ON WEDGE EXTENDABILITY OF CR-MEROMORPHIC FUNCTIONS

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ABSTRACT. In this article, we consider metrically thin singularities E of the solutions of the tangential Cauchy-Riemann operators on a $\mathcal{C}^{2,\alpha}$-smooth embedded Cauchy-Riemann generic manifold M (CR functions on $M \backslash E$) and more generally, we consider holomorphic functions defined in wedgelike domains attached to $M \backslash E$. Our main result establishes the wedge- and the L^1-removability of E under the hypothesis that the $(\dim M - 2)$-dimensional Hausdorff volume of E is zero and that M and $M \backslash E$ are globally minimal. As an application, we deduce that there exists a wedgelike domain attached to an everywhere locally minimal M to which every CR-meromorphic function on M extends meromorphically.

1. INTRODUCTION AND STATEMENT OF RESULTS

In continuation with our previous works [MP1,2,3], we study the wedge removability of metrically thin singularities of CR functions and its application to the local extendability of CR-meromorphic functions defined on CR manifolds of arbitrary codimension.

First we need to recall some fundamental notions concerning CR manifolds. For a detailed presentation we refer to [Bo]. Let M be a connected smooth CR generic manifold in $\mathbb{C}^{m+n}$ with $\text{CRdim}\, M = m \geq 1$, $\text{codim}_{\mathbb{R}}\, M = n \geq 1$, and $\dim_{\mathbb{R}} M = 2m + n$. We denote sometimes $N := m + n$. In suitable holomorphic coordinates $(w, z = x + iy) \in \mathbb{C}^{m+n}$, M may be represented as the graph of a differentiable vector-valued mapping in the form $x = h(w, y)$ with $h(0) = 0$, $dh(0) = 0$. The manifold M is called *globally minimal* if it consists of a single CR orbit. This notion generalizes the concept of *local minimality* in the sense of Tumanov, *cf.* [Trp], [Tu1,2], [J1,2], [M], [MP1]. A *wedge* $\mathcal{W}$ *with edge* $M' \subset M$ is a set of the form $\mathcal{W} = \{p + c : p \in M', c \in C\}$, where $C \subset \mathbb{C}^{m+n}$ is a truncated open cone with vertex in the origin. By a *wedgelike domain* $\mathcal{W}$ attached to M we mean a domain which contains for every point $p \in M$ a wedge with edge a neighborhood of p in M (*cf.* [MP1,2,3]).

A closed subset E of M is called *wedge removable* (briefly $\mathcal{W}$-removable) if for every wedgelike domain $\mathcal{W}_1$ attached to $M \backslash E$, there is a wedgelike domain $\mathcal{W}_2$ attached to M such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a holomorphic function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in some wedgelike open set $\mathcal{W}_3 \subset \mathcal{W}_1$ attached to $M \backslash E$. We say that E is L^1-*removable* if every

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locally integrable function f on M which is CR on $M\backslash E$ is CR on all of M (here, CR is understood in the distributional sense).

Let H^κ denote κ-dimensional Hausdorff measure, $\kappa \geq 0$. Our main result is :

Theorem 1.1. *Suppose M is $\mathcal{C}^{2,\alpha}$-smooth, $0 < \alpha < 1$. Then every closed subset E of M such that M and $M\backslash E$ are globally minimal and such that $H^{2m+n-2}(E) = 0$ is $\mathcal{W}$- and L^1-removable.*

(We shall say sometimes that E is of codimension 2^{+0} in M.) The hypersurface case of this statement follows from works of Lupacciolu, Stout, Chirka and others, with weaker regularity assumptions, M being $\mathcal{C}^2$-smooth, $\mathcal{C}^1$-smooth or even a Lipschitz graph (*see* [LS], [CS]), so Theorem 1.1 is new essentially in codimension $n \geq 2$. Recently, many geometrical removability results have been established in case the singularity E is a submanifold (*see* [St], [LS], [CS], [J2,3], [P1], [MP1,2,3], [JS], [P2], [MP4]) and Theorem 1.1 appears to answer one of the last open general questions in the subject (*see* also [J3], [MP4] for related open problems). As a rule L^1-removability follows once $\mathcal{W}$-removability being established (*see* especially Proposition 2.11 in [MP1]). In the case at hand we have already proved L^1-removability by different methods earlier (Theorem 3.1 in [MP3]) and also $\mathcal{W}$-removability if M is real analytic (*see* [MP2, Theorem 5.1], with M being $\mathcal{C}^\omega$-smooth and $H^{2m+n-2}(E) = 0$).

For the special case where M is $\mathcal{C}^3$-smooth and Levi-nondegenerate (*i.e.* the convex hull of the image of the Levi-form has nonempty interior), Theorem 1.1 is due to Dinh and Sarkis [DS]. It is known that this assumption entails the dimensional inequality $m^2 \geq n$. Especially, in the case of CR dimension $m = 1$, the abovementioned authors recover only the known hypersurface case ($n = 1$). We also point out a general restriction: by assuming that M is Levi-nondegenerate, or more generally that it is of Bloom-Graham finite type at *every* point of M, one would not take account of propagation aspects for the regularity of CR functions. For instance, it is well known that wedge extendability may hold despite of large Levi-flat regions in manifolds M consisting of a single CR orbit (*cf.* [Trp], [Tu1,2], [J1], [M]). For the sake of generality, this is why we only assume that M and $M\backslash E$ are globally minimal in Theorem 1.1.

A straightforward application is as follows. First, by [Trp], [Tu1,2], [J1], [M, Theorem 3.4], as $M\backslash E$ is globally minimal, there is a wedgelike domain $\mathcal{W}_0$ attached to $M\backslash E$ to which every continuous CR function (resp. CR distribution) f on $M\backslash E$ extends as a holomorphic function with continuous (resp. distributional) boundary value f. Then Theorem 1.1 entails that there exists a wedge $\mathcal{W}$ attached to M such that every such f extend holomorphically as an $F \in \mathcal{O}(\mathcal{W})$. There is *a priori* no growth control of F up to E. However, as proved in [MP1, Proposition 2.11], in the case where f is an element of $L^1(M)$ which is CR on $M\backslash E$, some growth control of Hardy-spaces type can be achieved on F to show that it admits a boundary value $b(F)$ over M (including E) which is L^1 and CR on M. This is how one may deduce L^1-removability from $\mathcal{W}$-removability in Theorem 1.1.

We now indicate a second application of Theorem 1.1 to the extension of CR-meromorphic functions. This notion was introduced for hypersurfaces by Harvey and Lawson [HL] and for generic CR manifolds by Dinh and Sarkis. Let f be a CR-meromorphic function, namely: **1.** $f : \mathcal{D}_f \to P_1(\mathbb{C})$ is a $\mathcal{C}^1$-smooth mapping defined over a dense open subset $\mathcal{D}_f$ of M with values in the Riemann sphere;

2. The closure Γ_f of its graph in $\mathbb{C}^{m+n} \times P_1(\mathbb{C})$ defines an oriented scarred $\mathcal{C}^1$-smooth CR manifold of CR dimension m (*i.e.* CR outside a closed thin set) and
3. We assume that $d[\Gamma_f] = 0$ in the sense of currents (*see* [HL], [Sa], [DS], [MP2] for further definition). According to an observation of Sarkis based on a counting dimension argument, the indeterminacy set Σ_f of f is a closed subset of empty interior in a two-codimensional scarred submanifold of M and its scar set is always metrically thin : $H^{2m+n-2}(Sc(\Sigma_f)) = 0$. Moreover, outside Σ_f, f defines a CR current in some suitable projective chart, hence it enjoys all the extendability properties of an usual CR function or distribution. However, the complement $M \backslash \Sigma_f$ need not be globally minimal if M is, and it is easy to construct manifolds M and closed sets $E \subset M$ with $H^{2m-1}(E) < \infty$ ($m = \dim_{CR} M$) which perturb global minimality (*see* [MP1], p. 811). It is therefore natural to make the additional assumption that M is minimal (locally, in the sense of Tumanov) at *every* point, which seems to be the weakest assumption which insures that $M \backslash E$ is globally minimal for arbitrary closed sets $E \subset M$ (even with a bound on their Hausdorff dimension). Finally, under these circumstances, the set Σ_f will be $\mathcal{W}$-removable: for its regular part $Reg(\Sigma_f)$, this already follows from Theorem 4 (ii) in [MP1] and for its scar set $Sc(\Sigma_f)$, this follows from Theorem 1.1 above. The removability of Σ_f means that the envelope of holomorphy of every wedge $\mathcal{W}_1$ attached to $M \backslash \Sigma_f$ contains a wedge $\mathcal{W}_2$ attached to M. As envelopes of meromorphy and envelopes of holomorphy of domains in $\mathbb{C}^{m+n}$ coincide by a theorem of Ivashkovich ([I]), we conclude :

Theorem 1.2. *Suppose M is $\mathcal{C}^{2,\alpha}$-smooth and locally minimal at every point. Then there exists a wedgelike domain $\mathcal{W}$ attached to M to which all CR-meromorphic functions on M extend meromorphically.*

The remainder of the paper is devoted to the proof of Theorem 1.1. We combine the *local* and the *global* techniques of deformations of analytic discs, using in an essential way two important papers of Tumanov [Tu1] and of Globevnik [G1]. In Sections 2 and 3, we first set up a standard local situation (*cf.* [MP1,2,3]). These preliminaries provide the necessary background for an informal discussion of the techniques of deformations of analytic discs we have to introduce. After these motivating remarks, a detailed presentation of the main part of the proof is provided in Section 4 (*see* especially Main Lemma 4.3).

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2. Localization

The following section contains important preliminary steps for the proof of Theorem 1.1 (*cf.* [MP1,2,3]).

As in [CS, p.96], we shall proceed by contradiction, since this strategy simplifies the general reasonings in the large. Also, in Section 3 below, we shall explain how to reduce the question to the simpler case where the functions which we have to extend are even holomorphic near $M \backslash E$. Whereas such a strategy is carried out in detail in [MP1] (with minor variations), we shall for completeness recall the complete reasonings briefly here, in Sections 2 and 3.

Thus, we fix $\mathcal{W}_1$ attached to $M\backslash E$ and say that an open submanifold $M' \subset M$ containing $M\backslash E$ enjoys the *$\mathcal{W}$-extension property* if there is a wedgelike domain $\mathcal{W}'$ attached to M' and a wedgelike set $\mathcal{W}_1' \subset \mathcal{W}' \cap \mathcal{W}_1$ attached to $M\backslash E$ such that, for each function $f \in \mathcal{O}(\mathcal{W}_1)$, its restriction to $\mathcal{W}_1'$ extends holomorphically to $\mathcal{W}'$.

This notion can be localized as follows. Let $E' \subset E$ be an arbitrary closed subset of E. We shall say that a point $p' \in E'$ is (locally) *removable* (with respect to E') if for every wedgelike domain $\mathcal{W}_1$ attached to $M\backslash E'$, there exists a neighborhood U of p' in M and a wedgelike domain $\mathcal{W}_2$ attached to $(M\backslash E')\cup U$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a holomorphic function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in some wedgelike open set $\mathcal{W}_3 \subset \mathcal{W}_1$ attached to $M\backslash E'$.

Next, we define the following set of closed subsets of E:

$$\mathcal{E} := \{E' \subset E \text{ closed} \ ; \ M\backslash E' \text{ is globally minimal} \\ \text{and has the } \mathcal{W}\text{-extension property}\}.$$

Then the residual set

$$E_{\rm nr} := \bigcap_{E' \in \mathcal{E}} E'$$

is closed. Here, the letters “nr” abbreviate “non-removable”, since one expects *a priori* that no point of $E_{\rm nr}$ should be removable in the above sense. Notice that for any two sets $E_1', E_2' \in \mathcal{E}$, $M\backslash E_1'$ and $M\backslash E_2'$ consist of a single CR orbit and have nonempty intersection. Hence $(M\backslash E_1') \cup (M\backslash E_2')$ is globally minimal and it follows that $M\backslash E_{\rm nr}$ is globally minimal.

Using Ayrapetian’s version of the edge of the wedge theorem (*see* also [Tu1, Theorem 1.2]), the different wedgelike domains attached to the sets $M\backslash E'$ can be glued (after appropriate contraction of their cone) to a wedgelike domain $\mathcal{W}_1$ attached to $M\backslash E_{\rm nr}$ in such a way that $M\backslash E_{\rm nr}$ enjoys the $\mathcal{W}$-extension property. Clearly, to establish Theorem 1.1, it is enough to show that $E_{\rm nr} = \emptyset$.

Let us argue indirectly (by contradiction) and assume that $E_{\rm nr} \neq \emptyset$. With respect to the ordering of sets by the inclusion relation, $E_{\rm nr}$ is then the minimal non-removable subset of E. In order to derive a contradiction to the minimality of $E_{\rm nr}$, it suffices therefore to remove one single point $p \in E_{\rm nr}$. More precisely one has to look for a neighborhood U_p of p such that $U_p \cup (M\backslash E_{\rm nr})$ is globally minimal and has the $\mathcal{W}$-extension property.

In order to achieve the first required property, it is very convenient to choose the point p such that locally the singularity $E_{\rm nr}$ lies behind a “wall” through p. More precisely we shall construct a generic hypersurface $M_1 \subset M$ containing p such that a neighborhood V of p in M writes as the disjoint union $M^+ \cup M^- \cup M_1$ of connected sets, where $M^\pm$ are two open “sides”, and the inclusion $E_{\rm nr} \cap V \subset M^- \cup \{p\}$ holds true. Since M_1 is a generic CR manifold, there is a CR vector field X on M defined in a neighborhood of p which is transverse to M_1. By integrating X, one easily finds a basis of neighborhoods U of p in M such that $U \cup (M\backslash E_{\rm nr})$ is globally minimal. Hence it remains to establish the $\mathcal{W}$-extension property at p, which is the main task.

For sake of completeness, we recall from [MP1] how to construct the generic wall M_1.

Lemma 2.1. *There is a point $p_1 \in E_{\rm nr}$ and a $\mathcal{C}^{2,\alpha}$-smooth generic hypersurface $M_1 \subset M$ passing through p_1 so that $E_{\rm nr}\backslash\{p_1\}$ lies near p_1 on one side of M_1* (see FIGURE 1).

Proof. Let $p \in E_{\rm nr} \neq \emptyset$ be an arbitrary point and let γ be a piecewise differentiable CR-curve linking p with a point $q \in M\backslash E_{\rm nr}$ (such a γ exists because M and $M\backslash E_{\rm nr}$ are globally minimal by assumption). After shortening γ, we may suppose that $\{p\} = E_{\rm nr} \cap \gamma$ and that γ is a smoothly embedded segment. Therefore γ can be described as a part of an integral curve of some nonvanishing $\mathcal{C}^{1,\alpha}$-smooth CR vector field (section of T^cM) L defined in a neighborhood of p.

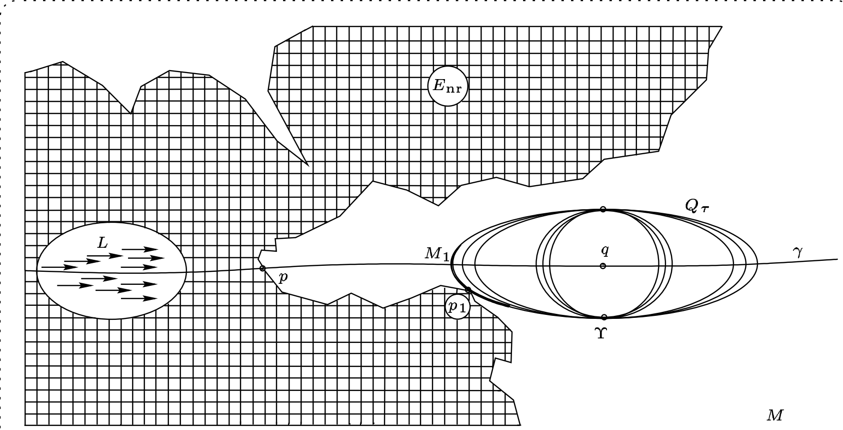

FIGURE 1: CONSTRUCTION OF M_1

Let $H \subset M$ be a small $(\dim M - 1)$-dimensional hypersurface of class $\mathcal{C}^{2,\alpha}$ passing through p and transverse to L. Integrating L with initial values in H we obtain $\mathcal{C}^{1,\alpha}$-smooth coordinates $(t,s) \in \mathbb{R} \times \mathbb{R}^{\dim M-1}$ so that for fixed s_0, the segments (t,s_0) are contained in the trajectories of L. After a translation, we may assume that $(0,0)$ corresponds to a point of γ close to p which is not contained in $E_{\rm nr}$, again denoted by q. Fix a small $\varepsilon > 0$ and for real $\tau \geq 1$, define the ellipsoids (*see* FIGURE 1 above)

$$Q_\tau := \{(t,s) : |t|^2/\tau + |s|^2 < \varepsilon\}.$$

There is a minimal $\tau_1 > 1$ with $\overline{Q_{\tau_1}} \cap E_{\rm nr} \neq \emptyset$. Then $\overline{Q_{\tau_1}} \cap E_{\rm nr} = \partial Q_{\tau_1} \cap E_{\rm nr}$ and $Q_{\tau_1} \cap E_{\rm nr} = \emptyset$. Observe that every ∂Q_τ is transverse to the trajectories of L out off the equatorial set $\Upsilon := \{(0,s) : |s|^2 = \varepsilon\}$ which is contained in $M\backslash E_{\rm nr}$. Hence ∂Q_{τ_1} is transverse to L in all points of $\partial Q_{\tau_1} \cap E_{\rm nr}$. So $\partial Q_{\tau_1}\backslash\Upsilon$ is generic in $\mathbb{C}^{m+n}$, since L is a CR field.

We could for instance choose a point $p_1 \in \partial Q_{\tau_1} \cap E_{\rm nr}$ and take for M_1 a neighborhood of p_1 in ∂Q_{τ_1}, but such an M_1 would be *only* of class $\mathcal{C}^{1,\alpha}$ and we want $\mathcal{C}^{2,\alpha}$-smoothness.

Therefore we fix a small $\delta > 0$ and approximate the family $\partial Q_\tau, 1 \leq \tau < \tau_1 + \delta$, by a nearby family of $\mathcal{C}^{2,\alpha}$-smooth hypersurfaces $\partial\widetilde{Q}_\tau, 1 \leq \tau < \tau_1 + \delta$. Clearly this can be done so that the $\partial\widetilde{Q}_\tau$ are still boundaries of increasing domains $\widetilde{Q}_\tau$

of approximately the same size as Q_τ and so that the points where the $\partial\widetilde{Q}_\tau$ are tangent to L are also contained in $M\backslash E_{\rm nr}$ near the equator Υ of Q_τ.

The same reasoning as above shows that there exist a real number $\widetilde{\tau}_1 > 1$, a point $p_1 \in E_{\rm nr}$ and a generic hypersurface M_0 passing through p_1 (which is a piece of $\partial\widetilde{Q}_{\widetilde{\tau}_1}$) such that $E_{\rm nr}$ lies in the left closed side $M_0^- \cup M_0$ in a neighborhood of p_1 (*see* FIGURE 1). We want more: $E_{\rm nr}\backslash\{p_1\} \subset M_1^-$. To achieve this last condition, it suffices to choose a $\mathcal{C}^{2,\alpha}$-smooth hypersurface M_1 passing through p_1 with $T_{p_1}M_0 = T_{p_1}M_1$ such that $M_1\backslash\{p_1\}$ is contained in M_0^+. □

3. Analytic discs

Let p_1 be as in Lemma 2.1. First, we can choose coordinates vanishing at p_1 and represent M near p_1 by the vectorial equation

$$x = h(w,y), \qquad w \in \mathbb{C}^m, \quad z = x + iy \in \mathbb{C}^n, \tag{3.1}$$

where $h = (h_1, \ldots, h_n)$ is of class $\mathcal{C}^{2,\alpha}$ and satisfies $h_j(0) = 0$ and $dh_j(0) = 0$.

Let us recall some generalities (*see* [Bo] for background). Denote by Δ the open unit disc in $\mathbb{C}$. An *analytic disc* attached to M is a holomorphic mapping $A : \Delta \to \mathbb{C}^N$ which extends continuously (or $\mathcal{C}^{k,\alpha}$-smoothly) up to the boundary $\partial\Delta$ and fulfills $A(\partial\Delta) \subset M$.

Discs of small size (for example with respect to the $\mathcal{C}^{2,\alpha}$-norm, $0 < \alpha < 1$) which are attached to M are then obtained as the solutions of the (modified) Bishop equation

$$Y = T_1[h(W(\cdot), Y(\cdot))] + y_0, \tag{3.2}$$

where T_1 denotes the harmonic conjugate operator (Hilbert transform on $\partial\Delta$) normalized at $\zeta = 1$, namely satisfying $T_1u(1) = 0$ for any $u \in \mathcal{C}^{2,\alpha}(b\Delta, \mathbb{R}^n)$. One verifies that every small $\mathcal{C}^{2,\alpha}$-smooth disc $A(\zeta) = (W(\zeta), Z(\zeta)) = (W(\zeta), X(\zeta) + iY(\zeta))$ attached to M satisfies (3.2). Conversely, for $W(\zeta)$ of small $\mathcal{C}^{2,\alpha}$-norm, equation (3.2) possesses a unique solution $Y(\zeta)$, and one easily checks that $A(\zeta) := (W(\zeta), h(W(\zeta), Y(\zeta)) + iY(\zeta))$ is then the unique disc attached to M with $Y(1) = y_0$ and w-component equal to $W(\zeta)$. According to an optimal analysis of the regularity of Bishop's equation due to Tumanov [Tu2] (and valid more generally in the classes $\mathcal{C}^{k,\alpha}$ for $k \geq 1$ and $0 < \alpha < 1$), $Y(\zeta)$ and then $A(\zeta)$ are of class $\mathcal{C}^{2,\alpha}$ over $\overline{\Delta}$.

After a linear transformation we can assume that the tangent space to M_1 is given by $\{x = 0,\ u_1 = 0\}$ and that T_0M^+ is given by $\{u_1 > 0\}$ near the origin. Let $\rho_0 > 0$ be small and let A be the analytic disc we obtain by solving

$$Y = T_1h[(W(\cdot), Y(\cdot))], \quad \text{with} \quad W(\zeta) := (\rho_0 - \rho_0\zeta, 0, \ldots, 0). \tag{3.3}$$

Notice that the disc $W_1(\zeta) := (\rho_0 - \rho_0\zeta)$ satisfies $W_1(1) = 0$ and $W_1(\overline{\Delta}\backslash\{1\}) \subset \{u_1 + iv_1 \in \mathbb{C} : u_1 > 0\}$. Elementary properties of Bishop's equation yield $A(\partial\Delta)\backslash\{1\} \subset M^+$ if $\rho_0 > 0$ is sufficiently small (*cf.* [MP1, Lemma 2.4]). FIGURE 2 below is devoted to provide a geometric intuition of the relative situation of the boundary of the disc A with respect to M_1.

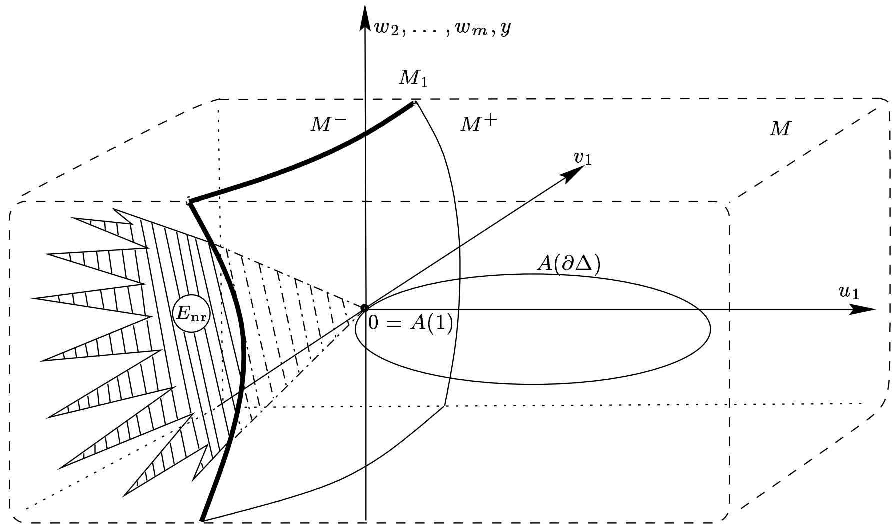

FIGURE 2: RELATIVE DISPOSITION OF $E_{\rm nr}$, M_1 AND $A(\partial\Delta)$ INSIDE M

At first, we explain how one usually constructs small wedges attached to M at p_1 by means of *deformations of analytic discs* and then in Sections 4, 5 and 6 below, we shall explain some of the modifications which are needed in the presence of a singularity $E_{\rm nr}$ in order to produce wedge extension at p_1. Following [MP3, pp. 863–864], we shall include (or say, "deform") A in a parametrized family $A_{\rho,s,v}$ with varying radius ρ plus supplementary parameters s, v and with $A_{\rho_0,0,0} = A$. During the construction, we shall sometimes permit ourselves to decrease parameters, related constants, neighborhoods and domains of existence without explicit mentioning. At present, our goal is to explain how we can add some conveninent extra simplifying assumptions to the hypotheses of Theorem 1.1, *see* especially conditions **1)**, **2)** and **3)** before Theorem 3.1 below.

Let $\mathcal{W}_1$ be the wedgelike domain attached to $M \backslash E_{\rm nr}$ constructed in Section 2 and let $f \in \mathcal{O}(\mathcal{W}_1)$. We want to extend f holomorphically to a wedge of edge a small neighborhood of the special point $p_1 \in E_{\rm nr}$ picked thanks to Lemma 2.1. Let $\mathcal{W}_2 \subset \mathcal{W}_1$ be a small wedge attached to a neighborhood of $A(-1)$ in M^+. As in [Tu1,2], [MP1,3], we can construct analytic discs $A_{\rho,s,v} = (W_{\rho,s,v}, Z_{\rho,s,v})$ attached to $M \cup \mathcal{W}_2$ with the following properties:

(1) The parameters s, v belong to neighborhoods U_s, U_v of 0 in $\mathbb{R}^{2m+n-1}$, $\mathbb{R}^{n-1}$ respectively and ρ belongs to the interval $[0, \rho_1)$, for some $\rho_1 > \rho_0$.

(2) The mapping $(\rho, s, v) \mapsto A_{\rho,s,v}$ is of class $\mathcal{C}^{2,\beta}$ for all $0 < \beta < \alpha$. For $\rho \neq 0$, these maps are embeddings of $\overline{\Delta}$ into $\mathbb{C}^{m+n}$. Finally, we have $A_{\rho_0,0,0} = A$ and the discs $A_{0,s,v}$ are constant.

(3) For every fixed $v_0 \in U_v$, the union $\bigcup_{s \in U_s} A_{\rho_0,s,v_0}(\{e^{i\theta} : |\theta| < \pi/4\})$ is an open subset of M containing the origin which is $\mathcal{C}^{2,\beta}$-smoothly foliated by the curves $A_{\rho_0,s,v_0}(\{e^{i\theta} : |\theta| < \pi/4\})$.

(4) The mapping $U_v \ni v \mapsto [\frac{d}{d\theta} A_{\rho_0,0,v}(e^{i\theta})]_{\theta=0} \in T_0M/T_0^cM \simeq \mathbb{R}^n$ has rank $n-1$ and its image is transverse to the vector $[\frac{d}{d\theta} A(e^{i\theta})]_{\theta=0} \in T_0M/T_0^cM \simeq$

$\mathbb{R}^n$. In geometric terms, this property means that the union of tangent real lines

$$\mathbb{R}\left[\frac{d}{dr}A_{\rho_0,0,v}(re^{i\theta})\right]_{\zeta=1} = -i\,\mathbb{R}\left[\frac{d}{d\theta}A_{\rho_0,0,v}(e^{i\theta})\right]_{\theta=0}$$

spans an open cone in the normal bundle to M, namely $T_0\mathbb{C}^{m+n}/T_0M \cong i(T_0M/T_0^cM)$.

(5) Let $\omega = \{\zeta \in \Delta : |\zeta - 1| < \delta\}$ be a neighborhood of 1 in Δ, with some small $\delta > 0$. It follows from properties **(3)** and **(4)** that the union $\mathcal{W} = \bigcup_{s\in U_s, v\in U_v} A_{\rho_0,s,v}(\omega)$ is an open wedge of edge a neighborhood of the origin in M which is foliated by the discs $A_{\rho_0,s,v}(\omega)$.

(6) The sets $D_{s,v} = \bigcup_{0\leq\rho<\rho_1, |\zeta|=1} A_{\rho,s,v}(\zeta)$ are *real* two-dimensional discs of class $\mathcal{C}^{2,\beta}$ embedded in M which are foliated (with a circle degenerating to a point for $\rho = 0$) by the circles $A_{\rho,s,v}(\partial\Delta)$.

(7) There exists a $(2m+n-2)$-dimensional submanifold H of $\mathbb{R}^{2m+n-1}$ passing through the origin such that for every fixed $v_0 \in U_v$, the union $\bigcup_{s\in H} D_{s,v_0}$ is a (dim M)-dimensional open box foliated by real 2-discs which is contained in M and which contains the origin. Intruitively, it is a stack of plates.

Let us make some commentaries. We stress that the family $A_{\rho,s,v}$ is obtained by solving the Bishop equation for explicitly prescribed data (*see* [MP3, p. 837] or [MP1, p. 863]; the important Lemma 2.7 in [MP1] which produces the parameter v satisfying **(4)** above is due to Tumanov [Tu1]). Since Bishop's equation is very flexible, this entails that every geometrical property of the family is stable under slight perturbation of the data. Notice for instance that as A is an embedding of $\overline{\Delta}$ into $\mathbb{C}^{m+n}$, all its small deformations will stay embeddings. In particular we get a likewise family $A^d_{\rho,s,v}$ if we replace M by a slightly deformed $\mathcal{C}^{2,\alpha}$-smooth manifold M^d (this corresponds to replacing h by a function h^d close to h in $\mathcal{C}^{2,\alpha}$-norm in (3.1), (3.2) and (3.3)).

Further remark. If A' is an arbitrary disc which is sufficiently close to A in $\mathcal{C}^{1,\beta}$-norm, for some $0 < \beta < \alpha$, we can also include A' in a similar $\mathcal{C}^{1,\gamma}$-smooth $(0 < \gamma < \beta)$ family $A'_{\rho,s}$, *without the parameter* v, which satisfies the geometric properties **(3)**, **(6)** and **(7)** above. This remark will be useful in the end of Section 4 below.

Using such a nice family $A_{\rho,s,v}$ which gently deforms as a family $A^d_{\rho,s,v}$ under perturbations, let us begin to remind from [MP1] how we can add three simplifying geometric assumptions to Theorem 1.1, without loss of generality.

First of all, using apartition of unity, we can perform arbitrarily small $\mathcal{C}^{2,\alpha}$-smooth deformations M^d of M leaving E_{nr} fixed and moving $M\backslash E_{\text{nr}}$ inside the wedgelike domain $\mathcal{W}_1$. Further, we can make M^d to depend on a single small real parameter $d \geq 0$ with $M^0 = M$ and $M^d\backslash E_{\text{nr}} \subset \mathcal{W}_1$ for all $d > 0$. Now, *the wedgelike domain* $\mathcal{W}_1$ *becomes a neighborhood of* M^d *in* $\mathbb{C}^{m+n}$. In the sequel, we shall denote this neighborhood by Ω. By stability of Bishop's equation, we obtain a deformed disc A^d attached to M^d by solving (3.3) with h^d in place of h. In the sequel, we will also consider a small neighborhood Ω_1 of $A^d(-1)$ in $\mathbb{C}^{m+n}$ which contains the intersection of the above wedge $\mathcal{W}_2$ with a neighborhood of $A(-1)$ in $\mathbb{C}^{m+n}$.

Again by stability of Bishop's equation, we also obtain deformed families $A^d_{\rho,s,v}$ attached to $M^d \cup \Omega_1$, satisfying properties **(1)-(7)** above. Recall that according to [Tu2], the mapping $(\rho, s, v, d) \mapsto A^d_{\rho,s,v}$ is $\mathcal{C}^{2,\beta}$-smooth for all $0 < \beta < \alpha$. In the core of the proof of our main Theorem 1.1 (Sections 4, 5 and 6 below), we will show that, for each sufficiently small fixed $d > 0$, we get holomorphic extension to the wedgelike set $\mathcal{W}^d = \bigcup_{s \in U_s, v \in U_v} A^d_{\rho_0,s,v}(\omega)$ attached to a neighborhood of 0 in M^d. But this implies Theorem 1.1: In the limit $d \to 0$, the wedges $\mathcal{W}^d$ tend smoothly to the wedge $\mathcal{W} := \mathcal{W}^0$ attached to a neighborhood of 0 in $M^0 = M$. As the construction depends smoothly on the deformations d, we derive univalent holomorphic extension to $\mathcal{W}$ thereby arriving at a contradiction to the definition of $E_{\rm nr}$.

As a summary of the above discussion, we formulate below the local statement that remains to prove. Essentially, we have shown that it suffices to prove Theorem 1.1 with the following three extra simplifying assumptions:

1) Instead of functions which are holomorphic in a wedgelike open set attached to $M \backslash E_{\rm nr}$, we consider functions which are holomorphic in a neighborhood of $M \backslash E_{\rm nr}$ in $\mathbb{C}^{m+n}$.

2) Proceeding by contradiction, we have argued that it suffices to remove at least one point of $E_{\rm nr}$.

3) Moreover, we can assume that the point $p_1 \in E_{\rm nr}$ we want to remove is behind a generic "wall" M_1 as depicted in FIGURE 2.

Consequently, from now on, we shall denote the set $E_{\rm nr}$ simply by E. We also denote M^d simply by M. We take again the disc A defined by (3.3) and its deformation $A_{\rho,s,v}$. The goal is now to show that holomorphic functions in a neighborhood of $M \backslash E$ in $\mathbb{C}^{m+n}$ extend holomorphically to a wedge at p_1, assuming the "nice" geometric situation of FIGURE 2. To be precise, we have argued that Theorem 1.1 is reduced to the following precise and geometrically more concrete statement.

Theorem 3.1. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic CR manifold in $\mathbb{C}^{m+n}$ of codimension n. Let $M_1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth generic CR manifold of dimension $2m+n-1$ and let $p_1 \in M_1$. Let M^+ and M^- denote the two local open sets in which M is divided by M_1, in a neighborhood of p_1. Suppose that $E \subset M$ is a* nonempty *closed subset with $p_1 \in E$ satisfying the Hausdorff condition $H^{2m+n-2}(E) = 0$ and suppose that $E \subset M^- \cup \{p_1\}$* (FIGURE 2)*. Let Ω be a neighborhood of $M \backslash E$ in $\mathbb{C}^{m+n}$, let A be the disc defined by (3.3), let Ω_1 be a neighborhood of $A(-1)$ in $\mathbb{C}^{m+n}$ which is contained in Ω and let $A_{\rho,s,v}$ be a family of discs attached to $M \cup \Omega_1$ with the properties* **(1)-(7)** *explained above. Then every function f which is holomorphic in Ω extends holomorphically to the wedge $\mathcal{W} = \bigcup_{s \in U_s, v \in U_v} A_{\rho_0,s,v}(\omega)$.*

Of course, Theorem 3.1 would be obvious if E would be empty, but we have to take account of E.

4. Proof of Theorem 3.1, part I

This section contains the part of the proof of Theorem 3.1 above which relies on constructions with the small discs $A_{\rho,s,v}$ attached to $M \cup \Omega$. Since we want the boundaries of our discs to avoid E, we shall employ the following elementary lemma several times, which is simply a convenient particularization of a general property of Hausdorff measures [C, Appendix A6].

Lemma 4.1. *Let N be a real d-dimensional manifold and let $E \subset N$ be a closed subset. Let U be a small neighborhood of the origin in $\mathbb{R}^{d-1}$ and let $\Phi : \partial\Delta \times U \to N$ (resp. $\Psi : (0,1) \times U \to M$) be an embedding.*

- **(i)** *If $H^{d-2}(E) = 0$, then the set of $x \in U$ for which $\Phi(\partial\Delta \times \{x\}) \cap E$ is nonempty (resp. $\Psi((0,1) \times \{x\}) \cap E \neq \emptyset$) is of zero $(d-2)$-dimensional Hausdorff measure.*
- **(ii)** *If $H^{d-1}(E) = 0$, then for almost every $x \in U$ in the sense of Lebesgue measure, we have $\Phi(\partial\Delta \times \{x\}) \cap E = \emptyset$ (resp. $\Psi((0,1) \times \{x\}) \cap E = \emptyset$).*

Proof of Theorem 3.1: We divide the proof in five steps.

Step 1: Holomorphic extension to a dense subset of $\mathcal{W}$. We shall start by constructing a holomorphic extension to an everywhere dense open subdomain of the wedge $\mathcal{W} = \bigcup_{s \in U_s, v \in U_v} A_{\rho_0, s, v}(\omega)$ by means of the disc technique (continuity principle).

For each fixed $v_0 \in U_v$, the first dimensional count of Lemma 4.1 (which applies by the foliation property **(3)** of the discs) yields a closed subset $\mathcal{S}_{v_0} \subset U_s$ depending on v_0 and satisfying $H^{2m+n-2}(\mathcal{S}_{v_0}) = 0$ such that for every $s \notin \mathcal{S}_{v_0}$ we have $A_{\rho_0, s, v_0}(\partial\Delta) \cap E = \emptyset$. Notice also that $\mathcal{S}_{v_0}$ does not locally disconnect U_s, for dimensional reasons ([C, Appendix A6]).

By property **(7)** of Section 3, the real two-dimensional discs D_{s,v_0} foliate an open subset of M, for s running in a manifold H of dimension $2m+n-2$. Consequently, for almost every $s \in H$, (in the sense of Lebesgue measure), we have $D_{s,v_0} \cap E = \emptyset$.

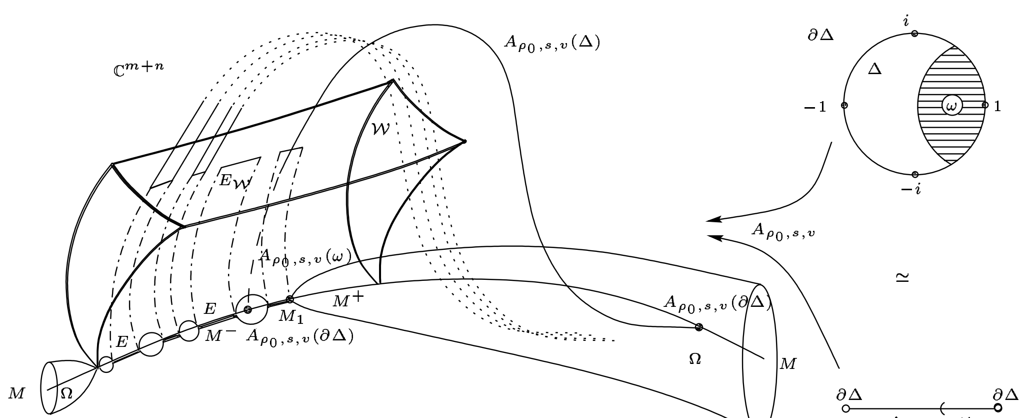

FIGURE 3: DESCRIPTION OF $E_{\mathcal{W}}$:= UNION OF ANALYTIC DISCS $A_{\rho_0,s,v}(\Delta)$ WHOSE BOUNDARY $A_{\rho_0,s,v}(\partial\Delta)$ MEETS THE SINGULARITY E

Since E is closed, we claim that for every $s \notin \mathcal{S}_{v_0}$, it follows that we can contract every boundary $A_{\rho_0,s,v_0}(\partial\Delta)$ which does not meet E, to a point in M without meeting E by an analytic isotopy (*cf.* [MP3, p. 864]). Indeed, by shifting s to some nearby s', we first move A_{ρ_0,s,v_0} into a disc A_{ρ_0,s',v_0} which also satisfies $A_{\rho_0,s',v_0}(\partial\Delta) \cap E = \emptyset$. Choosing well s', this boundary belongs to a real disc D_{s',v_0} satisfying $D_{s',v_0} \cap E = \emptyset$. This can be achieved with s' arbitrarily close to s, since $\mathcal{S}_{v_0}$ does not disconnect U_s. Then we contract in the obvious manner the disc A_{ρ_0,s',v_0} to the point $A_{0,s',v_0}(\overline{\Delta})$ by isotoping its boundary inside D_{s',v_0} (recall that D_{s',v_0} is a union of boundary of discs). Applying the continuity principle to this analytic isotopy of discs, we see that we can extend every function $f \in \mathcal{O}(\Omega)$ holomorphically to a neighborhood of $A_{\rho_0,s,v_0}(\overline{\Delta})$ in $\mathbb{C}^{m+n}$, for every $s \notin \mathcal{S}_{v_0}$ and for every $v_0 \in U_v$.

From the nice geometry **(5)** of the family $A_{\rho,s,v}$ one easily derives that the various local extensions near $A_{\rho_0,s,v_0}(\omega)$ for $s \notin \mathcal{S}_{v_0}$ fit in a univalent function $F \in \mathcal{O}(\mathcal{W}\backslash E_{\mathcal{W}})$, where $E_{\mathcal{W}} := \bigcup_{s\in\mathcal{S}_{v_0}, v_0\in U_v} A_{\rho_0,s,v_0}(\omega)$. Furthermore we observe that $E_{\mathcal{W}}$ is laminated by holomorphic discs and satisfies $H^{2m+2n-1}(E_{\mathcal{W}}) = 0$. This metrical property implies that $\mathcal{W}\backslash E_{\mathcal{W}}$ is locally connected. The remainder of the proof is devoted to show how to extend F through $E_{\mathcal{W}}$. This occupies the paper up to its end. The difficulty and the length of the proof comes from the fact that the disc method necessarily increases by a factor 1 the dimension of the singularity: it transforms a singularity set $E \subset M$ of codimension 2^{+0} into a bigger singularity set $E_{\mathcal{W}} \subset \mathcal{W}$ which is of codimension 1^{+0}.

Step 2: Plan for the removal of $E_{\mathcal{W}}$. Let us remember that our goal is to show that p_1 is $\mathcal{W}$-removable in order to achieve the final step in our reasoning by contradiction which begins in Section 2. To show that p_1 is removable, it suffices to extend F through $E_{\mathcal{W}}$. At first, we notice that because $H^{2m+2n-1}(E_{\mathcal{W}}) = 0$, it follows that $\mathcal{W}\backslash E_{\mathcal{W}}$ is locally connected, so the part of the envelope of holomorphy of $\mathcal{W}\backslash E_{\mathcal{W}}$ which is contained in $\mathcal{W}$ is not multisheeted: it is necessarily a subdomain of $\mathcal{W}$. In analogy with the beginning of Section 2, let us therefore denote by $E_{\mathcal{W}}^{\rm nr}$ the set of points of $E_{\mathcal{W}}$ through which our holomorphic function $F \in \mathcal{O}(\mathcal{W}\backslash E_{\mathcal{W}})$ does not extend holomorphically. If $E_{\mathcal{W}}^{\rm nr}$ is empty, we are done, gratuitously. As it might certainly be nonempty, we shall suppose therefore that $E_{\mathcal{W}}^{\rm nr} \neq \emptyset$ and we shall construct a contradiction in the remainder of the paper. Let $q \in E_{\mathcal{W}}^{\rm nr} \neq \emptyset$. To derive a contradiction, it suffices to show that F extends holomorphically through q. Philosophically again, it will suffice to remove one single point, which will simplify the presentation and the geometric reasonings. Finally, as $E_{\mathcal{W}}^{\rm nr} \neq \emptyset$ is contained in $E_{\mathcal{W}}$, there exist a point $\zeta_0 \in \partial\Delta$ and parameters (ρ_0, s_0, v_0) such that $q = A_{\rho_0,s_0,v_0}(\zeta_0)$. In the sequel, we shall simply denote the disc A_{ρ_0,s_0,v_0} by $A_{\rm nr}$. Obviously also, $H^{2m+2n-1}(E_{\mathcal{W}}^{\rm nr}) = 0$.

Step 3: Smoothing the boundary of the singular disc $A_{\rm nr}$ near $\zeta = -1$. In step 4 below, our goal will be to deform $A_{\rm nr}$ to extend F through q. As we shall need to glue a maximally real submanifold R_1 of M along $A_{\rm nr}(\partial\Delta\backslash\{|\zeta+1| < \varepsilon\})$ to some collection of maximally real *planes* along $A_{\rm nr}(\zeta)$ for $\zeta \in \partial\Delta$ near -1, and because $\mathcal{C}^{2,\beta}$-smoothness of $A_{\rm nr}$ will not be sufficient to keep the $\mathcal{C}^{2,\beta}$-smoothness of the glued object, it is convenient to smooth out first $A_{\rm nr}$ near $\zeta = -1$ (*see* especially Step 2 of Section 6 below). Fortunately, we can use the freedom Ω_1 (the small neighborhood of $A(-1)$ in Theorem 3.1) to modify the boundary of $A_{\rm nr}$. Thus, *for technical reasons only*, we need the following preliminary lemma, which is simply obtained by reparametrizing an almost full subdisc of $A_{\rm nr}$. This preparatory reparametrization is indispensible to state our Main Lemma 4.3 below correctly.

Lemma 4.2. *For every $\varepsilon > 0$, there exists an analytic disc A' satisfying*

- **(a)** *A' is a $\mathcal{C}^{2,\beta}$-smooth subdisc of $A_{\rm nr}$, namely $A'(\overline{\Delta}) \subset A_{\rm nr}(\overline{\Delta})$, such that moreover $A'(\overline{\Delta}) \supset A_{\rm nr}(\overline{\Delta} \backslash \{|\zeta+1| < 2\varepsilon\})$.*
- **(b)** *A' is real analytic over $\{\zeta \in \partial\Delta : |\zeta + 1| < \varepsilon\}$.*
- **(c)** *$||A' - A_{\rm nr}||_{\mathcal{C}^{2,\beta}} \leq \varepsilon$.*
- **(d)** *$A'(\partial\Delta) \subset M \cup \Omega_1$.*

Proof. Of course, **(d)** follows immediately from **(a)** and **(c)** if ε is sufficiently small. To construct A', we consider a $\mathcal{C}^\infty$-smooth cut-off function $\mu_\varepsilon : \partial\Delta \to [0,1]$ with $\mu_\varepsilon(\zeta) = 1$ for $|\zeta + 1| > 2\varepsilon$ and $\mu_\varepsilon(\zeta)$ equal to a constant $c_\varepsilon < 1$ with $c_\varepsilon > 1 - \varepsilon$ for $|\zeta + 1| < \varepsilon$. Let Δ_{μ_ε} be the (almost full) subdisc of Δ defined by $\{\zeta \in \Delta : |\zeta| < \mu_\varepsilon(\zeta/|\zeta|)\}$. Let ψ_ε be the Riemann conformal map $\Delta \to \Delta_{\mu_\varepsilon}$. We can assume that $\psi_\varepsilon(-1) = -c_\varepsilon \in \partial\Delta_{\mu_\varepsilon} \cap \mathbb{R}$. By Caratheodory's theorem and by the Schwarz symmetry principle, ψ_ε is $\mathcal{C}^\infty$-smooth up to the boundary and real analytic near $\zeta = -1$. If ε is sufficiently small and c_ε sufficiently close to 1, the stability of Riemann's uniformization theorem under small $\mathcal{C}^\infty$-smooth perturbations shows that the disc

$$A'(\zeta) := A_{\rm nr}(\psi_\varepsilon(\zeta)).$$

satisfies the desired properties, possibly with a slightly different small ε. □

Step 4: Variation of the singular disc. In the sequel, we shall constantly denote the disc of Lemma 4.2 by A'. We set $\zeta_q := \psi_\varepsilon^{-1}(\zeta_0)$, so that $A'(\zeta_q) = q$. Of course, after a reparametrization by a Blaschke transformation, we can (and we will) assume that $\zeta_q = 0$. By construction, $A'|_{\partial\Delta}$ is real analytic near -1 and the point $q = A'(0)$ is contained $E_{\mathcal{W}}^{\rm nr}$, the set through which our partial extension F does not extend *a priori*. To derive a contradiction, our next purpose is to produce a disc A'' close to A' and *passing through the fixed point* q such that q can be encircled by a small closed curve in $A''(\Delta) \backslash E_{\mathcal{W}}^{\rm nr}$, because in such a situation, we will be able to apply the continuity principle as in the typical local situation of Hartog's theorem (*see* **(4)** of Lemma 4.3 and Step 5 below).

At first glance it seems that we can produce A'' simply by turning A' a little around q: indeed, Lemma 4.1 applies, since $H^{2m+2n-1}(E_{\mathcal{W}}^{\rm nr}) = 0$. However, the difficult point is to guarantee that A'' is still attached to the union of M with the small neighborhood Ω_1 of $A(-1)$ in $\mathbb{C}^{m+n}$. The following key lemma asserts that these additional requirements can be fulfilled.

Main Lemma 4.3. *Let A' be the disc of Lemma 4.1, let $q = A'(0) \in E_{\mathcal{W}}^{\rm nr}$ and let $0 < \beta < \alpha$ be arbitrarily close to α. Then there exists a parameterized family $A'_{t'}$ of analytic discs with the following properties:*

- **(1)** *The parameter t' ranges in a neighborhood $U_{t'}$ of 0 in $\mathbb{R}^{2m+2n-1}$ and $A'_0 = A'$.*
- **(2)** *The mapping $U_{t'} \times \overline{\Delta} \ni (t', \zeta) \mapsto A'_{t'}(\zeta) \in \mathbb{C}^{m+n}$ is of class $\mathcal{C}^{1,\beta}$ and each $A'_{t'}$ is an embedding of $\overline{\Delta}$ into $\mathbb{C}^{m+n}$.*
- **(3)** *For all $t' \in U_{t'}$, the point $q = A'_{t'}(0)$ is fixed and $A'_{t'}(\partial\Delta) \subset M \cup \Omega_1$. Furthermore, there exists a small $\delta > 0$ such that the large boundary part $A'_{t'}(\partial\Delta \backslash \{|\zeta + 1| < \delta\})$ is attached to a fixed maximally real $(m + n)$-dimensional $\mathcal{C}^{2,\alpha}$-smooth submanifold R_1 of M.*

(4) *For every fixed $\rho_\varepsilon > 0$ which is sufficiently small and for t' ranging in a sufficiently small neighborhood of the origin, the union of circles*

$$\bigcup_{t'} \{A'_{t'}(\rho_\varepsilon e^{i\theta}) : \theta \in \mathbb{R}\}$$

foliates a neighborhood in $\mathbb{C}^{m+n}$ of the small fixed circle $\{A'(\rho_\varepsilon e^{i\theta}) : \theta \in \mathbb{R}\}$ which encircles the point q inside $A'(\Delta)$. Consequently, by Lemma 4.1, for almost all $t' \in U_{t'}$, the circle $\{A'_{t'}(\rho_\varepsilon e^{i\theta}) : \theta \in \mathbb{R}\}$ does not meet $E^{\rm nr}_{\mathcal{W}}$.

Let us make some explanatory commentaries. Notice that the discs are only $\mathcal{C}^{1,\beta}$-smooth, because the underlying method of Sections 5 and 6 (implicit function theorem in Banach spaces, *cf.* [G1]) imposes a real loss of smoothness. If we could have produce a $\mathcal{C}^{2,\beta}$-smooth family (assuming for instance that M was $\mathcal{C}^{3,\alpha}$-smooth from the beginning, or asking whether the regularity methods of [Tu2] are applicable to the *global* Bishop equation), we would have constructed a slightly different family and stated instead of **(4)** the following conic-like differential geometric property:

(4') The parameter t' ranges over a neighborhood $U_{t'}$ of the origin in $\mathbb{R}^{2m+2n-2}$ with $A'_{t'}(0) = q$ for all t' and the mapping

$$U_{t'} \ni t' \mapsto [\partial A'_{t'}/\partial\zeta](0) \in T_q\mathbb{C}^{m+n}$$

has maximal rank at $t' = 0$ with its image being transverse to the tangent space of $A'(\Delta)$ at q.

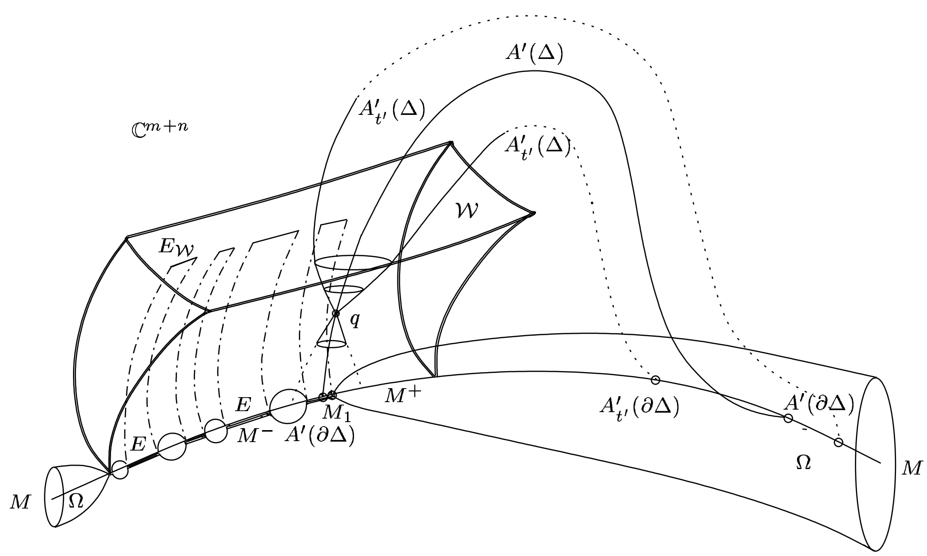

FIGURE 4: DEFORMATIONS OF A' BY SWEEPING OUT AN OPEN CONE WITH VERTEX q

In geometric terms, **(4')** tells that A' can be included in a family $A'_{t'}$ of discs passing through q which *sweeps out an open cone with vertex in q*. Using some basic differential geometric computations, the reader can easily check that the geometric property **(4')** implies **(4)** after adding one supplementary real parameter $t'_{2m+2n-1}$ corresponding to the radius $\rho = |\zeta|$ of the disc. Fortunately, for the needs of Step 5 below, the essential foliation property stated in **(4)** will be valuable with an only $\mathcal{C}^{1,\beta}$-smooth family and, as stated in the end of **(4)**, this family yields an appropriate disc $A'_{t'}$ with empty intersection with the singularity, namely $A'_{t'}(\{\rho_\varepsilon e^{i\theta} : \theta \in \mathbb{R}\}) \cap E^{\rm nr}_{\mathcal{W}} = \emptyset$. Using this Main Lemma 4.3, we can now accomplish the last step of the proof of Theorem 3.1.

Step 5: Removal of the point $q \in E^{\rm nr}_{\mathcal{W}}$. Let $A'_{t'}$ the family that we obtain by applying Main Lemma 4.3 to A'. According to the last sentence of Main Lemma 4.3, we may choose t' arbitrarily small and a positive radius $\rho_\varepsilon > 0$ sufficiently small so that the boundary of analytic subdisc $A'_{t'}(\{\rho_\varepsilon e^{i\theta} : \theta \in \mathbb{R}\})$ does not intersect $E^{\rm nr}_{\mathcal{W}}$. Let us denote such a disc $A'_{t'}$ simply by A'' in the sequel. Furthermore, we can assume that $A''(\{\rho_\varepsilon e^{i\theta} : \theta \in \mathbb{R}\})$ is contained in the small ball $B_\varepsilon := \{|z-q| \leq \varepsilon\}$ in which we shall localize an application of the continuity principle (*see* FIGURE 5). Thus, it remains essentially to check that F extends analytically to a neighborhood of q in $\mathbb{C}^{m+n}$ by constructing an analytic isotopy of A'' in $(\mathcal{W}\backslash E^{\rm nr}_{\mathcal{W}}) \cup \Omega$ and by applying the continuity principle.

One idea would be to translate a little bit in $\mathbb{C}^{m+n}$ the small disc $A''(\{\rho e^{i\theta} : \rho \leq \rho_\varepsilon, \theta \in \mathbb{R}\})$. However, there is *a priori* no reason for which such a small translated disc (which is of real dimension two) would avoid the singularity $E^{\rm nr}_{\mathcal{W}}$. Indeed, since we only know that $H^{2m+2n-1}(E^{\rm nr}_{\mathcal{W}}) = 0$, it is impossible in general that a two-dimensional manifold avoids such a "big" set of Hausdorff codimension 1^{+0}.

Of course, there is no surprise here: it is clear that functions which are holomorphic in the domain $\mathcal{W}\backslash E^{\rm nr}_{\mathcal{W}}$ do *not* extend automatically through a set with $H^{2m+2n-1}(E^{\rm nr}_{\mathcal{W}}) = 0$, since for instance, such a set $E^{\rm nr}_{\mathcal{W}}$ might contain infinitely many complex hypersurfaces, which are certainly not removable. So we really need to consider the whole disc A'' and to include it into another family of discs attached to $M \cup \Omega_1$ in order to produce an appropriate analytic isotopy.

The good idea is to include A'' in a family $A''_{\rho,s}$ similar to the one in Section 2 (with of course $A_{\rho_0,0} = A''$, but without the unnecessary parameter v), since we already know that for almost all $s \in U_s$, we can show as in Step 1 above that f (hence F too) extends holomorphically to a neighborhood of $A_{\rho_0,s}(\overline{\Delta})$ in $\mathbb{C}^{m+n}$.

To construct this family, we observe that A'' is not attached to M, but as A'' can be chosen arbitrarily close in $\mathcal{C}^{1,\beta}$-norm to the original disc A attached to M, it follows that A'' is certainly attached to some $\mathcal{C}^{1,\beta}$-smooth manifold M'' close to M which coincides with M except in a neighborhood of $A''(-1)$. Finally, the family $A''_{\rho,s}$ is constructed as in Section 2 (but without the parameter v, because in order to add the parameter v satisfying the second order condition **(4)** of Section 3, one would need $\mathcal{C}^{2,\beta}$-smoothness of the disc). By Tumanov's regularity theorem [Tu2], this family is again of class $\mathcal{C}^{1,\beta}$ for all $0 < \beta < \alpha$. Using properties **(3)** and **(6)** and reasoning as in Step 1 of this Section 4 (continuity principle), we deduce that the function f of Theorem 3.1 extends holomorphically to a neighborhood of $A_{\rho_0,s}(\overline{\Delta})$ in $\mathbb{C}^{m+n}$ for all $s \in U_s$, except those belonging to some closed thin set $\mathcal{S}$ with $H^{2m+n-2}(\mathcal{S}) = 0$. Since $\mathcal{S}$ does not locally disconnect U_s, such an extension necessarily coincides with the extension F in the intersection of their domains.

In summary, by using the family $A''_{\rho,s}$, we have shown that for almost all s, *the function F extends holomorphically to a neighborhood of* $A''_{\rho_0,s}(\overline{\Delta})$. We can therefore apply the continuity principle to *remove the point q*.

Indeed, we remind that $A'' = A''_{\rho_0,0}$ and that by construction the small boundary $A''_{\rho_0,0}(\{\rho_\varepsilon e^{i\theta} : \theta \in \mathbb{R}\})$ which encircles q does not intersect $E^{\rm nr}_{\mathcal{W}}$. It is now clear that the usual continuity principle along the family of small discs $A''_{\rho_0,s}(\{\rho e^{i\theta} : \rho < \rho_\varepsilon, \theta \in \mathbb{R}\})$ yields holomorphic extension of F at q (*see* again FIGURE 5).

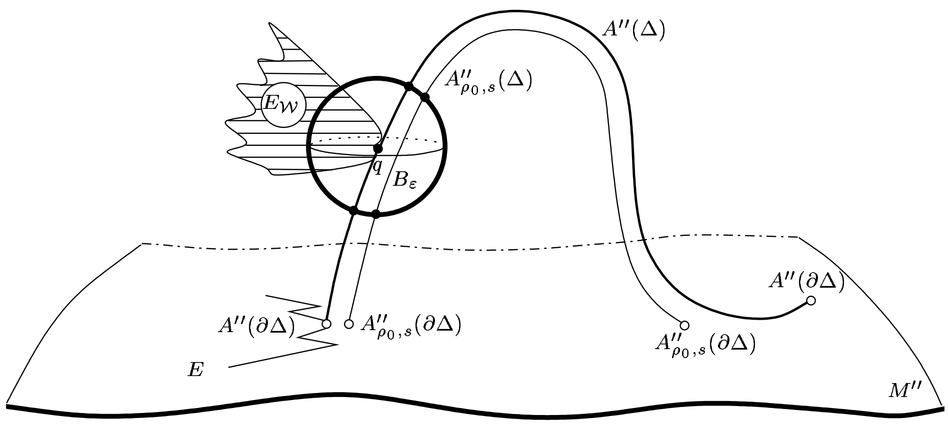

FIGURE 5: CONTINUITY PRINCIPLE IN B_ε

Finally, the proof of Theorem 3.1 is complete modulo the proof of Main Lemma 4.3, to which the remainder of the paper is devoted.

5. ANALYTIC DISCS ATTACHED TO MAXIMALLY REAL MANIFOLDS

A crucial ingredient of the proof of Theorem 3.1 is the description of a family of analytic discs which are close to the given disc A' of Main Lemma 4.3 and which are attached to a maximally real submanifold $R \subset M \cup \Omega_1$ (we shall construct such an R with $A'(\zeta) \in R$ for each $\zeta \in \partial\Delta$ in Section 6 below). This topic was developed by E. Bedford–B. Gaveau, F. Forstnerič in complex dimension two and generalized by J. Globevnik to higher dimensions. In this introductory section, we shall closely follow [G1,2].

We need the solution of the following more general distribution problem. Instead of a fixed maximally real submanifold R, we consider a smooth family $R(\zeta)$, $\zeta \in \partial\Delta$, of maximally real submanifolds of $\mathbb{C}^N$, $N \geq 2$, and we study the discs attached to this family which are close to an attached disc A' of reference, *i.e.* fulfilling $A'(\zeta) \in R(\zeta)$, $\forall\, \zeta \in \partial\Delta$. Let $\alpha > 0$ be as in Theorem 1.1 and let $0 < \beta < \alpha$ be arbitrarily close to α, as in Main Lemma 4.3.

Concretely, the manifolds $R(\zeta)$ are given by defining functions $r_j \in \mathcal{C}^{2,\beta}(\partial\Delta \times B, \mathbb{R})$, $j = 1, \ldots, N$, where $B \subset \mathbb{C}^N$ is a small open ball containing the origin, so that $r_j(\zeta, 0) = 0$ and $\partial r_1(\zeta, p) \wedge \cdots \wedge \partial r_N(\zeta, p)$ never vanishes for $\zeta \in \partial\Delta$ and $p \in B$. We would like to mention that in [G1, p. 289], the author considers the more general regularity $r_j \in \mathcal{C}^\beta(\partial\Delta, \mathcal{C}^2(B))$, but that for us, the simpler smoothness category $\mathcal{C}^{2,\beta}(\partial\Delta \times B, \mathbb{R})$ will be enough. Then we represent

$$R(\zeta) := \{p \in A(\zeta) + B : r_j(\zeta, p - A'(\zeta)) = 0, j = 1, \ldots, N\},$$

which is a $\mathcal{C}^{2,\beta}$-smooth maximally real manifold by the condition on ∂r_j. We suppose the given reference disc A' to be of class $\mathcal{C}^{2,\beta}$ up to the boundary. Following [G1], we describe the family of nearby attached disc as a $\mathcal{C}^{1,\beta}$-smooth submanifold of the space $\mathcal{C}^{2,\beta}(\partial\Delta, \mathbb{C}^N)$, with a loss of smoothness.

Remark. At first glance the transition from a fixed manifold to the family $R(\zeta)$ may appear purely technical. Nevertheless it gives in our application a decisive additional degree of freedom: If we had to construct a fixed manifold R containing the boundary of our given disc A', the boundary of A' would prescribe one direction of TR. It will prove very convenient to avoid this constraint by the transition to distributions $R(\zeta)$ and this freedom will be used in an essential way in Section 6 below.

It turns out that the problem is governed by an N-tuple $\kappa_1, \ldots, \kappa_N \in \mathbb{Z}$ of coordinate independent partial indices which are defined as follows. As in [G1], we shall always assume that the pull-back bundle $(A'|_{\partial\Delta})^*(TR(\zeta))$ is topologically trivial (this condition is dispensible, *see* [O]). For each $\zeta \in \partial\Delta$, let us denote by $L(\zeta)$ the tangent space to $R(\zeta)$ at $A'(\zeta)$. Then there is a $\mathcal{C}^{1,\beta}$-smooth map $G : \partial\Delta \to GL(N, \mathbb{C})$ such that for each $\zeta \in \partial\Delta$ the columns of G are a (real) basis of $L(\zeta)$. By results of Plemelj and Vekua, we can decompose the matrix function $B(\zeta) = G(\zeta)\overline{G(\zeta)^{-1}}, \zeta \in \partial\Delta$, as

$$B(\zeta) = F^+(\zeta)\Lambda(\zeta)F^-(\zeta),$$

with matrix functions

$$F^+ \in \mathcal{O}(\Delta, GL(N, \mathbb{C})) \cap \mathcal{C}^{1,\beta}(\overline{\Delta}, GL(N, \mathbb{C})),$$
$$F^- \in \mathcal{O}(\mathbb{C}\backslash\overline{\Delta}, GL(N, \mathbb{C})) \cap \mathcal{C}^{1,\beta}(\mathbb{C}\backslash\Delta, GL(N, \mathbb{C})),$$

and where $\Lambda(\zeta)$ is the matrix with powers ζ^{κ_j} on the diagonal and zero elsewhere. In [G1] it is shown that the matrix $B(\zeta)$ depends only on the family of maximally real linear space $L(\zeta)$ and that the κ^j are unique up to permutation. They are called the *partial indices* of R along $A'(\partial\Delta)$ and their sum $\kappa = \kappa_1 + \cdots + \kappa_N$ the *total index*. We stress that only κ is a topological invariant, in fact twice the winding number of $\det(G(\zeta))$ around the origin. In the literature on symplectic topology, κ is called *Maslov index* of the loop $\zeta \mapsto L(\zeta)$.

Building on work of Forstnerič [F], Globevnik [G1, Theorem 7.1] showed that the family of all analytic discs attached to $R(\zeta)$ which are $\mathcal{C}^{1,\beta}$-close to A' is a $\mathcal{C}^{1,\beta}$-smooth submanifold of $\mathcal{O}(\Delta, \mathbb{C}^N) \cap \mathcal{C}^{2,\beta}(\overline{\Delta}, \mathbb{C}^N)$ of dimension $\kappa + N$, if all κ_j are non-negative (by a result due to Oh [O], this is even true if $\kappa_j \geq -1$ for all j). Furthermore the result is stable with respect to small $\mathcal{C}^{2,\beta}$-smooth deformations of M.

We shall need some specific ingredients of Globevnik's construction. Since all our later arguments will exclude the appearance of odd partial indices and since the expression of the square root matrix $\sqrt{\Lambda}$ below is less complicated for even ones, we shall suppose from now on that $\kappa_j = 2m_j, j = 1, \ldots, N$.

Firstly one has to replace $G(\zeta)$ by another basis of $L(\zeta)$ which extends holomorphically to Δ. By [G1, Lemma 5.1], there is a finer decomposition

$$B(\zeta) = \Theta(\zeta)\Lambda(\zeta)\overline{\Theta(\zeta)^{-1}},$$

where $\Theta \in \mathcal{O}(\Delta, GL(N,\mathbb{C})) \cap \mathcal{C}^{1,\beta}(\overline{\Delta}, GL(N,\mathbb{C}))$. The substitute for $G(\zeta)$ is

$$\Theta(\zeta)\sqrt{\Lambda}(\zeta),$$

where $\sqrt{\Lambda}(\zeta)$ denotes the matrix with ζ^{m_j} on the diagonal. We denote by X_j (Y_j) the columns of $\Theta(\zeta)\sqrt{\Lambda}(\zeta)$ $(\sqrt{\Lambda}(\zeta))$ respectively. One can verify that the $X_j(\zeta)$ span $L(\zeta)$ ([G1, Theorem 5.1]). Observe $\Theta(\zeta)\sqrt{\Lambda}(\zeta) \in \mathcal{O}(\Delta, \mathbb{C}^N) \cap \mathcal{C}^{1,\beta}(\overline{\Delta}, \mathbb{C}^N)$.

Secondly one studies variations of $A'|_{\partial\Delta}$ as a function from $\partial\Delta$ to $\mathbb{C}^N$. Every nearby $\mathcal{C}^{1,\beta}$-smooth (not necessarily holomorphic) variation is a disc close to A' which can be written in the form ([F, p. 20])

$$G(u,f)(\zeta) = \sum_{j=1}^{N} u_j(\zeta)\, X_j(\zeta) + i \sum_{j=1}^{N} \{f_j(\zeta) + i\,(T_0 f_j)(\zeta)\}\, X_j(\zeta),$$

where $u_j, f_j \in \mathcal{C}^{1,\beta}(\partial\Delta, \mathbb{R})$, are uniquely determined by the variation. Here T_0 denotes the harmonic conjugation operator normalized at $\zeta = 0$. The condition $G(u,f)(\zeta) \in R(\zeta)$, $\forall\, \zeta \in \partial\Delta$ is equivalent to the validity of the system $r_j(\zeta)(G(u,f)(\zeta)) = 0$, $1 \leq j \leq N$. The implicit function theorem implies that this system can be solved for $f = \phi(u)$ for $\mathcal{C}^{1,\beta}$-small u with a $\mathcal{C}^{1,\beta}$-smooth mapping ϕ of Banach spaces $\mathcal{C}^{1,\beta}(\partial\Delta, \mathbb{R}^N) \to \mathcal{C}^{1,\beta}(\partial\Delta, \mathbb{R}^N)$. This follows from [G1, Theorem 6.1] by an application of the implicit function theorem in Banach spaces, except concerning the $\mathcal{C}^{1,\beta}$-smoothness, which, in our situation, is more direct and elementary than in [G1], since we have supposed that $r_j \in \mathcal{C}^{2,\beta}(\partial\Delta \times B, \mathbb{R})$.

Finally one has to determine for which choices of u the function $G(u, \phi(u))$ extends holomorphically to Δ. Writing

$$G(u,f)(\zeta) = \Theta(\zeta) \sum_{j=1}^{N} \{u_j(\zeta) + i\,[f_j(\zeta) + i\,(T_0 f_j)(\zeta)]\}\; Y_j,$$

we see that $G(u, \phi(u))$ extends holomorphically, if and only if

$$\Theta^{-1}(\zeta)\, G(u,\phi(u))(\zeta) = \sum_{j=1}^{N} \{u_j(\zeta) + i\,[\phi(u)_j(\zeta) + i\,(T_0\, \phi(u)_j)(\zeta)]\}\, Y_j \tag{5.1}$$

extends, *i.e.* if and only if the function $\zeta \mapsto \sum_{j=1}^{N} u_j(\zeta)\, Y_j(\zeta)$ extends. One can compute ([G1, p. 301]) that this is precisely the case, if $h_j(\zeta) = Y^{-1}(\zeta)\, u_j(\zeta)$ has polynomial components of the form

$$\begin{aligned} h_j(\zeta) = {} & t_1^j + i\, t_2^j + (t_3^j + i\, t_4^j)\, \zeta + \cdots + (t_{\kappa_{j-1}}^j + i\, t_{\kappa_j}^j)\, \zeta^{m_j - 1} + t_{\kappa_{j+1}}^j\, \zeta^{m_j} \\ & + (t_{\kappa_{j-1}}^j - i\, t_{\kappa_j}^j)\, \zeta^{m_j+1} + \cdots + (t_3^j - i\, t_4^j)\, \zeta^{\kappa_j - 1} + (t_1^j - i\, t_2^j)\, \zeta^{\kappa_j}, \end{aligned} \tag{5.2}$$

where all t_k^j are real. In total we get $\kappa_j + 1$ real parameters for the choice of h_j and hence $\kappa + N$ parameters for our local family of discs attached to $R(\zeta)$.

6. Proof of Theorem 3.1, part II

In this section we provide the final part of the proof of Theorem 3.1, namely Main Lemma 4.3, which relies essentially on global properties of analytic discs. The disc A' of Main Lemma 4.3 need not be attached to M but since it is close to A in $\mathcal{C}^{2,\beta}$-norm, it is certainly attached to a nearby manifold M' of class $\mathcal{C}^{2,\beta}$ which coincides with M except in Ω_1. The idea is now to first embed $A'(\partial\Delta)$ into a maximal real submanifold of $M \cup \Omega_1$ whose partial indices are easy to determine. Then we shall explain how to increase the partial indices separately by twisting R around $A'(\partial\Delta)$ inside Ω_1. The families of attached discs get richer with increasing indices and will eventually contain the required discs $A'_{t'}$ as a subfamily. We divide the proof in four essential steps.

Step 1: Construction of a first maximally real manifold R_1. Let h' be a defining function of M' as in (3.1). Then A' is the solution of a Bishop equation

$$Y' = T_1(h'(W', Y')) + y_0,$$

where $W' \in \mathcal{C}^{2,\beta}$ is the w-component of A' and $y_0 \in \mathbb{R}^n$ is close to 0. Recall that by construction, $W'(\zeta)$ is close to the w-component $(\rho_0 - \rho_0\zeta, 0, \dots, 0)$ of the disc A defined in (3.3). Let $A'_{u_*,y}$ be the discs defined by the perturbed equation

$$Y'_{u_*,y} = T_1(h'(W'_{u_*,y} + (0, u_*), Y'_{u_*,y})) + y_0 + y,$$

where $u_* := (u_2, \dots, u_m)$ is close to 0 and $y \in \mathbb{R}^n$ is close to 0. We have $A'_{0,0} = A'$. Since A defined by (3.3) and hence also A' are by construction almost parallel to the w_1-axis, the union

$$R_1 := \bigcup_{u_*,y} A'_{u_*,y}(\partial\Delta)$$

is a maximally real manifold of class $\mathcal{C}^{2,\beta}$ contained in M' and containing $A'(\partial\Delta)$. The explicit construction of R_1 allows an easy determination of the partial indices.

Lemma 6.1. *The partial indices of R_1 with respect to $A'(\partial\Delta)$ are $2, 0, \dots, 0$.*

Proof. We begin by constructing $N = m + n$ holomorphic vector fields along $A'(\partial\Delta)$ which generate (over $\mathbb{R}$) the tangent bundle of R_1. We denote $\zeta = e^{i\theta} \in \partial\Delta$ and define first $G_1(\zeta) := [\partial A'(e^{i\theta})/\partial\theta]$ as the push-forward of $\partial/\partial\theta$. Next, we put

$$\begin{cases} G_k(\zeta) := [\partial A'_{u_*,0}(\zeta)/\partial u_k]|_{u_*=0}, & \text{for } k = 2, \dots, m, \\ G_k(\zeta) := [\partial A'_{0,y}(\zeta)/\partial y_{k-m}]|_{y=0}, & \text{for } k = m+1, \dots, N. \end{cases}$$

For $k = 2, \dots, N$, G_k is the uniform limit of pointwise holomorphic difference quotients and therefore holomorphic itself. As $A'_{u_*,y}$ depends $\mathcal{C}^{2,\beta}$-smoothly on parameters, we obtain $G_k \in \mathcal{C}^{1,\beta}(\overline{\Delta}, \mathbb{C}^N)$, $k = 2, \dots, N$.

By [G1, Proposition 10.2], the maximal number of linearly independent holomorphically extendable sections equals the number of non-negative partial indices. Hence we deduce that all κ_j are non-negative.

Furthermore it is easy to see that the total index κ, which is twice the winding number of $\det G|_{\partial\Delta}$ around 0, equals 2. Indeed, A' is almost parallel to the w_1 axis, the direction in which G_1 has winding number 1, and the vector fields $G_2, \dots, G_N$ have a topologically trivial behaviour in the remaining directions.

This heuristic argument can be made precise in the following way. One easily can smoothly deform the complex coordinates z_j, w_k to (non-holomorphic coordinates) in which the matrix $G(\zeta)$ gets diagonal with diagonal entries $\zeta, 1, \dots, 1$. In the deformed coordinates the winding number of the determinant is obviously 1, and this remains unchanged when deforming back to the standard coordinates.

In summary the only possible constellations for the partial indices are $2, 0, \dots, 0$ and $1, 1, 0, \dots, 0$. But [G1, Proposition 10.1] excludes the second case as $\partial A'/\partial\theta$ does not vanish on $\overline{\Delta}$, which completes the proof. □

Step 2: Gluing R_1 with a family of maximally real planes. Our goal is to twist the manifold R_1 many times around the boundary of A' in the small neighborhood Ω_1 of $A'(-1)$ in order to *increase* its partial indices. Since it is rather easy to increase partial indices when a disc is attached to a family of *linear* maximally real subspaces of $\mathbb{C}^N$ (using Lemma 6.3 below, *see* the reasonings just after the proof), we aim to glue R_1 with its family of tangent planes $T_{A'(\zeta)}R_1$ for ζ near -1. Before proceeding, we have to take care of a regularity question: the family $\zeta \mapsto T_{A'(\zeta)}R_1$ being only of class $\mathcal{C}^{1,\beta}$, some preliminary regularizations are necessary. We remind that by Lemma 4.2 **(b)**, the disc A' is real analytic near $\zeta = -1$. This choice of smoothness is very adapted to our purpose. Indeed, using cut-off functions and the Weierstrass approximation theorem, we can construct a $\mathcal{C}^{2,\beta}$-smooth maximally real manifold R_2 to which A' is still attached and which is also *real analytic* in a neighborhood of $\{A'(\zeta) : |\zeta + 1| < \varepsilon/2\}$. Of course, this can be done with $||R_2 - R_1||_{\mathcal{C}^{2,\beta}}$ being arbitrarily small, so the partial indices of A' with respect to R_2 are still equal to $(2, 0, \dots, 0)$.

Using real analyticity, we can now glue R_2 with its family of maximally real tangent planes $T_{A'(\zeta)}R_2$ for $|\zeta + 1| < \varepsilon/4$ in a *smooth way* as follows. After localization near $A'(-1)$ using a cut-off function, the gluing problem is reduced to the following statement.

Lemma 6.2. *Let R be small real analytic maximally real submanifold of $\mathbb{C}^N$, let $p \in R$ and let $\gamma(s)$, $s \in (-\varepsilon, \varepsilon)$, be a real analytic curve in R passing through p. Then there exist smooth functions $r_j(s,z) \in \mathcal{C}^\infty((-\varepsilon,\varepsilon)\times B, \mathbb{R})$, for $j = 1, \dots, N$, where B is a small open ball centered at the origin in $\mathbb{C}^N$, such that*

- **(1)** $r_j(s, \gamma(s)) \equiv 0$.
- **(2)** $r_j(s, z) \equiv r_j(z) \equiv$ *the defining functions of R for $|s| \geq \varepsilon/2$.*
- **(3)** *For all s with $|s| \leq \varepsilon/4$, the set $\{z \in \mathbb{C}^N : r_j(s,z) = 0, j = 1, \dots, N\}$ coincides with the tangent space of R at $\gamma(s)$.*

Proof. Choosing coordinates $(z_1, \dots, z_N)$ vanishing at p, we can asssume that R is given by $r_j(z) := y_j - \varphi_j(x) = 0$ with $\varphi_j(0) = 0$ and $d\varphi_j(0) = 0$, and that $\gamma_j(s) = x_j(s) + iy_j(s)$, where $y_j(s) := \varphi_j(x(s))$. Let $\chi(s)$ be a $\mathcal{C}^\infty$-smooth cut-off function satisfying $\chi(s) \equiv 0$ for $|s| \leq \varepsilon/4$ and $\chi(s) \equiv 1$ for $|s| \geq \varepsilon/2$. We choose for $r_j(s,z)$ the following functions:

$$y_j - y_j(s) - \sum_{k=1}^{N} \frac{\partial \varphi_j}{\partial x_k}(x(s))\,[x_k - x_k(s)] - \chi(s)\left[\sum_{K\in\mathbb{N}^N, |K|\geq 2} \frac{\partial_x^K \varphi_j(x(s))}{K!}\,[x - x(s)]^K\right].$$

Clearly, the r_j are $\mathcal{C}^\infty$-smooth and **(3)** holds. As φ_j is real analytic in a neighborhood of γ, property **(2)** holds by Taylor's formula. □

In summary, we have shown that we can attach A' to some $\mathcal{C}^{2,\beta}$-smooth family $(R_3(\zeta))_{\zeta\in\partial\Delta}$ of maximally real submanifolds such that $R_3(\zeta)$ coincides with R_2 for $|\zeta+1|\geq\varepsilon/2$ and such that $R_3(\zeta)$ coincides with the maximally real plane $T_{A'(\zeta)}R_2$, for $|\zeta+1|\leq\varepsilon/4$. Clearly, the partial indices of A' with respect to the family $R_3(\zeta)$ are still equal to $(2,0,\dots,0)$.

Step 3: Increasing partial indices. This step is the crucial one in our argumentation. Recall that the partial indices are defined in terms of vector fields along $A'(\partial\Delta)$. In the previous section we have described how to select distinguished vector fields X_k as the columns of $\Theta\sqrt{\Lambda}$, where Θ,Λ were associated to a decomposition of $G_3(\zeta)\,\overline{G_3(\zeta)}^{-1}$, where the columns of the matrix $G_3(\zeta)$ span $T_{A'(\zeta)}\,R_3(\zeta)$. Our method is to modify the vector fields X_k by replacing them by products g_kX_k with the boundary values of certain holomorphic functions g_k. It turns out that the indices can be read from properties of the g_k. Here is how the g_k are constructed.

For convenience in the following lemma, we shall represent $\partial\Delta$ by the real closed interval $[-\pi,\pi]$ where π is identified with $-\pi$.

Lemma 6.3. *For every small $\varepsilon>0$, every integer $\ell\in\mathbb{N}$, there exists a holomorphic function $h\in\mathcal{O}(\Delta)\cap\mathcal{C}^\infty(\overline{\Delta})$ such that*

(1) *$h(\zeta)\neq 0$ for all $\zeta\in\overline{\Delta}$.*
(2) *The function $g(\zeta):=\zeta^\ell\,h(\zeta)$ is real-valued over $\{e^{i\theta}:|\theta|\leq\pi-\varepsilon/8\}$.*

It follows that the winding number of $g|_{\partial\Delta}$ around $0\in\Delta$ is equal to ℓ.

Proof. Let $v(\zeta)$ be an arbitrary $\mathcal{C}^\infty$-smooth 2π-periodic extension to $\mathbb{R}$ of the linear function $-\ell\theta$ defined on $[-\pi+\varepsilon/8,\pi-\varepsilon/8]$. Let T_0 be the harmonic conjugate operator satisfying $(T_0\,u)(0)=0$ for every $u\in L^2(\partial\Delta)$. Since T_0 is a bounded operator of the $\mathcal{C}^{k,\alpha}$ spaces of norm equal to 1, the function $T_0\,v$ is $\mathcal{C}^\infty$-smooth over $\partial\Delta$. It suffices to set $h:=\exp(-T_0\,v+iv)$. Indeed,

$$\zeta^\ell\,h(\zeta)=e^{i\ell\theta}\,e^{-T_0\,v+iv}$$

is real for $\theta\in[-\pi+\varepsilon/8,\pi-\varepsilon/8]$, as desired. □

Let $L(\zeta)$ denote the tangent space $T_{A'(\zeta)}\,R_3(\zeta)$ and let $X_k(\zeta)$ be $\mathcal{C}^{1,\beta}$-smooth vector fields as the columns of the matrix $\Theta\sqrt{\Lambda}$ constructed in Section 5 above. We remind that the $X_k(\zeta)$ span $L(\zeta)$. Further, as the partial indices of $A'(\zeta)$ with respect to $R_3(\zeta)$ are $(2,0,\dots,0)$, we have

$$\Theta(\zeta)\sqrt{\Lambda}(\zeta)=(\zeta\,\Theta_1(\zeta),\Theta_2(\zeta),\dots,\Theta_N(\zeta)).$$

Now, let us choose an arbitrary collection of nonnegative integers $\ell_1,\ell_2,\dots,\ell_N$ and associated functions $g_{\ell_1}(\zeta)=\zeta^{\ell_1}\,h_{\ell_1}(\zeta),\;\dots,\;g_{\ell_N}(\zeta)=\zeta^{\ell_N}\,h_{\ell_N}(\zeta)$ satisfying **(1)** and **(2)** of Lemma 6.3 above. With these functions, we define a new family of maximally real manifolds to which $A'(\zeta)$ is still attached as follows:

(a) For $|\theta|\leq\pi-\varepsilon/8$, $R_4(\zeta)\equiv R_3(\zeta)$.
(b) For $|\theta|\geq\pi-\varepsilon/8$, namely for ζ close to -1,

$$R_4(\zeta):=\mathrm{span}_{\mathbb{R}}\,(\zeta\,g_{\ell_1}(\zeta)\,\Theta_1(\zeta),g_{\ell_2}(\zeta)\,\Theta_2(\zeta),\dots,g_{\ell_N}(\zeta)\,\Theta_N(\zeta)).$$

It is important to notice that this definition yields a true $\mathcal{C}^{2,\beta}$-smooth family of maximally real manifolds, thanks to the fact that the family $R_3(\zeta)$ is already a family of *real linear spaces* for $|\zeta+1|\leq\varepsilon/4$, by construction. Interestingly, the partial indices have increased:

Lemma 6.4. *The partial indices of* $R_4(\zeta)$ *along* ∂A *are equal to* $2+2\ell_1, 2\ell_2, \dots, 2\ell_N$.

Proof. By construction, since the functions $g_{\ell_j}(\zeta)$ are real-valued over $\{e^{i\theta} : |\theta| \leq \pi - \varepsilon/8\}$, the tangent space $T_{A'(\zeta)}R_4(\zeta)$ is spanned *for all* $\zeta \in \partial\Delta$ by the N vectors

$$\zeta\, g_{\ell_1}(\zeta)\,\Theta_1(\zeta),\ g_{\ell_2}(\zeta)\,\Theta_2(\zeta),\ \dots,\ g_{\ell_N}(\zeta)\,\Theta_N(\zeta),$$

which form together a $N \times N$ matrix which we will denote by $G_4(\zeta)$. By Section 5, we can read directly from the matrix identity

$$\begin{aligned} G_4(\zeta)\,\overline{G_4(\zeta)}^{-1} &= (h_{\ell_1}(\zeta)\,\Theta_1(\zeta),\dots,h_{\ell_N}(\zeta)\,\Theta_N(\zeta))\ \times \\ &\times \operatorname{diag}(\zeta^{2+2\ell_1},\zeta^{2\ell_2},\dots,\zeta^{2\ell_N})\ \times\ \overline{(h_{\ell_1}(\zeta)\,\Theta_1(\zeta),\dots,h_{\ell_N}(\zeta)\,\Theta_N(\zeta))}^{-1} \end{aligned}$$

that the partial indices of $A'(\zeta)$ with respect to $R_4(\zeta)$ are equal to $(2+2\ell_1, 2\ell_2, \dots, 2\ell_N)$, as stated. □

Step 4: Construction of the family $A'_{t'}$. Now, as we need not very large partial indices, we choose $\ell_1 = 1$, $\ell_2 = 2, \dots, \ell_N = 2$, so the partial indices are simply $(4, 4, \dots, 4)$. Moreover, the matrix $Y(\zeta)$ is equal to the diagonal matrix $\operatorname{diag}(\zeta^2, \zeta^2, \dots, \zeta^2)$. Concerning the $5N$ parameters $(t_1^j, t_2^j, t_3^j, t_4^j, t_5^j)$ appearing in equation (5.2) above, we even choose $t_1^j = t_2^j = t_5^j = 0$. Then by the result of Globevnik, we thus obtain a family of discs depending on the $2N$-dimensional real parameter $t := (t_3^j + i\,t_4^j)_{1\leq j\leq N}$. The functions h_j and u_j defined in Section 5 are thus equal to

$$\begin{cases} h_j(t,\zeta) := (t_3^j + i\,t_4^j)\,\zeta + (t_3^j - i\,t_4^j)\,\zeta^3, \\ u_j(t,\zeta) := (t_3^j + i\,t_4^j)\,\bar{\zeta} + (t_3^j - i\,t_4^j)\,\zeta. \end{cases}$$

It remains to explain how we can extract the desired family $A'_{t'}$ by reducing this $(2m+2n)$-dimensional parameter space to some of dimension $(2m+2n-1)$ such that property **(4)** of Main Lemma 4.3 is satisfied. Let us denote by h_t and u_t the maps $\zeta \mapsto h(t,\zeta)$ and $\zeta \mapsto u(t,\zeta)$. By equation (5.1), we have

$$G(u_t, \phi(u_t))(\zeta) = \Theta(\zeta) \sum_{j=1}^{N} \{u_j(t,\zeta) + i\,[\phi(u_t)_j(\zeta) + i\,T_0\,\phi(u_t)_j(\zeta)]\}\ Y_j(\zeta).$$

and by Section 5, the $\mathcal{C}^{1,\beta}$-smooth discs

$$A'_t(\zeta) := A'(\zeta) + G(u_t, \phi(u_t))(\zeta)$$

are attached to $R_4(\zeta)$. By [G1, p. 299 top], the differential of ϕ at 0 is null: $D_u\phi(0) = 0$. It follows that

$$\frac{\partial}{\partial t}\left[\Theta(\zeta) \sum_{j=1}^{N} i\,[\phi(u_t)_j(\zeta) + i\,T_0\,\phi(u_t)_j(\zeta)]\ Y_j(\zeta)\right]_{t=0} \equiv 0.$$

So on the one hand, we can compute for $j = 1, \dots, N$

$$
\text{(6.1)} \qquad \begin{cases} \left[\dfrac{\partial A'_t}{\partial t_3^j}\right]_{t=0} = \rho e^{i\theta}\, \Theta(0)\, (0, \dots, 0, 1, 0, \dots, 0) + \mathrm{O}(\rho^2), \\ \left[\dfrac{\partial A'_t}{\partial t_4^j}\right]_{t=0} = \rho e^{i\theta}\, \Theta(0)\, (0, \dots, 0, i, 0, \dots, 0) + \mathrm{O}(\rho^2), \end{cases}
$$

where $\mathrm{O}(\rho^2)$ denotes a holomorphic disc in $\mathcal{O}(\Delta, \mathbb{C}^N) \cap \mathcal{C}^{0,\beta}(\overline{\Delta}, \mathbb{C}^N)$ vanishing up to order one at 0. For $\rho > 0$ small enough and θ arbitrary, it follows that these $2m + 2n$ vectors span $\mathbb{C}^{m+n}$. On the other hand, we compute

$$
\text{(6.2)} \qquad \left[\frac{\partial A'_t}{\partial \theta}\right]_{t=0} = \rho e^{i\theta}\, \Theta(0)\, (ia_1, \dots, ia_N) + \mathrm{O}(\rho^2),
$$

where the constants a_j are defined by $A'(\zeta) = (a_1\zeta, \dots, a_N\zeta) + \mathrm{O}(\zeta^2)$ and do not all vanish (since A' is an embedding).

Let us choose a $(2m+2n-1)$-dimensional real plane H which is supplementary to $\mathbb{R}\,\Theta(0)\,(ia_1, \dots, ia_N)$ in $\mathbb{C}^{m+n}$. Using (6.1), we can choose a $(2m + 2n - 1)$-dimensional real linear subspace $T' \subset \mathbb{R}^{2m+2n}$ and ρ_ε small enough such that, after restricting the family A'_t with $t' \in T'$, the $(2m+2n-1)$ vectors $[\partial A'_{t'}/\partial t'_j]_{t'=0}$, $j = 1, \dots, 2m+2n-1$, are linearly independent with the vector (6.2) for all $\zeta \in \Delta$ of the form $\zeta = \rho_\varepsilon e^{i\theta}$. It follows that the mapping

$$
(e^{i\theta}, t') \mapsto A'_{t'}(\rho_\varepsilon e^{i\theta})
$$

is a local embedding of the circle $\partial\Delta$ times a small neighborhood of the origin in $\mathbb{R}^{2m+2n-1}$, from which we see that the foliation property **(4)** of Main Lemma 4.3 holds.

This completes the proof of Step 4, the proof of Main Lemma 4.3, the proof of Theorem 3.1 and the proof of Theorem 1.1.

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