NOTE ON DOUBLE REFLECTION AND ALGEBRAICITY OF HOLOMORPHIC MAPPINGS

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ABSTRACT. In this note, our purpose is to establish shortly the algebraicity of a holomorphic mapping between two real algebraic CR manifolds under a double reflection condition which generalizes the classical single reflection. A complete study of various double reflection conditions illustrated by simple examples is also provided.

RÉSUMÉ. Dans cette note, notre but est d'établir brièvement l'algébricité d'une application holomorphe entre deux variétés CR réelles algébriques en supposant qu'une condition de réflexion double, généralisant la condition classique de réflexion simple, est satisfaite. Nous entreprenons une étude complète, supportée par des exemples élémentaires, de la combinatoire des différents théorèmes possibles.

The goal of this note is to understand double reflection for holomorphic mappings $f : M \to M'$ between real algebraic CR manifolds in complex spaces of different dimensions. We plan to understand it in the general case, that is with and without reducing, shrinking or stratifying the first family of equations "$f(z) \in r_{M'}(f(Q_{\bar z})) := \mathbb{V}'_z$" (see below) which comes from the first reflection.

Thus, let us quickly present the general problematics.

Let $f : M \to M'$ be a holomorphic map between two real algebraic CR manifolds in $\mathbb{C}^n$, $\mathbb{C}^{n'}$ given by the vanishing of real polynomial equations $\rho_j(z, \bar z) = 0$, $1 \leq j \leq d$, $\rho'_{j'}(z', \bar z') = 0$, $1 \leq j' \leq d'$, so that $\rho'(f(z), \bar f(\bar w)) = 0$ if $\rho(z, \bar w) = 0$, and let $Q_{\bar w}$, $Q'_{\bar w'}$ denote Segre varieties.

It appears that two crucial observations yield heuristic insights into the problem of finding sufficient conditions (C) such that: (C) $\Rightarrow$ f is algebraic.

First, starting form the natural observation:

$$\text{(I)} \qquad f(z) \in r_{M'}(f(Q_{\bar z})) := \{w' : Q'_{\bar w'} \supset f(Q_{\bar z})\} =: \mathbb{V}'_z$$

(intentionally, we do not specify $w' \in U'$ or $w' \in \mathbb{C}^{n'}$: these two possibilities will be studied hereafter), one is led to guess that the algebraic set $\mathbb{V}'_z$ (which is parametrized by z) is a *finite algebraic determinacy set* for the value of $f(z)$ if $\dim_{f(z)} \mathbb{V}'_z = 0 \ \forall z$. Effectively, this classical circumstance entails that f is algebraic (*see* [1,2,7,13]; the set $\mathbb{V}'_0$ is called the *characteristic variety of f at* 0 in [7]). As usual, this determinacy set $\mathbb{V}'_z$ can be constructed simply by applying the tangential Cauchy-Riemann fields of M to the equations $\rho'(f(z), \bar f(\bar w)) = 0$.

The second canonical observation, which is due to Zaitsev [14], is as follows:

$$\text{(II)} \qquad f(z) \in r_{M'}(f(Q_{\bar z})) \cap r^2_{M'}(f(Q_{\bar w})) = \mathbb{V}'_z \cap r_{M'}(\mathbb{V}'_w) =: \mathbb{X}'_{z,\bar w}.$$

Analogously, one is then led to predict that f is algebraic, provided $\dim_{f(z)} \mathbb{X}'_{z,\bar w} = 0 \ \forall z, w$ such that $\rho(z, \bar w) = 0$. We can justify our choice of notation $\mathbb{X}'_{z,\bar w}$ (*e.g.* instead of $\mathbb{X}'_{z,w}$) by the fact that the operator $r_{M'}$ conjugates the dependence with respect to parameters. We shall call identity (I) *first reflection* and identity (II) *second reflection*. For reasons explained below, third determination $r^3_{M'}$ or fourth $r^4_{M'}$, *etc.* provide no new information.

Also, let us mention a possible third approach ([8,14]). This approach consists in choosing *smaller* algebraic sets $\mathbb{W}'_z \subset \mathbb{V}'_z$ with nicer properties, in particular to insure the *holomorphic* dependence with respect to the parameter z, in order to compute the second reflection $r_{M'}(\mathbb{V}'_w)$ in an easier way. Of course, such a shrinking of $\mathbb{V}'_z$ in the form $\mathbb{W}'_z$ is (and in fact *must be*) constructive. Using only minors of matrices of holomorphic functions, we will in this paper

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provide a uniform manner of constructing such a set $\mathbb{W}'_z$ in an unambiguous way. The core of the article is to discuss such a shrinking.

This leads to a third type of determinacy:

$$\text{(III)} \qquad f(z) \in \mathbb{W}'_z \cap r_{M'}(\mathbb{W}'_w) =: \mathbb{Z}'_{z,\bar{w}}.$$

Through various statements and examples, our aim will be thus to compare the three conditions of determinacy of the value $f(z)$ by (I), (II) or (III), to ask which one is the strongest, to ask whether some are necessary, to ask whether it is sufficient to require $\dim_{f(p)}\mathbb{V}'_p = 0$ or $\dim_{f(p)}\mathbb{X}'_{p,\bar{p}} = 0$ or $\dim_{f(p)}\mathbb{Z}'_{p,\bar{p}} = 0$ for a single or all points $p \in M$, *etc.* There will appear a real *combinatorics of possible statements.* Among other things, we will mainly establish that:

1. *Determination of $f(z)$ by $\mathbb{W}'_z$ is strictly finer than by $\mathbb{V}'_z$.*

By this, we mean that $\dim_{f(z)}\mathbb{W}'_z = 0$, $\dim_{f(z)}\mathbb{V}'_z \geq 1 \; \forall z$ for some explicit examples of f, M, M', *idem* for **2**, **3**, **4**, **5**, **6** below. Such examples are constructed in the paper.

2. *Determination of $f(z)$ by $\mathbb{Z}'_{z,\bar{w}}$ is strictly finer than by $\mathbb{W}'_z$ (or by $\mathbb{V}'_z$).*

3. *Determination of $f(z)$ by $\mathbb{X}'_{z,\bar{w}}$ is strictly finer than by $\mathbb{V}'_z$.*

In other words, the inclusions $\mathbb{X}'_{z,\bar{w}} \subset \mathbb{V}'_z$ and $\mathbb{Z}'_{z,\bar{w}} \subset \mathbb{W}'_z \subset \mathbb{V}'_z$ are all strict in general. However, none of the inclusion $\mathbb{Z}'_{z,\bar{w}} \subset \mathbb{X}'_{z,\bar{w}}$ and $\mathbb{X}'_{z,\bar{w}} \subset \mathbb{Z}'_{z,\bar{w}}$ is true in general, because:

4. *Determination of $f(z)$ by $\mathbb{X}'_{z,\bar{w}}$ can be strictly finer than by $\mathbb{Z}'_{z,\bar{w}}$.*

5. *Determination of $f(z)$ by $\mathbb{Z}'_{z,\bar{w}}$ can be strictly finer than by $\mathbb{X}'_{z,\bar{w}}$.*

To summarize, we are led to define a fourth determinacy set:

$$\text{(IV)} \qquad f(z) \in \mathbb{W}'_z \cap r_{M'}(\mathbb{V}'_w) = \mathbb{X}'_{z,\bar{w}} \cap \mathbb{Z}'_{z,\bar{w}} =: \mathbb{M}'_{z,\bar{w}},$$

for which we can establish the following:

6. *Determination of $f(z)$ by $\mathbb{M}'_{z,\bar{w}}$ is strictly finer than by $\mathbb{X}'_{z,\bar{w}}$ or by $\mathbb{Z}'_{z,\bar{w}}$.*

In other words, the inclusions $\mathbb{M}'_{z,\bar{w}} \subset \mathbb{X}'_{z,\bar{w}}$ and $\mathbb{M}'_{z,\bar{w}} \subset \mathbb{Z}'_{z,\bar{w}}$ are all strict in general.

Our goal is to find examples which exhibit the nonanalytic behavior of $\mathbb{X}'_{z,\bar{w}}$ with respect to parameters and to explain why the various second reflection conditions (II), (III) and (IV) are inequivalent. Our work deliberately forsakes the point of view of jets, about which the reader is referred to the works [3,8,12,14].

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1. Introduction and statement of results

1.1. General assumptions. The general assumptions throughout this article are as follows. We denote by $\mathcal{V}_{\mathbb{C}^n}(p)$ a small open polydisc neighborhood of $p \in \mathbb{C}^n$ and by $\mathcal{V}_{\mathbb{C}^n}(M) := \cup_{q \in M}\mathcal{V}_{\mathbb{C}^n}(q)$. Our object of study is a local holomorphic map $f : \mathcal{V}_{\mathbb{C}^n}(M) \to \mathbb{C}^{n'}$ which induces a map $f : M \to M'$ between two real algebraic CR generic manifolds in $\mathbb{C}^n$, $\mathbb{C}^{n'}$. We assume that M is *minimal in the sense of Tumanov* at one of its points p (equivalently, (M,p) is of finite type in the sense of Bloom-Graham). Even if some of our statements remain true for M non-minimal, we shall not notify it for simplicity. Also, $n \geq 2$.

We set $m := \dim_{CR}M$, $d := \text{codim}_{\mathbb{R}}M$, $m' := \dim_{CR}M'$, $d' := \text{codim}_{\mathbb{R}}M'$, whence $m + d = n$, $m' + d' = n'$. We assume that $m \geq 1$ and $m' \geq 1$.

In suitable coordinates $z \in \mathbb{C}^n$, $z' \in \mathbb{C}^n$, we have $p = 0$, $f(p) = 0$, $M = \{z \in U : \rho(z,\bar{z}) = 0\}$ and $M' = \{z' \in U' : \rho'(z',\bar{z}') = 0\}$, where U and U' are small polydiscs centered at the origin, and where $\rho_j(z,\bar{z}) = \sum_{|\mu|,|\nu| \leq N} \rho_{j,\mu,\nu} z^\mu \bar{z}^\nu$, $1 \leq j \leq d$, $\rho'_{j'}(z',\bar{z}') = \sum_{|\mu'|,|\nu'| \leq N'} \rho_{j',\mu',\nu'} z'^{\mu'} \bar{z}'^{\nu'}$ $1 \leq j' \leq d'$ are *real polynomials* satisfying $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d(0) \neq 0$ and $\partial\rho'_1 \wedge \cdots \wedge \partial\rho'_{d'}(0) \neq 0$.

We can assume that $U = (\varepsilon\Delta)^n$, $\varepsilon > 0$ and $U' = (\varepsilon'\Delta)^{n'}$, $\varepsilon' > 0$, so $\overline{U}$ (conjugate set) $= U$ and $\overline{U}' = U'$.

1.2. Reflection operator. Let $Q_{\bar{w}} = \{z \in U : \rho(z, \bar{w}) = 0\}$ denote Segre variety (we maintain the bar on w in the notation $Q_{\bar{w}}$, *see* [11] for arguments). For every subset $E \subset U$, we define the action of the *reflection operator*:

$$r_M(E) := \{w \in U : Q_{\bar{w}} \supset E\} = \cap_{w \in E} Q_{\bar{w}}, \quad r_M^2(E) = r_M(r_M(E)), \tag{1.3}$$

say, the *first reflection* of E and its *second reflection* across M. Their basic properties are explained in Lemma 2.1:

1. $r_M(E) \cap E \subset M$.

2. $r_M(r_M(E)) \supset E$.

Observe that $\cap_{k \in \mathbb{N}} r_M^k(E) = E \cap r_M(E)$. A similar $r_{M'}$ is defined across M'. Now, let $z \in Q_{\bar{w}}$. Let $f : M \to M'$ as above. Then $f(Q_{\bar{z}}) \subset Q'_{\overline{f(z)}}$. Also:

1. $f(z) \in f(Q_{\bar{w}}) \subset r_{M'}^2(f(Q_{\bar{w}}))$.

2. $f(Q_{\bar{z}}) \subset Q'_{\overline{f(z)}}$ hence by (1.3)

$$f(z) \in r_{M'}(f(Q_{\bar{z}})) =: \mathbb{V}'_z. \tag{1.4}$$

Consequently also:

$$f(z) \in r_{M'}(f(Q_{\bar{z}})) \cap r_{M'}^2(f(Q_{\bar{w}})) = \mathbb{V}'_z \cap r_{M'}(\mathbb{V}'_w) =: \mathbb{X}'_{z,\bar{w}}. \tag{1.5}$$

Also, a last notification:

$$r_{M'}(f(Q_{\bar{z}})) \subset Q'_{\bar{f}(\bar{w})}, \qquad f(Q_{\bar{w}}) \subset r_{M'}^2(f(Q_{\bar{w}})) \subset Q'_{\bar{f}(\bar{w})}. \tag{1.6}$$

Observe that $r_{M'}^3, r_{M'}^4, \ldots$ offer nothing more, because $r_{M'}^{2k}(E') \supset E'$. Notice that the determination $f(z) \in r_{M'}(f(Q_{\bar{z}})) \cap Q'_{\bar{f}(\bar{w})}$ coincides with (1.4), since $r_{M'}(f(Q_{\bar{z}})) \subset Q'_{\bar{f}(\bar{w})}$ by definition.

1.7. Organization. This expanded Section 1 will now be divided in several paragraphs corresponding to various questions and answers that present themselves. We shall shortly present all the problems, all the results, all the technical lemmas, all the examples in the next paragraphs and we shall explain all the major links between them. Finally, the precise checking of all the remaining details and of all the technicalities will be postponed to Sections 2, 3 and 4.

1.8. The results. The fundamental observation is that, as M' is algebraic, both $r_{M'}(E')$ and $r_{M'}^2(E')$ are *complex algebraic* sets for *any* set E', since in (1.3) all the $Q'_{\bar{w}'}$ are. Therefore (1.5) should determine f as an algebraic map of z if $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$ for all z, w close to 0, $z \in Q_{\bar{w}}$, a result which is true and which was originally proved in [14] (preprint version) for $\mathbb{Z}'_{z,\bar{w}}$. An analogous result is due to Baouendi-Rothschild [1] and to Baouendi-Ebenfelt-Rothschild [2] with use of $\mathbb{V}'_z := r_{M'}(f(Q_{\bar{z}}))$ only.

Theorem 1.9. *If* $\dim_{f(z)} \mathbb{V}'_z = 0$ *or if* $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0 \; \forall \, z, w \in \mathcal{V}_{\mathbb{C}^n}(0)$ *with* $z \in Q_{\bar{w}}$, *then* f *is algebraic.*

Of course, the case $\dim_{f(z)} \mathbb{V}'_z = 0$ is contained in the more general case $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$, because we clearly have $\mathbb{X}'_{z,\bar{w}} \subset \mathbb{V}'_z$. Our proof of Theorem 1.9 will be achieved shortly, thanks to a partial algebraicity theorem proved in [10,13]. We shall indeed establish that the condition $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$, $\forall \, z, w \in \mathcal{V}_{\mathbb{C}^n}(0)$, $z \in Q_{\bar{w}}$ implies that f is complex algebraic on each Segre variety in some open set $V := \mathcal{V}_{\mathbb{C}^n}(0)$ and then apply:

Theorem 1.10. ([10]) *Let* $g \in \mathcal{O}(V, \mathbb{C})$ *and let* M *be minimal at* 0. *Then* g *is algebraic if and only if* $g|_{Q_{\bar{w}} \cap V}$ *is algebraic* $\forall \, w \in V$.

We also obtain an equivalent version of Theorem 1.9:

Theorem 1.11. ([10,14]) *If* $\dim_{f(z)} \mathbb{X}'_{z,\bar{z}} = 0 \; \forall \, z \in M \cap \mathcal{V}_{\mathbb{C}^n}(0)$, *then* f *is algebraic.*

Theorem 1.11 admits several applications and covers several known results (*e.g.* [1,2,13,14]). In truth, some unexpected phenomena and some subtle things are hidden behind Theorem 1.11. Our work is aimed to reveal most of them.

1.12. First remarks and questions. This Theorem 1.11 will be deduced from Theorem 1.9 by proving that there exist points $p \in M$ arbitrarily close to 0 such that $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$

$\forall z, w \in \mathcal{V}_{\mathbb{C}^n}(p)$, $z \in Q_{\bar{w}}$, *see* Proposition 1.23 below. The main difficulty here is that the set $\mathbb{X}'_{z,\bar{w}}$ is *not* in general holomorphically parametrized by $z, \bar{w} \in U$, $z \in Q_{\bar{w}}$, in the sense that there would exist analytic equations such that $\mathbb{X}'_{z,\bar{w}} = \{z' : \lambda_j(z, \bar{w}, z') = 0, 1 \leq j \leq J\}$. In fact, the existence of such equations would readily yield the following upper semi-continuity property:

$$\dim_{f(0)} \mathbb{X}'_{0,0} = 0 \Rightarrow \left\langle \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0, \ \forall z, w \in \mathcal{V}_{\mathbb{C}^n}(0), z \in Q_{\bar{w}} \right\rangle. \tag{1.13}$$

And (1.13) above would immediately yield that Theorem 1.9 implies Theorem 1.11. However, (1.13) fails to hold in general. Our goal is therefore to explore the properties of the map $(z, \bar{w}) \mapsto \mathbb{X}'_{z,\bar{w}}$. Let us denote $\mathcal{M} := \{z \in Q_{\bar{w}}\} = \{(z, \bar{w}) \in U \times U : \rho(z, \bar{w}) = 0\}$, which is a complex-algebraic d-codimensional submanifold of $U \times U$, called the *extrinsic complexification* of M. To be exhaustive, we wish also to compare the following twelve determinacy conditions:

$$\begin{array}{llll}
\mathcal{C}^4(\mathcal{M}) : & \dim_{f(z)} \mathbb{M}'_{z,\bar{w}} = 0 \ \ \forall (z, \bar{w}) \in \mathcal{M} & C^4(M) : & \dim_{f(p)} \mathbb{M}'_{p,\bar{p}} = 0 \ \ \forall p \in M \\
\mathcal{C}^3(\mathcal{M}) : & \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0 \ \ \forall (z, \bar{w}) \in \mathcal{M} & C^3(M) : & \dim_{f(p)} \mathbb{X}'_{p,\bar{p}} = 0 \ \ \forall p \in M \\
\mathcal{C}^2(\mathcal{M}) : & \dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = 0 \ \ \forall (z, \bar{w}) \in \mathcal{M} & C^2(M) : & \dim_{f(p)} \mathbb{Z}'_{p,\bar{p}} = 0 \ \ \forall p \in M \\
\mathcal{C}^1(\mathcal{M}) : & \dim_{f(z)} \mathbb{V}'_{z,\bar{w}} = 0 \ \ \forall (z, \bar{w}) \in \mathcal{M} & C^1(M) : & \dim_{f(p)} \mathbb{V}'_{p} = 0 \ \ \forall p \in M
\end{array}$$

and also:

$$C_p^4 : \dim_{f(p)} \mathbb{M}'_{p,\bar{p}} = 0, \quad C_p^3 : \dim_{f(p)} \mathbb{X}'_{p,\bar{p}} = 0,$$
$$C_p^2 : \dim_{f(p)} \mathbb{Z}'_{p,\bar{p}} = 0, \quad C_p^1 : \dim_{f(p)} \mathbb{V}'_{p} = 0.$$

As a preliminary in this exposition, we will first recall and state some of the nice properties of the mapping $z \mapsto \mathbb{V}'_z$.

1.14. Analytic dependence of $z \mapsto \mathbb{V}'_z$. By $\mathcal{A}_n(U)$, we denote the ring of polynomial mappings from U to $\mathbb{C}$, or more generally, of holomorphic algebraic functions over U (*see* [1,2]). By $\mathcal{O}_n(U)$, we denote the ring of plain holomorphic mapping from U to $\mathbb{C}$. By a slight abuse of notation, we denote by $\mathcal{O}_n(U) \times \mathcal{A}_n(U)$ the ring of functions $g(z, w)$ which are holomorphic with respect to z and polynomial with respect to w. Also, we write $\chi \in \overline{\mathcal{O}}_n(U)$ if χ is antiholomorphic with respect to the variable $w \in U$.

By the well-known process of applying the tangential Cauchy-Riemann operators to the identity $\rho'(f(z), \bar{f}(\bar{w})) = 0$ as $\rho(z, \bar{w}) = 0$, one can establish that the map $z \mapsto \mathbb{V}'_z$ is analytic (algebraic here because M, M' are algebraic):

Proposition 1.15. *There exist $J \in \mathbb{N}_*$ and functions $r_j(z, \bar{w}, z') \in \mathcal{A}_n(U) \times \overline{\mathcal{O}}_n(U) \times \mathcal{A}_{n'}(U')$, $1 \leq j \leq J$, such that $\forall$ $(z, w) \in U \times U$ with $z \in Q_{\bar{w}}$, then:*

$$r_{M'}(f(Q_{\bar{z}})) = \{z' \in U' : r_j(z, \bar{w}, z') = 0, 1 \leq j \leq J\} = \mathbb{V}'_z =: \mathbb{S}^1_{z,\bar{w}}. \tag{1.16}$$

Here, we write "vectorially" r for $(r_1, \ldots, r_J)$ and we have set:

$$\mathbb{S}^1 := \{(z, \bar{w}, z') \in U \times \overline{U} \times U' : r(z, \bar{w}, z') = 0\}. \tag{1.17}$$

Then in (1.16) above, $\mathbb{S}^1_{z,\bar{w}}$ simply denotes the fiber of $\mathbb{S}^1$. Although the equations (1.16) of $\mathbb{V}'_z$ do depend in general of some $\bar{w}$ such that $z \in Q_{\bar{w}}$, the zero-set $\mathbb{V}'_z$ appears to be *independent of* $\bar{w}$ provided $(z, \bar{w}) \in \mathcal{M}$. This is in fact clear because by its definition, it is the set equal to $r_{M'}(f(Q_{\bar{z}}))$. But the analytic equations $r(z, \bar{w}, z') = 0$ justify the notation $\mathbb{S}^1_{z,\bar{w}}$ (not to be confused with $\mathbb{V}'_z$).

Proposition 1.15 and the upper semi-continuity of the fiber dimension of a holomorphic map immediately yield the following:

(1.18)
$$C_0^1 \ \Rightarrow \ \mathcal{C}^1(\mathcal{M} \cap (V \times V)) \ \text{ and } \ C_0^1 \Rightarrow C^1(M \cap V) \ \text{ for some } \ V = \mathcal{V}_{\mathbb{C}^n}(0) \subset U.$$

Corollary 1.19. *If $\dim_{f(z)} \mathbb{S}^1_{z,\bar{w}} = 0$ for some $z \in U$ and some $w \in Q_{\bar{z}}$, then there exists $\mathcal{N} \subset \mathcal{M}$ a proper complex analytic subvariety such that the map*

$$\mathcal{V}_{\mathbb{C}^{n'}}(f(z)) \ni z' \mapsto r(z, \bar{w}, z') \in \mathbb{C}^J \tag{1.20}$$

is an immersion at $f(z)$, for all $(z, \bar{w}) \in \mathcal{M} \backslash \mathcal{N}$.

This is of course equivalent to the generic rank of the mapping

$$\mathcal{M} \times \mathbb{C}^{n'} \ni (z, \bar{w}, z') \mapsto (z, \bar{w}, r(z, \bar{w}, z')) \in \mathcal{M} \times \mathbb{C}^J \tag{1.21}$$

be maximal equal to $2m + d + n'$. Just one further remark. As $M \cong \{(z, \bar{z}) \in \mathcal{M}\}$ embeds as a real algebraic maximally real submanifold of $\mathcal{M}$, then $\mathcal{N} \cap M := N$ is also a proper real analytic subset of M. In particular, after applying the implicit function theorem to (1.20) near $(p, \bar{p})$, we obtain the existence of a neighborhood $V_p := \mathcal{V}_{\mathbb{C}^n}(p)$ and of holomorphic, partially algebraic functions $\Psi'_\nu(z, \bar{w}) \in \mathcal{A}_n(V_p) \times \overline{\mathcal{O}}_n(V_p)$, $1 \leq \nu \leq n'$ such that $f_\nu(z) = \Psi'_\nu(z, \bar{w})$, $\nu = 1, \ldots, n'$, $z \in Q_{\bar{w}}$ (using elimination theory, one can even assume that the Ψ'_ν are polynomial with respect to z). Fixing w, this shows that f is algebraic on Segre varieties and Theorem 1.10 then applies to show that f is algebraic. In particular, we recover the main theorem of [2] with a slight variation. For further properties and knowledge about the geometry of $\mathbb{S}^1$ (the first reflection variety), we refer to [1,2,3,4,6,7,10,14].

1.22. Almost everywhere analytic dependence of $(z, \bar{w}) \mapsto \mathbb{X}'_{z,\bar{w}}$. Now, we present the way how $(z, \bar{w}) \mapsto \mathbb{X}'_{z,\bar{w}}$ varies:

Proposition 1.23. *If $\dim_{f(z)} \mathbb{X}'_{z,\bar{z}} = 0$, $\forall\, z \in M$, then there exists a dense open subset $D_M \subset M$ such that*

$(*)$ $\forall p \in D_M$, $\exists\, U_p = \mathcal{V}_{\mathbb{C}^n}(p)$, $\exists\, U'_{p'} = \mathcal{V}_{\mathbb{C}^{n'}}(p' = f(p))$, $\exists\, K \in \mathbb{N}_*$, $\exists\, (s_k)_{1 \leq k \leq K}$, $s_k \in \overline{\mathcal{O}}_n(U_p) \times \mathcal{O}_n(U_p) \times \mathcal{A}_{n'}(U'_{p'})$ such that, if $\mathbb{S}^1 = \{r(z, \bar{w}, z') = 0\}$, then

$\forall\, z, w \in U_p$, $z \in Q_{\bar{w}}$, $\forall\, w, z_1 \in U_p$, $w \in Q_{\bar{z}_1}$,

$$U'_{p'} \cap \mathbb{X}'_{z,\bar{w}} \subset \{z' \in U'_{p'} : r(z, \bar{w}, z') = 0, s(\bar{w}, z_1, z') = 0\} =: \mathbb{S}^2_{z,\bar{w},z_1} \tag{1.24}$$

and such that the graph of f over $\mathcal{M} \sharp \mathcal{M} := \{(z, \bar{w}, z_1) : \rho(z, \bar{w}) = 0, \rho(w, \bar{z}_1) = 0\}$ intersected with $U_p \times \overline{U}_p \times U_p$ satisfies

$$\Gamma r(f) = \{(z, \bar{w}, z_1, f(z))\} = \{(z, \bar{w}, z_1, z') : z' \in \mathbb{S}^2_{z,\bar{w},z_1}\} =: \mathbb{S}^2. \tag{1.25}$$

Furthermore, there exist similar analytic equations $r(z, \bar{w}, z') = 0$, $s(\bar{w}, z_1, z') = 0$ such that

$(**)$ $\forall\, (z, \bar{w}, z_1) \in (\mathcal{M} \sharp \mathcal{M}) \cap (U_p \times \overline{U}_p \times U_p)$, the map

$$\mathcal{V}_{\mathbb{C}^{n'}}(f(z)) \ni z' \mapsto (r(z, \bar{w}, z'), s(\bar{w}, z_1, z')) \in \mathbb{C}^{J+K} \tag{1.26}$$

is an immersion at $f(z)$.

We invite the reader to notice that the dependence of $\mathbb{S}^2$ is holomorphic with respect to z and antiholomorphic with respect to w, which justifies and explains the notation $\mathbb{X}'_{z,\bar{w}}$. This technical proposition, whose proof is postponed to Section 3, appeals several remarks. The first one is: what is the structure of the closed set $M \backslash D_M \subset M$ exactly? Surprisingly, it is *not* an analytic set, it is in general a *subanalytic set*. Leaving this question for a while, we shall return to it in Examples 1.66 and 1.68 below. The second remark is: we prove Theorem 1.9 without using Proposition 1.23. This proposition is indeed used only to prove that Theorem 1.9 $\Rightarrow$ Theorem 1.11. Next, a third remark. As $\Gamma r(f) \equiv \mathbb{S}^2$, the projection $\pi : \mathbb{S}^2 (\subset U_p \times \overline{U}_p \times U_p \times U'_{p'}) \to U_p \times \overline{U}_p \times U_p$ is submersive. The zero locus $\mathbb{S}^2 \equiv \Gamma r(f)$ is smooth, but the equations defining $\mathbb{S}^2$ can be non-reduced. After taking the reduced complex space Red $\mathbb{S}^2$, we obtain the immersion property $(**)$ of Proposition 1.23 for $z, w, z_1 \in \mathcal{V}_{\mathbb{C}^n}(p)$. Of course, the equations $s(\bar{w}, z_1, z') = 0$ of which Proposition 1.23 asserts the existence are clearly those for $r^2_{M'}(f(Q_{\bar{w}}))$, while as before $r(z, \bar{w}, z')$ come for $r_{M'}(f(Q_{\bar{z}}))$. Now, we come to the most important remark. Thanks to the analytic parametrization (1.24), we have:

$$\left\langle \dim_{f(p)} \mathbb{S}^2_{p,\bar{p},p} = 0 \right\rangle \Rightarrow \left\langle \dim_{f(z)} \mathbb{S}^2_{z,\bar{w},z_1} = 0, \ \forall\, z, w, z_1 \in \mathcal{V}_{\mathbb{C}^n}(p) \subset U_p \right\rangle. \tag{1.27}$$

We therefore obtain the desired semi-continuity property:

(1.28)

$$\left\langle \dim_{f(z)} \mathbb{X}'_{z,\bar{z}} = 0, \ \forall\, z \in M \right\rangle \Rightarrow \left\langle \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0, \ \forall\, z, w \in \mathcal{V}_{\mathbb{C}^n}(p), z \in Q_{\bar{w}} \right\rangle.$$

In summary, the reduction of Theorem 1.11 to Theorem 1.9 *via* Proposition 1.23 is completed.

1.29. Solvability of f over a dense open set. The fundamental remark is that after solving $f(z)$ from $(**)$ of Proposition 1.23 at $q \in U_p$, $p \in D_M$, *i.e.* solving $f(z)$ from the collection of equations:

$$r_j(z, \bar{w}, f(z)) = 0, \quad 1 \leq j \leq J, \quad s_k(\bar{w}, z_1, f(z)) = 0, \quad 1 \leq k \leq K, \tag{1.30}$$

where $z \in Q_{\bar{w}}$, $w \in Q_{\bar{z}_1}$, we obtain:

Corollary 1.31. *$\forall\, p \in D_M$, $\forall\, q \in U_p = \mathcal{V}_{\mathbb{C}^n}(p)$, $\exists\, W_q = \mathcal{V}_{\mathbb{C}^n}(q)$, $\exists\, \Psi'_\nu(z, \bar{w}, z_1) \in \mathcal{A}_n(W_q) \times \overline{\mathcal{O}}_n(W_q) \times \mathcal{O}_n(W_q)$, $1 \leq \nu \leq n'$, such that*

$$f_1(z) = \Psi'_1(z, \bar{w}, z_1), \ldots, f_{n'}(z) = \Psi'_{n'}(z, \bar{w}, z_1), \tag{1.32}$$

$\forall\, z, w, z_1 \in W_q$, $z \in Q_{\bar{w}}$, $w \in Q_{\bar{z}_1}$.

Then equation (1.32) immediately shows that f is algebraic on each Segre variety $Q_{\bar{w}} \cap W_q$: just fix $\bar{w}, z_1$ in (1.32) and let $z \in Q_{\bar{w}}$ vary. Thus, again Theorem 1.10 applies, as in **1.14** above.

1.33. Algebraicity of f. Theorem 1.11 admits the following main corollary:

Theorem 1.34. ([6,10,14]) *If M' does not contain complex algebraic sets of positive dimension, then f is algebraic.*

Proof. Indeed, we have:

1. $\mathbb{X}'_{z,\bar{z}} = r_{M'}(f(Q_{\bar{z}})) \cap r^2_{M'}(f(Q_{\bar{z}})) \subset M'$ (by $r_{M'}(E') \cap r^2_{M'}(E') \subset M'\ \forall\, E'$).

2. $r_{M'}(f(Q_{\bar{z}}))$, $r^2_{M'}(f(Q_{\bar{z}}))$, $\mathbb{X}'_{z,\bar{z}}$ are complex algebraic sets through $f(z)$.

Consequently, $\dim_{f(z)} \mathbb{X}'_{z,\bar{z}} = 0\ \forall\, z \in M$ necessarily holds under the assumption of Theorem 1.34: Theorem 1.11 then applies. □

Remark. The author obtains a completely different proof of Theorem 1.34 in [10]. A similar proof is given in [6], but for M being Segre-transversal instead of being minimal.

1.35. Comparison of $\mathbb{V}'_z$, $\mathbb{X}'_{z,\bar{w}}$. We have now completed the presentation of the main steps in the proof of Theorem 1.11. Next, we come to the comparison between the zero dimension conditions about $\mathbb{V}'_z$ and $\mathbb{X}'_{z,\bar{w}}$. It is easy to see that there exist many examples of f, M, M', U, U' such that $\dim_{f(z)} \mathbb{V}'_z \geq 1$, $\forall\, z \in U$ and $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$, $\forall\, z, w \in U$, $z \in Q_{\bar{w}}$. Consequently the condition $\mathcal{C}^3(\mathcal{M})$ is strictly finer than $\mathcal{C}^1(\mathcal{M})$ (same fact about $\mathcal{C}^2(\mathcal{M})$ or $\mathcal{C}^4(\mathcal{M})$). Here is such an example (exercise).

Example 1.36. Take $M : z_4 = \bar{z}_4 + i z_1 \bar{z}_1$ in $\mathbb{C}^2_{(z_1, z_4)}$, $f(z_1, z_4) = (z_1, 0, 0, z_4) \in \mathbb{C}^4$, and the hypersurface

$$M' : \quad z'_4 = \bar{z}'_4 + i z'_1 \bar{z}'_1 + i {z'_1}^2 \bar{z}'_3 + i \bar{z'}^2_1 z'_3 + i z'_2 \bar{z}'_3 + i \bar{z}'_2 z'_3. \tag{1.37}$$

It is also known that the second reflection is superfluous when $n = n'$ and f is a biholomorphic map (*cf.* [1,2,3,14]) or if one assumes directly that $\dim_{f(z)} \mathbb{V}'_z = 0$ (*cf.* [7]).

1.38. Comparisons between $\mathbb{V}'_z, \mathbb{W}'_z, \mathbb{Z}'_{z,\bar{w}}, \mathbb{X}'_{z,\bar{w}}$. Yet another strategy (*cf.* [8,14]) consists in replacing (when possible) the set $\mathbb{S}^1 = \{(z, \bar{w}, z') : r(z, \bar{w}, z') = 0\}$ by some smaller complex analytic set $\widetilde{\mathbb{S}}^1 = \{(z, \bar{w}, z') : \tilde{r}(z, \bar{w}, z') = 0\} \subset \mathbb{S}^1$ such that:

1. $\tilde{r} \in \mathcal{A}_n(U) \times \overline{\mathcal{O}}_n(U) \times \mathcal{A}_{n'}(U')$.

2. $\Gamma r(f) =$ the graph of $f = \{(z, \bar{w}, f(z)) : (z, \bar{w}) \in \mathcal{M}\}$ is contained in $\widetilde{\mathbb{S}}^1$.

3. $\widetilde{\mathbb{S}}^1$ is obtained in a constructive way.

The set $\widetilde{\mathbb{S}}^1$ should really be given by means of an explicit construction, because $\mathbb{S}^1$ is the concrete datum from which one tries to deduce that z' is solvable in terms of $z, \bar{w}$. Constructing such a set $\widetilde{\mathbb{S}}^1$, one can hope that $\dim_{f(z)} \widetilde{\mathbb{S}}^1_{z,\bar{w}} = 0$. For instance, if $\Gamma r(f)$ is contained in the singular locus $\mathrm{Sing}(\mathbb{S}^1)$, which is computable in terms of $r(z, \bar{w}, z')$, since $\mathbb{S}_1$ is explicitely given, it is possible to shrink $\mathbb{S}^1$ and to replace it by $\widetilde{\mathbb{S}}^1 := \mathrm{Sing}(\mathbb{S}^1)$, obtaining *new, possibly finer* equations $\tilde{r}(z, \bar{w}, f(z)) = 0$. In [14] (preprint version), $\mathbb{S}^1$ is also shrunk more again, still in a constructive way, in order that $\widetilde{\mathbb{S}}^1$ becomes a "holomorphic family". Therefore, there might exist many different such $\widetilde{\mathbb{S}}^1$ depending on the way how $\mathbb{S}^1$ is shrunk in a constructive way. However in the end of Section 3 we propose a uniform unambiguous method, which uses only elementary tools: minors and the uniqueness principle (but not passing to the filtration by singular complex subspaces).

If $\dim_{f(z)}\widetilde{\mathbb{S}}^1_{z,\bar{w}} \geq 1$, denote $\mathbb{W}'_{z,\bar{w}} := \{z' \in U' : \tilde{r}(z,\bar{w},z') = 0\} = \widetilde{\mathbb{S}}^1_{z,\bar{w}}$ and

$$\mathbb{Z}'_{z,\bar{w}} := \mathbb{W}'_{z,\bar{w}} \cap r_{M'}(\mathbb{W}'_{w,\bar{z}_1}), \qquad z \in Q_{\bar{w}},\ w \in Q_{\bar{z}_1}. \tag{1.39}$$

Then $f(z) \in \mathbb{Z}'_{z,\bar{w}}$ because

$$\begin{aligned} & f(z) \in \mathbb{W}'_{z,\bar{w}} \subset \mathbb{V}'_z \\ & r_{M'}(\mathbb{W}'_{z,\bar{w}}) \supset r_{M'}(\mathbb{V}'_z) \text{ (by } r_{M'}(E') \supset r_{M'}(F') \text{ if } E' \subset F') \\ & r_{M'}(\mathbb{W}'_{w,\bar{z}_1}) \supset r_{M'}(\mathbb{V}'_w) = r^2_{M'}(f(Q_{\bar{w}})) \supset f(Q_{\bar{w}}) \ni f(z) \\ & f(z) \in \mathbb{W}'_{z,\bar{w}} \cap r_{M'}(\mathbb{W}'_{w,\bar{z}_1}) = \mathbb{Z}'_{z,\bar{w},z_1}. \end{aligned} \tag{1.40}$$

The gain in reducing $\mathbb{S}^1 \mapsto \widetilde{\mathbb{S}}^1$ lies in the fact that one can easily insure that $(w,\bar{z}_1) \mapsto r_{M'}(\mathbb{W}'_{w,\bar{z}_1})$ becomes an analytic parametrization (or a "holomorphic family") by having first a nice representation of $\mathbb{W}'_{z,\bar{w}}$:

Proposition 1.41. ([8,14]) *There exists $\mathcal{N} \subset \mathcal{M}$ a proper complex analytic subset such that $\forall p = (z_p, \bar{w}_p) \in \mathcal{M}\backslash\mathcal{N}$ $\exists \mathcal{U}_p = \mathcal{V}_{\mathcal{M}}(p)$, $\exists n'_1,\ n'_2 \in \mathbb{N}$, $n'_1 + n'_2 = n'$, $\exists\, \Phi'_\nu(z,\bar{w},z') \in \mathcal{A}_n(z) \times \mathcal{O}_n(\bar{w}) \times \mathcal{A}_{n'}(z'_1)$, $(z,\bar{w}) \in \mathcal{U}_p$, $(w,\bar{z}_1) \in \mathcal{U}_p$, $1 \leq \nu \leq n'_1$, such that*

$$\{(z,\bar{w},f(z)) : (z,\bar{w}) \in \mathcal{M}\} \subset \{(z,\bar{w},z') : z'_2 = \Phi'(z,\bar{w},z'_1)\} \subset \{r(z,\bar{w},z') = 0\}. \tag{1.42}$$

In [8], it is established that the representation (1.42) is unique: the set $\{z'_2 = \Phi'(z,\bar{w},z'_1)\}$ being the maximal for inclusion among all the sets of the form $\Lambda = \{z'_2 = \Psi'(z,\bar{w},z'_1)\}$ (for some splitting of the coordinates z') satisfying $\Gamma r(f) \subset \Lambda \subset \mathbb{S}^1$. Let now $p \in M\backslash(N := \mathcal{N} \cap M)$ and let $U_p = \mathcal{V}_{\mathbb{C}^n}(p)$ with $\mathcal{U}_p \subset U_p \times U_p$. The representation (4.2) yields after some easy work (*see* [8,14]) that there exist $K \in \mathbb{N}_*$, $(s_k)_{1\leq k\leq K} \in \overline{\mathcal{O}}_n(U_p) \times \mathcal{O}_n(U_p) \times \mathcal{A}_{n'}(U'_{p'})$ such that $\forall\, z, w \in U_p$, $z \in Q_{\bar{w}}$, $\forall\, w, z_1 \in U_p$, $w \in Q_{\bar{z}_1}$,

$$\mathbb{Z}'_{z,\bar{w},z_1} = \{z' \in U'_{p'} : r(z,\bar{w},z') = 0,\ s(\bar{w},z_1,z') = 0\}. \tag{1.43}$$

As in Proposition 1.23, we have got the analytic dependence of the map $(z,\bar{w}) \mapsto \mathbb{Z}'_{z,\bar{w}}$ (again, the set $\mathbb{Z}'_{z,\bar{w},z_1}$ does not depend as a set of z_1 if $(z,\bar{w},z_1) \in \mathcal{M}\sharp\mathcal{M}$ and it coincides with $\mathbb{Z}'_{z,\bar{w}}$ which was defined in a set theoretical way). We will come back later to the construction of Φ', *see* Proposition 1.74 below.

1.44. Fundamental remark. The constructiveness of a shrinking $\mathbb{S}^1 \mapsto \widetilde{\mathbb{S}}^1$ is essential. One is temptated to introduce $\mathbb{S}^1_{min} :=$ the minimal (for inclusion) $\mathcal{A}_n \times \overline{\mathcal{O}}_n \times \mathcal{A}_{n'}$-set contained in $\mathbb{S}^1$ satisfying $\Gamma r(f) \subset \mathbb{S}^1_{min} \subset \mathbb{S}^1$, *i.e.* the intersection of all $\widetilde{\mathbb{S}}^1$, even those which are not constructive, and to put ${\mathbb{Z}_{min,z,\bar{w},z_1}}' := \mathbb{S}^1_{min,z,\bar{w}} \cap r_{M'}(\mathbb{S}^1_{min,w,\bar{z}_1})$. However, the equations $r_{min}(z,\bar{w},z') = 0$ being not known from the datum $\mathbb{S}^1$ in general and not constructible in an explicit way, it is quite impossible to deduce from $f(z) \in {\mathbb{Z}_{min,z,\bar{w},z_1}}'$ anything. Not to mention that anyway if f was algebraic from the beginning, then $r_{min} := z' - f(z)$ would have been convenient and the condition $\dim_{f(z)}{\mathbb{Z}_{min,z,\bar{w},z_1}}' = 0$ (here $\dim_{f(z)}\mathbb{S}^1_{min,z,\bar{w},z_1} = 0$) then becomes surprisingly tautological!

1.45. Properties of $\mathbb{Z}'_{z,\bar{w}}$. Before entering into further discussions, let us summarize the properties of $\mathbb{Z}'_{z,\bar{w}}$ as follows (*see* also Proposition 1.74).

Theorem 1.46. **(1)** *The set of points where $(z,\bar{w}) \mapsto \mathbb{Z}'_{z,\bar{w}}$ is not holomorphic is a proper complex analytic subset $\mathcal{N}$ of $\mathcal{M}$. Let $N := \mathcal{N} \cap M$.*

(2) *If $\dim_{f(z)}\mathbb{Z}'_{z,\bar{z}} = 0$ $\forall\, z \in \mathcal{V}_{\mathbb{C}^n}(0) \cap M$, then $\dim_{f(z)}\mathbb{Z}'_{z,\bar{z}} = 0$ for z outside a proper real analytic subvariety N_1 of $M\backslash N$.*

(3) *If $\dim_{f(p)}\mathbb{Z}'_{p,\bar{p}} = 0$ at some point $p \in M\backslash(N \cup N_1)$, then f is algebraic.*

Remark. That the bad set $\mathcal{N}$ is analytic is a property which is specific to $\mathbb{Z}'_{z,\bar{w}}$. For $\mathbb{X}'_{z,\bar{w}}$, the bad set $M\backslash D_M$ is definitely not analytic, see Example 1.68 below.

1.47. Discussion. We can now summarize the main result in [14] (preprint version).

Theorem 1.48. ([14]) *If* $\dim_{f(p)} \mathbb{Z}'_{p,\bar{p}} = 0 \ \forall \, p \in \mathcal{V}_{\mathbb{C}^n}(0) \cap M$, *then* f *is algebraic.*

This theorem also implies Theorem 1.34 (which is the main application of double reflection). Indeed, a possible proof of Theorem 1.34 starting from Theorem 1.48 above can be achieved exactly as we did *supra* in **1.33**, because the intersection $\mathbb{Z}'_{p,\bar{p}} = \mathbb{W}'_{p,\bar{p}} \cap r_{M'}(\mathbb{W}'_{p,\bar{p}}) \subset M'$ must also be a zero-dimensional complex algebraic set.

1.49. Reverse inclusions. Now, we return to comparison of $\mathbb{X}'_{z,\bar{w}}$ with $\mathbb{Z}'_{z,\bar{w}}$. If $\dim_{f(z)} \mathbb{W}'_{z,\bar{w}} \geq 1$ $\forall \, z \in U$ (so second reflection is needed), one can expect that

$$\left\langle \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0, \ \forall \, z \right\rangle \Rightarrow \left\langle \dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = 0, \ \forall \, z \right\rangle, \tag{1.50}$$

i.e. that the study of $\mathbb{Z}'_{z,\bar{w}}$ is sufficient to get a complete proof of Theorem 1.9. Nevertheless, **two inclusions of opposite sense enter in competition**

$$\mathbb{W}'_{z,\bar{w}} \subset \mathbb{S}^1_{z,\bar{w}} \quad \text{and} \quad r_{M'}(\mathbb{W}'_{z,\bar{w}}) \supset r_{M'}(\mathbb{S}^1_{z,\bar{w}}) \tag{1.51}$$

so that it is not clear how

$$\mathbb{X}'_{z,\bar{w},z_1} = \mathbb{S}^1_{z,\bar{w}} \cap r_{M'}(\mathbb{S}^1_{w,\bar{z}_1}) \quad \text{and} \quad \mathbb{Z}'_{z,\bar{w},z_1} = \mathbb{W}'_{z,\bar{w}} \cap r_{M'}(\mathbb{W}'_{w,\bar{z}_1}) \tag{1.52}$$

could be comparable. Indeed, implication (1.50) is simply untrue:

Example 1.53. There exist f, M, M' such that:

$$\dim_{f(z)} \mathbb{X}'_{z,\bar{w},z_1} = 0, \ \forall \, z \quad \text{but} \quad \dim_{f(z)} \mathbb{Z}'_{z,\bar{w},z_1} \geq 1, \ \forall \, z. \tag{1.54}$$

Explicitely, take: $M : z_5 = \bar{z}_5 + i z_1 \bar{z}_1$ in $\mathbb{C}^2_{(z_1,z_5)}$, $f(z_1, z_5) = (z_1, 0, 0, 0, z_5)$, and:

$$M' : \quad z'_5 = \bar{z}'_5 + i z'_1 \bar{z}'_1 + i {z'_1}^2 \bar{z}'_3 \bar{z}'_4 + i \bar{z}'^2_1 z'_3 z'_4 + i {z'_3}^2 \bar{z}'^2_2 + i \bar{z}'^2_3 {z'_2}^2 + \\ + i \bar{z}'^3_3 z'^3_4 + i z'^3_3 \bar{z}'^3_4. \tag{1.55}$$

In conclusion, the determination of $f(z)$ by $\mathbb{X}'_{z,\bar{w},z_1}$ can be strictly finer than by $\mathbb{Z}'_{z,\bar{w},z_1}$ (for the details, see Section 4). And quite surprisingly, it is also true that determination of $f(z)$ by $\mathbb{Z}'_{z,\bar{w}}$ can be strictly finer than by $\mathbb{X}'_{z,\bar{w}}$.

Example 1.56. There exist f, M, M' such that $\dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = 0 \ \forall \, z$ but $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} \geq 1 \ \forall \, z$. To be explicit, take $M : z_4 = \bar{z}_4 + i z_1 \bar{z}_1$ in $\mathbb{C}^2_{(z_1,z_4)}$, $f(z_1, z_4) = (z_1, 0, 0, z_4) \in \mathbb{C}^4$, and:

$$M' : \qquad z'_4 = \bar{z}'_4 + i z'_1 \bar{z}'_1 + i {z'_1}^2 \bar{z}'_2 \bar{z}_3 + i \bar{z}'^2_1 z'_2 z'_3. \tag{1.57}$$

In summary:

$$\left\langle \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0, \ \forall \, z \right\rangle \not\Leftarrow \not\Rightarrow \left\langle \dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = 0, \ \forall \, z \right\rangle. \tag{1.58}$$

Consequently, it is justified to introduce:

$$\mathbb{M}'_{z,\bar{w}} := \mathbb{W}'_{z,\bar{w}} \cap r_{M'}(\mathbb{V}'_w), \tag{1.59}$$

where

$$\mathbb{W}'_{z,\bar{w}} = \widetilde{\mathbb{S}}^1_{z,\bar{w}}, \tag{1.60}$$

for a constructive shrinking $\widetilde{\mathbb{S}}^1$ of $\mathbb{S}^1$. (Of course, different such shrinkings may exist, which depend on the conditions that are imposed; the choice $\widetilde{\mathbb{S}}^1 = \mathbb{S}^1$ can always be done; our examples illustrate well the phenomenon.)

1.61. Comparison between $\mathbb{M}'_{z,\bar{w}}$ and $\mathbb{Z}'_{z,\bar{w}}$, $\mathbb{X}'_{z,\bar{w}}$. Notice that $\mathbb{M}'_{z,\bar{w}} = \mathbb{X}'_{z,\bar{w}} \cap \mathbb{Z}'_{z,\bar{w}}$. Now it is clear that:

$$\left\langle \dim_{f(z)} \mathbb{M}'_{z,\bar{w}} = 0, \ \forall \, z, w, z \in Q_{\bar{w}} \right\rangle \Leftarrow \left\langle \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0, \ \forall \, z, w, z \in Q_{\bar{w}} \right\rangle \tag{1.62}$$

$$\left\langle \dim_{f(z)} \mathbb{M}'_{z,\bar{w}} = 0, \ \forall \, z, w, z \in Q_{\bar{w}} \right\rangle \Leftarrow \left\langle \dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = 0, \ \forall \, z, w, z \in Q_{\bar{w}} \right\rangle. \tag{1.63}$$

Our examples also show that the reverse implications $\Rightarrow$ are both untrue.

1.64. Summarizing tabulae. It is time to give a complete link tabular between the twelve conditions

$$C_p^1, C_p^2, C_p^3, C_p^4, C^1(M), C^2(M), C^3(M), C^4(M), \mathcal{C}^1(\mathcal{M}), \mathcal{C}^2(\mathcal{M}), \mathcal{C}^3(\mathcal{M}), \mathcal{C}^4(\mathcal{M}).$$

Return to definitions. Here, $p \in M$ is a fixed chosen point, the origin in previous coordinates. The point p is chosen arbitrarily and is fixed. C_p^j denotes: $C_p^4 : \dim_{f(p)} \mathbb{M}'_{p,\bar{p}} = 0$, $C_p^3 : \dim_{f(p)} \mathbb{X}'_{p,\bar{p}} = 0$, $C_p^2 : \dim_{f(p)} \mathbb{Z}'_{p,\bar{p}} = 0$, $C_p^1 : \dim_{f(p)} \mathbb{V}'_p = 0$. Henceforth, $C^j(M)$ does not denote "$C_q^j \; \forall \, q \in M$, but $\forall \, q \in D_M$", *i.e.* over a dense open subset D_M of M. *Idem* for $\mathcal{C}^j(\mathcal{M})$. Consequently, the implication $C^j(M) \Rightarrow C_p^j$ is *a priori* untrue, since p can well belong to the set of points q where C_q^j is not satisfied.

First, we already know that if $C^j(M)$ is satisfied over an open dense subset of M, then $\mathcal{C}^j(\mathcal{M})$ is satisfied over an open dense subset of $\mathcal{M}$, for $j = 1, 2, 3, 4$, and conversely. We know this thanks to Propositions 1.15, 1.23 and 1.41. Therefore, if C_p denotes $(C_p^j)_{1 \leq j \leq 4}$, $C(M) = (C^j(M))_{1 \leq j \leq 4}$, $\mathcal{C}(\mathcal{M}) = (\mathcal{C}^j(\mathcal{M}))_{1 \leq j \leq 4}$, the comparison of our twelve conditions which could have been explored in a 12×12 tabular with 144 entries can be reduced to only three 4×4 tabulars:

$***$	C_p	$C(M)$	$\mathcal{C}(\mathcal{M})$
C_p	$\times$	$\times$	$\times$
$C(M)$	$\times$	$\times$	$\times$
$\mathcal{C}(\mathcal{M})$	$\times$	$\times$	$\times$

$\cong$ (

$*$	C_p
C_p	$\times$

,

$*$	$C(M)$
C_p	$\times$

,

$*$	$\mathcal{C}(\mathcal{M})$
$\mathcal{C}(\mathcal{M})$	$\times$

)

$***$	C_p^1	C_p^2	C_p^3	C_p^4
C_p^1	$\Leftarrow \ \Rightarrow$	$\not\Leftarrow \ \Rightarrow$	$\not\Leftarrow\Rightarrow$	$\not\Leftarrow \ \Rightarrow$
C_p^2	$\Leftarrow \ \not\Rightarrow$	$\Leftarrow \ \Rightarrow$	$\not\Leftarrow \ \not\Rightarrow$	$\not\Leftarrow \ \Rightarrow$
C_p^3	$\Leftarrow \ \not\Rightarrow$	$\not\Leftarrow \ \not\Rightarrow$	$\Leftarrow \ \Rightarrow$	$\not\Leftarrow \ \Rightarrow$
C_p^4	$\Leftarrow \ \not\Rightarrow$	$\Leftarrow \ \not\Rightarrow$	$\Leftarrow \ \not\Rightarrow$	$\Leftarrow \ \Rightarrow$

$***$	$C^1(M)$	$C^2(M)$	$C^3(M)$	$C^4(M)$
C_p^1	$\not\Leftarrow \ \Rightarrow$	$\not\Leftarrow \ \Rightarrow$	$\not\Leftarrow \ \Rightarrow$	$\not\Leftarrow \ \Rightarrow$
C_p^2	$\not\Leftarrow \ \not\Rightarrow$	$\not\Leftarrow \ \not\Rightarrow$ [1]	$\not\Leftarrow \ \not\Rightarrow$ [1]	$\not\Leftarrow\not\Rightarrow$ [1]
C_p^3	$\not\Leftarrow \ \not\Rightarrow$	$\not\Leftarrow \ \not\Rightarrow$ [1]	$\not\Leftarrow \ \not\Rightarrow$ [1]	$\not\Leftarrow \ \not\Rightarrow$ [2]
C_p^4	$\not\Leftarrow \ \not\Rightarrow$	$\not\Leftarrow \ \not\Rightarrow$ [1]	$\not\Leftarrow \ \not\Rightarrow$ [1]	$\not\Leftarrow \ \not\Rightarrow$ [1]

$***$	$\mathcal{C}^1(\mathcal{M})$	$\mathcal{C}^2(\mathcal{M})$	$\mathcal{C}^3(\mathcal{M})$	$\mathcal{C}^4(\mathcal{M})$
$\mathcal{C}^1(\mathcal{M})$	$\Leftarrow \ \Rightarrow$	$\not\Leftarrow$ [2] $\Rightarrow$	$\not\Leftarrow$ [2] $\Rightarrow$	$\not\Leftarrow$ [2] $\Rightarrow$
$\mathcal{C}^2(\mathcal{M})$	$\Leftarrow \ \not\Rightarrow$ [2]	$\Leftarrow \ \Rightarrow$	$\not\Leftarrow$ [3] $\not\Rightarrow$ [4]	$\not\Leftarrow$ [3] $\Rightarrow$
$\mathcal{C}^3(\mathcal{M})$	$\Leftarrow \ \not\Rightarrow$ [2]	$\not\Leftarrow$ [4] $\not\Rightarrow$ [3]	$\Leftarrow \ \Rightarrow$	$\not\Leftarrow$ [4] $\Rightarrow$
$\mathcal{C}^4(\mathcal{M}))$	$\Leftarrow \ \not\Rightarrow$ [2]	$\Leftarrow \ \not\Rightarrow$ [3]	$\Leftarrow \ \not\Rightarrow$ [4]	$\Leftarrow \ \Rightarrow$

Our examples are intended to explain only the main nontrivial (non) implication links above. Those not in the articles are easier to find.

1.65. Nonanalytic behavior of $(z, \bar{w}) \mapsto \mathbb{X}'_{z,\bar{w}}$. We give two examples of it.

Example 1.66. There exist f, M, M' with f nonalgebraic such that the function $\mathcal{M} \ni (z, \bar{w}) \mapsto \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} \in \mathbb{N}$ is not upper semi-continuous at 0. Explicitely, take $M : z_4 = \bar{z}_4 + i z_1 \bar{z}_1$ in $\mathbb{C}^2_{(z_1, z_4)}$,

$$M' : \quad z'_4 = \bar{z}'_4 + i z'_1 \bar{z}'_1 + i z'_3 \bar{z}'_2 + i \bar{z}'_3 z'_2 + i {z'}_1^2 \bar{z}'_3 \bar{z}'_2 + i \bar{z'}_1^2 z'_3 z'_2, \tag{1.67}$$

[1]Example 1.66 shows everything: $C_p^j \not\Rightarrow C^j$, $2 \leq j \leq 4$.

[2]Example 1.36 shows everything: $\mathcal{C}^j(\mathcal{M}) \not\Rightarrow \mathcal{C}^1(\mathcal{M})$, $2 \leq j \leq 4$.

[3]Example 1.53.

[4]Example 1.56.

and $f(z_1, z_4) = (z_1, z_4 \sin^3 z_1, 0, z_4)$. Here, $\dim_0 \mathbb{X}'_{0,0} = 0$, $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 1$ $\forall z \neq 0$, $z \in Q_{\bar{w}}$. (*Idem* for $\mathbb{M}'_{z,\bar{w}}$ instead.) See Section 4. This example therefore shows that $\mathbb{X}'_{z,\bar{w}}$ cannot be written as $\{z' : \lambda(z, \bar{w}, z') = 0\}$ for holomorphic $\lambda \in \mathcal{V}_{\mathbb{C}^n}(0) \times \mathcal{V}_{\mathbb{C}^n}(0) \times \mathcal{V}_{\mathbb{C}^{n'}}(0)$ in general.

Example 1.68. (*See* Section 4.) There exist f, M, M' all algebraic such that if Σ denotes the set of points $(z, \bar{w}, z_1) \in \mathcal{M}\sharp\mathcal{M}$ in a neighborhood of which $(*)$ of Proposition 1.23 is not satisfied, then Σ is not a complex analytic subset of $\mathcal{M}\sharp\mathcal{M}$ but a real analytic subset.

1.69. Globalization of $r_{M'}$, $r^2_{M'}$. First, we notice that $r_{M'}(E')(= r^{U'}_{M'}(E'))$ which we have localized in U' could have been defined globally as follows:

$$r^{\mathbb{C}^{n'}}_{M'}(E') = \{w' \in \mathbb{C}^{n'} : Q'_{\bar{w}'} \supset E'\}, \tag{1.70}$$

because the $\rho'_{j'}$ are polynomials. A variation of Theorem 1.9 would be:

Theorem 1.71. *If* $\dim_{f(z)}[r^{\mathbb{C}^{n'}}_{M'}(f(Q_{\bar{z}})) \cap (r^{\mathbb{C}^{n'}}_{M'})^2(f(Q_{\bar{w}}))] = 0$, $\forall z, w \in \mathcal{V}_{\mathbb{C}^n}(0)$, $z \in Q_{\bar{w}}$, *then f is algebraic.*

(Identical proof). Suprisingly, Theorem 1.71 can be more general than Theorem 1.9 because:

Example 1.72. There exist f, M, M', U, U' such that:

1. $\dim_{f(z)}[r^{U'}_{M'}(f(Q_{\bar{z}})) \cap (r^{U'}_{M'})^2(f(Q_{\bar{w}}))] \geq 1$, $\forall z, w \in U$, $z \in Q_{\bar{w}}$ and

2. $\dim_{f(z)}[r^{\mathbb{C}^{n'}}_{M'}(f(Q_{\bar{z}})) \cap (r^{\mathbb{C}^{n'}}_{M'})^2(f(Q_{\bar{w}}))] = 0$, $\forall z, w \in U$, $z \in Q_{\bar{w}}$.

Explicitely (*see* Section 4), take $U = \Delta^2$, $U' = \Delta^4$, the hypersurface $M : z_4 = \bar{z}_4 + iz_1\bar{z}_1$, $f(z_1, z_4) = (z_1, 0, 0, z_4) \in \mathbb{C}^4$, and

$$M' : z'_4 = \bar{z}'_4 + iz'_1\bar{z}'_1 + iz'^2_1(1 + \bar{z}'_3)\bar{z}'_3 + i\bar{z}'^2_1(1 + z'_3)z'_3 + iz'_2z'_3\bar{z}'^2_2 + i\bar{z}'_2\bar{z}'_3 z'^2_2. \tag{1.73}$$

Conversely, Theorem 1.9 can be more general than Theorem 1.71 (exercise left to the reader).

1.74. About $\mathbb{Z}'_{z,\bar{w}}$, $\mathbb{M}'_{z,\bar{w}}$. It is not difficult to see that all the three positive Theorems 1.9, 1.11 and 1.71 concerning $\mathbb{X}'_{z,\bar{w}}$ extend immediately to be satisfied by $\mathbb{Z}'_{z,\bar{w}}$ and by $\mathbb{M}'_{z,\bar{w}}$, once we have established the following result analogous to Proposition 1.23:

Proposition 1.75. *There exists a standard constructive way of finding a variety* $\widetilde{\mathbb{S}}^1 = \{(z, \bar{w}, z') : \tilde{r}(z, \bar{w}, z') = 0\}$ *contained in* $\mathbb{S}^1$ *with* $\tilde{r}_j(z, \bar{w}, z') \in \mathcal{A}_n(U) \times \overline{\mathcal{O}}_n(U) \times \mathcal{A}_{n'}(U')$, $1 \leq j \leq \tilde{J}$, $\tilde{J} \geq J$, *such that*

$$\Gamma r(f) = \{(z, \bar{w}, f(z)) : (z, \bar{w}) \in \mathcal{M}\} \subset \widetilde{\mathbb{S}}^1_{\mathcal{M}}, \tag{1.76}$$

$$\widetilde{\mathbb{S}}^1_{\mathcal{M}} = \{(z, \bar{w}, z') : \tilde{r}(z, \bar{w}, z') = 0, (z, \bar{w}) \in \mathcal{M}\}$$

and such that there exist a Zariski open subset $\mathcal{D}_{\mathcal{M}} := \mathcal{M} \backslash \mathcal{N}$ *of* $\mathcal{M}$, $\mathcal{N} \subset \mathcal{M}$ *complex analytic*, $\dim_{\mathbb{C}} \mathcal{N} \leq 2m + d - 1$, *and an integer* n'_1, $0 \leq n'_1 \leq n'$ *such that*

$(*)$ $\forall p \in \mathcal{D}_{\mathcal{M}}$, $\exists \mathcal{U}_p = \mathcal{V}_{\mathcal{M}}(p)$, $\exists U'_{p'} = \mathcal{V}_{\mathbb{C}^{n'}}(p' = f(p))$, *such that* $\widetilde{\mathbb{S}}^1_{\mathcal{U}_p} := (\mathcal{U}_p \times U'_{p'}) \cap \widetilde{\mathbb{S}}^1$ *is smooth and the projection* $\pi' : \widetilde{\mathbb{S}}^1_{\mathcal{U}_p} \to U'$ *is of constant rank* n'_1.

Consequently, $(*)$ *implies that*

$(**)$ $\forall p \in D_M := \mathcal{D}_{\mathcal{M}} \cap M$, $\exists U_p = \mathcal{V}_{\mathbb{C}^n}(p)$, $\exists U'_{p'} = \mathcal{V}_{\mathbb{C}^{n'}}(p' = f(p))$, $\exists K \in \mathbb{N}_*$, $\exists (s_k)_{1 \leq k \leq K}$, $s_k \in \overline{\mathcal{O}}_n(U_p) \times \mathcal{O}_n(U_p) \times \mathcal{A}_{n'}(U'_{p'})$ *such that* $\forall z_1 \in Q_{\bar{w}}$ *if* $\mathbb{W}'_{w,\bar{z}_1} := \widetilde{\mathbb{S}}^1_{w,\bar{z}_1} \cap (U_p \times \overline{U}_p \times U'_{p'})$ *for any* $(w, \bar{z}_1) \in \mathcal{M}$, *and if* $r^{U'_{p'}}_{M'}(E') := \{w' \in U'_{p'} : Q'_{\bar{w}'} \supset E'\}$, *then*

$$r^{U'_{p'}}_{M'}(\mathbb{W}_{w,\bar{z}_1}) = \{(\bar{w}, z_1, z') \in \overline{U}_p \times U_p \times U'_{p'} : s(\bar{w}, z_1, z') = 0\} \tag{1.77}$$

Proposition 1.75 shows that after shrinking the first reflection variety $\mathbb{S}^1$ to $\widetilde{\mathbb{S}}^1$, the crucial property $(*)$ above is satified. This property is appropriate to compute the second reflection $r_{M'}(\mathbb{W}'_{w,\bar{z_1}})$ *after localisation* in a smaller open subset $U'_{p'}$ because it yields the convenient analytic dependence with respect to the parameters $(w, \bar{z}_1)$, as we have written in (1.77). We would like to remind the reader that our examples show that there is a serious difference between

Proposition 1.23 and Proposition 1.75 and a serious difference between applying operators $r_{M'}^{U'_{p'}}$ or $r_{M'}^{U'}$ or $r_{M'}^{\mathbb{C}^{n'}}$. Finally, by applying Proposition 1.75, we clearly obtain the Theorems 1.9 and 1.11 with $\mathbb{Z}'_{z,\bar w}$ and with $\mathbb{M}'_{z,\bar w}$ instead of $\mathbb{X}'_{z,\bar w}$. The remainder of the paper is devoted to explore the technicalities.

2. Proof of Theorem 1.9

Lemma 2.1. *(i) For any set $E \subset U$, $E \cap r_M(E) \subset M$ and $E \subset r_M(r_M(E))$.*

(ii) $z \in Q_{\bar w}$ iff $w \in Q_{\bar z}$, $z \in Q_{\bar z}$ iff $z \in M$, $\{w \in U : Q_{\bar w} \supset E\} = \cap_{w\in E} Q_{\bar w}$.

(iii) $f(Q_{\bar z}) \subset Q'_{\overline{f(z)}}$.

(iv) $\rho(z, \bar w) = 0$ iff $\rho(w, \bar z) = 0$.

Proof. (ii), (iii) and (iv) are classical. Prove (i). If $e \in E$ and $e \in r_M(E) = \cap_{w\in E} Q_{\bar w}$ then $e \in Q_{\bar e}$, so $e \in M$ by (ii), *i.e.* $E \cap r_M(E) \subset M$. Furthermore, by construction of $r_M(E)$,

$$r_M(r_M(E)) = \cap\{Q_{\bar z} : z \in r_M(E)\} = \cap\{Q_{\bar z} : Q_{\bar z} \supset E\} \supset E. \qquad \square \tag{2.2}$$

Let $\mathcal{M} = \{(z, \bar w) \in U \times U : \rho(z, \bar w) = 0\}$. Let $\underline{\mathcal{L}}_l = \sum_{j=1}^n a_{j,l}(z, \bar w) \frac{\partial}{\partial \bar w_j}$, $1 \le l \le m$, be tangent vectors to $\mathcal{M}$ which are the complexifications of a basis of tangent vectors $\overline{L}_l = \sum_{j=1}^n a_{j,l}(z, \bar z) \frac{\partial}{\partial \bar z_j}$, $1 \le l \le m$ generating $T^{0,1}M$ with polynomial coefficients and which commute.

Lemma 2.3. *There exist $J \in \mathbb{N}_*$ and functions $r_j(z, \bar w, z') \in \mathcal{A}_n \times \overline{\mathcal{O}}_n \times \mathcal{A}_{n'}$, $1 \le j \le J$, such that $\forall\,(z, \bar w) \in \mathcal{M}$*

$$r_{M'}(f(Q_{\bar z})) = \{z' \in U' : r_j(z, \bar w, z') = 0, 1 \le j \le J\}. \tag{2.4}$$

Remark. Two sets $\{r_j(z, \bar w_1, z') = 0\}$ and $\{r_j(z, \bar w_2, z') = 0\}$ for different w_1, w_2 such that $z \in Q_{\bar w_1}$, $z \in Q_{\bar w_2}$, coincide and are equal to $r_{M'}(f(Q_{\bar z}))$.

Proof. By definition, $r_{M'}(f(Q_{\bar z})) = \{w' \in U' : \rho'(f(w), \bar w') = 0, \forall\, w \in Q_{\bar z}\}$. Using Lemma 2.1 (iv), $r_{M'}(f(Q_{\bar z})) = \{z' \in U' : \rho'(z', \bar f(\bar w)) = 0, \forall\, w \in Q_{\bar z}\}$. Equivalently, $Q_{\bar z} \ni w \mapsto \rho'(z', \bar f(\bar w)) \in \mathbb{C}^{d'}$ vanishes identically as an antiholomorphic map of w defined on the complex algebraic manifold $Q_{\bar z}$. Thanks to the identity principle, this is equivalent to $\underline{\mathcal{L}}^\gamma(\rho'(z', \bar f(\bar w))) = 0$, $\forall\, \gamma \in \mathbb{N}^m$. Put $r_\gamma(z, \bar w, z') := \underline{\mathcal{L}}^\gamma(\rho'(z', \bar f(\bar w)))$. Then $r_\gamma \in \mathcal{A}_n \times \overline{\mathcal{O}}_n \times \mathcal{A}'_{n'}$. By noetherianity, a finite subcollection of the r_γ's defines $r_{M'}(f(Q_{\bar z}))$. $\square$

Lemma 2.5. *There exist $K \in \mathbb{N}_*$ and polynomials $s_k(z')$ (depending on $\bar w$), $1 \le k \le K$, such that $r^2_{M'}(f(Q_{\bar w})) = \{z' \in U' : s_k(z') = 0, 1 \le k \le K\}$.*

Proof. Simply because $r^2_{M'}(f(Q_{\bar w}))$ is algebraic, by the definition (1.3). $\square$

End of proof of Theorem 1.9. Fix $\bar w$. We prove that $f|_{Q_{\bar w} \cap V}$ is algebraic. Indeed

1. $\forall\, z \in Q_{\bar w}$, $r_j(z, \bar w, f(z)) \equiv 0$, $1 \le j \le J$ and $s_k(f(z)) \equiv 0$, $1 \le k \le K$.

2. The set $\Phi := \{z' : r_j(z, \bar w, z') = 0, 1 \le j \le J$ and $s_k(z') = 0, 1 \le k \le K\}$ is zero dimensional at each point $f(z)$, $z \in V$, $z \in Q_{\bar w}$.

By Theorem 5.3.9 in [4] (in the algebraic case), there exist Weierstrass polynomials $P_j(z, z'_j) = {z'_j}^{N_j} + \sum_{1\le k\le N_j} A_{k,j}(z) {z'_j}^{N_j-k}$, $A_{k,j} \in \mathbb{C}[z]$, $1 \le j \le n'$, such that Φ is contained in $\Psi = \{(z, z') \in V \times V' : P_j(z, z'_j) = 0, 1 \le j \le n'\}$. As $(z, f(z)) \in \Phi$, we obtain that

$$f_j(z)^{N_j} + \sum_{1\le k\le N_j} A_{k,j}(z) f_j^{N_j-k}(z) \equiv 0, \qquad z \in Q_{\bar w} \cap V,\ 1 \le j \le n'. \tag{2.6}$$

Equation (2.6) above yields at once that each map $Q_{\bar w} \cap V \ni z \mapsto f_j(z) \in \mathbb{C}$ is holomorphic *algebraic*, $1 \le j \le n'$. To conclude, apply Theorem 1.10. $\square$

3. Proof of Proposition 1.23

Proposition 1.23 relies upon the following statement (denseness of D_M is then clear and $(*) \Rightarrow (**)$ also):

Proposition 3.1. *If $\dim_{f(z)} \mathbb{X}'_{z,\bar{z}} = 0$, $\forall\, z \in M \cap \mathcal{V}_{\mathbb{C}^n}(0) =: M \cap V$, then there exists $p \in M \cap V$ arbitrarily close to 0 such that $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$, $\forall\, z, w \in \mathcal{V}_{\mathbb{C}^n}(p)$, $z \in Q_{\bar{w}}$ and $(*)$ of Proposition 1.23 holds in $U_p = \mathcal{V}_{\mathbb{C}^n}(p) \subset V$.*

As we have already observed, the main difficulty here is that there does not necessarily exist holomorphic equations $\lambda(z, \bar{w}, z')$ such that $\mathbb{X}'_{z,\bar{w}} = \{z' \in U' : \lambda(z, \bar{w}, z') = 0\}$ as for example like there existed some for $r_{M'}(f(Q_{\bar{z}})) = \{z' \in U' : r(z, \bar{z}, z') = 0\}$, $z \in M$, $z \in Q_{\bar{z}}$. To get such a local parametrized family, we shall have to shift $p \in M$ from a certain number of images by holomorphic maps of complex analytic sets. Our proof shows that the set of $p \in M$ in a neighborhood of which $\mathbb{X}'_{z,\bar{w}}$ should be holomorphically parametrized is a dense open subset of M. It will also clearly show that the bad set can be at least as worst as a subanalytic set.

We will prove Proposition 3.1 with M, M' of class $\mathcal{C}^\omega$. For that purpose, let $\mathbb{V}'_z := r_{M'}(f(Q_{\bar{z}})) = \{z' \in U' : r(z, \bar{w}, z') = 0\}$ (in vectorial notations, $r = (r_1, \ldots, r_J)$) so that $r^2_{M'}(f(Q_{\bar{w}})) = r_{M'}(\mathbb{V}'_w)$. Let us recall that the representation of $\mathbb{V}'_z$ by holomorphic equations $r(z, \bar{w}, z') = 0$ gives the same set $\mathbb{V}'_z$ for any choice of $(z, \bar{w}) \in \mathcal{M}$ (*cf.* Lemma 2.3). Therefore the introduction of a third point $z_1 \in Q_{\bar{w}}$ yields a representation $\mathbb{V}'_w = \{z' \in U' : r(w, \bar{z}_1, z') = 0\}$.

From now on, we let $z, w, z_1 \in U$, $z \in Q_{\bar{w}}$, $w \in Q_{\bar{z}_1}$, and we denote $\mathbb{S}^1_{z,\bar{w}}$, $\mathbb{S}^1_{w,\bar{z}_1}$ instead of $\mathbb{V}'_z$, $\mathbb{V}'_w$. This is justified by the fact that although the set $\{z' \in U' : r(z, \bar{w}, z') = 0\}$ does not depend on $\bar{w}$, the *equations* $r_\gamma(z, \bar{w}, z') = \underline{\mathcal{L}}^\gamma[\rho'(z', \bar{f}(\bar{w}))] = 0$ *do really depend* on $\bar{w}$. (Inspect for instance the identity map $\mathbb{C}^2 \to \mathbb{C}^2$, $M \to M$, $M = \{z_2 = \bar{z}_2 + i z_1 \bar{z}_1\}$.) The notation $\mathbb{S}^1_{z,\bar{w}}$ simply means a fiber over $(z, \bar{w})$ of the set $\mathbb{S}^1$ (even if $(z, \bar{w}) \notin \mathcal{M}$). We then write

$$r_{M'}(\mathbb{S}^1_{w,\bar{z}}) = \{z' \in U' : \rho'(z', \overline{\zeta}') = 0 \ \forall \zeta' \text{ satisfying } r(w, \bar{z}_1, \zeta') = 0\}. \tag{3.2}$$

We shall establish that there exist points $p = (z_p, \bar{z}_p) \in M$ arbitrarily close to 0, neighborhoods $U_p = \mathcal{V}_{\mathbb{C}^n}(p)$, $U'_{p'} = \mathcal{V}_{\mathbb{C}^{n'}}(p' = f(p))$ and holomorphic functions $s(\bar{w}, z_1, z')$ near $(z_p, \bar{z}_p, f(z_p))$ in $U_p \times \overline{U}_p \times U'_{p'}$ such that:

1. $r_{M'}(\mathbb{S}^1_{w,\bar{z}_1}) \subset \{z' \in U'_{p'} : s(\bar{w}, z_1, z') = 0\}$, $\forall\, w, z_1 \in U_p$, $w \in Q_{\bar{z}_1}$, and:

2. $r_{M'}(\mathbb{S}^1_{w,\bar{w}}) = \{z' \in U'_{p'} : s(\bar{w}, w, z') = 0\}$, $\forall\, w \in U_p \cap M$.

Lemma 3.3. *If the above conditions* **1-2** *are fulfilled, then $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$, $\forall\, (z, \bar{w}) \in \mathcal{V}_{\mathcal{M}}(p)$, if $\dim_{f(p)} \mathbb{X}'_{p,\bar{p}} = 0$.*

Proof. Indeed, let $\mathbb{Y}'_{z,\bar{w},z_1} = \{z' \in U' : r(z, \bar{w}, z') = 0, s(\bar{w}, z_1, z') = 0\} \supset \mathbb{X}'_{z,\bar{w}} \ni f(z)$. By assumption, $\dim_{f(p)} \mathbb{Y}'_{p,\bar{p},p} = 0$. Since $\mathbb{Y}'_{z,\bar{w},z_1}$ is holomorphically parametrized, then $\dim_{f(z)} \mathbb{Y}'_{z,\bar{w},z_1} = 0$, $\forall\, z, w, \bar{z}_1$ in some small neighborhood of p in $\mathbb{C}^n$ with $z \in Q_{\bar{w}}$, $w \in Q_{\bar{z}_1}$. Therefore $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$ also. □

Let us assume for a while that we have obtained a dense open set D_M where the above conditions **1** and **2** are fulfilled. To complete $(*)$ of Proposition 1.23, it suffices to take $\mathbb{S}^2 :=$ the irreducible component of $\mathbb{Y}' = \{(z, \bar{w}, z_1, z') \in ((\mathcal{M} \sharp \mathcal{M}) \cap (U_p \times \overline{U}_p \times U_p)) \times U'_{p'} : r(z, \bar{w}, z') = 0, s(\bar{w}, z_1, z') = 0\}$ containing the graph of f over $(\mathcal{M} \sharp \mathcal{M}) \cap (U_p \times \overline{U}_p \times U_p)$. $\mathbb{S}^2$ is defined by similar partially polynomial equations $r \in \mathcal{A}_n(U_p) \times \overline{\mathcal{O}}(U_p) \times \mathcal{A}_{n'}(U'_{p'})$, $s \in \overline{\mathcal{O}}_n(U_p) \times \mathcal{O}_n(U_p) \times \mathcal{A}_{n'}(U'_{p'})$. The graph of f is in fact a local irreducible component of $\mathbb{S}^2$, for reasons of dimension. Therefore, $\mathbb{S}^2$ is smooth there and (if the equations $r = s = 0$ are reduced) the rank of the mapping (1.26) equals n' almost everywhere, which completes the proof of Proposition 1.23. □

Consequently, it remains to establish **1** and **2** above. In fact, this can be reduced to a statement which we now formalize in an independent fashion, abandoning Segre varieties, *see* Lemma 3.7. Let Δ be the unit disc in $\mathbb{C}$. Let $\kappa \in \mathbb{N}_*$, $n \in \mathbb{N}_*$, $\nu_1 \in \mathbb{N}_*$, $g : \Delta^\kappa \times \Delta^n \to \mathbb{C}^{\nu_1}$ be a holomorphic power series mapping converging normally in $(2\Delta)^{\kappa+n}$, $(t, z) \mapsto g(t, z)$, let

$$F := \{(t, z) \in \Delta^\kappa \times \Delta^n : g(t, z) = 0\}. \tag{3.4}$$

Assume that there exists a holomorphic map $\lambda : \Delta^\kappa \to \Delta^n$ converging normally in $(2\Delta)^\kappa$ such that $(t, \lambda(t)) \in F$, $\forall\, t \in \Delta^\kappa$, hence $\pi(F) = \Delta^\kappa$, where $\pi : \Delta^\kappa \times \Delta^n \to \Delta^\kappa$, $(t, z) \mapsto t$. Let $n' \in \mathbb{N}_*$, $\nu_2 \in \mathbb{N}_*$, let $\rho : \Delta^{n'} \times \Delta^n \to \mathbb{C}^{\nu_2}$ be a holomorphic series converging normally in $(2\Delta)^{n'+n}$ and denote for $t \in \Delta^\kappa$

$$G_F[t] := \{z' \in \Delta^{n'} : \rho(z', z) = 0, \ \forall\, z \ \ s.t. \ \ g(t, z) = 0\}. \tag{3.5}$$

Let $I^\kappa \subset \Delta^\kappa$ be the maximally real set $I^\kappa = (-1, 1)^\kappa$, $I = (-1, 1)$. Introduce the complex filtration $F = F_1 \supset F_2 \supset F_3 \supset \cdots \supset F_{a+1} = \emptyset$, $F_a \neq \emptyset$, $a \geq 1$, of F by singular subspaces: $F_{i+1} = F_{i,sing}$. Assume also that:

$$F_\alpha = \{(t, z) \in \Delta^\kappa \times \Delta^n : g_\alpha(t, z) = 0\}, \qquad 1 \leq \alpha \leq a, \tag{3.6}$$

with $g_\alpha : (2\Delta)^{\kappa+n} \to \mathbb{C}^{\nu_{1\alpha}}$ converging normally and assume that all irreducible components of F_α are defined analogously. We denote by $\Delta_\kappa(\underline{t}, \underline{\varepsilon})$, $\underline{\varepsilon} > 0$, the polydisc of center $\underline{t}$, radius $\underline{\varepsilon}$, with $\underline{t} \in \Delta^\kappa$, $\underline{\varepsilon} << \text{dist}(\underline{t}, b\Delta^\kappa)$. It remains to establish:

Lemma 3.7. *There exist $\underline{t} \in I^\kappa$, $\underline{\varepsilon} > 0$ and holomorphic equations $s(t, z')$ in $\Delta_\kappa(\underline{t}, \underline{\varepsilon}) \times \Delta^{n'}$ such that:*

1. $G_F[t] \subset \{z' \in \Delta^{n'} : s(t, z') = 0\}$, $\quad \forall\, t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon})$, *and:*
2. $G_F[t] = \{z' \in \Delta^{n'} : s(t, z') = 0\}$, $\quad \forall\, t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon}) \cap I^\kappa$.

Proof. First, $G_F[t] = G_{F_{1,reg}}[t] \cap G_{F_{2,reg}}[t] \cap \cdots \cap G_{F_{a,reg}}[t]$ $(F_a = F_{a,reg})$. Also, $\forall\, \alpha$, $1 \leq \alpha \leq a$, $G_F[t] = G_{F_{1,reg}}[t] \cap \cdots \cap G_{F_{\alpha-1,reg}}[t] \cap G_{F_\alpha}[t]$. Let $F_\alpha = \cup_{\beta=1}^{b_\alpha} F_{\alpha\beta}$, $b_\alpha \in \mathbb{N}_*$, denote the decomposition of F_α into irreducible components. Then also:

$$G_F[t] = \bigcap_{1 \leq \alpha \leq a, 1 \leq \beta \leq b_\alpha} G_{F_{\alpha\beta}}[t]. \tag{3.8}$$

We have (see [C], Chapter 1):

$F_{\alpha\beta,reg} \backslash (\cup_{\gamma \neq \beta} F_{\alpha\gamma}) \subset F_{\alpha,reg}$,
$F_{\alpha\beta,reg} \cap (\cup_{\gamma \neq \beta} F_{\alpha\gamma})$ is a proper analytic subset of $F_{\alpha\beta,reg}$,
$F_{\alpha,reg} = \cup_{1 \leq \beta \leq b_\alpha} (F_{\alpha\beta,reg} \backslash (\cup_{\gamma \neq \beta} F_{\alpha\gamma}))$ and
$G_{F_{\alpha,reg}}[t] \cap G_{F_{\alpha+1}}[t] \subset \cap_{1 \leq \beta \leq b_\alpha} G_{F_{\alpha\beta,reg}}[t]$.

This yields

$$G_F[t] = \bigcap_{1 \leq \alpha \leq a, 1 \leq \beta \leq b_\alpha} G_{F_{\alpha\beta,reg}}[t]. \tag{3.9}$$

Denote now by $F_1, \ldots, F_c$, $c = b_1 + \cdots + b_a \in \mathbb{N}_*$ the $F_{\alpha\beta}$'s which are irreducible. So $G_F[t] = \cap_{1 \leq \gamma \leq c} G_{F_{\gamma,reg}}[t]$. Let F be one of the F_γ's, $1 \leq \gamma \leq c$. Now, we come to a dichotomy. Either the generic rank satisfies

$$\text{gen rk}_{\mathbb{C}}(\pi|_{F_{reg}}) = \kappa \qquad \text{or} \qquad \text{gen rk}_{\mathbb{C}}(\pi|_{F_{reg}}) < \kappa. \tag{3.10}$$

Lemma 3.11. *Let $F :=$ one of the F_γ's. If* gen $\text{rk}_{\mathbb{C}}(\pi|_{F_{reg}}) < \kappa$, *then the* closed *set* $\overline{\pi}(F) := \pi(\overline{F} \subset \overline{\Delta}^\kappa \times \overline{\Delta}^n) \subset \overline{\Delta}^\kappa$ *(here, $\bar{\cdot}$ denotes closure) is contained in a countable union $\cup_{\nu \in \mathbb{N}_*} A_\nu$ of analytic sets $A_\nu \cong \Delta^{\lambda_\nu}$ with $0 \leq \lambda_\nu < \kappa$, $\nu \in \mathbb{N}_*$.*

Proof. Let F be a F_γ with gen $\text{rk}_{\mathbb{C}}(\pi|_{F_{reg}}) < \kappa$. Since F is defined over $(2\Delta)^{\kappa+n}$ and irreducible, paragraph 3.8 in [C] applies. □

Hence the Lebesgue measure $\lambda_{2\kappa}(\overline{\pi}(F)) = 0$. Furthermore, $\forall\, \nu$, $\lambda_\kappa(\overline{I}^\kappa \cap A_\nu) = 0$, since I^κ is maximally real. Hence $\lambda_\kappa(\overline{\pi}(F) \cap I^\kappa) = 0$. Thus there exists an open dense subset B_F of I^κ such that for all $\underline{t} \in B_F$, there exists an open neighborhood $\mathcal{V}_{\Delta^\kappa}(\underline{t})$ with $\mathcal{V}_{\Delta^\kappa}(\underline{t}) \cap \overline{\pi}(F) = \emptyset$. *Consequently, all irreducible components F_γ such that* gen $\text{rk}_{\mathbb{C}}(\pi|_{F_{\gamma,reg}}) < \kappa$ *can be forgotten.* Indeed, for almost all $t \in \Delta^\kappa$, $F_\gamma[t] = \emptyset$, so for such t, F_γ makes no contribution to the set $G_F[t]$ defined by intersecting the sets $\{\rho(z', z) = 0\}$ over those $z \in F_\gamma[t]$. But of course, since there exist $\lambda : \Delta^\kappa \to \Delta^n$ such that $(t, \lambda(t)) \in F = F_1 \cup \cdots \cup F_c$, $\forall\, t \in \Delta^\kappa$, there exists at least one γ such that gen $\text{rk}_{\mathbb{C}}(\pi|_{F_{\gamma,reg}}) = \kappa$. Let now Υ denote the dense open set of $t \in I^\kappa$ such that $F_\gamma[t] = \emptyset$ for the γ's with gen $\text{rk}_{\mathbb{C}}(\pi|_{F_{\gamma,reg}}) < \kappa$. We proceed with gen $\text{rk}_{\mathbb{C}}(\pi|_{F_{\gamma,reg}}) = \kappa$,

$\forall\,\gamma = 1, \ldots, c$ after forgetting other component and renumbering the remaining ones. Fix $F :=$ a F_γ. Let $C :=$ critical locus of $\pi|_{F_{reg}}$. Denote $\widetilde{F} := F_{reg} \backslash C$. It is known that C extends as a complex analytic subset of F itself and that $\mathrm{rk}_{\mathbb{C}}(\pi|_C) < \kappa$ ([C], *ibidem*). Again for an open dense set Υ of t (still denoted by Υ), we have $\overline{\pi}(C) \not\ni t$ (Lemma 3.11).

Let $A_F := \pi(\widetilde{F})$, $B_F := \pi(\widetilde{F}) \cap I^\kappa$. Clearly, A_F is a nonempty subdomain of Δ^κ (since $\widetilde{F}$ is connected).

If $B_F = \emptyset$, $\widetilde{F} \cap \pi^{-1}(t)$ makes no contribution to $G_{F_{reg}}[t]$, if $t \in I^\kappa$. We can forget those components F since according to the desired conditions 1-2 of Lemma 3.7, it is harmless to add equations to $G_{F_{reg}}[t]$ for some other $t \in A_F$ that are close to I^κ but do not belong to I^κ.

Assume therefore that $B_F \neq \emptyset$. Again, by $\pi(\Gamma r(\lambda)) = \Delta^\kappa$, there must exist at least one F such that $B_F \neq \emptyset$. Let $m_1 := \dim_{\mathbb{C}} F$. Choose $\underline{t} \in B_F \cap \Upsilon$, which is possible since B_F is open and Υ is dense open, choose $\underline{\varepsilon} > 0$ with $\Delta_\kappa(\underline{t}, \underline{\varepsilon}) \cap I^\kappa \subset\subset A_F \cap \Upsilon$. For all $t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon})$, $(\pi|_{\widetilde{F}})^{-1}(t)$ consists of finitely many $(m_1 - \kappa)$-dimensional complex submanifolds of $\widetilde{F}$, since $\pi^{-1}(\underline{\Delta}_\kappa(\underline{t}, \underline{\varepsilon})) \cap C = \emptyset$, whence π has constant rank κ over $\tilde{F} \cap (\underline{\Delta}_\kappa(\underline{t}, \underline{\varepsilon}) \times \Delta^n)$ and since

$$(\pi|_{\widetilde{F}})^{-1}(t) \subset (\pi|_F)^{-1}(t) \tag{3.12}$$

and the latter has a finite number of connected components. This number can only increase locally as t moves. It is bounded on $\overline{\Delta}_\kappa(\underline{t}, \underline{\varepsilon}) \cap \overline{I}^\kappa$. Hence we can find a new $\underline{\underline{t}} \in I^\kappa \cap \Delta_\kappa(\underline{t}, \underline{\varepsilon})$ in a neighborhood of which this number of connected components is constant, say in $\Delta_\kappa(\underline{\underline{t}}, \underline{\underline{\varepsilon}}) \cap I^\kappa$. Denote again simply this polydisc by $\Delta_\kappa(\underline{t}, \underline{\varepsilon})$.

Recall also that $\pi(C) \cap \Delta_\kappa(\underline{t}, \underline{\varepsilon}) = \emptyset$, so $G_{F_{reg}}[t] = G_{\widetilde{F}}[t]$, $\forall\, t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon})$.

Lemma 3.13. *Let $\underline{t} \in I^\kappa$ such that there exists $\underline{\varepsilon} > 0$ such that the number of connected components of $(\pi|_{\widetilde{F}})^{-1}(t)$ is constant equal to $\delta \in \mathbb{N}_*$ for all $t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon}) \cap I^\kappa$ and with $\Delta_\kappa(\underline{t}, \underline{\varepsilon}) \cap I^\kappa \subset \Upsilon$.* Then there exist holomorphic equations $s(t, z')$ in $\Delta_\kappa(\underline{t}, \underline{\varepsilon}) \times \Delta^{n'}$ such that

1. $G_{F_{reg}}[t] = G_{\widetilde{F}}[t] \subset \{z' \in \Delta^{n'} : s(t, z') = 0\} \quad \forall\, t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon})$.
2. $G_{F_{reg}}[t] = G_{\widetilde{F}}[t] = \{z' \in \Delta^{n'} : s(t, z') = 0\} \quad \forall\, t \in \Delta_\kappa(\underline{t}, \underline{\varepsilon}) \cap I^\kappa$.

Assume for a while that Lemma 3.13 is proved. Then Lemma 3.7 holds for one irreducible component F of the F_γ's. Pick a second component. Letting t vary now in $\Delta_\kappa(\underline{t}, \underline{\varepsilon})$ (instead of Δ^κ), we can repeat the above argument a finite number of steps and get Lemma 3.7 as desired. □

For short, let us denote $\underline{\Delta}_\kappa := \Delta_\kappa(\underline{t}, \underline{\varepsilon})$.

Proof of Lemma 3.13. Let $D_1, \ldots, D_\delta$ be the components of $(\pi|_{\widetilde{F}})^{-1}(\underline{t})$. These are $(m_1 - \kappa)$-dimensional connected complex submanifolds of $\widetilde{F}$ (because $\pi : \widetilde{F} \to \Delta^\kappa$ is submersive). Let $p_1, \ldots, p_\delta \in D_1, \ldots, D_\delta$ be points, let $U_1, \ldots, U_\delta$ be neighborhoods of $p_1, \ldots, p_\delta$ in $\widetilde{F}$ with maps $\Phi_j : \underline{\Delta}^\kappa \times \Delta^{m_1 - \kappa} \to U_j$ such that $\Phi_j(0 \times \Delta^{m_1 - \kappa}) = D_j \cap U_j$, $\Phi_j(q \times \Delta^{m_1 - \kappa})$ is the fiber $\pi^{-1}(\pi(\Phi_j(q \times 0))) \subset \tilde{F}$, $\forall\, 1 \leq j \leq \delta$, $q \in \underline{\Delta}^\kappa$ and such that $\pi(\Phi_j(q \times \Delta^{m_1 - \kappa})) = q$.

After all the above reductions and simplifications, we now can prove the main step in two lemmas:

Lemma 3.14. *1.* $G_{F_{reg}}[t] \subset G_{U_1}[t] \cap \cdots \cap G_{U_\delta}[t] \ \ \forall\, t \in \underline{\Delta}_\kappa$ *and*
2. $G_{F_{reg}}[t] = G_{U_1}[t] \cap \cdots \cap G_{U_\delta}[t] \ \ \forall\, t \in \underline{\Delta}_\kappa \cap I^\kappa$.

Lemma 3.15. *Each $G_{U_j}[t]$ is equal to a set $\{z' \in U' : s_j(t, z') = 0\}$, where $s_j =$ a finite set of holomorphic functions.*

Proof of Lemma 3.14. Let $D_1[t], \ldots, D_\delta[t]$ denote the connected components of $(\pi|_{\widetilde{F}})^{-1}(t)$, $t \in \underline{\Delta}_\kappa \cap I^\kappa$.

Then $D_j[t] \cap U_j = \Phi_j(t \times \Delta^{m_1 - \kappa}) := U_j[t]$ and

$$G_{F_{reg}}[t] = G_{\widetilde{F}}[t] = G_{D_1[t]}[t] \cap \cdots \cap G_{D_\delta[t]}[t] \ \ \forall\, t \in \underline{\Delta}_\kappa \cap I^\kappa \tag{3.16}$$

Now, if $\rho(z', z) = 0\ \forall\, z \in U_j[t]$, by the uniqueness principle, then $\rho(z', z) \equiv 0$ on the connected complex manifold $D_j[t]$, so $G_{\widetilde{F}}[t] = G_{U_1[t]}[t] \cap \cdots G_{U_\delta[t]}[t]$, $\forall\, t \in \underline{\Delta}_\kappa \cap I^\kappa$. If $t \in \underline{\Delta}_\kappa \backslash I^\kappa$, the cardinal of the set of connected components of $(\pi|_{\widetilde{F}})^{-1}$ can be $> \delta$, so $G_{\widetilde{F}}[t]$ diminishes, $G_{\widetilde{F}}[t] \subset G_{U_1[t]}[t] \cap \cdots \cap G_{U_\delta[t]}[t]$. □

Proof of Lemma 3.15. Let $E_j := \Phi_j(\underline{\Delta}_\kappa \times 0)$ given by a holomorphic graph $z = \tilde{\omega}(t)$ over $\underline{\Delta}_\kappa$. Then E_j is a transverse manifold to the fibers of π. For each j, there exist vector fields $L_1^j, \ldots, L_{m_1-\kappa}^j$ over U_j with holomorphic coefficients in (t, z) commuting with each other with integral manifolds $\Phi_j(t \times \Delta^{m_1-\kappa})$. Then $\rho(z', z) = 0 \,\forall\, z \in (\pi|_{U_j})^{-1}(t)$ if and only if $(\mathrm{L}^j)^\gamma \rho(z', z)|_{z=\tilde{\omega}(t)} = 0$ $\forall\, \gamma \in \mathbb{N}^{m_1-\kappa}$. Put $s_j(t, z') := ((\mathrm{L}^j)^\gamma \rho(z', z)|_{z=\tilde{\omega}(t)})_{\gamma \in \mathbb{N}^{m_1-\kappa}}$ and use noetherianity. □

Proof of Proposition 1.75. Starting with $\mathbb{S}^1_{\mathcal{M}} := \{(z, \bar{w}, z') : (z, \bar{w}) \in \mathcal{M}, r(z, \bar{w}, z') = 0\}$ and $\Gamma r(f) = \{(z, \bar{w}, f(z)) : (z, \bar{w}) \in \mathcal{M}\} \subset \mathbb{S}^1_{\mathcal{M}}$, we can again formalize the data as follows. We take coordinates on $\mathcal{M} \cong \Delta^\kappa$, $\kappa = 2m + d$. Let $\kappa \in \mathbb{N}_*$, $n \in \mathbb{N}_*$, $J \in \mathbb{N}_*$, $r : \Delta^\kappa \times \Delta^n \to \mathbb{C}^J$, $(t, z) \mapsto r(t, z)$ be a holomorphic power series mapping converging normally in $(2\Delta)^\kappa$, assume $r_j \in \mathcal{O}_\kappa(\Delta^\kappa) \times \mathcal{A}_n(\Delta^n)$, let

$$S = \{(t, z) \in \Delta^\kappa \times \Delta^n : r(t, z) = 0\}. \tag{3.17}$$

Assume that there exists $\lambda : \Delta^\kappa \times \Delta^n \to \Delta^n$ holomorphic, converging in $(2\Delta)^\kappa$ such that $\Gamma r(\lambda) \subset S$, let $\pi : \Delta^\kappa \times \Delta^n \to \Delta^n$ be the projection.

Let us inductively define a collection of S_α's, $\alpha \in \mathbb{N}_*$. First $S_1 = S$. Next, $S_\alpha = \{(t, z) \in \Delta^\kappa \times \Delta^n : r_\alpha(t, z) = 0\}$, $r_\alpha : \Delta^\kappa \times \Delta^n \to \mathbb{C}^{J_\alpha}$, $J_\alpha \in \mathbb{N}_*$, $J_\alpha \geq J_{\alpha-1}$, $r_{\alpha,j} = r_{\alpha-1,j}$ $\forall\, 1 \leq j \leq J_{\alpha-1}$, $r_{\alpha,j} \in \mathcal{O}_\kappa(\Delta^\kappa) \times \mathcal{A}_n(\Delta^n)$ and $\Gamma r(\lambda) \subset S_\alpha$.

The construction of $S_{\alpha+1}$ consists in forming the Jacobian matrix of the $r_{\alpha,j}$'s with respect to z, $H_\alpha = (\frac{\partial r_{\alpha,j}}{\partial z_k})^{1 \leq j \leq J_\alpha}_{1 \leq k \leq n}$, in taking $(r_{\alpha+1,j})_j :=$ the collection of all the minors $\delta_{\alpha,j}(t, z)$, $1 \leq j \leq e_\alpha$, of maximal generic rank over $\Delta^\kappa \times \Delta^n$ of this matrix, where $e_\alpha =: J_{\alpha+1} - J_\alpha \in \mathbb{N}_*$ is the number of such minors. Then put $(r_{\alpha+1,j})_{1 \leq j \leq J_{\alpha+1}} := ((r_{\alpha,j})_{1 \leq j \leq J_\alpha}, (\delta_{\alpha,j-J_\alpha})_{J_\alpha+1 \leq j \leq J_{\alpha+1}}$ and put

$$S_{\alpha+1} := \{(t, z) \in \Delta^\kappa \times \Delta^n : r_{\alpha+1,j}(t, z) = 0, 1 \leq j \leq J_{\alpha+1}\}. \tag{3.18}$$

Of course, $r_{\alpha+1,j} \in \mathcal{O}_\kappa(\Delta^\kappa) \times \mathcal{A}_n(\Delta^n)$, $\forall\, j = J_\alpha + 1, \ldots, J_{\alpha+1}$. Also if we were starting with $(z, \bar{w}) = t \in \mathcal{M}$, we would have got some $r_{\alpha+1,j}(z, \bar{w}, z')$ depending on the two variables $(z, \bar{w})$ even if we let $(z, \bar{w})$ vary only in $\mathcal{M}$, hence getting new equations like the $\tilde{r}_j$ of Proposition 1.75.

Then $S_{\alpha+1} \subsetneq S_\alpha$. Indeed by construction $J_{\alpha+1} > J_\alpha$ and the zero-locus of equations from a minor $\delta_{\alpha,j}$ of maximal generic rank coincides with S_α at each point (t_p, z_p) where $\delta_\alpha(t_p, z_p) \neq 0$ but $S_{\alpha+1}$ does not contain S_α in a neighborhood of such a point.

Thus there exists an integer $a \in \mathbb{N}_*$ such that $S_{a+1} = S_a$ and $S_{a+1} \supset \Gamma r(\lambda)$ or $S_a \supset \Gamma r(\lambda)$ and there exists a minor $\delta_{a,j}(t, z)$ such that $\Gamma r(\lambda) \not\subset \{\delta_{\alpha,j} = 0\}$. The case $S_{a+1} = S_a$ and $\Gamma r(\lambda) \subset S_{a+1}$ is impossible because then $\dim_{\mathbb{C}} S_{a+1} \geq \kappa \geq 1$ and therefore its minors are nontrivial which implies that $S_{a+1} \subsetneq S_a$ by the above remark.

Therefore $S_a \supset \Gamma r(\lambda)$ and $\Gamma r(\lambda) \not\subset \{\delta_{a,j} = 0\}$.

At each point of the Zariski open subset $\{\delta_{a,j} \neq 0\} \cap \Gamma r(\lambda)$ of $\Gamma r(\lambda)$, locally S_a is given by equations of the form $z_2 = \Phi(t, z_1)$, $(z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_1}$, $n_1 + n_2 = n$, because of the constant rank theorem. This proves $(*)$ of Proposition 1.75 in this context. Notice that we make localization in a smaller open set, which is a neighborhood of some point $(\underline{t}, \lambda(\underline{t})) \in \Gamma r(\lambda) \cap \{\delta_{a,j} \neq 0\}$.

Next, we compute $G_F[t]$ in case $F (= S_a)$ is given by $\{(t, z) \in \Delta^\kappa \times \Delta^n : z_2 = \Phi(t, z_1)\}$ to get $(**)$. This is a particular case of Lemma 3.15: let $L = \frac{\partial}{\partial z_1} + \Phi_{z_1}(t, z_1)\frac{\partial}{\partial z_2}$ be in vectorial notation the basis of vector fields tangent to F. Then $\rho'(z', z_1, \Phi(t, z_1)) = 0 \;\forall\, z_1$ if and only if $L^\gamma \rho'(z', 0, \Phi(t, 0)) = 0 \;\forall\, \gamma \in \mathbb{N}^{n_1}$: these define analytic equations $s(t, z')$, which completes the proof of Proposition 1.75. Notice that we make localization before computing $G_F[t]$: this corresponds to taking $\mathbb{W}'_{w,\bar{z}_1} = \widetilde{\mathbb{S}}^1_{w,\bar{z}_1} \cap (U_p \times \overline{U}_p \times U'_{p'})$ and then $r_{M'}^{U'_{p'}}(\mathbb{W}'_{w,\bar{z}_1})$. □

4. Examples

The general idea of all of these examples is to construct M, M', f with the reflection set $\mathbb{S}^1 = \{(z, \bar{w}, z') : r(z, \bar{w}, z') = 0\}$ containing two or more irreducible components and to exploit this fact in order to exhibit rather disharmonious phenomena.

Check of Example 1.53. Let $z \in Q_{\bar{w}}$,

$$z_5 = \bar{w}_5 + iz_1\bar{w}_1. \tag{4.1}$$

Then $z' \in r_{M'}(f(Q_{\bar{z}}))$ if and only if

$$\rho'(z', (\bar{w}_1, 0, 0, 0, z_5 - iz_1\bar{w}_1)) = 0 \qquad \forall\, \bar{w}_1 \in \mathbb{C}, i.e. \tag{4.2}$$

$$z'_5 - [z_5 - iz_1\bar{w}_1 + i\bar{w}_1^2 z'_3 z'_4 + iz'_1\bar{w}_1] = 0 \qquad \forall\, \bar{w}_1 \in \mathbb{C}.$$

From this follows $z'_5 = z_5$, $z'_1 = z_1$, $z'_3 z'_4 = 0$. Therefore

$$\mathbb{S}^1 = \{(z, \bar{w}, z') : z'_5 = z_5, z'_1 = z_1, z'_3 z'_4 = 0\} \tag{4.3}$$

$$\mathbb{V}'_z = \{(z_1, \zeta'_2, \zeta'_3, 0, z_5) : \zeta'_2, \zeta'_3 \in \mathbb{C}\} \cup \{(z_1, \zeta'_2, 0, \zeta'_4, z_5) : \zeta'_2, \zeta'_4 \in \mathbb{C}\}. \tag{4.4}$$

Now, it is clear that $\Gamma r(f)$ is contained in

$$\mathbb{S}^1_{sing} = \{(z, \bar{w}, z') : z'_5 = z_5, z'_1 = z_1, z'_3 = 0, z'_4 = 0\} := \widetilde{\mathbb{S}}^1 \tag{4.5}$$

$$\widetilde{\mathbb{S}}^1_{z,\bar{w}} = \{(z_1, \zeta'_2, 0, 0, z_5) : \zeta'_2 \in \mathbb{C}\}. \tag{4.6}$$

To compute $r_{M'}(\widetilde{\mathbb{S}}^1_{w,\bar{z}_1})$, we write

$$\rho'(w', \bar{w}_1, \bar{\zeta}'_2, 0, 0, \bar{w}_5) = 0 \quad \forall\, \bar{\zeta}'_2, i.e. \tag{4.7}$$

$$w'_5 - \bar{w}_5 - i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_3 w'_4 + \bar{\zeta}'^2_2 w'^2_3] = 0 \quad \forall\, \bar{\zeta}'_2.$$

From this follows $w'_3 = 0$, $w'_5 = \bar{w}_5 + iw'_1\bar{w}_1$. Therefore

$$r_{M'}(\widetilde{\mathbb{S}}^1_{w,\bar{z}_1}) = \{(w'_1, w'_2, 0, w'_4, \bar{w}_5 + iw'_1\bar{w}_1) : w'_1, w'_2, w'_4 \in \mathbb{C}\} \tag{4.8}$$

and

$$\widetilde{\mathbb{S}}^1_{z,\bar{w}} \cap \widetilde{\mathbb{S}}^1_{w,\bar{z}_1} = \{(z_1, \zeta'_2, 0, 0, z_5) : \zeta'_2 \in \mathbb{C}\}. \tag{4.9}$$

In conclusion, $\dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = \forall\, z$. On the other hand,

$$\mathbb{V}'_w = \{(w_1, \zeta'_2, \zeta'_3, 0, w_5) : \zeta'_2, \zeta'_3 \in \mathbb{C}\} \cup \{(w_1, \zeta'_2, 0, \zeta'_4, w_5) : \zeta'_2, \zeta'_3 \in \mathbb{C}\} \tag{4.10}$$

and the equations of $r_{M'}(\mathbb{V}'_w)$ are given by

$$\rho'(w', (\bar{w}_1, \bar{\zeta}'_2, \bar{\zeta}'_3, 0, \bar{w}'_5)) = 0 \qquad \forall\, \bar{\zeta}'_2 \forall\, \bar{\zeta}'_3, i.e. \tag{4.11}$$

$$w'_5 - [\bar{w}_5 + i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_3 w'_4 + w'_3\bar{\zeta}'^2_2 + w'^2_2\bar{\zeta}'^2_3 + w'^3_4\bar{\zeta}'^3_3]] = 0 \qquad \forall\, \bar{\zeta}'_2 \forall\, \bar{\zeta}'_3$$

$$\rho'(w', (\bar{w}_1, \bar{\zeta}'_2, 0, \bar{\zeta}'_4, \bar{w}_5)) = 0 \qquad \forall\, \bar{\zeta}'_2 \forall\, \bar{\zeta}'_4, i.e. \tag{4.12}$$

$$w'_5 - [\bar{w}_5 + i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_3 w'_4 + w'^2_3\bar{\zeta}'^2_2 + \bar{\zeta}'^3_4 w'^3_3]] = 0 \qquad \forall\, \bar{\zeta}'_2 \forall\, \bar{\zeta}'_4.$$

From (4.11) we deduce $w'_3 = 0$, $w'_2 = 0$, $w'_4 = 0$, $w'_5 = \bar{w}_5 + iw'_1\bar{w}_1$. From (4.12) we deduce $w'_3 = 0$, $w'_5 = \bar{w}_5 + iw'_1\bar{w}_1$. Therefore

$$r_{M'}(\mathbb{V}'_w) = \{(w'_1, 0, 0, 0, \bar{w}_5 + iw'_1\bar{w}_1) : w'_1 \in \mathbb{C}\} = f(Q_{\bar{w}}) \tag{4.13}$$

and finally

$$\mathbb{V}'_z \cap r_{M'}(\mathbb{V}'_w) = \{(z_1, 0, 0, 0, z_5)\} = \{f(z)\}. \tag{4.14}$$

In conclusion, $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0$. This completes Example 1.53. □

Check of Example 1.56. Here, if $z_4 = \bar{w}_4 + i\bar{w}_1 z_1$,

$$\mathbb{S}^1 = \{(z, \bar{w}, z') : z'_4 = z_4, z'_1 = z_1, z'_2 z'_3 = 0\}. \tag{4.15}$$

Now, it is clear that $\Gamma r(f)$ is contained in $\mathbb{S}^1_{sing}$

$$\mathbb{S}^1_{sing} = \{(z, \bar{w}, z') : z'_4 = z_4, z'_1 = z_1, z'_2 = 0, z'_3 = 0\}. \tag{4.16}$$

Whence

$$\widetilde{\mathbb{S}}^1_{z,\bar{w}} = \{(z_1, 0, 0, 0, z_5)\} = \mathbb{W}'_{z,\bar{w}} \tag{4.17}$$

and finally

$$\dim_{f(z)} \mathbb{W}'_{z,\bar{w}} = \dim_{f(z)} \mathbb{Z}'_{z,\bar{w}} = 0 \qquad \forall\, z. \tag{4.18}$$

On the other hand,

$$\mathbb{S}^1_{z,\bar{w}} = \{(z_1, \zeta'_2, 0, z_4) : \zeta'_2 \in \mathbb{C}\} \cup \{(z_1, 0, \zeta'_3, z_4) : \zeta'_2 \in \mathbb{C}\} = \mathbb{V}'_z \tag{4.19}$$

and the equations of $r_{M'}(\mathbb{V}'_w)$ are given by

$$\rho'(w', (\bar{w}_1, 0, \bar{\zeta}'_3, \bar{w}_4)) = w'_4 - [\bar{w}_4 + i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_2 w'_3]] = 0 \qquad \forall\, \bar{\zeta}'_3 \tag{4.20}$$

$$\rho'(w', (\bar{w}_1, \bar{\zeta}'_2, 0, \bar{w}_4)) = w'_4 - [\bar{w}_4 + i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_2 w'_3]] = 0 \qquad \forall\, \bar{\zeta}'_2. \tag{4.21}$$

It follows only the equation $w'_4 = \bar{w}_4 + i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_2 w'_3]$. Therefore

$$r_{M'}(\mathbb{V}'_w) = \{(w'_1, w'_2, w'_3, \bar{w}_4 + i[w'_1\bar{w}_1 + \bar{w}_1^2 w'_2 w'_3]) : w'_1, w'_2, w'_3 \in \mathbb{C}\} \tag{4.22}$$

(4.23)

$$\mathbb{V}'_z \cap r_{M'}(\mathbb{V}'_w) = \{(z_1, w'_2, 0, z_4) : w'_2 \in \mathbb{C}\} \cup \{(z_1, 0, w'_3, z_4) : w'_3 \in \mathbb{C}\} = \mathbb{V}'_z,$$

whence $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 1 \; \forall\, z$. Example 1.56 is complete. □

Check of Example 1.66. Let us establish:

The function $\mathcal{M} \ni (z, \bar{w}) \mapsto \dim_{f(z)} \mathbb{X}'_{z,\bar{w}} \in \mathbb{N}$ *is neither upper semi continuous nor lower semi continuous in general.*

Proof. First, whenever $\dim_{f(0)} r_{M'}(f(Q_0)) = 0$, then $\dim_{f(z)} r_{M'}(f(Q_{\bar{z}})) = 0$ too for $z \in \mathcal{V}_{\mathbb{C}^n}(0)$ because of Lemma 2.3 and so there exists $V = \mathcal{V}_{\mathbb{C}^n}(0)$ such that $\dim_{f(z)} \mathbb{X}'_{z,\bar{w}} = 0 \; \forall\, z, w \in U$, $z \in Q_{\bar{w}}$. For instance, $M = M'$, $f = \mathrm{Id}$, $M = \{z_2 = \bar{z}_2 + i z_1 \bar{z}_1\}$.

Therefore $(z, \bar{w}) \mapsto \dim_{f(z)} \mathbb{X}'_{z,\bar{w}}$ could be continuous.

This is false. Indeed, let $M = M' = \{z_3 = \bar{z}_3 + i z_1 \bar{z}_1 (1 + z_2 \bar{z}_2)\} \subset \mathbb{C}^3$, $f = \mathrm{Id}$. First, M is Levi-nondegenerate at every point of $\mathbb{C}^3 \backslash \{z_1 = 0\}$, so $\dim_{f(p)} \mathbb{X}'_{p,\bar{p}} = 0$ at those points. Let $Q_0 = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{C}\}$, $r_{M'}(Q_0) = \{(0, z_2, 0) : z_2 \in \mathbb{C}\} = \{q \in \mathbb{C}^3 : Q_{\bar{q}} = Q_0\}$, so $r_{M'}(Q_0) = Q_0 = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{C}\}$, so $r_{M'}(Q_0) \cap r^2_{M'}(Q_0) = \{(0, z_2, 0) : z_2 \in \mathbb{C}\}$ has dimension 1.

Therefore $(z, \bar{w}) \mapsto \dim_{f(z)} \mathbb{X}'_{z,\bar{w}}$ can be at best upper semi-continuous.

This is false. Indeed, let $M = \{(z_1, z_4) \in \mathbb{C}^2 : z_4 = \bar{z}_4 + i z_1 \bar{z}_1\}$, let

$$M' = z'_4 = \bar{z}'_4 + i z'_1 \bar{z}'_1 + i z'_3 \bar{z}'_2 + i \bar{z}'_3 z'_2 + i z'^2_1 \bar{z}'_3 \bar{z}'_2 + i \bar{z}'^2_1 z'_3 z'_2 \tag{4.24}$$

and

$$f(z_1, z_4) = (z_1, z_4 \sin^3 z_1, 0, z_4), \qquad f(M) \subset M'. \tag{4.25}$$

Then $Q_0 = \{(z_1, 0)\}$, $f(Q_0) = \{(z_1, 0, 0, 0)\}$. We claim that

1. $r_{M'}(f(Q_0)) \cap r_{M'}(f(Q_0)) = \{(0, 0, 0, 0)\}$ and
2. $r_{M'}(f(Q_{\bar{z}})) \cap r^2_{M'}(f(Q_{\bar{w}})) = \{(z_1, z'_2, 0, z_4) : z'_2 \in \mathbb{C}\} \; \forall\, z, w,\; z \in Q_{\bar{w}},\; z \neq 0$.

This will show that $(z, \bar{w}) \mapsto \dim_{f(z)} \mathbb{X}'_{z,\bar{w}}$ cannot be upper semicontinuous in general.

Indeed,

(4.25)

$$r_{M'}(f(Q_0)) = \{(0, z_2, 0, 0)\} \cup \{(0, 0, z_3, 0)\} \;\text{ and }\; r^2_{M'}(f(Q_0)) = \{(z_1, 0, 0, 0)\},$$

so 1 holds.

Let $(z_1, z_4) \neq (0,0)$, let $z \in Q_{\bar{w}}$, $w = (w_1, w_4)$, $z_4 = \bar{w}_4 + iz_1\bar{w}_1$. Then $f(Q_{\bar{z}}) = \{(w_1, (\bar{z}_4 + iw_1\bar{z}_1)\sin^3 w_1, 0, \bar{z}_4 + iw_1\bar{z}_1) : w_1 \in \mathbb{C}\}$. By definition, $r_{M'}(f(Q_{\bar{z}})) = \{z' : \rho'(z', \bar{f}(\bar{w})) = 0 \ \forall w \in Q_{\bar{z}}\}$. Write

$$\rho'(z', \bar{f}(\bar{w})) = 0 \qquad \forall \bar{w}_1, \ i.e. \tag{4.27}$$

$$z'_4 - [z_4 - i\bar{w}_1 z_1 + iz'_1\bar{w}_1 + iz'_3(z_4 - i\bar{w}_1 z_1)\sin^3 \bar{w}_1 + iz'_3 z'_2 \bar{w}_1^2] = 0 \qquad \forall \bar{w}_1.$$

We deduce equations $z'_4 = z_4$, $z'_1 = z_1$, $z'_3 z'_2 = 0$, $z'_3 z_4 = 0$, $z'_3 z_1 = 0$. Therefore if $(z_1, z_4) \neq (0,0)$, then

$$r_{M'}(f(Q_{\bar{z}})) = \{(z_1, z'_2, 0, z_4) : z'_2 \in \mathbb{C}\}. \tag{4.28}$$

Next, $r^2_{M'}(f(Q_{\bar{w}}))$ is given by

$$\rho'(z', (\bar{w}_1, \bar{\zeta}'_2, 0, \bar{w}_4)) = z'_4 - [\bar{w}_4 + iz'_1\bar{w}_1 + iz'_3 z'_2 \bar{w}_1^2 + iz'_3\bar{\zeta}'_2] = 0 \qquad \forall \bar{\zeta}'_2. \tag{4.29}$$

We deduce equations $z'_3 = 0$, $z'_4 = \bar{w}_4 + iz'_1\bar{w}_1$, so

$$r^2_{M'}(f(Q_{\bar{w}})) = \{(z'_1, z'_2, 0, \bar{w}_4 + iz'_1\bar{w}_1) : z'_1, z'_2 \in \mathbb{C}\}, \qquad (z_1, z_4) \neq (0,0). \tag{4.30}$$

Finally for such $(z_1, z_4) \neq (0,0)$,

$$r_{M'}(f(Q_{\bar{z}})) \cap r^2_{M'}(f(Q_{\bar{w}})) = \{(z_1, z'_2, 0, z_4) : z'_2 \in \mathbb{C}\}, \tag{4.31}$$

which shows that 2 above holds. This completes Example 1.66. □

Example 1.66 already shows that $\mathbb{X}'_{z,\bar{w}}$ is not analytically parametrized by $(z, \bar{w})$. Example 1.68 also provides a supplementary reason.

Check of Example 1.68. First, let us take in (3.4): $\kappa = 2$, $n = 3$, $\lambda(t_1, t_2) = (t_1, t_2, 0)$,

$$F = \{(t,z) \in \Delta^2 \times \Delta^3 : (t_1 z_3 - t_2^2)(z_1 - t_1) = 0, (t_1 z_3 - t_2^2)(z_2 - t_2) = 0, \tag{4.32}$$
$$(t_1 z_3 - t_2^2) z_3 = 0\} = F_1 \cup F_2 = \Gamma r(\lambda) \cup \{t_1 z_3 - t_2^2 = 0\}.$$

Then the fibers $F_2[t] = \emptyset$ if $|t_2^2| \geq |t_1|$, say if $t \in T_c := \Delta^2 \cap \{|t_2^2| \geq |t_1|\}$, and $F_2[t] = \{(z_1, z_2, z_3) \in \Delta^3 : z_3 = t_2^2/t_1\}$ if $t \in T_0 := \Delta^2 \backslash T_c$. Clearly

$$G_F[t] = \{z' \in \Delta^{n'} : \rho(z', (t_1, t_2, 0)) = 0\} \qquad \forall t \in \Delta^2 \tag{4.33}$$

and

$$G_{F_2}[t] = \{z' \in \Delta^{n'}\} \tag{4.34}$$

if $t \in T_c$ and

$$G_{F_2}[t] = \{z' \in \Delta^{n'} : (\partial_{z_1}^{k_1}\partial_{z_2}^{k_2}\rho)(z', 0, 0, t_2^2/t_1) = 0 \ \forall k_1 \forall k_2 \in \mathbb{N}\} \tag{4.35}$$

if $t \in T_0$. The border equals $\overline{T}_0 \cap T_c = \{t \in \Delta^2 : |t_1| = |t_2^2|\}$. It is real analytic, not complex.

Next, we build a mapping on the basis of this example. Let $n = 3$, $n' = 4$, $M : z_4 = \bar{z}_4 + iz_1\bar{z}_1 + iz_2\bar{z}_2$,

$$M' : \quad z'_4 = \bar{z}'_4 + i[z'^2_1 \bar{z}'_3(\bar{z}'^2_2 - \bar{z}'_1\bar{z}'_3) + \bar{z}'^2_1 z'_3(z'^2_2 - z'_1 z'_3) + z'_1\bar{z}'_1 + z'_2\bar{z}'_2], \tag{4.36}$$

$f(z_1, z_2, z_4) = (z_1, z_2, 0, z_4)$. Then one can check that the equations of $\mathbb{S}'_{z,\bar{w}} = \mathbb{V}'_z$ are: $z'_4 = z_4$, $z'_1 = z_1$, $z'_2 = z_2$, $z'_3(z_2^2 - z_1 z_3) = 0$, from which Example 1.68 follows. □

Check of Example 1.72. Consider $f : \mathbb{C}^2 \ni (z_1, z_4) \mapsto (z_1, 0, 0, z_4) \in \mathbb{C}^4$, $M : z_4 = \bar{z}_4 + iz_1\bar{z}_1$ and

$$M' : \ z'_4 = \bar{z}'_4 + iz'_1\bar{z}'_1 + iz'^2_1(1 + \bar{z}'_3)\bar{z}'_3 + i\bar{z}'^2_1(1 + z'_3)z'_3 + iz'_2 z'_3 \bar{z}'^2_2 + i\bar{z}'_2\bar{z}'_3 z'^2_2. \tag{4.37}$$

Identify M with $f(M) = \{(z_1, 0, 0, z_4) : (z_1, z_4) \in M\}$. Take $U = \Delta^2 \cong \Delta \times 0 \times 0 \times \Delta$, $U' = \Delta^4$. We will first check 1 and 2 for $z = \bar{w} = 0 \in \mathbb{C}^2$.

First, compute $r_{M'}(Q_0)$ by writing $\rho'(z', \bar{f}(\bar{z}_1, 0)) = z'_4 - [i\bar{z}_1 z'_1 + i\bar{z}_1^2(1 + z'_3)z'_3]$ so that equations of $r_{M'}(Q_0)$ are $z'_4 = 0$, $z'_1 = 0$, $(1 + z'_3)z'_3 = 0$, whence

$$r^{U'}_{M'}(Q_0) = \{(0, z'_2, 0, 0) : z'_2 \in \Delta\} \tag{4.38}$$

$$r_{M'}^{\mathbb{C}^{n'}}(Q_0) = \{(0, z_2', 0, 0) : z_2' \in \mathbb{C}\} \cup \{(0, z_2', 1, 0) : z_2' \in \mathbb{C}\} := \mathbb{A}_0^1 \cup \mathbb{A}_0^2. \tag{4.39}$$

To compute $(r_{M'}^{U'})^2(Q_0)$, write $\rho'(z', (0, \bar{w}_2', 0, 0)) = z_4' - i[\bar{w'}_2^2 z_2' z_3']$ so that equations of $(r_{M'}^{U'})^2(Q_0)$ are $z_4' = 0$, $z_2' z_3' = 0$, whence

$$(r_{M'}^{U'})^2(Q_0) = \{(z_1', z_2', 0, 0) : z_1', z_2' \in \Delta\} \cup \{(z_1', 0, z_3', 0) : z_1', z_3' \in \Delta\} \tag{4.40}$$

$$(r_{M'}^{U'})(Q_0) \cap (r_{M'}^{U'})^2(Q_0) = \{(0, z_2', 0, 0) : z_2' \in \Delta\}. \tag{4.41}$$

On the other hand, $(r_{M'}^{\mathbb{C}^{n'}})^2(Q_0) = r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_0^1) \cap r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_0^2)$, where as above

$$r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_0^1) = \{(z_1', z_2', 0, 0) : z_1', z_2' \in \mathbb{C}\} \cup \{(z_1', 0, z_3', 0) : z_1', z_3' \in \mathbb{C}\}. \tag{4.42}$$

To compute $r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_0^2)$, write $\rho'(z', (0, \bar{w}_2', 1, 0)) = z_4' - i[\bar{w'}_2^2 z_2' z_3' + \bar{w}_2' z'^2_2] = 0 \ \forall \bar{w}_2'$ so that its equations are $z_4' = 0$, $z'^2_2 = 0$, $z_2' z_3' = 0$, whence $r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_0^2) = \{(z_1', 0, z_3', 0) : z_1' \in \mathbb{C}, z_2' \in \mathbb{C}\}$ and

$$r_{M'}^{\mathbb{C}^{n'}}(Q_0) \cap (r_{M'}^{\mathbb{C}^{n'}})^2(Q_0) = \{(0, 0, 0, 0)\} \cup \{(0, 0, 1, 0)\}. \tag{4.43}$$

In conclusion, for $z = \bar{w} = 0 \in M$, $r_{M'}^{\mathbb{C}^{n'}}(Q_0) \cap (r_{M'}^{\mathbb{C}^{n'}})^2(Q_0)$ is finite whereas $\dim_0[r_{M'}^{U'}(Q_0) \cap (r_{M'}^{U'})^2(Q_0)] = 1$. Now, let $z \in Q_{\bar{w}}$, $z, w \in \Delta \times 0 \times 0 \times \Delta$, $(z_1, 0, 0, z_4) \in r_{M'}^{U'}(Q_{\bar{z}_1, 0, 0, \bar{z}_4}) \cap (r_{M'}^{U'})^2(Q_{\bar{w}_1, 0, 0, \bar{w}_4})$. As above,

$$r_{M'}^{U'}(Q_{\bar{z}}) = \{(z_1, z_2', 0, z_4) : z_2' \in \Delta\} \tag{4.44}$$

$$r_{M'}^{\mathbb{C}^{n'}}(Q_{\bar{z}}) = \{(z_1, z_2', 0, z_4) : z_2' \in \mathbb{C}\} \cup \{(z_1, z_2', 1, z_4) : z_2' \in \mathbb{C}\} := \mathbb{A}_z^1 \cup \mathbb{A}_z^2. \tag{4.45}$$

To compute $(r_{M'}^{U'})^2(Q_{\bar{w}}) = \{w' \in \Delta^4 : \rho'(w', \bar{z}') = 0 \ \forall z' \in r_{M'}^{U'}(Q_{\bar{w}}\})$, write first

$$\rho'(w', (\bar{w}_1, \bar{z}_2', 0, \bar{w}_4)) = w_4' - [w_4 + i w_1' \bar{w}_1 + i \bar{w}_1^2 (1 + w_3') w_3' + i w_2' w_3' \bar{z'}_2^2] \tag{4.46}$$

whence

$$\begin{aligned}(r_{M'}^{U'})^2(Q_{\bar{w}}) = \{(w_1', w_2', 0, \bar{w}_4 + i w_1' \bar{w}_1) : w_1', w_2' \in \Delta, |\bar{w}_4 + i w_1' \bar{w}_1| < 1\} \cup \\ \cup \{(w_1', 0, w_3', \bar{w}_4 + i w_1' \bar{w}_1) : w_1', w_2' \in \Delta, |\bar{w}_4 + i w_1' \bar{w}_1| < 1\}.\end{aligned} \tag{4.47}$$

$$r_{M'}^{U'}(Q_{\bar{z}}) \cap (r_{M'}^{U'})^2(Q_{\bar{w}}) = \{(z_1, w_2', 0, z_4) : w_2' \in \Delta\}. \tag{4.48}$$

On the other hand analogously

$$r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_w^1) = \{(z_1, w_2', 0, z_4) : w_2' \in \mathbb{C}\}. \tag{4.48}$$

To compute $r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_w^2) = \{w' \in \mathbb{C}^4 : \rho'(w', \bar{z}') = 0 \ \forall z' \in \mathbb{A}_w^2\}$, write

$$\begin{aligned}\rho'(w', (w_1, z_2', 1, w_4)) = \\ = w_4' - [w_4 + i w_1' \bar{w}_1 + i \bar{w}_1^2 (1 + w_3') w_3' + i w_2' w_3' \bar{z'}_2^2 + i w'^2_2 \bar{z}_2']\end{aligned} \tag{4.50}$$

whence

$$r_{M'}^{\mathbb{C}^{n'}}(\mathbb{A}_w^2) = \{(w_1', 0, w_3', \bar{w}_4 + i w_1' \bar{w}_1) : w_1', w_2' \in \mathbb{C}\} \tag{4.51}$$

$$r_{M'}^{\mathbb{C}^{n'}}(Q_{\bar{z}}) \cap (r_{M'}^{\mathbb{C}^{n'}})^2(\mathbb{A}_w^1) \cap (r_{M'}^{\mathbb{C}^{n'}})^2(\mathbb{A}_w^2) = \{(z_1, 0, 0, z_4)\} \cup \{(z_1, 0, 1, z_4\}. \tag{4.52}$$

This completes Example 1.72. □

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