of W' implies that if P has the same 1-jet as an element of U , then the intersection of P with S can only be the origin.

This shows that the planes parametrised by U satisfy the part of the (t^1) condition concerned with S .

The proof that the part of the (t^1) condition concerning $X - S$ holds follows from the proof of Theorem 3.3 of [G4]. We review the idea of this proof.

We know from Theorem 2.13 that P is (t^1) for $X - S$ if for any lift P' to \tilde{X} , P' is (t_0) for $\beta^{-1}(X - S)$.

Consider the inclusions

$$(4.3.1) \quad \left. \frac{\partial F}{\partial t_i} \right|_P \in \overline{(JM(F)_P)} \subset \mathcal{O}^{n+k-d}_{X \cap P} \quad \forall i$$

$$(4.3.2) \quad \left. \frac{\partial F}{\partial t_i} \circ \beta \right|_{a_{i,j}=a_{0i,j}} \in \overline{\left(\frac{\partial F}{\partial z_j} \circ \beta + \sum_i a_{0i,j} \frac{\partial F}{\partial t_i} \circ \beta \right)} \Big|_{a_{i,j}=a_{0i,j}} \quad \forall i$$

$$(4.3.3) \quad \left. \frac{\partial G}{\partial a_{i,j}} \right|_{a_{i,j}=a_{0i,j}} \in \overline{(m_n JM_z(G_{a_{0i,j}}))} \quad \forall i, j$$

In these inclusions we assume that P is defined by $\{y_1 = \sum_i a_{01,i} z_i, \dots, y_k = \sum_i a_{0k,i} z_i\}$, where y_i are coordinates on Y . The inverse image of P on the Grassman modification is just $(z, a_{01,1}, \dots, a_{0k,n})$. The proof of 2.13 shows that these inclusions are equivalent.

The constancy of the multiplicity of $JM(F)_P$ implies that the multiplicity of the module $\left(\frac{\partial F}{\partial z_j} \circ \beta + \sum_i a_{0i,j} \frac{\partial F}{\partial t_i} \circ \beta \right) \Big|_{a_{i,j}=a_{0i,j}}$ is constant since the induced map of quotient modules is an isomorphism. Further, the inclusion 4.3.2 holds generically since the Whitney conditions imply that 4.3.3 holds generically. Now we can apply the principle of specialization of integral dependence to the sheaf of modules $(\frac{\partial F}{\partial z_j} \circ \beta + \sum_i a_{i,j} \frac{\partial F}{\partial t_i} \circ \beta)$ and to the sections $\frac{\partial F}{\partial t_i} \circ \beta$. It follows that the inclusion 4.3.2 holds at the level of the total space, hence so does 4.3.3, hence the family $\{X \cap P\}$ is Whitney equisingular along U at the point of U corresponding to P , and (t^1) holds for P at the origin.

On the other hand, if we assume that (t^1) holds for P at the origin, then $\{X \cap P_u\}$ is Whitney equisingular at the point corresponding to P , then since the Milnor numbers of $\{X \cap P_u\}$ and its plane slices are invariants of Whitney equisingularity type, we have that the multiplicity of $JM(F, f_{P_u})$, which is the sum of the Milnor number of $X \cap P_u$ and of a hyperplane slice by [G2], is independent of u . Hence the multiplicity of $JM(F)_{P_u}$ is independent of u . This finishes the case $r = 1$.

The idea of the rest of the proof uses the relation proved earlier linking (t^r) on X and (t^{r-1}) on \tilde{X} .

Varying the r -jet of P with $(r-1)$ -jet fixed amounts to varying the $(r-1)$ -jet of \tilde{P} with the $(r-2)$ -jet fixed. So by induction we are done. \square

Corollary 4.4. *If we fix an $(r-1)$ -jet, then the set of r -jets which satisfy (t^r) is Zariski open and dense.*

Corollary 4.5. *Consider the family obtained by fixing an $(r-1)$ -jet and varying the r -jet. Then the family has Verdier equisingular intersection with X exactly for those parameter values for which the multiplicity takes its generic value.*

Proof. This follows from Theorem 4.3 and Corollary 2.28. \square

By Theorem 4.3 we know that if P is a direct transversal such that the multiplicity, $e(JM(F)_{P_r}, \mathcal{O}_{X \cap P_r})$ has the generic value e among all such modules associated to r -jets with the same $(r-1)$ -jet as P , then X is (t^r) for P_r , hence for P . Call the multiplicity $e(JM(F)_P, \mathcal{O}_{X \cap P})$, the associated multiplicity of the pair (X, P) . It is not difficult to

see that if X is (t^r) for P , then the associated multiplicity is the generic value e . For the deformation $P(t)$ to P_r has constant r -jet, and X is (t^r) for P_r , hence by Corollary 2.28 the deformation is a Verdier equisingular family of isolated complete intersection singularities, hence by Theorem 1 and Proposition 2.6 of [G2], the multiplicity of the Jacobian module of $X \cap P(t)$ is constant. The multiplicity of this Jacobian module is the same as the associated multiplicity, because the quotients of these two modules are isomorphic as $\mathcal{O}_{X \cap P}$ -modules. If X is not (t^r) for P , the associated multiplicity of (X, P_r) must be greater than the generic value by Theorem 4.3. It may be that the associated multiplicity of (X, P) is less than that of (X, P_r) ; can it be the generic value of e or even less? The next lemma deals with this question.

Lemma 4.6. *Suppose (t^r) fails for a direct transversal P . Then the value of $e(JM(F)_P, \mathcal{O}_{X \cap P})$ is strictly greater than e , the generic value among all P'_r such that $j^{r-1}P = j^{r-1}P'$.*

Proof. By the modification theorem (2.13) and the techniques used in the Genericity Theorem (4.3) it suffices to prove the lemma for (t^1) . Let the equations for P be $\{y_i - \sum p_{i,j}(x)x_j = 0\}$. Consider the family of direct transversals given by $\{y_i - \sum(a_{i,j} + p_{i,j}(x))x_j = 0\}$ in \mathbf{C}^{n+k} . Denote by P^a the transversal determined by $a = (a_{i,j})$. We can study the resulting family of sections by the following construction. Let $\tilde{\beta} : \mathbf{C}^n \times \mathbf{C}^{nk} \rightarrow \mathbf{C}^{n+k}$ be given by

$$\tilde{\beta}(x_1, \dots, x_n, a_{1,1}, \dots, a_{k,n}) = (x_1, \dots, x_n, \sum_i (a_{1,i} + p_{1,i}(x))x_i, \dots, \sum_i (a_{k,i} + p_{k,i}(x))x_i).$$

Let \tilde{X} denote the set defined by $F \circ \tilde{\beta}$. Given $\phi : \mathbf{C}, 0 \rightarrow \tilde{X}, P \times \{0\}$, then $\tilde{\beta} \circ \phi$ is tangent to P at the origin, and only curves tangent to P at zero lift to \tilde{X} . Tracing through the proof of Theorem 2.13, it is easy to see that (t^1) holds for P^a iff $\tilde{\beta}^* JM(F)_{P^a}$ is a reduction of $\tilde{\beta}^* JM(F)$. Since (t^1) holds for P^a for generic a , this inclusion is true generically. At these generic a values, the value of the associated multiplicity of (X, P^a) is the generic value for 1-jets at the origin of X , since (t^1) holds. This shows that the associated multiplicity of (X, P) must be at least the generic value since by [G-K], Proposition 1.1, the multiplicity is upper semicontinuous. Now suppose the associated multiplicity has the generic value of the associated multiplicities of 1-jets. Then on \tilde{X} in a neighborhood of $(0, 0)$ the associated multiplicities are constant, and the Principle of Specialization of Integral Dependence shows that $\tilde{\beta}^* JM(F)_{P^0}$ is a reduction of $\tilde{\beta}^* JM(F)$. Thus P is (t^1) , which is a contradiction. \square

Theorem 4.7. *Suppose we have a family $P(t)$ of direct transversals with fixed $(r-1)$ -jet, and for some parameter value $e(JM(F)_{P(t)}, \mathcal{O}_{X \cap P(t)})$ has the generic value e of the multiplicity. Then the family has Verdier equisingular intersection with X on the Zariski open, dense subset of values where $e(JM(F)_{P(t)}, \mathcal{O}_{X \cap P(t)}) = e$ and it fails to have Verdier equisingular intersection with X where $e(JM(F)_{P(t)}, \mathcal{O}_{X \cap P(t)}) > e$.*

Proof. From the hypotheses we know that the set of parameter values where the associated multiplicity has the generic value e is a non-empty Zariski open set U . By Lemma 4.6, we know that (t^r) holds for $P(t)$ exactly on this Zariski open set. Then the restriction of the family of sections to U is Verdier equisingular. At points on the complement of U , the associated multiplicity jumps, so the multiplicity of the Jacobian module of the section jumps, and Verdier equisingularity fails by Theorem 1 of [G2]. \square

We have seen that if we fix P_{r-1} and vary the r jet, then only one invariant, the associated multiplicity, need be constant to ensure Verdier equisingularity, provided this invariant takes its generic value. What if the value of the associated multiplicity is greater than the generic value? If we have some control over the plane sections of P , then it is still possible to say something. Given an isolated complete intersection singularity $X, 0 \subset \mathbf{C}^n$ we can consider the μ_* sequence of X . This is the sequence of Milnor numbers $\mu_0(X) = \mu(X), \dots, \mu_i(X) = \mu(X \cap H_i), \dots, \mu_{d+1}(X) = 1$, where $d = \dim X$ and H_i is a generic linear space of codimension i . Sometimes authors index by the dimension of H , so $\mu^d(X) = \mu(X \cap H^d)$. There is a nice connection between the associated multiplicity of (X, P) and the μ_* numbers of $X \cap P$. If $X \cap P$ is an isolated complete intersection singularity, then the associated multiplicity of (X, P) is the sum of $\mu(X \cap P)$ and $\mu((X \cap H) \cap (P \cap H))$. This follows from the fact that the associated multiplicity is the multiplicity of the Jacobian module of $X \cap P$, Proposition 2.6 of [G2], and the theorem of Lê and Greuel (cf. p. 211 of [G2], material after 1.2). Coupling this connection with Theorems 1 and 2 of [G-2] shows that a family of isolated complete intersection singularities is Verdier equisingular iff the μ_* invariants are independent of parameter.

Theorem 4.8. *Suppose X is a complete intersection defined by F . Suppose $P(t)$ is a family of direct transversals which intersect X in the expected dimension, $P(t) \cap S(X) = 0$, and the associated multiplicity of $(X, P(t))$ is constant. Suppose further that $\mu_1(X \cap P(t)) = \mu_{k+1}(X)$, and $\mu_2(X \cap P(t)) = \mu_{k+2}(X)$. Then the family $\{X \cap P(t)\}$ is Verdier equisingular.*

Proof. Because $\mu_1(X \cap P(t)) = \mu_{k+1}(X)$, and $\mu_2(X \cap P(t)) = \mu_{k+2}(X)$, it follows that the associated multiplicity of $(X \cap H, P(t) \cap H)$ is the generic value for planes of the same dimension as $(P(t) \cap H)_1$, H a generic hyperplane. It follows from Lemma 4.6, that $X \cap H$ is (t^1) for $P(t) \cap H$, hence $(X \cap H) \cap (P(t) \cap H)$ and its sections are Verdier equisingular. Since the associated multiplicity of $(X, P(t))$ is constant, it follows that the Milnor number of $X \cap P(t)$ is constant as well. Since the μ_* invariants are constant, the family $X \cap P(t)$ is Verdier equisingular as well. \square

We can also prove analogues of the theorems of this section for the ambient (t^r) conditions and the (w_f) condition. In the ambient (t^r) set-up, we use the multiplicity of $\epsilon(JM(F)_{P_r}, \mathcal{O}_{X \cap P_r})$. If $X = \mathbf{C}^n$, then this is just the Milnor number of $f|_{P_r}$. A typical result is the following:

Theorem 4.9. *Suppose we have a family $P(t)$ of direct transversals with fixed $(r-1)$ -jet, and for some parameter value $\mu(f|_{P(t)})$ has the generic value μ of the Milnor number. Then the family of functions has a rugose trivialization on the Zariski open, dense subset of values where $\mu(f|_{P(t)}) = \mu$ and it fails to have such a trivialization where $\mu(f|_{P(t)}) > \mu$.*

Again in the hypersurface case, we can do better than 4.8 because we are able to operate in the ambient space, not just on the variety. Again the invariant of interest is the multiplicity of the ideal $J(f|_P)$ which is just the Milnor number of $f|_P$.

Theorem 4.10. *Suppose X is a hypersurface defined by f , $X \subset \mathbf{C}^n \times \mathbf{C}^k$. Suppose $P(t)$ is a family of direct transversals which intersect X in the expected dimension, and*

$P(t) \cap S(X) = 0$. Suppose also $\mu(X \cap P(t))$ is constant and $\mu_1(X \cap P(t)) = \mu_{k+1}(X)$. Then the family $\{X \cap P(t)\}$ is Verdier equisingular.

Proof. Let H_t be a generic hyperplane so that $\mu(X \cap P(t) \cap H_t) = \mu_{k+1}(X)$. Note that, for a generic plane H' of codimension $k+1$, $\mu_{k+1}(X) = \mu(X \cap H')$ iff $\overline{J(f)}|_{H'} = \overline{J(f)}_{H'}$, and in the above case $e(J(f)|_{H'}) = \text{col}(J(f)_{H'}, I(H'))$ (col is the colength). Now $\mu(X \cap P(t) \cap H_t) = \text{col}(J(f)_{P(t) \cap H_t}, I(P(t) \cap H_t))$. Here we use the ambient genericity lemma: if $\mu_{k+1}(X) = \mu(X \cap H')$, then the ambient (t^1) holds for H' , i.e. $\mathfrak{m}_n J_y(F) \subseteq \overline{\mathfrak{m}_n J(F)_{H'} + I(H') J_y(F)}$. Then a proof similar to that of Lemma 4.6 shows that $\mu(X \cap P_1(t) \cap H_t) = \mu_{k+1}(X)$. At this point the proof proceeds as in Theorem 4.8. \square

Example 4.11. Suppose X is a hypersurface in \mathbf{C}^n . If $n = 3$ and $P_t = H_t$ is a family of hyperplanes with $H_t \cap X$ having constant Milnor number, then the family is Verdier equisingular because it is a family of plane curves. If $n = 4$, then we need $\mu(H_t \cap X)$ and $\mu_1(H_t \cap X)$ constant. If $n = 5$, then we need $\mu(H_t \cap X)$ constant and $\mu_1(H_t \cap X) = \mu_2(X)$.

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