ON REMOVABLE SINGULARITIES FOR CR FUNCTIONS IN HIGHER CODIMENSION

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In recent years, several papers (for a complete reference list, see Chirka and Stout [3]) have been published on the subject of removable singularities for the boundary values of holomorphic functions on some domains or hypersurfaces in the complex euclidean space.

In this paper, we study the higher codimensional case. Our results for the hypersurface case are weaker than those in [3] and [4], for the smoothness assumption.

Note with $T_{z_0}M$ the usual tangent space of a real manifold $M \subset \mathbf{C}^n$ at $z_0 \in M$ and by $T^c_{z_0}M = T_{z_0}M \cap JT_{z_0}M$ its complex tangent space, where J denotes the complex structure on $T\mathbf{C}^n$. M is said to be generic if $T_zM + JT_zM = T_z\mathbf{C}^n$ for all $z \in M$. We consider continuous distributional solutions of the tangential Cauchy-Riemann equations on M, which will be referred as *CR functions* on M. Let $z_0 \in M$. By a wedge of edge M at z_0, we mean an open set in $\mathbf{C}^n$ of the form

$$\mathcal{W} = \{z + \eta;\ z \in U, \eta \in C\},$$

for some open neighborhood U of z_0 in M and some convex truncated open cone C in $T_{z_0}\mathbf{C}^n/T_{z_0}M$, *i.e.* the intersection of a convex open cone with a ball centered at 0.

Let now $N \ni z_0$ be a proper C^1 submanifold of M and assume that M is minimal at z_0. Our submanifolds will always be assumed to be *embedded submanifolds.* Call N *removable at* z_0 if there exists a wedge $\mathcal{W}$ of edge M at z_0 with CR functions on $M \backslash N$ extending holomorphically into $\mathcal{W}$ and continuously in $\mathcal{W} \cup (M \backslash N)$.

Our main results are the following.

THEOREM 1. *Let M be a $C^{2,\alpha}$-smooth $(0 < \alpha < 1)$ generic manifold in $\mathbf{C}^n (n \geq 2)$ minimal at every point, with CRdim $M = p \geq 1$. Then every C^1 submanifold $N \subset M$ with codim$_M N \geq 3$ and $T_zN \not\supset T^c_zM$ for every $z \in N$ is removable.*

THEOREM 2. *Let M be a $C^{2,\alpha}$-smooth $(0 < \alpha < 1)$ generic manifold in $\mathbf{C}^n (n \geq 3)$ minimal at every point, with CRdim $M = p \geq 2$. Then every connected C^1 submanifold $N \subset M$ with codim$_M N = 2$ and $T_zN \not\supset T^c_zM$ for every $z \in N$ is removable provided N does not consist of a CR manifold with CRdim $N = p - 1$.*

Minimality is understood in the sense of Tumanov. The following theorem is due to Jöricke, via a minimalization theorem [5]. We recall her proof and give another one.

THEOREM 3. *Let M be as in Theorem 2 and let N be a connected C^2 submanifold with codim$_M N = 1$ which is generic in $\mathbf{C}^n$. Let $z_0 \in N$, let Φ be a closed subset of N with $\Phi \neq N$ and $b\Phi \ni z_0$. If Φ does not contain the germ through z_0 of a CR manifold Σ with CRdim $\Sigma = p - 1$, Φ is removable at z_0.*

It should be noted that, though not being a CR manifold, a two-codimensional manifold N can contain proper submanifolds Σ with CRdim $\Sigma = p - 1$ and $2p - 2 \leq$ dim $\Sigma \leq 2p + q - 2$. Hence our Theorem 2 is far from being a corollary of Theorem 3, even in the hypersurface case.

Instead of considering continuous CR functions or CR distributions, one is led in the study of removable singularities in complex analysis to consider functions holomorphic into wedges without growth condition at all. Thus another notion of removability is as follows. By a wedge *attached* to a generic manifold M, we mean an open *connected* set which contains a wedge of edge M at every point of M with a continuously varying direction in the normal bundle to M. Let $\Phi \subset M$ be a proper closed subset of M. Call Φ *removable* if, given a wedge $\mathcal{W}_0$ attached to $M\backslash\Phi$, there exists a wedge $\mathcal{W}$ attached to M such that holomorphic functions into $\mathcal{W}_0$ extend holomorphically into $\mathcal{W}$. (For a precise definition, see Proposition 5.5).

We point out that the merit of our "deformation philosophy" is to show that the two notions of removability are rather one and the same. Theorems 1, 2 and 3 hold for both.

THEOREM 4. *Let M, N be as in theorems 1,2,3 respectively and let $\mathcal{W}_0$ be a wedge attached to $M\backslash N$. Then there exists a wedge $\mathcal{W}$ of edge M at z_0 such that holomorphic functions into $\mathcal{W}_0$ extend holomorphically into $\mathcal{W}$.*

Remark 1. We give a proof of Theorem 4 with an everywhere minimal manifold M, but it is possible to prove removability of N in the wedge sense without any minimality assumption on the base manifold.

Remark 2. Let M, N, Φ be as in Theorem 3. We choose a formulation avoiding the notion of orbits, but we shall obtain a more global result, if N is assumed to be a closed embedded submanifold: *Φ is fully removable if Φ does not contain any CR orbit of N.*

Remark 3. The condition that the tangent space to N does not contain the full complex tangent space to M at z_0 means, roughly, that the CR geometry is not entirely absorbed at that point. Such a hypothesis seems unavoidable and is commonly made in the hypersurface case [3] [4]. However, in the real analytic category, we can dispense ourselves of it in Theorem 1 (Corollary 4.3) and presumably also in Theorem 2.

Remark 4. We shall also derive in Theorem 5.2 below removability of a proper closed subset Φ in a connected C^1 manifold N of codimension two in M with the condition on the tangent spaces, for general $p \geq 1$, thus extending Theorem 1. When $p \geq 2$ and N is not a $(p-1)$-CR manifold, the set $N\backslash N^{CR}$ of generic points is nonempty; these points are naturally shown to be removable, since one can prove there that CR functions on $M\backslash N$ are locally uniformly approximable on compact subsets of $M\backslash N$ by holomorphic polynomials (Proposition 5.B) and then apply the propagation method of Trépreau or the one of Tumanov (both can be applied). To finish out the proof of Theorem 2, one then invokes Theorem 5.2 with the set $\Phi = N^{CR} \neq N$ of $(p-1)$-CR points of N.

Remark 5. The content of the sufficient condition in Theorem 2 can be explained as follows, at least locally. For every holomorphic function g near z_0 with $g(z_0) = 0$ and $\partial g(z_0) \neq 0$, the complex hypersurface $\Sigma_g = \{g = 0\}$ intersects M along a two-codimensional submanifold $N = \Sigma_g \cap M$, whenever Σ_g is transversal to M in $\mathbf{C}^n$ at z_0, *i.e.* $T_{z_0}\Sigma_g + T_{z_0}M = T_{z_0}\mathbf{C}^n$. Then the restriction of $1/g$ to $M\backslash N$ does not extend holomorphically into any wedge of edge M at z_0, since $T_{z_0}\Sigma_g$ absorbs the whole of the normal bundle to M at z_0. In that case, N is a CR manifold with CRdim $N = p - 1$, since it is generic in Σ_g.

One can prove a converse statement when N is CR and minimal.

Recall that an analytic wedge $\mathcal{W}^{an}$ with edge a CR manifold N is a complex manifold with edge N, smooth up to N, with dimension equal to the rank of the bundle $TN + JTN$.

THEOREM 5. *Let M be a $C^{2,\alpha}$-smooth generic manifold in $\mathbf{C}^n (n \geq 3)$ with CRdim $M = p \geq 2$*

and let $z_0 \in M$. Let $N \ni z_0$ be a $C^{2,\alpha}$ CR submanifold of M with $codim_M N = 2$, CRdim $N = p-1$, $T_{z_0}N + T^c_{z_0}M = T_{z_0}M$ which is minimal everywhere. Then there exists a wedge $\mathcal{W}$ of edge M at z_0 and an analytic wedge $\mathcal{W}^{an}$ with edge N which is a closed complex hypersurface in $\mathcal{W}$ such that every CR function on $M\backslash N$ extends holomorphically into $\mathcal{W}\backslash\mathcal{W}^{an}$.

Here is a description of the content of the paper. In Sections 1 and 2, we recall the notion of the defect of an analytic disc attached to a generic manifold. This will be useful to insure the existence of a good disc. In Section 3, we introduce isotopies of analytic discs and delineate a continuity principle. In Section 4 and 5, we complete the proof of Theorems 1 and 2 by using normal deformations of analytic discs in the main Proposition 4.1.

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1. Preliminaries. This paragraph is concerned with results of Tumanov [9] and extracted from Baouendi, Rotshchild and Trépreau [2].

Let M be a CR generic submanifold of the complex euclidean space, $T_{z_0}M$ its tangent space at $z_0 \in M$ and $T^c_{z_0}M = T_{z_0}M \cap JT_{z_0}M$ its complex tangent space, where J denotes the standard complex structure on $T\mathbf{C}^n$. Set $p = \text{CRdim } M$ and $q = \text{codim } M$ and assume that $p > 0$ and $q > 0$. In the following, M will always be of smoothness class $C^{2,\alpha}$. An *analytic disc* in $\mathbf{C}^n$ is a continuous mapping $A : \overline{\Delta} \to \mathbf{C}^n$ which is holomorphic in Δ, where Δ is the open unit disc in $\mathbf{C}$, $\Delta = \{\zeta \in \mathbf{C};\ |\zeta| < 1\}$, $\overline{\Delta} = \Delta \cup b\Delta$ and $b\Delta$ is the unit circle in $\mathbf{C}$. We say that A is *attached to M through z_0* if $A(b\Delta) \subset M$ and $A(1) = z_0$. We shall always assume all analytic discs to be in the Banach space $\mathcal{B} = (C^{1,\alpha}(\overline{\Delta}) \cap \mathcal{O}(\Delta))^n$ and we set $\mathcal{B}_0 = \{A \in \mathcal{B};\ A(1) = z_0\}$ or $\mathcal{B}_0 = (C^{1,\alpha}_0(\overline{\Delta}) \cap \mathcal{O}(\Delta))^n$, where $C^{1,\alpha}_0(\overline{\Delta})$ denotes the set of $C^{1,\alpha}$ functions on $\overline{\Delta}$ vanishing at 1.

Note with z_0 the constant disc and consider, for $\varepsilon > 0$ the neighborhood

$$\mathcal{B}_{z_0,\varepsilon} = \{A \in (C^{1,\alpha}(\overline{\Delta}) \cap \mathcal{O}(\Delta))^n;\ A(1) = z_0, ||A - z_0||_{C^{1,\alpha}} < \varepsilon\} \subset \mathcal{B} \tag{1}$$

If $r \in C^{2,\alpha}(U, \mathbf{R}^q)$ is a definig function for M in a neighborhood U of z_0 in $\mathbf{C}^n$, we introduce the mapping $R : \mathcal{B}_{z_0,\varepsilon} \to \mathcal{F}$ defined by

$$A \mapsto (\zeta \mapsto r(A(\zeta))), \tag{2}$$

where $\mathcal{F} = C^{1,\alpha}_0(b\Delta, \mathbf{R}^q)$ is the subspace consisting of $\mathbf{R}^q$-valued functions of class $C^{1,\alpha}$ on $b\Delta$ vanishing at 1.

With these notations, the subset $\mathcal{A} \subset \mathcal{B}_{z_0,\varepsilon}$ of discs that are attached to M is given by

$$\mathcal{A} = \mathcal{A}_{z_0,\varepsilon} = \{A \in \mathcal{B}_{z_0,\varepsilon};\ R(A) = 0\}. \tag{3}$$

Lemma 1.1. *$\mathcal{A}$ is a Banach submanifold of $\mathcal{B}_{z_0,\varepsilon}$ parameterized by $(C^{1,\alpha}_0(\overline{\Delta}) \cap \mathcal{O}(\Delta))^p$.*

Proof. By virtue of the implicit function theorem in Banach spaces, it is sufficient to prove that the differential mapping $R'(z_0) : \mathcal{B}_0 \to \mathcal{F}$ possesses a continuous right inverse S.

The matrix $r_z = (\frac{\partial r_j}{\partial z_k}(z_0))_{1\le j\le q, 1\le k\le n}$ has rank q, since M is generic. Let D be a $n \times q$ matrix such that $r_z(z_0)D = I_{q\times q}$. For $\widehat{A} \in \mathcal{B}_0$ and $A \in \mathcal{B}_{z_0,\varepsilon}$ we have

$$[R'(A)\widehat{A}](\zeta) = 2\text{Re } (r_z(A(\zeta))\widehat{A}(\zeta)), \tag{4}$$

and if we set, for $f \in \mathcal{F}$

$$S(f) = \frac{1}{2} D(f + iT_1 f), \tag{5}$$

where T_1 denotes the Hilbert transform of f vanishing at 1, we get

$$\begin{aligned} 2R'(z_0)S(f) &= 2r_z(z_0)S(f) + 2\overline{r_z(z_0)S(f)} \\ &= r_z(z_0)D(f + iT_1 f) + r_{\bar z}(z_0)\overline{D}(f - iT_1 f) \\ &= 2f. \end{aligned}$$

$S : \mathcal{F} \to \mathcal{B}_0$ is the continuous right inverse of $R'(z_0)$ we searched for. Furthermore, the closed subspace $Ker\ R'(z_0) = E_0$ which parameterizes $\mathcal{A}$ in a neighborhood of z_0 consists of discs $\widehat{A} \in \mathcal{B}_0$ such that $2\mathrm{Re}\ r_z(z_0)\widehat{A} \equiv 0$ on $b\Delta$. E_0 is isomorphic to $(C_0^{1,\alpha}(\overline{\Delta}) \cap \mathcal{O}(\Delta))^p$, since $rg\ r_z(z_0) = q$ and, for every function f, holomorphic in Δ and continuous up to $b\Delta$ such that $2\mathrm{Re}\ f \equiv 0$ on $b\Delta$, there exists a constant $c \in \mathbf{R}$ such that $f \equiv ic$ in $\overline{\Delta}$ (if $f(1) = 0$, necessarily $c = 0$). This completes the proof.

Defect of an analytic disc. Let M be generic as before. We shall define the defect of an analytic disc $A \in \mathcal{A}$ as follows. Let $\Sigma(M) \subset \Lambda^{1,0}(T^*\mathbf{C}^n)$ denote the conormal bundle to M in $\mathbf{C}^n$. Its fiber at $z \in M$ are given by

$$\Sigma_z(M) = \{\omega \in \Lambda_z^{1,0}(T^*\mathbf{C}^n);\ \mathrm{Im}\ <\omega, X> = 0\ \forall X \in T_zM\}.$$

We consider analytic discs $B : \overline{\Delta} \to \Lambda^{1,0}\mathbf{C}^n$ attached to $\Sigma(M)$. Choosing $(z_1, ..., z_n, \mu_1, ..., \mu_n)$ as holomorphic coordinates on $\Lambda^{1,0}\mathbf{C}^n$, $B(\zeta) = (A(\zeta), \mu(\zeta))$ is attached to $\Sigma(M)$ if and only if

$$A(b\Delta) \subset M \qquad \text{and} \qquad \sum_j \mu_j(\zeta) dz_j \in \Sigma_{A(\zeta)}(M) \qquad \zeta \in b\Delta. \tag{6}$$

The set V_A of discs that are attached to $\Sigma(M)$ can be equipped with a vector space structure on the fiber component. For $\zeta \in b\Delta$, we set

$$V_A(\zeta) = \{\omega \in \Sigma_{A(\zeta)}M;\ \omega = \sum_{j=1}^n \mu_j(\zeta) dz_j,\ (A(\zeta), \mu(\zeta)) \in V_A\}. \tag{7}$$

Definition 1.2. *If $\zeta \in b\Delta$ and A is a disc attached to M, the defect $def_\zeta A$ of A at ζ is the dimension of the vector space $V_A(\zeta)$.*

The canonical identification between $\Lambda^{1,0}\mathbf{C}^n$ and $T^*\mathbf{C}^n$ enables one to identify also $\Sigma(M)$ to the characteristic bundle of the CR vector fields on M, $(T^cM)^\perp \subset T^*M$ (see [2] or [6]). Let $\mathbf{R}^{q*}$ denote the dual space to $\mathbf{R}^q$.

Proposition 1.3. *If $\varepsilon > 0$ is sufficiently small and $A \in \mathcal{A}_{z_0,\varepsilon} = \mathcal{A}$,* the defect $def_\zeta A$ is independent of $\zeta \in b\Delta$. *More precisely,,*

$$V_A(\zeta_0) = \{\xi \in T^*_{A(\zeta_0)}M,\ \xi = ib\nu(\zeta_0)\partial r(A(\zeta_0));$$

$$b \in \mathbf{R}^{q*} \text{ and } \zeta \mapsto b\nu(\zeta) r_z(A(\zeta)) \text{ extends holomorphically to } \Delta\},$$

where $\nu(\zeta)$ is the unique $q \times q$ invertible matrix with coefficients in $C^{1,\alpha}(b\Delta, \mathbf{R})$ such that $\nu(1) = I_{q\times q}$ and $\zeta \mapsto \nu(\zeta) r_z(A(\zeta))D$ extends holomorphically to Δ.

Proof. A disc $B(\zeta) = (A(\zeta), \mu(\zeta))$ is attached to $\Sigma(M)$ if and only if

$$\sum_j \mu_j(\zeta) dz_j = it(\zeta)\partial r(A(\zeta)) \tag{8}$$

with $\mu_j(\zeta)$ extending holomorphically to Δ and $b\Delta \ni \zeta \mapsto t(\zeta) \in \mathbf{R}^{q*}$ being of class $C^{1,\alpha}(b\Delta)$. As a consequence, the mapping

$$\zeta \mapsto it(\zeta)r_z(A(\zeta))D \in \mathbf{C}^q \tag{9}$$

extends holomorphically to Δ. The existence and uniqueness of a matrix $\nu(\zeta)$ such that $\zeta \mapsto \nu(\zeta)r_z(A(\zeta))D$ extends holomorphically to Δ can be established, using elementary Banach space techniques, by noting that $r_z(A(\zeta))D$ is close to the identity if the size ε of A is sufficiently small.
Then $t(\zeta)r_z(A(\zeta))D(\nu(\zeta)r_z(A(\zeta))D)^{-1} = t(\zeta)\nu(\zeta)^{-1}$ extends holomorphically to Δ. Since t and ν are real and $\nu(1) = I_{q\times q}$, $t(\zeta) = t(1)\nu(\zeta)$ for $\zeta \in b\Delta$ and the form for $V_A(\zeta_0)$ follows if one sets $b = t(1)$.

The proof of Proposition 1.3 is complete.

For a fixed $\zeta_0 \in b\Delta$, we define the *evaluation map* $\mathcal{F}_{\zeta_0} : \mathcal{B} \to \mathbf{C}^n$ given by

$$\mathcal{F}_{\zeta_0}(A) = A(\zeta_0).$$

Also, for $A \in \mathcal{B}$, the *tangential direction* mapping at $\zeta = 1$,

$$A \mapsto \mathcal{G}(A) = \frac{\partial A}{\partial \theta}(1).$$

If $\mathcal{F}_{\zeta_0}$ and $\mathcal{G}$ are restricted to the Banach submanifold $\mathcal{A}$ constructed above, their differentials at $A \in \mathcal{A}$ are linear applications

$$\mathcal{F}'_{\zeta_0}(A) : T_A\mathcal{A} \to T_{A(\zeta_0)}M \qquad \mathcal{G}'(A) : T_A\mathcal{A} \to T_{z_0}M.$$

Notice that $\mathcal{F}'_1(A) \equiv 0$, since $\mathcal{F}_1(A) = z_0$ for each $A \in \mathcal{A}$.

THEOREM 1.4. *Let $\zeta_0 \in b\Delta, \zeta_0 \neq 1$. If ε is sufficiently small, $A \in \mathcal{A} = \mathcal{A}_{z_0,\varepsilon}$ and $V_A(\zeta_0) \subset T^*_{A(\zeta_0)}M$ is defined by (7),*
(i) $\mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A} = V_A(\zeta_0)^\perp \quad \subset T_{A(\zeta_0)}M$
(ii) $\mathcal{G}'(A)T_A\mathcal{A} = V_A(1)^\perp \quad \subset T_{z_0}M$,
where orthogonality is taken in the sense of duality between TM and T^*M.

Since $V_A(\zeta_0)$ can be viewed as a subspace of $(T^c_{A(\zeta_0)}M)^\perp \subset T^*_{A(\zeta_0)}M$, we have the following corollary of Theorem 1.4.

COROLLARY 1.5. *Under the assumptions of Theorem 1.4, the codimension of $\mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A}$ in $T_{A(\zeta_0)}M$ and the codimension of $\mathcal{G}'(A)T_A\mathcal{A}$ in $T_{z_0}M$ coincide and are equal to the defect of A. Furthermore, the following inclusion holds*

$$T^c_{A(\zeta_0)}M \subset \mathcal{F}'_{\zeta_0}(A)T_A\mathcal{A} \quad \textit{and} \quad T^c_{z_0}M \subset \mathcal{G}'(A)T_A\mathcal{A}.$$

For the proof of Theorem 1, which is due in substance to Tumanov, see Baouendi, Rothschild and Trépreau [2]. In Section 2 below, we plain to use Corollary 1.5 and the independence of the defect of a disc on everywhere minimal manifolds.

Remark. A better regularity of discs attached to $C^{2,\alpha}$-smooth manifolds can be obtained by analysing Bishop's equation as, for example, in [10].

2. Existence of a disc. In sections 4 and 5, we shall need an embedded disc $A \in \mathcal{A}$ such that $A(1) \in N$, $A(b\Delta\backslash\{1\}) \subset M\backslash N$ and $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) \notin T_{A(1)}N$ to perform removal of singularities along that disc. In the present section, we show how we can derive the existence of such a disc from the hypothesis that M is minimal at every point.

In the rest of the paper, the word neighborhood always means *open* neighborhood.

Let $N \subset M$ be a C^1 submanifold through z_0 with $\text{codim}_M N \geq 2$, $T_{z_0}N \not\supset T^c_{z_0}M$ and pick a C^1 generic manifold $M_1 \subset M$ containing N with $codim_M M_1 = 1$. We can assume that $z_0 = 0$ is our reference point in a coordinate system $(w,z) = (w_1,...,w_p,z_1,...,z_q), w = u+iv \in \mathbf{C}^p, z = x+iy \in \mathbf{C}^q$ such that $T^c_0M = \mathbf{C}^p_w \times \{0\}$, $T_0M = \mathbf{C}^p_w \times \mathbf{R}^q_x$ and $T_0M_1 = \{y=0, v_1=0\}$, $T^c_0M \cap T_0M_1 = \{z=0, v_1=0\}$. Then M is given by the q smooth scalar equations

$$y = h(w,x) \qquad h \in C^{2,\alpha}, \qquad h(0) = dh(0) = 0. \tag{10}$$

Using the solution of Bishop's equation given in [10], one can see that for sufficiently small $c > 0$, the embedded disc $A_c : \zeta \mapsto A_c(\zeta) = (w_c(\zeta), z_c(\zeta))$ with w-component $w_c(\zeta) = (c(1-\zeta),0,...,0)$ and z-component satisfying the functional relation (called Bishop's equation)

$$x_c(\zeta) = -[T_1(h(w_c(.),x_c(.)))](\zeta) \qquad \zeta \in b\Delta,$$

meets M_1 at two points exactly along its boundary, namely $A_c(1) = 0$ and another, say $A_c(e^{i\theta_c})$, where $e^{i\theta_c} \in b\Delta$ is close to -1. (T_1 denotes the Hilbert transform on the unit circle normalized by the condition $(T_1u)(1) = 0$, $T_1u = Tu - u(1)$.)

Indeed, for every $\zeta \in b\Delta$, $\frac{d}{dc}|_{c=0}A_c(\zeta) \in T^c_0M$, since $X_c(\zeta) := \frac{d}{dc}|_{c=0}x_c(\zeta)$ satisfies the functional relation $X = -T_1(2\text{Re}((1-.)ch_{w_1}(0,X)))$ on $b\Delta$, hence $X \equiv 0$, by the uniqueness in the solutions of Bishop's equation, and $Y = T_1X = 0$ also. According to [10], given $\beta < \alpha$ with $\beta > 0$, there exists $c_0 > 0$ such that for $c \leq c_0$, the mapping $(c,\zeta) \mapsto A_c(\zeta)$ is $C^{2,\beta}$-smooth. Since $\frac{d}{dc}|_{c=0}\frac{d}{d\theta}|_{\theta=0}z_c(e^{i\theta}) = 0$ and $\frac{d}{dc}|_{c=0}z_c(1) \equiv 0$, we obtain $|\frac{\partial^2 z_c}{\partial c \partial\theta}(\zeta)| \leq C(c^\beta + |1-\zeta|^\beta)$ for some constant $C > 0$ depending on the second derivatives of h at 0, hence $|\frac{dz_c}{d\theta}(e^{i\theta})| \leq 3cC$ and also $|z_c(\zeta)| \leq C(c|1-\zeta|(c^\beta + |1-\zeta|^\beta))$. As a consequence, one can realize $A_c(b\Delta)$ as a graph over $w_c(b\Delta), A_c(b\Delta) = \{(w,\psi(w));\ w \in w_c(b\Delta)\}$ where ψ is $C^{2,\beta}$-smooth and satisfies $|\frac{d\psi}{dw}(w)| \leq 3C$. Since M_1 has codimension one in M and the boundary of the disc w_c in $\mathbf{C}^p_w$ meets $T^c_0M \cap T_0M_1 = \{z=0, v_1=0\}$ at two points $w_c(1)$ and $w_c(-1)$, the lifting bA_c of bw_c to $\mathbf{C}^n \cap M$ necessarily meets M_1 at exactly two points, $A_c(1)$ and a second one, $A_c(e^{i\theta_c})$ with $e^{i\theta_c}$ close to -1 in $b\Delta$, if c_0 is sufficiently small in order that $A_{c_0}(b\Delta)$ is contained in a neighborhood U of 0 in M with $M_1 \cap U$ very close to T_0M_1.

Remark. The property that bA_c meets M_1 at exactly two points is stable under $C^{2,\beta}$ perturbations of A_c.

Since N is contained in M_1, bA_c can meet N by at most two points in $b\Delta$. If bA_c meets N only at $\zeta = 1$, the disc satisfies our requirement.

If not, we shall obtain a good disc by slightly perturbing A_c. For $\delta > 0$, let $\mathcal{V}(A_c,\delta)$ denote the ball of radius δ and center A_c in $(C^{1,\alpha}(\overline{\Delta}) \cap \mathcal{O}(\Delta))^n$.

PROPOSITION 2.1. *Assume that M is minimal at every point and $codim_M N \geq 2$. Then, for each $\delta > 0$ and $\beta < \alpha$, there exists a $C^{2,\beta}$-smooth embedded disc $A \in \mathcal{A} \cap \mathcal{V}(A_c,\delta)$ such that $A(1) = z_0$, $A(b\Delta\backslash\{1\}) \subset M\backslash N$, $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) = v_0 \notin T^c_{z_0}M$ and $v_0 \notin T_{z_0}N$.*

Proof. Choose a disc $A \in \mathcal{V}(A_c,\delta)$ of minimal possible defect, $A \in \mathcal{A}$ attached to M. If δ is sufficiently small, bA will meet M_1 at only two points, $A(1) = 0 \in M_1$ and some

$A(\zeta_1) = p_1 \in M_1$, ζ_1 being close to -1 in $b\Delta$. Indeed, discs in $\mathcal{V}(A_c, \delta) \cap \mathcal{A}$ are close to A_c in C^1 norm. If $p_1 \in M_1 \backslash N$, we are done. So, suppose that $p_1 \in N$. Since M is minimal at the point p_1 and A is of minimal possible defect, A is of defect 0. Indeed, according to Theorem 1.4, the evaluation map $\mathcal{F}_{\zeta_1}$ is of constant rank $2p + q - def A$ in a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$. Then the image of $\mathcal{U}$ by $\mathcal{F}_{\zeta_1}$ is a submanifold Σ of M containing p_1 with $dim\ \Sigma = 2p + q - def A$ for which $T_z\Sigma \supset T_z^c M$ for all $z \in \Sigma$ by Corollary 1.5. Since M is minimal at p_1, necessarily $def A = 0$.

Shrinking $\mathcal{U}$, we can assume that all boundaries of discs in $\mathcal{U}$ also meet M_1 at exactly two points where their tangent direction is not tangent to M_1. According to the above, there exists a $(2p + q - 1)$-dimensional manifold $T \subset \mathcal{U}$ through A such that $\mathcal{F}_{\zeta_1}$ is a diffeomorphism of T onto a neighborhood of p_1 in M_1. Since N is a proper submanifold of M_1, there exist many discs A_t in $T \subset \mathcal{A}$ with boundary satisfying $A_t(b\Delta \backslash \{1\}) \subset M \backslash N$.

According to Theorem 1.4, one can also find such a disc A_t with $\frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) \notin T_{z_0}^c M$.

Finally, the $C^{2,\beta}$-smoothness of the mapping $(t, \zeta) \mapsto A_t(\zeta)$ is a consequence of the estimates in [10], if we consider $(C^{2,\alpha}(\overline{\Delta}) \cap \mathcal{O}(\Delta))^p$ instead of $(C^{1,\alpha}(\overline{\Delta}) \cap \mathcal{O}(\Delta))^p$ as a parameter space to $\mathcal{A}$. The proof of Proposition 2.1 is complete.

3. Isotopies of analytic discs and a continuity principle. Let $T_M\mathbf{C}^n = T\mathbf{C}^n|_M/TM$ denote the normal bundle to M and let $\eta_0 \in T_M\mathbf{C}^n[z_0]$. By a *wedge of edge M at (z_0, η_0)*, we mean an open set of the form

$$\mathcal{W} = \mathcal{W}(U, z_0) = \{z + \eta; z \in U, \eta \in C\},$$

where C is a conic neighborhood of some *nonzero* representative of η_0 in $T_{z_0}\mathbf{C}^n$, identified with $\mathbf{C}^n$. The definition is independent, in the germs, of the choice of the representative, in the sense that each wedge contains another wedge for another choice. By definition, such a wedge $\mathcal{W}$ contains a neighborhood of z_0 in $\mathbf{C}^n$ if $\eta_0 = 0$ in $T_M\mathbf{C}^n[z_0]$.

The proof of Theorem 1 and Theorem 2 starts as follows. Recall that we assumed that M is minimal at every point. Then, according to the theorem of Tumanov, CR functions on $M\backslash N$ are wedge extendible at every point of $M\backslash N$. The following definition will be convenient for our purpose.

Definition 3.1. *An open* connected *set $\mathcal{W}_0$ will be called a* wedge attached to M *if there exists a* continuous *section $\eta : M \to T_M\mathbf{C}^n$ of the normal bundle to M and $\mathcal{W}_0$ contains a wedge $\mathcal{W}_z$ of edge M at $(z, \eta(z))$ for every z in M.*

Applying if necessary the edge of the wedge theorem ([1]) at points where the direction of extendibility varies discontinuously, the hypothesis on M implies that there exists a wedge $\mathcal{W}_0$ attached to $(M\backslash N)$ to which CR functions on $M\backslash N$ have a holomorphic extension.

Heuristics. The aim, to achieve the proof of the two theorems, is to show that such holomorphic functions extend holomorphically into a wedge of edge M at z_0, under the various hypotheses on N. Notice that we cannot a priori know wether CR functions on $M\backslash N$ are approximable by holomorphic polynomials on small compact subsets of $M\backslash N$. This is why we are forced to use instead the continuity principle by first deforming M into $\mathcal{W}_0$.

Fix a function $f \in CR(M\backslash N)$. It extends into some open wedge $\mathcal{W}_0$ attached to $M\backslash N$. Using a C^3-smooth partition of unity on $(M\backslash N) \cap V$, for some neighborhood V of z_0 in M, we can smoothly deform M into $\mathcal{W}_0$ leaving N fixed and then replace f by the restriction of its extension to the $C^{2,\alpha}$ deformation M^d of M. Indeed, smooth deformations d of M into $\mathcal{W}_0$ arbitrarily close to M in $C^{2,\alpha}$ norm and fixing N are possible, since $\mathcal{W}_0$ has a continuously varying direction over $M\backslash N$. Then, instead of a function $f \in CR(M\backslash N)$, we get a function f, holomorphic into a neighborhood ω ($\equiv \mathcal{W}_0$) of $M^d\backslash N$ in $\mathbf{C}^n$. The aim will subsequently be to prove that such holomorphic functions extend into a wedge of edge M at z_0.

A natural tool in describing envelopes of holomorphy of general open sets in $\mathbf{C}^n$ is the *continuity principle.* Our version is the following.

If $E \subset \mathbf{C}^n$ is any set, let $\mathcal{V}(E, r)$ denote $\{z \in \mathbf{C}^n;\ \mathrm{dist}(E, z) < r\}$ where dist denotes the polycircular distance. Then $B(z, r) = \mathcal{V}(\{z\}, r)$ is the polydisc of center z and radius r.

Note with $\mathcal{O}(\omega)$ the ring of holomorphic functions into the open set ω.

LEMMA 3.2. *Let $\omega \subset \mathbf{C}^n$ be an open connected set and let $A : \overline{\Delta} \to \mathbf{C}^n$ be an analytic disc such that $c|\zeta - \zeta'| < |A(\zeta) - A(\zeta')| < C|\zeta - \zeta'|$ for some constants $0 < c < C$ and every $\zeta, \zeta' \in \overline{\Delta}$. Set $r = dist(A(b\Delta), b\omega))$ and $\sigma = \frac{rc}{2C}$. Then for every holomorphic function $f \in \mathcal{O}(\omega)$, there exists a function $F \in \mathcal{O}(\mathcal{V}(A(\overline{\Delta}), \sigma))$ with $F = f$ on $\mathcal{V}(A(b\Delta), \sigma)$.*

Proof. For $z_0 = A(\zeta_0), \zeta_0 \in \overline{\Delta}$, let $\sum_{k\in\mathbf{N}^n} f_k(z - z_0)^k$ denote the germ at z_0 of the converging Taylor series defining f there. Take $\rho > 0$ with $\rho < r$. (3.2 is trivial, if $r = 0$). Then the maximum principle and Cauchy's inequalities yield

$$|f_k| = \frac{1}{k!}|D^k f(z_0)| \leq \frac{1}{k!}\sup_{z\in bA}|D^k f(z)| \leq \frac{M_\rho(f)}{\rho^k},$$

where $M_\rho(f) = \sup\{|f(z)|;\ z \in \mathcal{V}(A(b\Delta), \rho)\} < \infty$. This proves that the Taylor series of f converges in $B(z_0, r)$ for every $z_0 \in A(\overline{\Delta})$, defining an element $f_{z_0,r} \in \mathcal{O}(B(z_0, r))$. Now, if $B(z, \sigma) \cap B(z', \sigma) \neq \emptyset, z = A(\zeta), z' = A(\zeta')$ and $\tau \in [\zeta, \zeta']$, then $|\tau - \zeta| \leq |\zeta' - \zeta| < \frac{2\sigma}{c}$ and

$$|A(\tau) - A(\zeta)| < C|\tau - \zeta| < \frac{2C\sigma}{c} = r.$$

Therefore $A([\zeta, \zeta']) \subset B(z, r) \cap B(z', r)$ and the two holomorphic funcions $f_{z,r}$ and $f_{z',r}$ coincide on the connected intersection, hence define a function holomorphic into $B(z, r) \cup B(z', r)$. This proves that the $f_{z,\sigma} = f_{z,r}|_{B(z,\sigma)}$ stick together in a well-defined holomorphic function into $\mathcal{V}(A(\overline{\Delta}), \sigma)$.

The proof of Lemma 3.2 is complete.

PROPOSITION 3.3. *Let ω be an open connected set and let $A_s : \overline{\Delta} \to \mathbf{C}^n, b\Delta \to \omega, 0 \leq s \leq 1$ be a continuous family of analytic discs such that, for some constants $0 < c_s < C_s$, $c_s|\zeta - \zeta'| < |A_s(\zeta) - A_s(\zeta')| < C_s|\zeta - \zeta'|$. Assume $A_1(\overline{\Delta}) \subset\subset \omega$ and set $r_s = dist(A_s(b\Delta), b\omega), \sigma_s = \frac{r_s c_s}{2C_s}$. Then, for every holomorphic function $f \in \mathcal{O}(\omega)$, there exist functions $F_s \in \mathcal{O}(\mathcal{V}(A_s(\overline{\Delta}), \sigma_s))$ such that $F_s \equiv f$ on $\mathcal{V}(A_s(b\Delta), \sigma_s)$.*

Proof. Let $I_0 \subset [0, 1]$ denote the connected set of $s_0 \in [0, 1]$ such that the statement holds true for every $s_0 \leq s \leq 1$. $1 \in I_0$, according to Lemma 3.2. Let $s_1 < s_0$ be such that $A_{s_1}(\overline{\Delta}) \subset \mathcal{V}(A_{s_0}(\overline{\Delta}), \sigma_{s_0})$. Since $F_{s_0} \equiv f$ into $\mathcal{V}(A_{s_0}(b\Delta), \sigma_{s_0})$ and $\mathrm{dist}(A_{s_1}(b\Delta), b\omega) = r_{s_1}$, the Taylor series of f at points $A_{s_1}(\zeta_1) = z_1, \zeta_1 \in b\Delta$, converges in each polydisc $B(z_1, r_{s_1})$. As in the proof of Lemma 3.2, this proves that there exists a function $F_{s_1} \in \mathcal{O}(\mathcal{V}(A_{s_1}(\overline{\Delta}), \sigma_{s_1}))$ with $F_{s_1} \equiv f$ in $\mathcal{V}(A_{s_1}(b\Delta), \sigma_{s_1})$. Thus I_0 is both open and closed, hence $I_0 = [0, 1]$.

The proof of Proposition 3.3 is complete.

Let Φ be a proper closed subset of M. Then $M\backslash\Phi$ is a smooth generic manifold, which will play, in the sequel, the role of the M in Definition 3.4 below.

Definition 3.4. *Let M be generic. An embedded analytic disc A attached to M is said to be* analytically isotopic to a point in M *if there exists a C^1-smooth mapping $(s,\zeta) \mapsto A_s(\zeta), 0 \leq s \leq 1, \zeta \in \overline{\Delta}$, such that $A_0 = A$, each A_s is an embedded analytic disc attached to M for $0 \leq s < 1$ and A_1 is a constant mapping $\overline{\Delta} \to \{pt\} \in M$.*

In the next section, to derive a proof of Theorem 1, we shall use isotopies of embedded analytic discs without needing a particular control of the size of the open neighborhood arising in our continuity principle Proposition 3.3.

Corollary 3.5. *Let M be generic, $C^{2,\alpha}$, let Φ be a proper closed subset of M and let ω be a neighborhood of $M\backslash\Phi$ in $\mathbf{C}^n$. If an embedded disc A attached to $M\backslash\Phi$ is analytically isotopic to a point in $M\backslash\Phi$, there exists a connected open neighborhood ω_A of $A(\overline{\Delta})$ in $\mathbf{C}^n$ such that for every holomorphic function $f \in \mathcal{O}(\omega)$ there exists a holomorphic function $f_A \in \mathcal{O}(\omega_A)$ with $f_A \equiv f$ in a neighborhood of $A(b\Delta)$ in $\mathbf{C}^n$.*

Finally, the following proposition gives an auxiliary tool in place of the approximation theorem of Baouendi and Treves.

Proposition 3.6. *Let M, Φ be as above. Assume that $M\backslash\Phi$ has the wedge extension property and let $z_0 \in b\Phi$. Let $A \in \mathcal{A}_{z_1,\varepsilon}$ for some $z_1 \in M\backslash\Phi$, $|z_1 - z_0| << \varepsilon$ be an embedded disc attached to $M\backslash\Phi$. If A is analytically isotopic to a point in $M\backslash\Phi$, each A_s of the isotopy being of size ε, then $f \circ A|_{b\Delta}$ extends holomorphically to Δ for every function $f \in CR(M\backslash\Phi)$.*

Proof. Fix a function $f \in CR(M\backslash\Phi)$. It extends holomorphically and continuously into some wedge $\mathcal{W}_0$ attached to $M\backslash\Phi$. Since the isotopy is small, we can follow it on small deformations M^d of M into $\mathcal{W}_0$ that fix Φ. Indeed, the solvability of the equation $R(A) = 0$ as in Lemma 1.1 is stable under perturbations of R, since R is a submersion. Moreover, we may assume that M^d depends on a small real parameter $d \geq 0$, $M^0 = M$ and the solution $(A_s)^d$ depends smoothly on (s,d). Indeed, an implicit function theorem with parameters is valid on Banach spaces. If the deformation d is sufficiently small, this yields an analytic isotopy $(A_s)^d$ of A^d to a point in $M^d\backslash\Phi$. According to Proposition 3.3, $f \circ A^d|_{b\Delta}$ then extends holomorphically to Δ. Letting d tend to 0, we get the desired result by the continuity of f on $\mathcal{W}_0 \cup (M\backslash\Phi)$ and the smoothness of $d \mapsto A^d$.

The proof of Proposition 3.6 is complete.

Let A_c denote the disc introduced in Section 2, c small. In Propositions 3.7 and 3.8 below, we give conditions which insure that discs close to A_c and attached to M minus a singularity set N are analytically isotopic to a point in $M\backslash N$.

Proposition 3.7. *Let M be generic, $C^{2,\alpha}$-smooth, let N be a C^1-smooth proper submanifold of M with $codim_M N = 3$, let $z_0 \in N$ and assume that $T_{z_0}N \not\supset T_{z_0}^c M$. If $c > 0$ is small enough, if $A_1 \in \mathcal{V}(A_c, \delta) \cap \mathcal{A}$ for $\delta > 0, \delta << c$ small enough satisfies $A_1(b\Delta\backslash\{1\}) \subset M\backslash N$, there exists $\delta' > 0$, $\delta' << \delta$ such that each disc $A \in \mathcal{V}(A_1, \delta')$ attached to $M\backslash N$ is analytically isotopic to a point in $M\backslash N$.*

More generally, the following proposition holds.

Proposition 3.8. *Let M be generic, $C^{2,\alpha}$-smooth, let N be a C^1-smooth proper submanifold of M with $codim_M N = 2$, let $z_0 \in N$ and assume that $T_{z_0}N \not\supset T_{z_0}^c M$. Let Φ be a proper closed subset of N with $z_0 \in b\Phi$. If $c > 0$ is small enough, if $A_1 \in \mathcal{V}(A_c, \delta) \cap \mathcal{A}$ for $\delta > 0, \delta << c$ small enough satisfies $A_1(b\Delta\backslash\{1\}) \subset M\backslash N$, there exists $\delta' > 0$, $\delta' << \delta$ such that each disc $A \in \mathcal{V}(A_1, \delta')$ attached to $M\backslash\Phi$ is analytically isotopic to a point in $M\backslash\Phi$.*

Proof. Suppose first that $T_0N \cap \mathbf{C}_{w_1} = \{0\}$. We shall show that the following stronger statement holds: if $c > 0$ is sufficiently small, there exists $\delta > 0$, $\delta << c$ such that each disc $A \in \mathcal{V}(A_c, \delta)$ attached to $M\backslash\Phi$ is analytically isotopic to a point in $M\backslash\Phi$.

Let $\Gamma \subset M$ be a convex open conic neighborhood of the positive u_1-axis with vertex 0 such that $\overline{\Gamma} \cap N = \{0\}$. Choose a similar cone Γ' with $\overline{\Gamma'} \cap S^{2n-1} \subset\subset \overline{\Gamma} \cap S^{2n-1}$, where S^{2n-1} denotes the unit sphere in $\mathbf{R}^{2n}$.

If c is sufficiently small, $A_c(b\Delta) \cap M_1$ consists of two points, 0 and a second one, say p_1 with $|p_1 - p_1^0| < O(c^2)$, where p_1^0 denotes the point in the u_1-axis with u_1 coordinate equal to $2c$ (in coordinates (w, x) on M). Take $\delta > 0, \delta << c$ such that for each disc $A \in \mathcal{V}(A_c, 2\delta) \cap \mathcal{A}$, $A(b\Delta) \cap M_1$ also consists of two points, 0 and a second one in Γ' close to p_1^0. If δ is small enough, we can furthermore insure that all discs in $\mathcal{V}(A_c, 2\delta)$ attached to M are embedded discs and intersect M_1 along their boundaries at two points, one of which is (2δ)-close to 0 and a second one in Γ.

Fix a disc $A \in \mathcal{V}(A_c, \delta)$ attached to $M\backslash\Phi$ and assume that $A(1) \in M_1\backslash\Phi$ is δ-close to 0, which can be done modulo an automorphism of Δ. Write $A(\zeta) = (w(\zeta), z(\zeta))$. Since N has codimension one in M_1 and Φ does not disconnect M_1 near 0, there exists a C^1-smooth curve $\mu : [0,1] \to M_1\backslash\Phi$ with $\mu(0) = A(1) \in M_1\backslash\Phi, \mu(1) \in \Gamma', |\mu(s)| < \delta$ and $\mu(s) \in M_1\backslash\Phi$ for each $0 \leq s \leq 1$. Consider the disc $A_s(\zeta) = (w(\zeta) + \mu_w(s), z_s(\zeta))$ where

$$x_s = -T_1 h(w + \mu_w(s), x_s) + \mu_x(s) \qquad \text{on } b\Delta.$$

Then $A_s(1) = \mu_s(1)$ and each A_s is an embedded disc belonging to $\mathcal{V}(A_c, 2\delta)$, hence attached to $M\backslash\Phi$. Therefore A is analytically isotopic in $M\backslash\Phi$ with a disc $A_1 = B$ such that $(w_1, x_1 + ih(w_1, x_1)) = B(1) \in \Gamma' \cap M_1$ and $B(b\Delta)$ meets M_1 at another point in Γ. Since the pure holomorphic w-component $w(\zeta)$ of B is (2δ)-close to $w_c(\zeta) = c(1-\zeta)$ in $C^{1,\alpha}$-norm, and since z_s is close to z_c which satisfies the estimates $|z_c(\zeta)| = O(c|1-\zeta|(c^\beta + |1-\zeta|^\beta))$ and $|\frac{dz_c}{d\theta}(e^{i\theta}) = O(c)|$, the analytic isotopy $B_s(\zeta) = ((1-s)w(\zeta) + w_1, z_{B,s}(\zeta))$, where

$$x_{B,s} = -T_1 h((1-s)w + w_1, x_{B,s}) + x_1$$

for $0 \leq s \leq 1$ joins B with the point $B(1)$ in $M\backslash\Phi$, each B_s being an embedded disc attached to $M\backslash N$, $0 \leq s < 1$. Indeed, each B_s also meets M_1 along its boundary at two points contained in Γ for $0 \leq s < 1$ according to the position of M_1 and since, w_1 being close to $w_{c,1}$, $\{w_1(\zeta);\ \zeta \in b\Delta\} \subset \mathbf{C}$ is contained in a set of the form $\{\text{Re } w_1 > -a|\text{Im } w_1|\}$, $a > 0$ small.

Assume now that $T_0N \cap \mathbf{C}_{w_1} = \mathbf{R}_{u_1}$ and let $A_1 \in \mathcal{V}(A_c, \delta) \cap \mathcal{A}$ with $A_1(b\Delta\backslash\{1\}) \subset M\backslash N$, $A_1(1) = z_0$. If c and $\delta << c$ are sufficiently small, each disc $A \in \mathcal{V}(A_c, 2\delta)$ attached to M is an embedded disc whose boundary meets M_1 at two points exactly. Since N is one-codimensional in M_1, N divides M_1 in two connected components, M_1^+ and M_1^-. Therefore, we can assume that $bA_1 \cap M_1$ consists of $A_1(1) = 0$ and a point $A_1(\zeta_1) = p_1 \in M_1^+$. Let $\delta' > 0, \delta' << \delta$ and fix a disc $A \in \mathcal{V}(A_1, \delta')$ attached to $M\backslash\Phi$, let $A(1) \in M_1\backslash\Phi$ be the point in $M_1\backslash\Phi$ δ'-close to 0. Since Φ does not disconnect M_1 near z_0, there exists a C^1-smooth curve $\mu : [0,1] \to M_1\backslash\Phi$ with $\mu(0) = A(1), \mu(1) \in M_1^+$, $|\mu(s)| < \delta'$ and $\mu(s) \in M_1\backslash\Phi$ for $0 \leq s \leq 1$. If $\delta' <<$ dist (p_1, N), perturbing the base point of A along the curve μ enables one to make an analytic isotopy in $M\backslash\Phi$ of A with an embedded disc B attached to $M\backslash\Phi$ with the property that $bB \cap M_1$ consists of two points in M_1^+.

Then $B(\zeta) = (w(\zeta) + w_1, z(\zeta))$ with $w(\zeta)$ being (2δ)-close to $w_c(\zeta)$ in $C^{1,\alpha}$-norm, $w(1) = 0$ and $w(\zeta)$ is the holomorphic w-component of A. Now we can push away B from N by using a isotopy along a curve $\gamma : [0,1] \to M_1^+$ such that $\gamma(s)$ belongs to a one dimensional manifold $\Lambda \subset M_1$ with $\Lambda \cap N = \{z_1\}$ and $T_{z_1}N + T_{z_1}\Lambda = T_{z_1}M_1$, some z_1 close to $B(1)$ in M_1. $B_s(\zeta)$ is obtained as the perturbation of B keeping the same pure holomorphic w-component $w(\zeta)$ and satisfying $B_s(1) = \gamma(s)$, $\gamma(0) = B(1)$. In other words, $B_s(\zeta) = (w(\zeta) + \gamma_w(s), z_s(\zeta))$ where

$$x_s = -T_1 h(w + \gamma_w(s), x_s) + \gamma_x(s).$$

Differentiating the equation with respect to s, it is possible to check that $\frac{dB_s}{ds}(\zeta)$ is close to $\frac{d\gamma}{ds}(s)$ uniformly in ζ for small c, δ, δ' and then independent of $\frac{dB_s}{d\theta}(e^{i\theta})$, which is close to be contained in the $\mathbf{C}_{w_1}$-axis. Therefore the embedded disc B_s goes away from N in the direction of $\Lambda \cap M_1^+$. Since w-embedded discs which are sufficiently far away from N are analytically isotopic to a point in $M\backslash N$, B and then A are analytically isotopic to a point in $M\backslash\Phi$.

The proof of Proposition 3.8 is complete.

4. Proof of theorem 1.

The following proposition together with Proposition 2.1 and Proposition 3.7 proves that the envelope of holomorphy of arbitrarily thin neighborhoods of $M\backslash N$ in $\mathbf{C}^n$ always cover a wedge of edge M at every point of N, if N is a three-codimensional submanifold of M as in Theorem 1.

PROPOSITION 4.1. *Let M be generic, $C^{2,\alpha}$-smooth, let $z_0 \in M$, let $N \ni z_0$ be a C^1 submanifold with $codim_M N = 3$ and $T_{z_0}N \not\supset T_{z_0}^c M$ and let ω be a neighborhood of $M\backslash N$ in $\mathbf{C}^n$. Assume there exists a sufficiently small embedded analytic disc $A \in C^{2,\beta}(\overline{\Delta})$ attached to M, $A(1) = z_0$, with $A(b\Delta\backslash\{1\}) \subset M\backslash N$, $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) = v_0 \notin T_{z_0}^c M$, $v_0 \notin T_{z_0}N$ and all discs in $\mathcal{V}(A,\delta)$ attached to $M\backslash N$ are analytically isotopic to a point in $M\backslash N$, for some $\delta > 0$. Then there exists a wedge $\mathcal{W}$ of edge M at (z_0, Jv_0) such that for every holomorphic function $f \in \mathcal{O}(\omega)$ there exists a function $F \in \mathcal{O}(\mathcal{W})$ with $F = f$ in the intersection of $\mathcal{W}$ with a neighborhood of $M\backslash N$ in $\mathbf{C}^n$.*

Remark. M is not supposed to be minimal at any point.

Proof. Fix a function $f \in \mathcal{O}(\omega)$. We shall construct deformations of our given original disc A as in [11] with boundaries in $M \cup \omega$ to show that the envelope of holomorphy of ω contains a (very thin) wedge of edge M at $A(1)$. Instead of appealing to a Baouendi-Treves approximation theorem, a version of which being not a priori valuable here, we shall naturally deal with the help of the so-called continuity principle.

We can assume that $A(1) = 0$ and that M is given in a coordinate system as in (10). Set $A(\zeta) = (w(\zeta), x(\zeta) + iy(\zeta))$. Since $T_0N \not\supset T_0^c M$ and $v_0 = \frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) \notin T_0N$, we can choose a C^1 generic one-codimensional submanifold $K \subset M$ in $\mathbf{C}^n$ containing N with tangent space T_0K not containing v_0. Making a unitary transformation in the w-space, we get that $T_0K \cap T_0^c M$ is given by $\{z = 0, v_1 = 0\}$ and then making a complex linear transformation in $\mathbf{C}^n$ stabilizing $\{y = 0\}$, we can assume that K has equation $v_1 = k(u_1, w_2, ..., w_p, x)$, where $k(0) = 0, dk(0) = 0$. Since $v_0 \notin T_0K$, the projection on the v_1-axis of v_0 is nonzero.

Let $\mu = \mu(w,x)$ be a C^∞, $\mathbf{R}$-valued function with support near the point $(w(-1), x(-1))$ that equals 1 there and let $\kappa : \mathbf{R}^q \to \mathbf{R}^q$ be a C^∞ function with $\kappa(0) = 0$ and $\kappa'(0) = Id$. We can assume that the supports of μ and κ are sufficiently concentrated in order that every manifold M_t with equation

$$y = H(w,x,t) = h(w,x) + \kappa(t)\mu(w,x) \tag{11}$$

is contained in ω and the deformation is localized in a neighborhood of $A(-1)$ in $\mathbf{C}^n$. Let $\chi = \chi(\zeta)$ be a smooth function on the unit circle supported in a small neighborhood of $\zeta = -1$ that will be chosen later. For every small t, we shall consider the natural perturbation A_t of the given disc A with the same holomorphic w-component as A that is attached to the union of the M_t. More precisely, $A_t(\zeta) = (w(\zeta), x_t(\zeta) + iy_t(\zeta))$ is obtained by solving the following equation on the unit circle

$$x_t = -T_1 H(w, x_t, t\chi). \tag{12}$$

According to [10], the solution of (12) exists and depends in a $C^{2,\beta}$ fashion with respect to all variables, no matter $0 < \beta'$ is $< \beta$, provided everything is very small. Rename β' as β. We let Π denote the canonical bundle epimorphism $\Pi : T\mathbf{C}^n|_M \to T\mathbf{C}^n|_M/TM$ and consider the $C^{1,\beta}$ mapping

$$D: \qquad \mathbf{R}^q \ni t \longmapsto \Pi\left(-\frac{\partial A_t}{\partial \zeta}(1)\right) \in T_0\mathbf{C}^n/T_0M \simeq \mathbf{R}^q. \tag{13}$$

LEMMA 4.A. *χ can be chosen in order that rk $D'(0) = q$.*

Proof. We, for completeness, include a proof of this result, originally due to Tumanov in [11]. Set $t = (t_1, ..., t_q)$. Differentiating the equation $y_t(\zeta) = H(w(\zeta), x_t(\zeta), t\chi(\zeta)), \zeta \in b\Delta$, with respect to $t_j, j = 1, ..., q$, we obtain that the holomorphic disc $\frac{\partial}{\partial t_j}|_{t=0} A_t(\zeta) = \dot{A}(\zeta) = (0, \dot{X}(\zeta) + i\dot{Y}(\zeta))$ satisfies the following equation on the unit circle

$$\dot{Y} = H_x \circ A \dot{X} + \chi H_{t_j} \circ A. \tag{14}$$

We also introduce some notations. For a $C^{1,\beta}$-smooth function $g(\zeta)$ on the unit circle with $g(1) = 0$, we write

$$\mathcal{J}(g) = \frac{1}{\pi}\int_0^{2\pi} \frac{g(e^{i\theta})}{|e^{i\theta} - 1|^2} d\theta,$$

where the integral is understood in the sense of principal value. Then, if $g \in C^{1,\beta}(\overline{\Delta})$ is holomorphic in Δ and vanishes at 1, we have

$$\mathcal{J}(g) = -\frac{\partial g}{\partial \zeta}(1) = i\frac{d}{d\theta}|_{\theta=0} g(e^{i\theta}). \tag{15}$$

Notice also that for $C^{1,\beta}$ real-valued functions g, g' with $g(1) = g'(1) = 0$, applying (15) to the holomorphic function $(g + iT_1 g)(g' + iT_1 g')$ vanishing to second order at 1, we obtain

$$\mathcal{J}(gg' - T_1 g T_1 g') = 0. \tag{16}$$

Associate with A and H a $q \times q$ matrix-valued function $G(\zeta)$ on the unit circle as a solution to the equation

$$G = I + T_1(GH_x \circ A). \tag{17}$$

The definition of G implies that $G(1) = I$ and $T_1 G = -GH_x \circ A + H_x \circ A(1) = -GH_x \circ A$, since $A(1) = 0$ and $h(0) = 0, dh(0) = 0$. Using (14) and G, we can write on the unit circle

$$\begin{aligned} G\chi H_{t_j} \circ A &= G(\dot{Y} - H_x \circ A\dot{X}) \\ &= G\dot{Y} - (T_1 G)(T_1 \dot{Y}) \\ &= \dot{Y} + (G - I)\dot{Y} - T_1(G - I)T_1\dot{Y}. \end{aligned}$$

By virtue of (16),

$$\mathcal{J}(G\chi H_{t_j}\circ A)=\mathcal{J}(\dot{Y}). \tag{18}$$

On the other hand, according to (15) and the fact that $\dot{X}=-T_1\dot{Y}$,

$$\mathcal{J}(\dot{Y})+i\mathcal{J}(T_1\dot{Y})=i\frac{\partial\dot{Z}}{\partial\zeta}(1)=\frac{d}{d\theta}|_{\theta=0}(\dot{X}+i\dot{Y}). \tag{19}$$

Identifying the imaginary part of the two extreme terms and taking (14) into account, we have

$$\mathcal{J}(T_1\dot{Y})=\frac{d}{d\theta}|_{\theta=0}\dot{Y}=\frac{d}{d\theta}|_{\theta=0}(H_x\circ A\dot{X}+\chi H_{t_j}\circ A)=0, \tag{20}$$

if we choose χ in order that χ is equal to zero near $\zeta=1$ and since $\dot{X}(1)=0$, $dH(0)=0$. (18), (19) and (20) therefore yield

$$i\frac{\partial\dot{Z}}{\partial\zeta}(1)=\mathcal{J}(G\chi H_{t_j}\circ A).$$

Natural coordinates on $T_0\mathbf{C}^n/T_0M$ being given by $y_1,...,y_q$, we obtain in these coordinates

$$\frac{\partial D}{\partial t_j}(0)=\Pi\left(-\frac{\partial\dot{Z}}{\partial\zeta}(1)\right)=\mathcal{J}(G\chi H_{t_j}\circ A). \tag{21}$$

Furthermore, choose χ in order that $\mathcal{J}(\chi)=1$ and the support of χ is concentrated near $\zeta=-1$ so that the vectors $\mathcal{J}(G\chi H_{t_j}\circ A)$ are close to the vectors $G(-1)H_{t_j}\circ A(-1)$ and linearly independent, for $j=1,...,q$ respectively. This is possible, since G is non singular at every point on the unit circle and the $H_{t_j}\circ A(-1)$, $j=1,...,q$ are linearly independent by the choice of κ. This completes the proof of lemma 4.A.

We deduce:

COROLLARY 4.B. *There exists a neighborhood $\mathcal{T}$ of 0 in $\mathbf{R}^q$ such that $\{sD(t);\ s>0, t\in\mathcal{T}\}$ contains Γ_0 an open cone with vertex 0 in $\mathbf{R}^q$ and $\Pi(-\frac{\partial A}{\partial\zeta}(1))\in\Gamma_0$.*

We shall need another type of deformations of our disc. Let τ denote a supplementary real parameter, $\tau_0>0$ and $\mathcal{V}$ a neighborhood of 0 in $\mathbf{C}^{p-1}$. For $t\in\mathcal{T}$, $|\tau|<\tau_0$, $a\in\mathcal{V}$, we consider the disc

$$A_{t,\tau,a}(\zeta)=(e^{i\tau}w_1(\zeta),w_2(\zeta)+a_2(\zeta-1),...,w_p(\zeta)+a_p(\zeta-1),x_{t,\tau,a}(\zeta)+iy_{t,\tau,a}(\zeta)) \tag{22}$$

where $x_{t,\tau,a}$ is the $C^{2,\beta}$-smooth in all variables solution of the following equation on $b\Delta$

$$x_{t,\tau,a}=-T_1H(e^{i\tau}w_1(\zeta),w_2(\zeta)+a_2(\zeta-1),...,w_p(\zeta)+a_p(\zeta-1),x_{t,\tau,a},t\chi). \tag{23}$$

Introduce also the smooth mapping

$$E:\qquad \mathcal{T}\times I_{\tau_0}\times\mathcal{V}\ni(t,\tau,a)\ \longmapsto\ \frac{d}{d\theta}|_{\theta=0}A_{t,\tau,a}(e^{i\theta})\in T_0M. \tag{24}$$

According to Lemma 4.A, we have that $rk\ E'(0)=2p+q-1$ for a good choice of the function χ, since we defined $w_{t,\tau,a}(\zeta)$ not depending on t and the partial rank of E with respect to (τ,a)

is equal to $(2p-1)$. So is the rank of the mapping derived from E by taking normalized tangent vectors with respect to a hermitian structure on $\mathbf{C}^n$.

Therefore we obtained

LEMMA 4.C. *χ can be chosen in order that the following holds: there exist $\tau_0 > 0$, $\mathcal{T}$ a neighborhood of 0 in $\mathbf{R}^q$ and $\mathcal{V}$ a neighborhood of 0 in $\mathbf{C}^{p-1}$ such that the set*

$$\hat{\Gamma}_0 = \{s\frac{dA_{t,\tau,a}}{d\theta}(1);\ s<0 \text{ or } s>0, t\in\mathcal{T}, \tau\in I_{\tau_0}, a\in\mathcal{V}\} \tag{25}$$

is a $(2p+q)$-dimensional open connected bicone with vertex 0 in the $(2p+q)$-dimensional space T_0M containing $v_0 = \frac{d}{d\theta}|_{\theta=0}A(e^{i\theta})$.

What we call a bicone is a union of two linear cones with same vertex and symmetric opposite directions. As a consequence, we can read Lemma 4.C inside the base manifold as follows.

LEMMA 4.D. *Shrinking the open neighborhoods $\mathcal{T}, I_{\tau_0}, \mathcal{V}$ if necessary, there exists an open connected arc $1\in I_1\subset b\Delta$ such that the set*

$$\hat{\Gamma} = \{A_{t,\tau,a}(\zeta);\ t\in\mathcal{T}, \tau\in I_{\tau_0}, a\in\mathcal{V}, \zeta\in I_1\} \tag{26}$$

is a $(2p+q)$-dimensional closed connected (nonlinear) truncated bicone contained in M.

We now introduce a third deformation of A, consisting in varying the base point. We let A_{t,τ,a,p_0} denote the disc attached to $M\cup\omega$ which is the perturbation of $A_{t,\tau,a}$ that passes through the point $p_0\in K$ with coordinates $(u_1^0, w_2^0, ..., w_p^0, x^0)$ on K, *i.e.* $A_{t,\tau,a,p_0}(1) = p_0$ and

$$A_{t,\tau,a,p_0}(\zeta) = (e^{i\tau}w_1(\zeta)+u_1^0+iv_1^0, w_2(\zeta)+a_2(\zeta-1)+w_2^0, ..., w_p(\zeta)+a_p(\zeta-1)+w_p^0, x_{t,\tau,a,p_0}(\zeta)+iy_{t,\tau,a,p_0}(\zeta)) \tag{27}$$

where x_{t,τ,a,p_0} is the solution of Bishop's equation with parameters

$$x_{t,\tau,a,p_0} = -T_1H(e^{i\tau}w_1+u_1^0+iv_1^0, w_2+a_2(.-1)+w_2^0, ..., w_p+a_p(.-1)+w_p^0, x_{t,\tau,a,p_0}, t\chi)+x^0, \tag{28}$$

and where $v_1^0 = k(u_1^0, w_2^0, ..., w_p^0, x_1^0, ..., x_q^0)$, so $A_{t,\tau,a,p_0}(1) = p_0\in K$.

For convenience, we shall allow us to shrink a finite number of times the open set $\mathcal{T}\times I_{\tau_0}\times\mathcal{V}\times\mathcal{K}$ *without explicit mention* in the rest of the proof of Proposition 4.1.

Notice that if we let $\tau=0$ and $a=0$, the set of points $A_{t,0,0,p_0}(\zeta)$ for ζ in the interior of a neighborhood Δ_1 of 1 in $\overline{\Delta}$ always cover a wedge of edge M at 0 in the direction Jv_0 when t varies in a neighborhood $\mathcal{T}$ of 0 in $\mathbf{R}^q$ and p_0 in a neighborhood $\mathcal{K}$ of 0 in K, according to Corollary 4.B and the fact that $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta})\notin T_0K$. In the non singular case, when a local approximation theorem is valid for CR functions, this leads to propagation of wedge extendibility along a disc, as in [11].

Since, however, our $f\in\mathcal{O}(\omega)$ possibly has singularities on N, a different argument is needed. Notice that every embedded disc A_{t,τ,a,p_0} with sufficiently small parameters (t,τ,a,p_0) is close to A and is analytically isotopic to a point in ω for $p_0\in\mathcal{K}\backslash N$. Indeed, A_{t,τ,a,p_0} is first analytically isotopic in ω to A_{0,τ,a,p_0} by construction and A_{0,τ,a,p_0} is attached to $M\backslash N$. Clearly, since $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) = v_0$ is not tangent to N, and $A = A_{0,0,0,0}$ satisfies $A(b\Delta\backslash\{1\})\subset M\backslash N$, each disc A_{0,τ,a,p_0} with p_0 not in N cannot meet N along its boundary, provided all the parameters are small. On the contrary, discs in the family with base point $p_0\in\mathcal{K}\cap N$ meet N, but only at p_0. Then the fact that for $p_0\in\mathcal{K}\backslash N$, A_{0,τ,a,p_0} is analytically isotopic to a point in $M\backslash N$

is a consequence of our hypothesis that all discs in $\mathcal{V}(A,\delta)$ attached to $M\backslash N$ are analytically isotopic to a point in $M\backslash N$, if $\delta>0$ is very small.

Since our discs with base point $p_0\in\mathcal{K}\backslash N$ are analytically isotopic in ω to a point, our given function $f\in\mathcal{O}(\omega)$ can be analytically extended in a neighborhood ω_A of $A_{t,\tau,a,p_0}(\overline{\Delta})$ in $\mathbf{C}^n$, if we make use of the so-called continuity principle along the isotopy (Proposition 3.3). Precisely, there exist a neighborhood ω_A of $A(\overline{\Delta})$ in $\mathbf{C}^n$ and $f_A\in\mathcal{O}(\omega_A)$ with $f_A\equiv f$ in a neighborhood of $A_{t,\tau,a,p_0}(b\Delta)$. On the contrary, discs meeting N necessarily lack a similar extension property.

We obtained that for each $p_0\in\mathcal{K}\backslash N$, every CR function $f\in \mathrm{CR}(M\backslash N)$ has the property that $f\circ A_{t,\tau,a,p_0}|_{b\Delta}$ extends holomorphically into Δ.

Let $v_0\in\Gamma$ be a q-dimensional proper linear bicone in the $(2p+q)$-dimensional space T_0M and contained in $\widehat{\Gamma}_0$ such that the projection $T_0\Gamma\to T_0M/T_0^cM$ is surjective and $\overline{\Gamma}\cap T_0^cM=\{0\}$. Let $\mathcal{P}$ denote the set of parameters

$$\mathcal{P}=\{(t,\tau,a)\in\mathcal{T}\times I_{\tau_0}\times\mathcal{V};\ \frac{d}{d\theta}A_{t,\tau,a}(1)\in\Gamma\}.$$

Then $\mathcal{P}$ is a C^1-smooth $(q-1)$-dimensional submanifold of $\mathcal{T}\times I_{\tau_0}\times\mathcal{V}$. Shrinking $\mathcal{T},I_{\tau_0},\mathcal{V},\mathcal{K}$ and Δ_1 if necessary, the set

$$\mathcal{W}=\{A_{t,\tau,a,p_0}(\zeta);\ (t,\tau,a)\in\mathcal{P},p_0\in\mathcal{K},\zeta\in\overset{\circ}{\Delta}_1\}$$

contains a wedge of edge M at 0. Furthermore, the mapping

$$\mathcal{P}\times\mathcal{K}\times\overset{\circ}{\Delta}_1\ni(t,\tau,a,p_0,\zeta)\mapsto A_{t,\tau,a,p_0}(\zeta)\in\mathbf{C}^n\backslash M$$

becomes a smooth embedding. Since the mapping remains injective on $\mathcal{K}\backslash N$, we can set unambiguously

$$F(z):=\frac{1}{2i\pi}\int_{b\Delta}\frac{f\circ A_{t,\tau,a,p_0}(\eta)}{\eta-\zeta}d\eta$$

as a value at points $z=A_{t,\tau,a,p_0}(\zeta)$ for an extension of $f|_{M\backslash N}$, $p_0\in\mathcal{K}\backslash N,(t,\tau,a)\in\mathcal{P},\zeta\in\overset{\circ}{\Delta}_1$. Since f extends holomorphically to the interior of these discs, we get a continuous extension F on each $A_{t,\tau,a,p_0}(\Delta_1)$, $p_0\in\mathcal{K}\backslash N$. Thus, the extension F of $f|_{M\backslash N}$ also becomes continuous on

$$(\mathcal{W}\backslash N_{\mathcal{P}})\cup(M\backslash N),$$

where

$$N_{\mathcal{P}}=\{A_{t,\tau,a,p_0}(\zeta);\ (t,\tau,a)\in\mathcal{P},p_0\in\mathcal{K}\cap N,\zeta\in\overset{\circ}{\Delta}_1\}.$$

Since $f|_{M\backslash N}$ extends analytically to a neighborhood of $A_{t,\tau,a,p_0}(\overline{\Delta})$, F is holomorphic into $\mathcal{W}\backslash N_{\mathcal{P}}$. Indeed, fix a point $(\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0)\in\mathcal{P}\times(\mathcal{K}\backslash N)$ and let $\tilde{\mathcal{P}}\times\tilde{\mathcal{K}}$ be a neighborhood of $(\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0)$ in $\mathcal{P}\times(\mathcal{K}\backslash N)$ such that for each $(t,\tau,a,p_0)\in\tilde{\mathcal{P}}\times\tilde{\mathcal{K}}$, $A_{t,\tau,a,p_0}(\overline{\Delta})$ is contained in some neighborhood $\tilde{\omega}$ of $A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(\overline{\Delta})$ in $\mathbf{C}^n$ such that there exists a holomorphic function $\tilde{f}\in\mathcal{O}(\tilde{\omega})$ with $\tilde{f}$ equal to f near $A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(b\Delta)$. Let $\tilde{\zeta}\in\overset{\circ}{\Delta}_1$ and $\tilde{z}=A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(\tilde{\zeta})$. To check that the previously defined function F is holomorphic in a neighborhood of $\tilde{z}$, we note that for $z=A_{t,\tau,a,p_0}(\zeta)$, $(t,\tau,a,p_0)\in\tilde{\mathcal{P}}\times\tilde{\mathcal{K}}$, ζ in some neighborhood $\tilde{\Delta}_1$ of $\tilde{\zeta}$ in $\overset{\circ}{\Delta}_1$, $\tilde{f}(z)$ is given by the Cauchy integral formula

$$\tilde{f}(z)=\frac{1}{2i\pi}\int_{b\Delta}\frac{\tilde{f}\circ A_{t,\tau,a,p_0}(\eta)}{\eta-\zeta}d\eta=\frac{1}{2i\pi}\int_{b\Delta}\frac{f\circ A_{t,\tau,a,p_0}(\eta)}{\eta-\zeta}d\eta=F(z).$$

As a consequence, $\tilde{f}(z) = F(z)$ for z in a small neighborhood of $\tilde{z}$ in $\mathbf{C}^n$, since the mapping $(t, \tau, a, p_0, \zeta) \mapsto A_{t,\tau,a,p_0}(\zeta)$ from $\tilde{\mathcal{P}} \times \tilde{\mathcal{K}} \times \tilde{\Delta}_1$ to $\mathbf{C}^n$ has rank $2n$ at $(\tilde{t}, \tilde{\tau}, \tilde{a}, \tilde{p}_0, \tilde{\zeta})$.

This proves that F is holomorphic into $\mathcal{W}\backslash N_{\mathcal{P}}$.

By shrinking ω near 0, which does not modify the possible disc deformations, we can insure that

$$\omega \cap \mathcal{W}$$

is connected, since $\overline{\Gamma} \cap T_0^c M = \{0\}$ and then also

$$\omega \cap (\mathcal{W}\backslash N_{\mathcal{P}}),$$

since $N_{\mathcal{P}}$ is a closed two-codimensional submanifold of $\mathcal{W}$. Therefore $f \in \mathcal{O}(\omega)$ and $F \in \mathcal{O}(\mathcal{W}\backslash N_{\mathcal{P}})$ stick together in a single holomorphic function in $\omega \cup (\mathcal{W}\backslash N_{\mathcal{P}})$, since both are continuous up to $M\backslash N$, which is a uniqueness set, and coincide there.

However, the proof of Proposition 4.1 will not be finished until we get rid of $N_{\mathcal{P}}$. This is why we introduced the supplementary parameters τ and a.

Since $\hat{\Gamma}_0$ is a $(2p+q)$-dimensional bicone in the $(2p+q)$-dimensional space T_0M, we can choose two q-dimensional bicones $v_0 \in \Gamma_2$ and Γ_2' contained in $\hat{\Gamma}_0$ with the properties that

> (i) The projections $T_0\Gamma_2 \to T_0M/T_0^cM$ and $T_0\Gamma_2' \to T_0M/T_0^cM$ are surjective, $\overline{\Gamma_2}, \overline{\Gamma_2}' \cap T_0^cM = \{0\}$. Moreover, for $v_2 \in \Gamma_2$ there exists $v_2' \in \Gamma_2'$ such that $v_2 - v_2' \in T_0^cM/(T_0^cM \cap T_0N)\backslash\{0\}$ and vice versa.
> (ii) There exist open bicones $\hat{\Gamma}_2$ and $\hat{\Gamma}_2'$ in $\hat{\Gamma}_0$ with $\Gamma_2 \subset \hat{\Gamma}_2, \Gamma_2' \subset \hat{\Gamma}_2'$ and $\hat{\Gamma}_2 \cap \hat{\Gamma}_2' = \emptyset$.

Set $\mathcal{P}_2 = \{(t, \tau, a); \frac{d}{d\theta} A_{t,\tau,a}(1) \in \Gamma_2\}$ and similarly for $\mathcal{P}_2'$. Then condition (i) insures that the two following wedges with edge M at $(0, \eta_0)$, $\eta_0 = Jv_0 \bmod T_0M$,

$$\mathcal{W}_2 = \{A_{t,\tau,a,p_0}(\mathring{\Delta}_1);\ (t,\tau,a) \in \mathcal{P}_2, p_0 \in \mathcal{K}\} \qquad \mathcal{W}_2' = \{A_{t,\tau,a,p_0}(\mathring{\Delta}_1);\ (t,\tau,a) \in \mathcal{P}_2', p_0 \in \mathcal{K}\} \quad (29)$$

contain a wedge $\mathcal{W}$ of edge M at $(0, \eta_0)$. According to the above, one can construct two holomorphic extensions F_2 and F_2' of f into $\mathcal{W}_2\backslash N_{\mathcal{P}_2}$ and $\mathcal{W}_2'\backslash N_{\mathcal{P}_2'}$ respectively, where $N_{\mathcal{P}_2}$ and $N_{\mathcal{P}_2'}$ denote the unattainable sets,

$$N_{\mathcal{P}_2} = \{A_{t,\tau,a,p_0}(\zeta); (t,\tau,a) \in \mathcal{P}_2, p_0 \in \mathcal{K} \cap N, \zeta \in \mathring{\Delta}_1\}$$

and similarly for $N_{\mathcal{P}_2'}$. Since the rank of the mapping

$$\mathcal{P} \times (\mathcal{K}\backslash N) \times \mathring{\Delta}_1 \ni (t, \tau, a, p_0, \zeta) \mapsto A_{t,\tau,a,p_0}(\zeta) \in \mathbf{C}^n\backslash M$$

is equal to $(2n-2)$, $N_{\mathcal{P}_2}$ and $N_{\mathcal{P}_2'}$ are closed two-codimensional submanifolds of $\mathcal{W}_2, \mathcal{W}_2'$ respectively. This implies that

$$(\mathcal{W}\backslash N_{\mathcal{P}_2}) \cap (\mathcal{W}\backslash N_{\mathcal{P}_2'}) = \mathcal{W}\backslash(N_{\mathcal{P}_2} \cup N_{\mathcal{P}_2'})$$

is an open connected set. The restrictions of F_2 and F_2' to $(\mathcal{W}\backslash N_{\mathcal{P}_2})$ and $(\mathcal{W}\backslash N_{\mathcal{P}_2'})$ therefore stick together in a function F that is holomorphic into

$$\mathcal{W}\backslash(N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'}).$$

Indeed, both are continuous up to $M\backslash N$, which is a uniqueness set, and coincide there with $f|_{M\backslash N}$ by construction. Assuming again that ω is thin near 0, we insure that

$$\omega \cap (\mathcal{W}\backslash(N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'}))$$

is a connected open set. Therefore f and F stick together in a well-defined holomorphic function into $\omega \cup (\mathcal{W}\backslash(N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'}))$.

According to condition (ii),

$$\overline{N}_{\mathcal{P}_2} \cap \overline{N}_{\mathcal{P}_2'} \cap M = N.$$

Furthermore, let $\tilde{z} = A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(\tilde{\zeta})$ be a point in $N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'} \cap \mathcal{W}$ (if nonempty). Then, according to conditions (i) and (ii), $T_{\tilde{z}}N_{\mathcal{P}_2}$ and $T_{\tilde{z}}N_{\mathcal{P}_2'}$ intersect transversally in $T_{\tilde{z}}\mathbf{C}^n$, since these are close to $T_0N + JT_0\overline{\Gamma}_2$ and $T_0N + JT_0\overline{\Gamma}_2'$ respectively. Therefore, in a neighborhood $\tilde{\mathcal{W}}$ of $\tilde{z}$ in $\mathcal{W}$, $L_{\tilde{z}} = N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'} \cap \tilde{\mathcal{W}}$ consists of a four-codimensional manifold in $\tilde{\mathcal{W}}$. $L_{\tilde{z}}$ is removable for functions which are holomorphic into $\tilde{\mathcal{W}}\backslash L_{\tilde{z}}$. We therefore showed that $N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'}$ is in fact removable for functions holomorphic into $\mathcal{W}\backslash(N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'})$. So F extends holomorphically through $N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'}$ as a function $F \in \mathcal{O}(\mathcal{W})$ with $F = f$ in the intersection of $\mathcal{W}$ with a neighborhood of $M\backslash N$ in $\mathbf{C}^n$.

The proof of Proposition 4.1 is complete.

To achieve the proof of Theorem 1, we shall remark that in the proof of Proposition 4.1, we only used the part of ω lying near $A(-1)$ to perform deformations of A. Therefore

PROPOSITION 4.2. *Let M be generic, $C^{2,\alpha}$-smooth, let $z_0 \in M$, let $N \ni z_0$ be a C^1 submanifold with $\mathrm{codim}_M N = 3$ and $T_{z_0}N \not\supset T^c_{z_0}M$. Assume there exists a sufficiently small embedded analytic disc $A \in C^{2,\beta}(\overline{\Delta})$ attached to M, $A(1) = z_0$, with $A(b\Delta\backslash\{1\}) \subset M\backslash N$, $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) = v_0 \notin T^c_{z_0}M$, $v_0 \notin T_{z_0}N$ and all discs in $\mathcal{V}(A,\delta)$ attached to $M\backslash N$ are analytically isotopic to a point in $M\backslash N$, for some $\delta > 0$ and let $\mathcal{W}_0$ be an open wedge attached to $M\backslash N$ containing a neighborhood of $A(-1)$ in $\mathbf{C}^n$. Then there exists a wedge $\mathcal{W}$ of edge M at (z_0, Jv_0) such that for every continuous function f in $\mathcal{W}_0 \cup (M\backslash N)$ which is CR on $M\backslash N$ and holomorphic into $\mathcal{W}_0$, there exists a holomorphic function F in $\mathcal{O}(\mathcal{W})$ continuous up to $M\backslash N$ with $F = f$ in a neighborhood of z_0 in $M\backslash N$.*

Proof. We only have to check that the extension $F(z)$ defined unambiguously in terms of a Cauchy integral on boundaries of discs becomes holomorphic into the wedge $\mathcal{W}$ foliated by images of the interior of Δ_1. One can introduce very small deformations M^d of M into $\mathcal{W}_0$ depending on a real parameter $d \geq 0$ with $M^0 = M$, apply the continuity principle argument to conclude F^d is holomorphic and let d tend to 0.

The proof of Theorem 1 can be completed as follows. Proposition 2.1 and Proposition 3.7 insure the existence of an embedded disc A meeting N only at one point $A(1)$ of its boundary with nearby discs being analytically isotopic to a point in $M\backslash N$. Slightly deform M near $A(-1)$ in a manifold M^d by pushing it into the open wedge $\mathcal{W}_0$ of automatic extension, the deformation A^d of A remaining a disc with all similar properties. Then Proposition 4.2 yields a holomorphic extension in a wedge of edge M^d at z_0. Since $M^d \equiv M$ near z_0, this gives a wedge of edge M at z_0.

The proof of Theorem 1 is complete.

COROLLARY 4.3. *Let M be a real analytic generic manifold in $\mathbf{C}^n$ $(n \geq 2)$ of finite type at every point with CRdim $M = p \geq 1$. Then every real analytic subset $A \subset M$ with $\mathrm{codim}_M A \geq 3$ is removable.*

Proof. Let $A \subset M$ be a real analytic subset with dim $A < 2p$ and let $M^d \subset \mathbf{C}^n$ be a C^∞ generic manifold of finite type at every point with $M^d \supset A$ and $TM^d|_A = TM|_A$. Fix an open neighborhood V of A in M^d and $\varepsilon > 0$. We shall show that there exists a C^∞ generic manifold M^{d_m} with $M^{d_m} \cap (M^d \backslash V) = M^d \backslash V$ and $||M^{d_m} - M^d||_{C^\infty} < \varepsilon$ such that for each function $f \in \mathrm{CR}(M^d \backslash N)$, there exists a function $f^{d_m} \in \mathrm{CR}(M^{d_m})$ with $f^{d_m} = f$ on $M^d \backslash V$. According to the Deformation Lemma below, this will give the desired result on removability of A.

A admits a stratification with the property that the closure of each stratum only intersects strata of smaller dimension. Take N' a connected stratum of maximal possible dimension. Then N' is an embedded submanifold of $M^d \backslash (A \backslash N')$ to which Theorem 1 applies, since dim $N' < 2p$ implies $T_z N' \not\supset T_z^c M^d$ for every $z \in N'$. Since N' is removable, there exists a C^∞ deformation M^{d_1} over N' of M^d with support in V such that $||M^{d_1} - M^d||_{C^\infty} < \varepsilon_1 < \varepsilon/2$ and $M^{d_1} \cap (M^d \backslash V) = M^d \backslash V$ and $f^{d_1} \in \mathrm{CR}(M^{d_1} \backslash (A \backslash N'))$ with $f^{d_1} = f$ in $M^d \backslash V$. For ε_1 small enough, M^{d_1} is of finite type at every point. Since the stratum of maximal possible dimension in the set of remaining strata always looks like a locally embedded submanifold, all strata of A can be successively removed on the successive deformations of M^d. Hence A is removable.

Assume by induction that given a real analytic subset $A' \subset M$ with dim $A' \leq k \leq \dim M - 4$ and a C^∞ everywhere of finite type generic manifold in $\mathbf{C}^n$, $M^d \supset A'$, with $TM^d|_{A'} = TM|_{A'}$, A' is removable for CR functions on $M^d \backslash A'$. Let $A \subset M$ be a stratified real analytic subset, dim $A = k + 1$ and let M^d be a C^∞ everywhere of finite type generic manifold in $\mathbf{C}^n$ with $M^d \supset A$ and $TM^d|_A = TM|_A$. Let V be a neighborhood of A in M^d and let $\varepsilon > 0$ be arbitrary. Choose N a connected stratum of A of maximal dimension. Then the set

$$N^c = \{z \in N;\ T_z N \supset T_z^c M^d = T_z^c M\}$$

is a proper subanalytic set of N. Indeed, since M^d is minimal at every point, M^d does not contain germs of CR manifolds Σ with CRdim Σ = CRdim M^d. A relatively open set in N contained in N^c would be such a CR manifold by definition of N^c. Therefore, the dense open subset of N

$$N \backslash N^c = \{z \in N;\ T_z N \not\supset T_z^c M\}$$

is removable, by virtue of Theorem 1.

Fix a function $f \in \mathrm{CR}(M^d \backslash N)$. Smoothly deform M^d in a manifold M^{d_1} of class C^∞ and of finite type at every point over $N \backslash N^c$ by pushing it into the wedge where removability of $N \backslash N^c$ holds and keeping both $TM^d|_{N^c} = TM^{d_1}|_{N^c}$ and $M^{d_1} \cap (M^d \backslash V) = (M^d \backslash V)$, with $||M^{d_1} - M^d||_{C^\infty} < \varepsilon_1 < \varepsilon/2$. Take the restriction f^{d_1} of f to $M^{d_1} \backslash N^c$. According to the induction hypothesis, N^c is removable on M^{d_1}, hence there exists a smooth deformation M^{d_2} of M^{d_1} with $||M^{d_2} - M^{d_1}||_{C^\infty} < \varepsilon_2 < \varepsilon/4$, $TM^{d_2}|_{A \backslash N} = TM^d|_{A \backslash N}$, $M^{d_2} \cap (M^d \backslash V) = M^d \backslash V$ such that for each function $f \in \mathrm{CR}(M^d \backslash A)$ there exists $f^{d_2} \in \mathrm{CR}(M^{d_2} \backslash (A \backslash N))$ with $f^{d_2} = f$ on $M^d \backslash V$.

All strata N can be successively removed by applying the induction hypothesis to N^c.

The proof finishes out with the following.

Deformation Lemma. *Let M be minimal at $z_0 \in M$, Φ a proper closed subset with $b\Phi \ni z_0$. Assume that for each neighborhood V of Φ in M, each $\varepsilon > 0$, there exists a manifold M^d with $M^d \cap (M \backslash V) = M \backslash V$, $||M^d - M||_{C^{2,\alpha}} < \varepsilon$ such that for each function $f \in CR(M \backslash \Phi)$ there exists a function $f^d \in CR(M^d)$ with $f^d \equiv f$ on $M \backslash V$. Then there exists a wedge $\mathcal{W}$ of edge M at z_0 with $CR(M \backslash \Phi)$ extending holomorphically into $\mathcal{W}$ and continuously in $\mathcal{W} \cup (M \backslash \Phi)$.*

Proof. The result follows from the existence of a disc with defect 0 attached to M^d when ε is small enough, obtained as a perturbation of a disc with defect 0 attached to the manifold M, minimal at z_0. Each f^d extends holomorphically into a wedge $\mathcal{W}^d$ of edge M^d which stabilizes as ε tends to 0 and gives a wedge $\mathcal{W}$ attached to M as V shrinks to Φ.

The proof of Corollary 4.3 is complete.

5. Proof of Theorem 2. The meaning of Theorem 2 is that, when the Cauchy-Riemann dimension of M is greater or equal to 2, some two-codimensional singularities are locally removable. However, this is not generically true in CR dimension equal to 1. Take for example M, a hypersurface in $\mathbf{C}^2$ and g a holomorphic function in a neighborhood of some $z_0 \in M$ in $\mathbf{C}^2$ such that $g(z_0) = 0$ and $dg(z_0) \neq 0$. Whenever $\Sigma_g = \{g = 0\}$ is not tangent to M at z_0, $\frac{1}{g}|_{M\backslash N}$ defines a nonextendible CR function in any side, and $N = \Sigma_g \cap M$ is locally a two-codimensional submanifold in M with $T_{z_0}N \cap T^c_{z_0}M = \{0\}$. The difference between CR dimension one and CR dimension greater than two is explained in the two proofs of the following theorem.

THEOREM 5.A. *Let M be a $C^{2,\alpha}$-smooth $(0 < \alpha < 1)$ generic manifold in $\mathbf{C}^n (n \geq 3)$ with CRdim $M = p \geq 2$ and $N \subset M$ a $C^{2,\alpha}$ submanifold of M with $codim_M N = 2$ that is generic in $\mathbf{C}^n$. Assume that there exists an open neighborhood V of N in M such that $(M\backslash N) \cap V$ has the wedge extension property. Then there exists an open wedge $\mathcal{W}$ attached to $M \cap V$ such that every continuous CR function on $(M\backslash N) \cap V$ extends holomorphically into $\mathcal{W}$ and continuously in $(M\backslash N) \cup \mathcal{W}$.*

Proof. The condition that N is a generic manifold in $\mathbf{C}^n$ means that CRdim $(T_nN \cap T^c_nM) = p-2, \forall n \in N$. Therefore, N cannot contain germs of CR manifolds with CR dimension equal to $p-1$. In two senses of speaking, Theorem 5.A treats the generic case of Theorem 2.

To obtain Theorem 5.A, we shall use a natural and beautiful deformation result due to Jöricke, obtained in [5] as a tool in her deriving a proof of a conjecture of Trépreau.

THEOREM. (JÖRICKE [5]). *Let $z_0 \in M$ be generic, $C^{2,\alpha}$-smooth $(0 < \alpha < 1)$ in $\mathbf{C}^n$ and let C be a truncated open convex cone in $\mathbf{C}^n$ with vertex z_0 and some $v \in T^c_{z_0}M\backslash\{0\}$ in C. Then for every neighborhood $\omega \subset C$ of $C \cap M$ in $\mathbf{C}^n$, every $\beta, 0 < \beta < \alpha$, there exists a $C^{2,\beta}$-smooth generic manifold $M^d \subset M \cup \omega$ with $M^d\backslash\omega = M\backslash\omega$ such that M^d is minimal at z_0.*

Let $z_0 \in N$ and let $U \subset V$ be a small neighborhood of z_0 in M. Let M_1 be a closed $C^{2,\alpha}$ one codimensional connected generic submanifold of $M \cap U$ containing $N \cap U$. Since N itself is generic, there exists C a truncated open convex cone in $\mathbf{C}^n$ with vertex z_0 and some $v \in T^c_{z_0}M_1\backslash\{0\}$ in C such that $C \cap N = \emptyset$.

Since $(M \cap U)\backslash N$ has the wedge extension property, there exists a wedge $\mathcal{W}_0$ attached to $(M \cap U)\backslash N$ to which CR functions on $M\backslash N$ holomorphically extend. We can first perform a small $C^{2,\alpha}$ deformation M^d of M into $\mathcal{W}_0$ leaving $M\backslash C$ fixed in order that $\mathcal{W}_0$ becomes a neighborhood ω of $(M_1)^d \cap C$ in $\mathbf{C}^n$. Moreover, we can assume that the tangent spaces and the complex tangent spaces to M and M_1 at z_0 are fixed under d. Thus, $v \in T^c_{z_0}(M_1)^d$ too.

Secondly, minimalize $(M_1)^d$ at z_0 by applying the deformation theorem of Jöricke: we get a manifold $(M_1)^{d_2}$ contained in $\omega \cup (M_1\backslash C)$ such that $(M_1)^{d_2} \cap (M_1\backslash C) = M_1\backslash C$ and $(M_1)^{d_2}$ is minimal at z_0. Furthermore, this deformation can be extended in a smooth deformation M^{d_2} of M with support in $\overline{C \cap M}$, *i.e.* $M^{d_2} \cap (M\backslash C) = M\backslash C$.

We shall now show that there exists a neighborhood W of z_0 in M such that every point in $(W \cap N)\backslash\{z_0\}$ is removable, that is, for each point $z \in (W \cap N)\backslash\{z_0\}$ there exists a wedge $\mathcal{W}_z$ of edge M at z with $CR((M\backslash N) \cap V)$ extending holomorphically into $\mathcal{W}_z$.

Indeed, let f be a function in $\mathrm{CR}((M\backslash N)\cap V)$. Since f extends holomorphically into $\mathcal{W}_0$, we get a function, still denoted by f, which is CR on $M^{d_2}\backslash N$. We shall show in Lemma 5 below that f extends holomorphically into a wedge of edge M^{d_2} at z_0. Since $M \equiv M^{d_2}$ in a neighborhood of each point $z \in N$ with $z \neq z_0$, this gives the desired result.

LEMMA. (CHIRKA and STOUT [3]). *Let M be generic, $C^{2,\alpha}$ and let $M_1 \subset M$ be a $C^{2,\beta}$-smooth one codimensional submanifold, generic and minimal at $z_0 \in M_1$ in $\mathbf{C}^n$. Let $\Phi \subset M_1$ be a proper closed subset of M_1, $b\Phi \ni z_0$. Then Φ is removable at z_0.*

Proof. Fix a function $f \in \mathrm{CR}(M\backslash\Phi)$. Include M_1 in a regular one parameter family $M_{1,s}, |s| < \delta, \delta > 0$ of $C^{2,\beta}$ manifolds contained in M, such that $M_{1,0} = M_1$, $M_{1,s} \cap \Phi = \emptyset, s \neq 0$ and $\cup_s M_{1,s} = M$ near z_0. Using cartesian equations for both M_1 and M, these manifolds $M_{1,s}$ can be defined as translation-like modifications of M_1. Furthermore, we can impose that all manifolds $M_{1,s}$ contain some fixed point $z_1 \in M_1\backslash\Phi$. The method of sweeping out by wedges can be applied ([3]). Since existence of a disc with minimal defect is a stable property under $C^{2,\beta}$ deformations, Tumanov's theorem gives: every $f_s = f|_{M_{1,s}}$ is wedge extendible into a wedge $\mathcal{W}_s$ of edge $M_{1,s}$ for $s \in (-\delta', \delta')\backslash\{0\}$, $0 < \delta' \leq \delta$, and the wedges depend smoothly on s. By the uniqueness theorem and the fact that $M\backslash\Phi$ is connected near z_0, all $M_{1,s}$ containing $z_1 \in M_1\backslash\Phi$, holomorphic extensions obtained constitute a single function $\tilde{f}$ holomorphic into the union $\mathcal{W} = \cup_{s\neq 0}\mathcal{W}_t$. By construction, the set $\mathcal{W}$ contains a nontrivial wedge of edge M at z_0. Moreover, $\tilde{f}$ is continuous up to $M\backslash\Phi$ and equals f there.

The proof of Theorem 5.A is complete.

Remark. The method of sweeping out by wedges can only be applied when CRdim $M \geq 2$, since otherwise any generic manifold M_1 as above is totally real in $\mathbf{C}^n$ hence has no CR structure.

Remark. The method of proof of Theorem 5.A cannot be used to derive Theorem 1 or Theorem 2 in full generality for the following reason. If N contains a CR manifold Σ through z_0 with CRdim $\Sigma = p - 1$, no one codimensional submanifold M_1 of M containing N that is generic in $\mathbf{C}^n$ can be minimal, since then Σ contains the local CR orbit $\mathcal{O}_{CR}^{loc}(z_0, M_1)$. Notice that Σ can be a proper submanifold of N, $2p - 2 \leq dim\ \Sigma \leq dim\ N = 2p + q - 2$ or $2p + q - 3$.

A second proof of Theorem 5.A can be obtained as follows and allows N to be of class C^1.

PROPOSITION 5.B. *Let M be a C^2-smooth generic manifold in $\mathbf{C}^n$, CRdim $M = p \geq 2$, let $N \subset M$ be a C^1-smooth* generic *submanifold with $codim_M N = 2$ and let $z_0 \in N$. Then there exist two neighborhoods $U, V \subset\subset U$ of z_0 in M such that each function $f \in CR(U\backslash N)$ can be uniformly approximated on compact subsets of $V\backslash N$ by holomorphic polynomials.*

Proof. This is an adaptation of the approximation theorem of Baouendi and Treves. Let L_0 be a maximally real submanifold through z_0 with $T_{z_0}L_0 = \mathbf{R}^n$ contained in N, let H be a C^2 manifold through z_0 with $\dim H + \dim L_0 = \dim M$ and $T_{z_0}L_0 + T_{z_0}H = T_{z_0}M$. There are maximally real manifolds L_h through $h \in H$, closed in a fixed neighborhood U of z_0 in M, with the properties that $M \cap V$ is the disjoint union of the $L_h \cap V$, for some neighborhood V of z_0 in M, the L_h are uniformly close in C^1 norm to L_0, $L_h \subset N$ if $h \in N$ and $L_h \cap N = \emptyset$ if $h \notin N$. Fix a manifold L_1 contained in $M\backslash N$ and set, if $f \in \mathrm{CR}(M\backslash N)$ and $\hat{z} \in V\backslash N$

$$G_\tau f(\hat{z}) = \left(\frac{\tau}{\pi}\right)^{n/2} \int_{L_1} e^{-\tau(z-\hat{z})^2} f(z)dz,$$

where $(z - \hat{z})^2 = (z_1 - \hat{z}_1)^2 + \cdots + (z_n - \hat{z}_n)^2$, $\tau > 0$ and $dz = dz_1 \wedge \cdots \wedge dz_n$.

If $\widehat{z} \in V\backslash N$, $\widehat{z}$ belongs to some manifold $L_{\hat{h}}$ and we have, if f is of class C^1

$$G_\tau f(\widehat{z}) = \left(\frac{\tau}{\pi}\right)^{n/2} \int_{L_{\hat{h}}} e^{-\tau(z-\widehat{z})^2} f(z)dz + \left(\frac{\tau}{\pi}\right)^{n/2} \int_{\Sigma} d(e^{-\tau(z-\widehat{z})^2} f(z)dz) = \left(\frac{\tau}{\pi}\right)^{n/2} \int_{L_{\hat{h}}} e^{-\tau(z-\widehat{z})^2} f(z)dz,$$

by Stokes' theorem. The last equality holds since f is CR on $M\backslash N$, hence $d(f(z)dz) = 0$, and since Σ with $b\Sigma = L_1 - L_{\hat{h}}$ can be chosen to be contained in $M\backslash N$. When f is only continous, the middle equality has to be interpreted in the distribution sense.

Analysing the real and imaginary parts of the phase function $-\tau(z-\widehat{z})^2$ on $L_{\hat{h}}$, one can show that the last integral tends to $f(\widehat{z})$ as τ tends to ∞ if the $L_{\hat{h}}$ are sufficiently close to L_0 in C^1 norm. Then the expression defining $G_\tau f(\widehat{z})$ by integration on L_1 gives converging polynomial sequences using a truncated developement of the exponential in power series.

The proof of Proposition 5.B is complete.

Remark. The approximation property is still valid for CR functions extending holomorphically into some wedge $\mathcal{W}_1$ of edge V at $z_1 \in V$, on compact subsets of $(V\backslash N) \cup \mathcal{W}_2, \mathcal{W}_2 \subset \mathcal{W}_1$. Indeed, some uniformity is allowed in choosing the L_h filling $(V\backslash N) \cup \mathcal{W}_2$.

Here is a second version of Theorem 5.A, whose proof uses deformations of discs instead of the minimalization theorem and the sweeping out by wedges lemma.

THEOREM 5.A.1 *Let M be a $C^{2,\alpha}$-smooth $(0 < \alpha < 1)$ generic manifold in $\mathbf{C}^n (n \geq 3)$ with CRdim $M = p \geq 2$ and $N \subset M$ a C^1 submanifold of M with $codim_M N = 2$ that is generic in $\mathbf{C}^n$. Assume there exists an open neighborhood V of N in M such that $(M\backslash N) \cap V$ has the wedge extension property. Then there exists an open wedge $\mathcal{W}$ attached to $M \cap V$ such that every continuous CR function on $(M\backslash N) \cap V$ extends holomorphically into $\mathcal{W}$ and continuously in $(M\backslash N) \cup \mathcal{W}$.*

Proof. According to Proposition 5.B, CR functions on $M\backslash N$ are locally uniformly approximable on compact subsets of $M\backslash N$ by holomorphic polynomials. However, embedded discs attached to $M\backslash N$ are not in general analytically isotopic to a point in $M\backslash N$, since the first homology groups $H_1(V\backslash N) \equiv \mathbf{Z}$, for V a small open ball of center $z_0 \in N$.

Theorem 5.A.1 will be a consequence of Proposition below and existence of a good disc. Existence is checked as follows. There are holomorphic coordinates (w, z) on $\mathbf{C}^n$ as in (10) such that $T_0 N = \{v_1 = v_2 = 0\}$. Then for small $c > 0$, the disc A_c with holomorphic w-component $w_c(\zeta) = (c(1-\zeta), ic(1-\zeta), 0, ..., 0)$ and z-component satisfying the Bishop equation $x_c = -T_1 h(w_c, x_c)$ is attached to M and satisfies $A_c(1) = 0 \in N, A_c(b\Delta\backslash\{1\}) \subset M\backslash N, v_0 = \frac{d}{d\theta}|_{\theta=0} A_c(e^{i\theta}) \notin T_0 N$.

PROPOSITION 5.1 *Let M be generic, $C^{2,\alpha}$-smooth, $p \geq 1$, let $z_0 \in M$, let $N \ni z_0$ be a C^1 submanifold with $codim_M N = 2$ and $T_{z_0} N \not\supset T^c_{z_0} M$ and let V be a neighborhood of z_0 in M. Assume that there exists a sufficiently small analytic disc $A \in C^{2,\beta}(\overline{\Delta})$ attached to M with $A(1) = z_0, A(b\Delta\backslash\{1\}) \subset V\backslash N, \frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) = v_0 \notin T^c_{z_0} M, v_0 \notin T_{z_0} N$. Let $\mathcal{W}_1$ be a wedge of edge M at $A(-1)$ and assume that CR functions on $M\backslash N$ extending holomorphically into $\mathcal{W}_1$ are uniformly approximable on compact subsets of $(V\backslash N) \cup \mathcal{W}_2, \mathcal{W}_2 \subset \mathcal{W}_1$, by holomorphic polynomials. Then for each $\varepsilon > 0$, there exists $v_{00} \in T_{z_0} M$ with $|v_{00} - v_0| < \varepsilon$, $v_{00} \notin T_{z_0} N$, $v_{00} \notin T^c_{z_0} M$ and a wedge $\mathcal{W}$ of edge M at (z_0, Jv_{00}) such that, if a function $f \in CR(M\backslash N)$ extends holomorphically into $\mathcal{W}_1$, it extends to be holomorphic into $\mathcal{W}$.*

Remark. The proposition holds even if N is not generic. Thus, in a generic manifold with $M\backslash N$ having the wedge extension property, a C^1 two-codimensional singularity N with

$T_{z_0}N \not\supset T^c_{z_0}M$ is removable at z_0 *if and only if* CR functions on $(M\backslash N) \cup \mathcal{W}_1$ are locally uniformly approximable by holomorphic polynomials on compact subsets of $(M\backslash N) \cup \mathcal{W}_2$ near z_0.

Proof. The proof uses the same deformations of discs as in Section 4, but holomorphic extendibility into the sets $\mathcal{W}\backslash N_{\mathcal{P}}$ now is a direct consequence of the uniform approximability of CR$(M\backslash N)$ by holomorphic polynomials, and the maximum principle, as usual in the field [7] [9].

If $v_0 \in T^c_{z_0}M$, choose a disc $A_{00} = A_{t_1,\tau_1,a_1}$ in the family of deformed discs constructed in (22) with $v_{00} = \frac{d}{d\theta}|_{\theta=0} A_{00}(e^{i\theta}) = v_{00} \notin T^c_{z_0}M$, $v_{00} \notin T_{z_0}N$ and $|v_{00} - v_0| < \varepsilon$.

The argument of overlapping wedges gives the following. Condition (i) insures that $\mathcal{W}_2$ is a wedge of edge M at $(0, Jv_{00})$ and

$$N_{\mathcal{P}_2} = \{A_{t,\tau,a,p_0}(\overset{\circ}{\Delta}_1);\ (t,\tau,a,p_0) \in \mathcal{P}_2, p_0 \in \mathcal{K} \cap N\}$$

is a closed one-codimensional conic submanifold in $\mathcal{W}_2$ and similar properties hold for $\mathcal{W}_2'$, $N'_{\mathcal{P}_2}$. Connectedness arguments run without modifications, since $Jv_{00} \notin T_0M$.

Given a function $f \in$ CR$(M\backslash N)$, one obtains a function F that is holomorphic into

$$(\mathcal{W}_2 \cup \mathcal{W}_2')\backslash(N_{\mathcal{P}_2} \cap N'_{\mathcal{P}_2})$$

and continuous up to $M\backslash N$, with $F = f$ near 0 on $M\backslash N$.

Condition (i) insures that the two wedges $\mathcal{W}_2$ and $\mathcal{W}_2'$ contain a wedge $\mathcal{W}$ of edge M at $(0, Jv_{00})$. Condition (ii) together with the facts that $T_0N \not\supset T^c_0M$ and $v_{00} \notin T_0N$ imply that $M_1 = N_{\mathcal{P}_2} \cap N'_{\mathcal{P}_2}$ is locally (*i.e.* near a point in $\mathcal{W}$, see the proof of Proposition 4.1) a C^1 two-codimensional submanifold which is generic in $\mathbf{C}^n$.

A generic locally closed two -codimensional manifold M_1 in an open set $\mathcal{W}$ is removable for functions which are holomorphic into $\mathcal{W}\backslash M_1$.

Indeed, according to Proposition 5.B, CR (*i.e. holomorphic*) functions on $\mathcal{W}\backslash M_1$ are locally uniformly approximable by holomorphic polynomials. Therefore, given a function $f \in \mathcal{O}(\mathcal{W}\backslash M_1)$ and a small disc A attached to $\mathcal{W}\backslash M_1$, $f \circ A|_{b\Delta}$ extends holomorphically into Δ. Let $z_0 \in M_1 \cap \mathcal{W}$ and choose a holomorphic coordinate system at z_0 such that $T_{z_0}M_1 = \{y_1 = y_2 = 0\}$. Then the discs $A_{c,p_0} = (c\zeta, ic\zeta, 0, ..., 0) + p_0, c > 0, p_0 \in M_1, |p_0| << c$ satisfy $A_{c,p_0}(1) = p_0 \in M_1, A_{c,p_0}(b\Delta) \subset \mathcal{W}\backslash M_1$. As a consequence of the existence of such a family, each function $f \in \mathcal{O}(\mathcal{W}\backslash M_1)$ extends continuously, hence holomorphically, through M_1, as a $F \in \mathcal{O}(\mathcal{W})$ with $F = f$ on $\mathcal{W}\backslash M_1$, and with

$$F(p_0) = \frac{1}{2i\pi} \int_{b\Delta} \frac{f \circ A_{c,p_0}(\eta)}{\eta - \zeta} d\eta$$

for $p_0 \in M_1 \cap \mathcal{W}$.

The proof of Proposition 5.1 is complete.

Return to the proof of Theorem 2. Since N does not consist of a CR manifold with CRdim $N = p - 1$, all points where N is generic in $\mathbf{C}^n$ are removable. Therefore, the set

$$N^{CR} = \{n \in N;\ \text{CRdim}\ T_nN = p - 1\}$$

is a proper closed subset of N. The following Theorem finishes out the proof of Theorem 2.

THEOREM 5.2. *Let M be a $C^{2,\alpha}$-smooth* $(0 < \alpha < 1)$ *generic manifold in* $\mathbf{C}^n (n \geq 3)$ *with* $p = CRdim\ M \geq 2$, $N \subset M$ *a connected* C^1 *submanifold of* M *with* $codim_M N = 2$ *such that* $T_z N \not\supset T_z^c M$ *for each* $z \in N$ *and let* Φ *be a proper closed subset of* N. *Assume that there exists a neighborhood* V *of* N *in* M *such that* M *is minimal at every point of* V. *Then there exists an open wedge* $\mathcal{W}$ *attached to* $M \cap V$ *such that every continuous CR function on* $(M \backslash \Phi) \cap V$ *extends holomorphically into* $\mathcal{W}$ *and continuously in* $(M \backslash \Phi) \cup \mathcal{W}$.

Proof. Fix a function $f \in CR(M \backslash \Phi)$. $(M \backslash \Phi) \cap V$ has the wedge extension property, since M is minimal at every point of V. Furthermore, f extends holomorphically into some open wedge $\mathcal{W}_0$ attached to $(M \backslash \Phi_{nr}) \cap V$, where Φ_r denotes the set of removable points of Φ and $\Phi_{nr} = \Phi \backslash \Phi_r$ denotes the set of nonremovable points.

According to Theorem 5.A.1, Φ_r contains all points of Φ where N is generic in $\mathbf{C}^n$. By definition, Φ_r is a relatively open subset of Φ and Φ_{nr} is a proper closed subset of N. Assume that Φ_{nr} is nonempty. We shall reach a contradiction.

Since N is connected, there exists a point z_0 in the relative boundary of Φ_{nr} with respect to N.

Smoothly deform M in a $C^{2,\alpha}$-smooth manifold M^d by pushing it slightly into $\mathcal{W}_0$, the deformation being sufficiently small in order that there still exists a good disc attached to M^d at z_0, as constructed in Section 2 on everywhere minimal CR manifolds.

According to the Deformation Lemma, Proposition 2.1 and Proposition 3.8, removability of z_0 will be a consequence of the following.

PROPOSITION 5.3. *Let M be generic, $C^{2,\alpha}$-smooth, let* $z_0 \in M$, *let* $N \ni z_0$ *be a* C^1 *submanifold with* $codim_M N = 2$ *and* $T_{z_0} N \not\supset T_{z_0}^c M$ *and let* ω *be a neighborhood of* $M \backslash \Phi$ *in* $\mathbf{C}^n$ *for some proper closed subset* Φ *of* N *with* $z_0 \in b\Phi$. *Assume there exists a sufficiently small embedded analytic disc* $A \in C^{2,\beta}(\overline{\Delta})$ *attached to* M, $A(1) = z_0$, *with* $A(b\Delta \backslash \{1\}) \subset M \backslash N$, $\frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) = v_0 \notin T_{z_0}^c M$, $v_0 \notin T_{z_0} N$ *and all discs in* $\mathcal{V}(A, \delta)$ *attached to* $M \backslash \Phi$ *are analytically isotopic to a point in* $M \backslash \Phi$, *for some* $\delta > 0$. *Then there exists a wedge* $\mathcal{W}$ *of edge* M *at* (z_0, Jv_0) *such that for every holomorphic function* $f \in \mathcal{O}(\omega)$ *there exists a function* $F \in \mathcal{O}(\mathcal{W})$ *with* $F = f$ *in the intersection of* $\mathcal{W}$ *with a neighborhood of* $M \backslash N$ *in* $\mathbf{C}^n$.

Proof. The proof uses same deformations of discs as in Section 4 until we reach the argument of overlapping wedges. We take the notations of Section 4.

Notice that condition (i) insures that $\mathcal{W}_2$ is a wedge of edge M at $(0, \eta_0), \eta_0 = Jv_0 \bmod T_0 M$ and

$$N_{\mathcal{P}_2} = \{A_{t,\tau,a,p_0}(\mathring{\Delta}_1);\ (t, \tau, a) \in \mathcal{P}_2, p_0 \in \mathcal{K} \cap N\}$$

is a closed one-codimensional conic submanifold in $\mathcal{W}_2$ (in other words, a CR-wedge over N or a *manifold with edge* N), $\Phi_{\mathcal{P}_2} = \{A_{t,\tau,a,p_0}(\Delta_1);\ (t, \tau, a) \in \mathcal{P}_2, p_0 \in \mathcal{K} \cap \Phi\}$ being a proper closed subset of $N_{\mathcal{P}_2}$. Since $\mathcal{W}_2 \backslash \Phi_{\mathcal{P}_2}$ is therefore connected, the continuity principle argument and Cauchy's integral yield a function F_2 that is holomorphic into $\mathcal{W}_2 \backslash \Phi_{\mathcal{P}_2}$, and similarly also $F_2' \in \mathcal{O}(\mathcal{W}_2' \backslash \Phi_{\mathcal{P}_2}')$. By shrinking ω near 0 and $\mathcal{P}_2, \mathcal{P}_2'$ if necessary, we can insure that all the open sets

$$\omega \cap (\mathcal{W}_2 \backslash \Phi_{\mathcal{P}_2}) \qquad \omega \cap (\mathcal{W}_2' \backslash \Phi_{\mathcal{P}_2'}) \qquad \omega \cap ((\mathcal{W}_2 \cup \mathcal{W}_2') \backslash (\Phi_{\mathcal{P}_2} \cap \Phi_{\mathcal{P}_2'})) \tag{30}$$

are connected. Indeed, for the first two, this is true if ω is sufficiently thin near 0 and for the third, condition (i) forces the two connected wedges $\mathcal{W}_2$ and $\mathcal{W}_2'$ to be with nonempty intersection. Since F_2 and F_2' by construction assume the values of f on $M \backslash \Phi$, and since $M \backslash \Phi$

is a uniqueness set, we have shown that there exists a function F that is holomorphic into

$$\omega \bigcup ((\mathcal{W}_2 \cup \mathcal{W}_2')\backslash(\Phi_{\mathcal{P}_2} \cap \Phi_{\mathcal{P}_2'})) \tag{31}$$

and $F|_\omega = f$.

Condition (i) insures that the two wedges $\mathcal{W}_2$ and $\mathcal{W}_2'$ contain a wedge $\mathcal{W}$ of edge M at $(0, \eta_0)$. Take for convenience the restriction of F to $\omega \cup \mathcal{W}\backslash(\Phi_{\mathcal{P}_2} \cap \Phi_{\mathcal{P}_2'})$ and still denote it by F. Let $\tilde{z} = A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(\tilde{\zeta})$ be a point in $\Phi_{\mathcal{P}_2} \cap \Phi_{\mathcal{P}_2'} \cap \mathcal{W}$ (if such exists). Let $\tilde{\mathcal{W}}$ be a neighborhood of $\tilde{z}$ in $\mathcal{W}$. Then, according to conditions (i) and (ii) together with the facts that $T_0N \not\supset T_0^cM$ and the projection on the v_1-axis of $v_0 = \frac{d}{d\theta}A_{0,0,0,0}(1)$ is nonzero, $N_{\mathcal{P}_2} \cap N_{\mathcal{P}_2'} \cap \tilde{\mathcal{W}}$ is contained in a two-codimensonal generic manifold $L_{\tilde{z}}$ passing through $\tilde{z}$. $L_{\tilde{z}}$ being generic, we can remove it for CR functions which are holomorphic into $\tilde{\mathcal{W}}\backslash L_{\tilde{z}}$. Indeed, this is done as in the proof of Proposition 5.1. Hence also $\Phi_{\mathcal{P}_2} \cap \Phi_{\mathcal{P}_2'} \cap \mathcal{W}$ is removable. We therefore showed that F extends holomorphically through $\Phi_{\mathcal{P}_2} \cap \Phi_{\mathcal{P}_2'}$ as a function $F \in \mathcal{O}(\mathcal{W})$ continuous up to $M\backslash N$ with $F|_{M\backslash N} = f|_{M\backslash N}$.

The proof of Theorem 5.2 is complete.

Proof of Theorem 3. Theorem 3 is a corollary of the following.

THEOREM 5.4. *Let M be a $C^{2,\alpha}$-smooth* $(0 < \alpha < 1)$ *generic manifold in* $\mathbf{C}^n$ $(n \geq 3)$*, let N be a closed connected C^2 generic submanifold of M with $codim_M N = 1$ and let $\Phi \subset N$ be a proper closed subset of N. Assume that $CR(M\backslash\Phi)$ has the wedge extension property and let Φ_r denote the set of removable points in Φ. Then $\Phi_{nr} = \Phi\backslash\Phi_r$ has the following structure: $b\Phi_{nr}$ is a union of CR orbits of N.*

Proof. Assume on the contrary that there exists $z_0 \in b\Phi_{nr}$ and a point $z_1 \in \mathcal{O}_{CR}(N, z_0)$ with $z_1 \notin b\Phi_{nr}$. Either there exists such a z_1 with $z_1 \in M\backslash N$ or $\mathcal{O}_{CR}(N, z_0) \subset \Phi_{nr}$ and $z_1 \in \text{Int } \Phi_{nr}$. In the latter case, moving backwards along some perturbation of the piecewise smooth integral curve of T^cM joining n with z_1, one obtains that there exists a C^2 integral curve $s \mapsto X_s(z_2)$, $s \in [0, s_0]$, $s_0 > 0$ of a T^cM-tangent vector field X with $z_2 \in M\backslash N$, $X_s(z_2) \in M\backslash N$, $s \in [0, s_0)$ and $z = X_{s_0}(z_2) \in b\Phi_{nr}$. This is also true in the first case.

Though $b\Phi_{nr}$ can make a too thin étroiture at z, there exist points $z_0 \in b\Phi_{nr}$ close to z such that there exists a truncated convex open cone Γ at $(z_0, -X(z_0))$ contained in N with $\Gamma \cap \Phi_{nr} = \emptyset$. Indeed, choose a real euclidean coordinate system $(n_1, ..., n_k)$, $k = \dim N$ on N near z such that integral curves of X correspond to lines $n_2 = ct, ..., n_k = ct$. Let $s_1 < s_0$ be close to s_0, let $r > 0$ be so small that the closed ball $B_1 = \{z \in N;\ (n_1 - n_1^1)^2 + \cdots + (n_k - n_k^1)^2 \leq r^2\}$ is contained in $N\backslash\Phi$ where $z_1 = X_{s_1}(z_2)$ has coordinates $(n_1^1, ..., n_k^1)$. When $\delta \geq 1$ increases, the domains $B_\delta = \{z \in N;\ (n_1 - n_1^1)^2/\delta^2 + \cdots + (n_k - n_k^1)^2 \leq r^2\}$ increase and first touch $b\Phi_{nr}$ for $\delta_{sup} > 1$ at points $z_0 \in N$ with $n_1(z_0) \neq n_1^1$. Therefore, $B_{\delta_{sup}}$ contains open convex cones Γ at $(z_0, -X(z_0))$ and since $\text{Int } B_{\delta_{sup}} \subset N\backslash\Phi_{nr}$, $\Gamma \cap \Phi_{nr} = \emptyset$.

We shall show that such a point z_0 is removable, thus deriving a contradiction, firstly using the minimalization theorem and secondly using the deformation of discs technique.

According to the Deformation Lemma, or Proposition 5.5 below, we can assume that we are given a continuous function $f \in \text{CR}(M\backslash\Phi_{nr})$ which extends holomorphically into a wedge $\mathcal{W}_0$ attached to $M\backslash\Phi_{nr}$.

There exists a truncated open convex cone C in $\mathbf{C}^n$ with vertex z_0 such that $C \cap N = \Gamma$. If N is $C^{2,\beta}$, $\beta > 0$, minimalize N into C in a manifold M_1 as in the proof of Theorem 5.A and apply the argument of sweeping out by wedges. Then $b\Phi_{nr}\backslash\{z_0\}$ is removable near z_0, and therefore also z_0 by Theorem 1, a contradiction.

Second, assume N is C^2-smooth. Since there exists a cone $\Gamma \subset N$ at $(z_0, -X(z_0))$ and $X(z_0) \in T^c_{z_0}N$, there exists a disc A attached to N with $A(1) \in N$ and $A(\gamma) \subset \Gamma$ for some open arc $\gamma \subset b\Delta$ with $-1 \in \gamma$. Indeed, choosing a holomorphic coordinate system (w, z) at z_0 as in (10) with $z_0 = 0$ and $T_{z_0}N = \{y = 0, v_1 = 0\}$ and $X(0)$ directed in the positive u_2-axis, this is true for the disc with $(w_2, ..., w_p)$ ζ-holomorphic component equal $(c(1-\zeta), 0, ..., 0)$, $c > 0$ small and $(w_1, z)(\zeta)$ satisfying Bishop's equation relative to N.

One can introduce a manifold $K \subset N$ with $0 \in K$, $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) \notin T_0K$, $\text{codim}_N K = 1$ and deformations A_t of A in the normal space to N at $A(-1)$ depending on a real parameter $t \in \mathcal{T} \subset \mathbf{R}^q$, $q = \text{codim } N - 1$, and on $p_0 \in \mathcal{K} \subset K$, such that each A_{t,p_0} is attached to $N \cup \mathcal{W}_0$ and $\mathcal{W}_A = \{A_{t,p_0}(\mathring{\Delta}_1);\ t \in \mathcal{T}, p_0 \in \mathcal{K}\}$ generates a wedge of edge N at 0. Include N in a $C^{2,\alpha}$-smooth one-parameter family N_s, $|s| < \delta, \delta > 0$ of $C^{2,\alpha}$ generic manifolds N_s contained in M such that $N_0 = N$, $N_s \cap \Phi_{nr} = \emptyset$ for $s \neq 0, \cup_s N_s = M$ near 0 and the N_s contain some fixed point in Γ. Then, according to the smooth dependence of the solutions of Bishop's equation on parameters, for each small $s \neq 0$, the sets $\mathcal{W}_s = \{A_{s,t,p_0}(\mathring{\Delta}_1); t \in \mathcal{T}, p_0 \in \mathcal{K}\}$ are wedges of edge N_s to which $f|_{N_s}$, which is CR on N_s, holomorphically extend and whose direction depends smoothly on s. As in the proof of the sweeping out by wedges lemma, the union of the $\mathcal{W}_s$ for $s \neq 0$ generates a wedge $\mathcal{W}$ of edge M at z_0 and the extensions obtained stick in a well-defined holomorphic function into $\mathcal{W}$.

The proof of Theorem 5.4. is complete.

Remark. The second proof of Theorem 5.4 could also provide a second proof of Theorem 5.A. with N of class C^2.

Proof of Theorem 4. The main observation is resumed in Proposition 5.5 below, according to which all notions of removability considered during the course are in fact one and the same.

Given two wedges $\mathcal{W}_1$ and $\mathcal{W}_2$, one sets $\mathcal{W}_1 \subset\subset \mathcal{W}_2$ if their cones satisfy $C_1 \cap S^{2n-1} \subset\subset C_2 \cap S^{2n-1}$, where S^{2n-1} denotes the unit sphere in $\mathbf{C}^n$ identified with R^{2n}. Given two wedges $\mathcal{W}_0$ and $\mathcal{W}_0'$ attached to M, one sets $\mathcal{W}_0 \subset\subset \mathcal{W}_0'$ if $\mathcal{W}_{0,z} \subset\subset \mathcal{W}_{0,z'}$ for each z. Notice that one has $\mathcal{W}_0 \subset\subset \mathcal{W}_0'$ provided $\mathcal{W}_0 \cap \mathcal{W}_0'$ contains a nonempty wedge attached to M, even if $\mathcal{W}_0 \not\subset \mathcal{W}_0'$ This is because a wedge attached to M is not supposed to be exactly a wedge of edge M at each point of M.

PROPOSITION 5.5. *Let M be generic, $C^{2,\alpha}$-smooth, let Φ be a proper closed subset of M, $z_0 \in b\Phi$ and assume that M is minimal at z_0. Then the following are equivalent.*

(i) Given a wedge $\mathcal{W}_0$ attached to M, $\exists U \ni z_0$, $\exists \mathcal{W}$ attached to U such that $\forall f \in \mathcal{O}(\mathcal{W}_0)$, $\exists F \in \mathcal{O}(\mathcal{W})$ with $F = f$ into a subwedge $\mathcal{W}_1 \subset\subset \mathcal{W}_0 \cap \mathcal{W}$ attached to $U \backslash \Phi$.

(ii) Given a wedge $\mathcal{W}_0$ attached to $M \backslash \Phi$, $\exists U \ni z_0, \exists \mathcal{W} = \mathcal{W}(U, z_0)$, such that $\forall f \in \mathcal{O}(\mathcal{W}_0)$ continuous up to $M \backslash \Phi$, $\exists F \in \mathcal{O}(\mathcal{W})$ continuous up to $U \backslash \Phi$ with $F = f$ in $U \backslash \Phi$.

(iii) Given a wedge $\mathcal{W}_0$ attached to $M \backslash \Phi$, $\exists U \ni z_0$, $\forall$ V neighborhood of $\Phi \cap U$ in U, $\forall \varepsilon > 0$, $\exists U^d$ $C^{2,\alpha}$ deformation of U, $U^d \cap (U \backslash V) = U \backslash V$, $||U^d - U||_{C^{2,\alpha}} < \varepsilon$ such that $\forall f \in \mathcal{O}(\mathcal{W}_0)$ continuous up to $M \backslash \Phi$, $\exists f^d \in CR(U^d)$ with $f^d = f$ on $U \backslash V$.

When these properties hold, there exists a neighborhood $\mathcal{U}$ of z_0 in $\mathbf{C}^n$ and $\mathcal{W}$ a wedge attached to $(M \backslash \Phi) \cup (\mathcal{U} \cap M)$ such that $\mathcal{W} \equiv \mathcal{W}_0$ outside $\mathcal{U}$, $\mathcal{W} \cap \mathcal{W}_0$ contains a wedge $\mathcal{W}_1$ attached to $M \backslash \Phi$ and for each function $f \in \mathcal{O}(\mathcal{W}_0)$ there exists $F \in \mathcal{O}(\mathcal{W})$ with $F = f$ into $\mathcal{W}_1$. Moreover, these properties are equivalent to $(i)'$, $(ii)'$, $(iii)'$ where one replaces $\mathcal{W}_0$ with a neighborhood ω of $M \backslash \Phi$ in $\mathbf{C}^n$.

Proof. $(i) \Rightarrow (ii)$. Let $\mathcal{W}_0$ be wedge attached to $M\backslash\Phi$. By (i), there exists a wedge $\mathcal{W} = \mathcal{W}(U', z_0)$ such that $\forall f \in \mathcal{O}(\mathcal{W}_0)$, $\exists F \in \mathcal{O}(\mathcal{W})$, $F = f$ into a subwedge $\mathcal{W}_1 \subset\subset \mathcal{W}_0 \cap \mathcal{W}$ attached to $U\backslash\Phi$. Since $\mathcal{W}_1 \subset\subset \mathcal{W}_0$, for such f continous up to $M\backslash\Phi$, $f|_{\mathcal{W}_1}$ admits a continuous limit up to $U\backslash\Phi$, and then also $F|_{\mathcal{W}_1} = f|_{\mathcal{W}_1}$, proving that F is continous up to $U\backslash\Phi$. Hence (ii) holds.

$(ii) \Rightarrow (iii)$. Let $\mathcal{W}_0$ be a wedge attached to $M\backslash\Phi$. By (ii), there exists a wedge $\mathcal{W} = \mathcal{W}(U', z_0)$ over U' at z_0 such that $\forall f \in \mathcal{O}(\mathcal{W}_0)$ continous up to $M\backslash\Phi$, $\exists F \in \mathcal{O}(\mathcal{W})$ continuous up to $U'\backslash\Phi$ with $F = f$ in $U'\backslash\Phi$. Let $U \subset\subset U'$ be a subneighborhood, let V be a neighborhood of $\Phi \cap U$ in U and $\varepsilon > 0$ arbitrary. Since $\mathcal{W}(U', z_0)$ is a wedge of edge U' at z_0, there exsists a $C^{2,\alpha}$ deformation $U^d \subset \mathcal{W} \cup (U\backslash V)$ of U with $U^d \equiv U$ in $U\backslash V$ and $||U^d - U||_{C^{2,\alpha}} < \varepsilon$. For each function f extending as a $F \in \mathcal{O}(\mathcal{W})$, one sets $f^d := F|_{U^d}$ which is CR without singularities on U^d and satisfies $f^d = f$ on $U\backslash V$. Hence (iii) holds.

$(iii) \Rightarrow (i)$. Let $\mathcal{W}_0$ be a wedge attached to $M\backslash\Phi$. Choose a wedge $\mathcal{W}_1 \subset \mathcal{W}_0$ attached to $M\backslash\Phi$ with $\mathcal{W}_1 \subset\subset \mathcal{W}_0$ near z_0, a $C^{2,\alpha}$ varying direction on $M\backslash\Phi$ near z_0 and $\mathcal{W}_1 \equiv \mathcal{W}_0$ away from z_0. Deform first M in a one-parameter family of $C^{2,\alpha}$ manifolds $M^{d_1}, d_1 \geq 0, M^{d_1} \subset \mathcal{W}_1 \cup \Phi$ with $||M^{d_1} - M||_{C^{2,\alpha}} < \varepsilon(d_1) < \delta/2, \delta > 0, \varepsilon(d_1) \to 0$ as $d_1 \to 0$ and for each function $f \in \mathcal{O}(\mathcal{W}_0)$ take $f^{d_1} = f|_{M^{d_1}} \in \mathrm{CR}(M^{d_1}\backslash\Phi)$. Then the f^{d_1} are in fact defined in a neighborhood $\mathcal{W}_1$ of $M^{d_1}\backslash\Phi$ in $\mathbf{C}^n$ and holomorphic there. Choose $\delta > 0$ so small that for each $C^{2,\alpha}$ manifold M' with $||M' - M||_{C^{2,\alpha}} < \delta$, there exists a wedge $\mathcal{W}(A')$ associated with a disc of zero defect A' attached to M' which is a smooth perturbation of a disc A attached to M through z_0 of zero defect and the size U' of the base of $\mathcal{W}(A')$ in M' satisfies $\mathrm{dist}(bU', z_0') \geq \kappa > 0$ for every M'.

According to (iii), there exists a neighborhood $U^{d_1} \ni z_0$ such that, given V a neighborhood of $\Phi \cap U^{d_1}$ in U^{d_1} and $\varepsilon_2 > 0$ arbitrary, there exists a $C^{2,\alpha}$ deformation $(U^{d_1})^{d_2}$ of U^{d_1}, $d_2 \geq 0, (U^{d_1})^0 = U^{d_1}$ with $||(U^{d_1})^{d_2} - U^{d_1}||_{C^{2,\alpha}} < \varepsilon_2 < \delta/2$ such that $\forall f \in \mathcal{O}(\mathcal{W}_1), \exists f^{d_2} \in \mathrm{CR}((U^{d_1})^{d_2})$ with $f^{d_2} \equiv f^{d_1}$ in $U^{d_1}\backslash V = (U^{d_1})^{d_2}\backslash V$. Since f^{d_2} is CR and we can assume that the size of A^{d_1} is smaller than U^{d_1}, so there exist perturbations $(A^{d_1})^{d_2}$ of A of zero defect attached to $(U^{d_1})^{d_2}$, therefore f^{d_2} extends holomorphically into a wedge $\mathcal{W}((A^{d_1})^{d_2})$ of edge $(U^{d_1})^{d_2}$ near $(z_0^{d_1})^{d_2}$. Letting d_2, ε_2 tend to zero and V shrink to Φ, this shows that $f^{d_1} \in \mathrm{CR}(U^{d_1}\backslash\Phi)$ extends holomorphically into $\mathcal{W}(A^{d_1})$ as a function $F^{d_1} \in \mathcal{O}(\mathcal{W}(A^{d_1}))$ continuous up to $U^{d_1}\backslash\Phi$. In fact, we just showed that $(iii) \Rightarrow (ii)$.

Then the F^{d_1} stick together in a well-defined holomorphic function in $\mathcal{W}_2 = \bigcup_{d_1>0} \mathcal{W}(A^{d_1})$. Indeed, the wedges $\mathcal{W}(A^{d_1})$ varying differentiably as $\frac{\partial A^{d_1}}{\partial \varsigma}(1)$, they make successive connected intersection and one obtains a function $F^{d_2} \in \mathcal{O}(\mathcal{W}_2)$. Shrinking the height of $\mathcal{W}_1$, *i.e.* the troncature of the cones defining $\mathcal{W}_1$ at each point, one can insure that $\mathcal{W}_2 \cap \mathcal{W}_1$ has as many connected components as $U\backslash\Phi$, for some neighborhood $U \ni z_0$ in M. Thus, $f|_{\mathcal{W}_1}$ and F_2 stick in a well-defined function $F \in \mathcal{O}(\mathcal{W}), \mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2$. Hence (i) holds.

The functions $f \in \mathcal{O}(\mathcal{W}_1)$ and $F \in \mathcal{O}(\mathcal{W})$ stick together in a holomorphic function in the wedge $\mathcal{W}_1 \cup \mathcal{W}$ attached to $(M\backslash\Phi) \cup U_1$, since $f \equiv F$ in each connected component of $\mathcal{W}_1$.

The equivalence of $(i)', (ii)'$ and $(iii)'$ is proved in exactly the same way. (iii) implies $(iii)'$ and conversely, in the proof of $(iii) \Rightarrow (i)$, it was in fact proved that $(iii)' \Rightarrow (i)$.

The proof of Proposition 5.5 is complete.

We can also translate the equivalences of Proposition 5.5 when one assumes that only a CR function is previously given.

COROLLARY 5.6. *Let M be generic, $C^{2,\alpha}$-smooth, let $\Phi \subset M$ be a proper closed subset of M, $z_0 \in b\Phi$ and assume that M is minimal at z_0. Then the following are equivalent.*

(iv) $\exists\ \mathcal{W}$ *of edge* M *at* z_0 *such that* $\forall f \in CR(M\backslash\Phi)$ $\exists F \in \mathcal{O}(\mathcal{W})$ *continuous up to* $M\backslash\Phi$ *with* $F = f$ *there.*

(v) $\exists U \ni z_0$ *such that* $\forall V$ *neighborhood of* $\Phi \cap U$ *in* U, $\forall \varepsilon > 0$, $\exists U^d$ $C^{2,\alpha}$, $U^d \cap (U\backslash V) = U\backslash V$, $||U^d - U||_{C^{2,\alpha}} < \varepsilon$ *such that* $\forall f \in CR(M\backslash\Phi)$, $\exists f^d \in CR(U^d)$, $f^d = f$ *on* $U\backslash V$.

If $CR(M\backslash\Phi)$ *has the wedge extension property at every point of* $M\backslash\Phi$*, these are still equivalent to* $(i), (ii), (iii)$ *above.*

Remark. Assume now that $\Phi = N$ is a proper at least one-codimensional submanifold of M. We can avoid assuming that M is minimal at z_0 in the equivalence above. Using normal deformations of discs as in the proof of Theorem 5.4, we could prove also:

COROLLARY 5.7. *Let* M *be generic,* $C^{2,\alpha}$*-smooth, let* $N \subset M$ *be a submanifold,* $codim_M N \geq 1$*, let* $z_0 \in N$ *and assume that* $T_{z_0}N \not\supset T^c_{z_0}M$*. Then the equivalences of Proposition 5.5 hold with* $\Phi = N$.

Proof. We only suggest that, when assuming that $T_{z_0}N \not\supset T^c_{z_0}M$, we implicitely assume that there exists a disc A attached to M with $A(1) = z_0 \in N$ and $A(\gamma) \cap N = \emptyset$ for some open arc $\gamma \ni -1$ in $b\Delta$. Then, (iii) enables one to consider CR functions on a deformation of M without singularities and the deformations of A in the normal space to M at $A(-1)$ in $\mathcal{W}_0$ render it possible that A plays the role of a disc of zero defect as in the proof of 5.5, $(iii) \Rightarrow (i)$.

The proof of Corollary 5.7 is complete.

Proof of Theorem 5. Since $T_{z_0}N + T^c_{z_0}M = T_{z_0}M$ and $\text{codim}_M N = 2$, we can assume that N is given in a coordinate system as (10) by

$$y = h(w, x) \qquad w_p = g(w', x),$$

where $w' = (w_1, ..., w_{p-1})$, $h(0) = 0, dh(0) = 0$ and $g(0) = 0$. Since N is CR and has CRdim $N = p-1$, g is CR on the manifold $N^\pi \subset \mathbf{C}^{p-1}$ with equation $y = H(w', x)$, where $H(w', x) = h(w', g(w', x), x)$. Since N^π is minimal at 0, there exists a family A^π_{t,n_0} of discs attached to N^π in $\mathbf{C}^{n-1}$ such that for t in a neighborhood $\mathcal{T}$ of 0 in $\mathbf{R}^{q-1}$, n_0 in a one codimensional manifold $\mathcal{K} \subset N^\pi$ through 0 and $\zeta \in \overset{\circ}{\Delta}_1$, the set of points $A^\pi_{t,n_0}(\zeta)$ spans a wedge $\mathcal{W}^\pi$ of edge N^π in $\mathbf{C}^{n-1}$. We can assume that the A^π_{t,n_0} have a w'-component which embeds $\overline{\Delta}$ into $\mathbf{C}^{n-1}$ and $\frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) \not\in T^c_0 N^\pi$, since N^π is minimal at every point (see the argument in the proof of Proposition 2.1). Set $A^\pi_{t,n_0}(\zeta) = (w'_{t,n_0}(\zeta), z_{t,n_0}(\zeta))$. Since g is CR, $A_{t,n_0}(\zeta) = (w'_{t,n_0}(\zeta), g(w'_{t,n_0}(\zeta), x_{t,n_0}(\zeta)), z_{t,n_0}(\zeta))$ is a holomorphic disc in ζ attached to N. For a in a neighborhood $\mathcal{V}$ of 0 in $\mathbf{C}$, consider the analytic disc attached to M $A_{t,n_0,a}(\zeta) = (w'_{t,n_0}(\zeta), g(w'_{t,n_0}(\zeta), x_{t,n_0}(\zeta)) + a, z_{t,n_0,a}(\zeta))$ where $x_{t,n_0,a} = -T_1 h(w'_{t,n_0}, g(w'_{t,n_0}, x_{t,n_0}) + a, x_{t,n_0,a}) + x_{t,n_0}(1)$. Since $h(0) = 0, dh(0) = 0$, we have the estimate $|x_{t,n_0,a} - x_{t,n_0}| < \varepsilon|a|$, $\varepsilon << 1$ and this proves that $A_{t,n_0,a}(b\Delta) \cap N = \emptyset$ when $a \neq 0$. Furthermore, if the size of A^π_{t,n_0} is sufficiently small, by taking $a = a(s) \neq 0$, $0 \leq s \leq 1$, one can check that the discs $A_{t,n_0,a}$ are analytically isotopic to a point in $M\backslash N$. Since $\dim T^c_{z_0}M/(T^c_{z_0}M \cap T_{z_0}N) = 2$, $M\backslash N$ is globally minimal around z_0, hence CR functions on $M\backslash N$ are wedge extendible at every point of $M\backslash N$ near z_0 [6]. Therefore the argument in the proof of Proposition 4.2 can be repeated here: CR functions are holomorphically extendible into the wedge open set

$$\mathcal{W}\backslash\mathcal{W}^{an} = \{A_{t,n_0,a}(\zeta);\ t \in \mathcal{T}, n_0 \in \mathcal{K}, a \in \mathcal{V}\backslash\{0\}, \zeta \in \overset{\circ}{\Delta}_1\} \tag{32}$$

minus the analytic wedge

$$\mathcal{W}^{an} = \{A_{t,n_0}(\zeta);\ t \in \mathcal{T}, n_0 \in \mathcal{K}, \zeta \in \overset{\circ}{\Delta}_1\}.$$

The proof of Theorem 5 is complete.

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