

ON REMOVABLE SINGULARITIES
FOR INTEGRABLE CR FUNCTIONS

Joël Merker and Egmont Porten

The subject of removable singularities for the boundary values of holomorphic functions of several variables has been intensively studied in recent years, see works: [1], [14], [17], [18], [19], [21], [23], [24], [25], [26], [28], [32], [34], and the excellent surveys [11], [40].

The present paper is devoted to a further step into this direction. Its main intention is to illustrate the way how, on a CR-manifold M of arbitrary codimension, removable singularity theorems may be understood in terms of the analysis of CR-orbits of M .

Let M be a locally embeddable CR manifold, $\dim_{CR} M = m$, $\text{codim}_{\mathbb{C}^n} M = n$, $\dim_{\mathbb{R}} M = d = 2m + n$. Locally, $M \subset \mathbb{C}^{m+n}$. Let $\Phi \subset M$ be a closed set.

We give various conditions in order that Φ is L^1 -removable, *i.e.*

$$(1) \quad L^1_{loc}(M) \cap L^1_{loc,CR}(M \setminus \Phi) = L^1_{loc,CR}(M)$$

Let us recall here that $\Phi \subset\subset M$ is called $(L^p, \bar{\partial}_b)$ -removable if $L^p_{loc,CR}(M \setminus \Phi) \cap L^p_{loc}(M) = L^p_{loc,CR}(M)$, *i.e.* any $f \in L^p_{loc}(M)$ satisfying $\int_{M \setminus \Phi} f \bar{\partial} \varphi = 0$ for each $(m+n, m-1)$ -form $\varphi \in \mathcal{C}^\infty_c(\mathbb{C}^{m+n} \setminus \Phi)$ satisfies $\int_M f \bar{\partial} \psi = 0$ for each $(m+n, m-1)$ -form $\psi \in \mathcal{C}^\infty_c(\mathbb{C}^{m+n})$.

Let H^κ denote κ -dimensional Hausdorff measure. Our main results are the following. Since L^p_{loc} embeds in L^1_{loc} for $p \geq 1$, we state them for $p = 1$. We give explanations below.

THEOREM 1. ([20], [32]) *If M is \mathcal{C}^3 , a function $f \in L^1_{loc}(M)$ is CR if and only if $f|_{\mathcal{O}}$ belongs to $L^1_{loc,CR}(\mathcal{O})$ for almost every CR orbit \mathcal{O} .*

COROLLARY 1. *If $\Phi = \cup_{a \in A} \mathcal{O}_a$ is of zero d -dimensional measure, then (1) is satisfied.*

Theorem 1 reduces the problem to the case where M is a single CR orbit, *i.e.* M is globally minimal [27]. All the results also hold for $p \geq 1$ equally.

THEOREM 2. *Let M be $\mathcal{C}^{2,\alpha}$, $0 < \alpha < 1$, $\dim_{CR} M = m \geq 1$. Every closed subset E of M such that M and $M \setminus E$ are globally minimal and such that $H^{d-3}_{loc}(E) < \infty$ is L^1 -removable.*

The notion of wedge- (\mathcal{W} -) removability is defined here in higher codimension (see below).

THEOREM 3. ([29]) *Let M be \mathcal{C}^ω , $\dim_{CR} M = m \geq 1$. Every closed set $E \subset M$ such that M and $M \setminus E$ are globally minimal and such that $H^{d-2}(E) = 0$ is \mathcal{W} - and L^1 -removable.*

THEOREM 4. *Let M be $\mathcal{C}^{2,\alpha}$, $0 < \alpha < 1$, $m \geq 1$, and let N be a connected \mathcal{C}^2 submanifold of M such that M and $M \setminus N$ are globally minimal.*

- (i) *If $\text{codim}_M N \geq 3$, then N is \mathcal{W} - and L^1 -removable;*
- (ii) *Every closed set $\Phi \subset N$ is \mathcal{W} - and L^1 -removable if $\Phi \neq N$, $\text{codim}_M N = 2$ and $m \geq 1$;*
- (iii) *N is \mathcal{W} - and L^1 -removable if N is generic at one point, $\text{codim}_M N = 2$ and $m \geq 2$.*

Remark. One of the main feature of our results is that they are stated in analogy with known results in the hypersurface case ($n = 1$) with variations on this theme. Also, they

become classical and easy in case $n = 0$, *i.e.* M is an open set Ω in \mathbb{C}^m , $m \geq 2$. Finally, there is obstruction to removability in case N is not generic at any point, similar to N being a complex hypersurface in Ω if $n = 0$.

Remark. After the reduction to M being globally minimal, the assumption that $M \setminus \Phi$ is globally minimal too is essential and cannot be dropped (see below).

Remark. Theorem 3 is treated in [29] and the following will appear in [33]:

(iv) *If $m \geq 2$, $\text{codim}_M N = 1$ and N is generic, then every closed set $K \subset N$ which does not contain any CR orbit of N is L^1 -removable.*

Finally, using the theory of CR orbits, we extend also [24]. A set $S \subset M$ is called a \mathcal{C}^λ peak set, $0 < \lambda < 1$, if there exists a *nonconstant* function $\varpi \in \mathcal{C}_{CR}^\lambda(M)$ such that $S = \{\varpi = 1\}$ and $|\varpi| \leq 1$.

THEOREM 5. *Let M be $\mathcal{C}^{2,\alpha}$ globally minimal. Then every \mathcal{C}^λ peak set S satisfies $H^d(S) = 0$ and is L^1 -removable.*

COROLLARY 2. *Let M be \mathcal{C}^3 . Then a \mathcal{C}^λ peak set S is L^1 -removable if $H^d(\cup_{\mathcal{O} \subset S} \mathcal{O}) = 0$.*

Now, we explain the terminology, compare our results to the codimension one case and give some motivations.

The general feature of our work is that L^p -removability is linked with \mathcal{W} -removability, *i.e.* with envelope of holomorphy results.

a. About the methods. Of course, different approaches for proving $(L^p, \bar{\partial}_b)$ -removability are conceivable, and the distinguished role of holomorphic hulls in our context is not clear yet.

For instance, one could consider the problem from the viewpoint of the general theory of removable singularities for solutions of linear partial differential operators. Let $\Omega \subset \mathbb{R}^n$ be a domain, let $K \subset\subset \Omega$ be a compact and let $P = \sum_{|\beta| \leq e} a_\beta(x) \partial_x^\beta$ be such an operator. K is called (L^p, P) -removable if any $u \in L^p(\Omega)$ with $Pu \equiv 0$ on $\Omega \setminus K$ satisfies $Pu \equiv 0$ on Ω (in the distributional sense). Of course, the notion makes sense by replacing $L^p(\Omega)$ with other differentiability classes, *e.g.* $\mathcal{C}^0(\Omega)$, $\mathcal{C}^k(\Omega)$, or even $\mathcal{D}'(\Omega)$.

Harvey and Polking [14] have proved removable singularity theorems for general P , but they give results only in case $p > 1$. Indeed, their main theorem 4.1 [14], states that K is $(L^p, \bar{\partial}_b)$ -removable if the Hausdorff measure $H^{n-p'}(K) < \infty$, $p' = p/(p-1)$, $e = \deg P$. The authors further point out that this result cannot be improved in terms of Hausdorff measures in the class of all first order differential operators. Especially information about L^1 -removability is never available on this level (more precisely, in our special setting results for $p < e/(e-1)$ cannot be derived from the above theorem).

In fact, one of the main argument here (*cf.* [14], [31], [24], [20], [21]) is to write

$$\int_{\Omega} f^\tau t P \varphi = \int_{\Omega \setminus K} f^\tau P((1 - \chi_\varepsilon)\varphi) + \int_{K^\varepsilon} f^\tau P(\chi_\varepsilon \varphi) = \int_{K^\varepsilon} f^\tau P(\varphi \chi_\varepsilon),$$

where $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, $\chi_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $|\nabla \chi_\varepsilon| \leq C/\varepsilon$, $\chi_\varepsilon \equiv 1$ on K , $\chi_\varepsilon \equiv 0$ on $\Omega \setminus K^\varepsilon$, $K^\varepsilon = \{p \in \mathbb{R}^n : \text{dist}(p, K) < \varepsilon\}$, ${}^\tau P$ is the transpose of P , and to find conditions on K , P , f in order that $\int_{K^\varepsilon} f^\tau P(\varphi \chi_\varepsilon)$, when $\varepsilon \rightarrow 0$. For instance, one can prove along these lines that, given $S \subset M$ a \mathcal{C}^1 submanifold with $\text{codim}_M S = 1$, then $\mathcal{C}_{CR}^0(M \setminus S) \cap \mathcal{C}^0(M) = \mathcal{C}_{CR}^0(M)$ [31] and also that $L_{CR}^p(M \setminus S) \cap L_{loc}^p(M) = L_{loc, CR}^p(M)$ under the additional condition that S is a complex hypersurface in M [24].

Actually, these kinds of arguments do not even answer the following very simple question, which was the model for our work on Theorem 2: Can one always remove for L^1_{CR} functions a *single point* of a *minimal* hypersurface?

EXAMPLE 1. Let $M \subset \mathbb{C}^{m+1}$, $K = \{p\} \subset M$. First, it is clear that if p belongs to a complex hypersurface $S \subset M$, then p is L^1 -removable ([24] again). If $M = b\Omega$, $\Omega \subset \subset \mathbb{C}^{m+1}$, there is a trick ([1], p.115). Let φ be a smooth $(m+1, m-1)$ -form with compact support. Then the form $\psi := \varphi - \varphi(p)$ satisfies that $\text{supp } \psi \cap b\Omega$ is still compact, since $\overline{\Omega}$ is compact, and $\bar{\partial}\varphi = \bar{\partial}\psi$, so that, in proving $\int_{b\Omega} f \bar{\partial}\varphi = 0$, one can assume that $\varphi(p) = 0$. But then $\int_{\{p\}^\varepsilon} f \bar{\partial}(\chi_\varepsilon \varphi) = \int_{\{p\}^\varepsilon} f \varphi \bar{\partial}\chi_\varepsilon + \int_{\{p\}^\varepsilon} f \chi_\varepsilon \bar{\partial}\varphi$ tends to zero, because $\varphi(p) = 0$, $|\nabla\chi_\varepsilon| \leq C/\varepsilon$ and $\|f\|_{L^1(\{p\}^\varepsilon)} \rightarrow 0$. *This argument fails in case M is a local piece of a minimal hypersurface.* \square

b. Obstructions to removability. Let us consider the archetypal non-removable singularity: Assume that a generic CR manifold M intersects transversely a smooth analytic hypersurface X given as the vanishing-locus of a holomorphic function f . Then it is easily seen that the intersection $M \cap X$ is a smooth submanifold of M of codimension 2 and that $f|(M \setminus (M \cap X))$ is locally integrable but not CR. We note that $M \cap X$ is itself a CR manifold of CR dimension $\text{CRdim}M - 2$ and of course nowhere generic: obstruction to 4 (iii).

Theorem 4 is a partial generalization to higher dimensions of a theorem of Jöricke [21] which states that *every connected, 2-codimensional, smooth subvariety of a boundary $M \subset \mathbb{C}^n$, $n \geq 3$ is removable if it is not maximally complex in the sense of Harvey and Lawson*. This paper gave evidence of the fact that, excepted for boundaries in \mathbb{C}^2 , the removal of singularities in CR-manifolds of CRdimension 1 should be especially hard.

We explain a second obstruction, due to orbits, after Theorem 6.

c. The strategy. Here, a good strategy would be to understand *a priori* the *a posteriori* fact that, if $\{p\}$ was L^1 -removable, if M is minimal at p , then $L^1_{CR}(M \setminus \{p\})$ would extend holomorphically to one side of M at p . In other words, we are looking for a holomorphic extension to a wedge attached to a full neighborhood of p in the hope to re-obtain the original function around p as a L^1_{CR} boundary value of the extension.

This strategy has been endeavoured by Jöricke [21] for hypersurfaces or in CR dimension $m \geq 2$ by the second author [32].

Our main objective in this article is to push forward in any codimension $n \geq 1$ and in any CR dimension $m \geq 1$ the theory of L^p -removability for the induced $\bar{\partial}$ in the context of the *extension theory* of Trépreau and Tumanov. This theory has acquired recently its final level of maturity, but there is still no complete survey about its global aspects.

d. CR orbits. So, let us first recall the fundamental picture of a CR manifold M , its decomposition in the so-called CR orbits ([41], [43], [51], [20], [21], [27]). An immersed connected submanifold $S \subset M$ is called a *CR-integral* submanifold if $T_p S \supset T_p^c M$, for all $p \in S$. A fundamental result of Sussmann, applied to our context for the first time by Treves, shows that M carries a finest partition into CR-submanifolds - the so called *CR-orbits* - which are complete with respect to their manifold-topology. The description of CR orbits as subsets of M is very simple. A point q is in the CR orbit of $p \in M$, *i.e.* $q \in \mathcal{O}_{CR}(M, p)$, if and only if there exists a piecewise smooth integral curve of $T^c M$ with origin p and target q . The content of Sussmann's Theorem 4.1 [41] is that *these sets have a structure of smooth immersed submanifolds of M .*

ANALOGY. The CR orbits are kinds of *irreducible components* of CR manifolds, like irreducible components of analytic sets, but in general, of transfinite cardinal.

We emphasize that the behaviour of CR-functions is dominated by the CR-orbits of the supporting manifold. First, Theorem 1 asserts that the obstructions for the construction of integrable CR-functions may be expressed in terms of CR-orbits and splits up into the differential aspect, which is completely localized on the orbits, and some global part depending on the overall geometry of the decomposition into orbits.

On a much more advanced level, CR-orbits appear as propagators of structural properties of CR-objects, like vanishing in a neighborhood of a point, or being CR extendable to a manifold with boundary. For our purposes the theory of analytic extensions into wedges, as developed by Trépreau and Tumanov ([43], [49]) is of paramount importance. Their work culminated in the following theorem, conjectured by Trépreau and proved independently by B. Jöricke and the first author ([27], [20]): For each CR orbit \mathcal{O}_{CR} , there exists an analytic wedge \mathcal{W}^{an} attached to \mathcal{O}_{CR} , *i.e.* a conic complex manifold with edge \mathcal{O}_{CR} and with $\dim_{\mathbb{C}}\mathcal{W}^{an} = \dim_{\mathbb{R}}\mathcal{O}_{CR} - \dim_{CR}M$, such that each continuous CR function on \mathcal{O}_{CR} admits a holomorphic extension to \mathcal{W}^{an} . This justifies that we work with these concepts.

Basically, it is this extension phenomenon which allows us to derive non-local results. So, in the Theorems 2, 3 and 4, one assumes explicitly that M and $M \setminus E$ or $M \setminus N$ are globally minimal.

e. \mathcal{W} -removability. Now, we need to explain the notion of removability which generalizes the one-sided removability of the hypersurface case and which will be our major tool in proving our L^1 -removability results.

In \mathbb{C}^{m+n} , one-sided neighborhoods are replaced by *wedges*, *i.e.* open sets in \mathbb{C}^{m+n} of the form $\mathcal{W}_p = \{z + \eta : z \in U, \eta \in C\}$, for open $p \in U \subset M$ and open truncated cone $C \subset T_p\mathbb{C}^{m+n} \cap \mathcal{V}_{\mathbb{C}^n}(p)$. Let $\Phi \subset M$ be a closed set, $p \in b\Phi$. Then an open connected set $\mathcal{W}_0 = \mathcal{W}_0(M \setminus \Phi)$ is called a *wedge attached to $M \setminus \Phi$* if there exists a continuous section $\eta : M \rightarrow T_M\mathbb{C}^{m+n} := T\mathbb{C}^{m+n}|_M/TM$ of the normal bundle to M and \mathcal{W}_0 contains a wedge at $(p, \eta(p))$, for all $p \in M \setminus \Phi$. This is a one-sided neighborhood of $M \setminus \Phi$ in case $n = 1$. Thus a closed set $\Phi \subset M$ is called *\mathcal{W} -removable* if, given a wedge $\mathcal{W}_0 = \mathcal{W}_0(M \setminus \Phi)$ attached to $M \setminus \Phi$, there exists a wedge $\mathcal{W} = \mathcal{W}(M)$ attached to M with holomorphic functions in \mathcal{W}_0 extending holomorphically to \mathcal{W} [ME2], [MP2].

We localize these notions as follows. We say that a point $p \in \Phi$ (here Φ has no interior points) is *\mathcal{W} - or L^1 -removable* if there exists a small neighborhood V of p in M such that L^1_{CR} or wedge extendable functions over $M \setminus \Phi$ extend to be L^1_{CR} or wedge extendable over V .

f. Description of the proofs. As the proofs to be explained in the following section are long and complicated we conclude the introduction by explaining the underlying scheme of the proof Theorem 4.

Subsequently, we will briefly mention the difference to the proof of Theorem 2.

Step 1: Reduction to a single CR orbit. Another version of Theorem 2 would be

THEOREM 2'. *Let M be $\mathcal{C}^{2,\alpha}$, $0 < \alpha < 1$, $\dim_{CR}M = m \geq 1$. Let E be a closed subset of M such that $H_{loc}^{d-3}(E) < \infty$. If, for almost every CR orbit \mathcal{O}_{CR} , $\mathcal{O}_{CR} \setminus (\mathcal{O}_{CR} \cap E)$ is globally minimal, then E is L^1 -removable.*

Thanks to Theorem 1, it is sufficient to remove $E \cap \mathcal{O}_{CR}$ for almost all CR orbits \mathcal{O}_{CR} (it can be easily shown that $H_{loc}^{e-3}(E \cap \mathcal{O}_{CR}) < \infty$ for almost all CR orbits \mathcal{O}_{CR} , $e = \dim_{\mathbb{R}}\mathcal{O}_{CR}$).

Replacing M by any $\mathcal{O}_{CR} = \text{new } M$, and E by $E \cap \mathcal{O}_{CR}$, we can assume that M is globally minimal in Theorem 2', *i.e.* prove only Theorem 2.

Step 2: Wedge extension over $M \setminus \Phi$. Here we intend to apply the following finest possible *extension theorem* (solution of Trépreau's conjecture) to $M \setminus \Phi$:

THEOREM. ([43], [49], [27], [20].) *If M is a globally minimal locally embeddable generic $\mathcal{C}^{2,\alpha}$ manifold, there exists a wedge \mathcal{W}_0 attached to M such that $\mathcal{C}_{CR}^0(M)$, $L_{loc,CR}^1(M)$, $\mathcal{D}'_{CR}(M)$, extend holomorphically to \mathcal{W}_0 .*

The desired application is possible as soon as we know that $M \setminus \Phi$ also is a single orbit. Even if Φ is contained in a codimension 2 submanifold, it is in general not true that global minimality of M implies global minimality of $M \setminus \Phi$, as shown by the following example.

EXAMPLE 2. A typical obstruction is where $\Phi = bS$ bounds a proper closed CR manifold $S \subset (M \setminus \Phi)$, with $\dim_{CR} S = \dim_{CR} M$ and $\bar{S} = S \cup \Phi$, Φ a smooth submanifold of M with $\dim_{\mathbb{R}} \Phi = \dim_{\mathbb{R}} S - 1$ and $M \setminus (S \cup \Phi)$, and S is a single CR orbit of $M \setminus \Phi$. For instance, in $\mathbb{C}_{w,z}^2$, (w, z) , $w = u + iv$, $z = x + iy$, the hypersurface $M : y = w\bar{w}\varphi(u) + xw\bar{w}$, where $\varphi \equiv 0$ on $\{u \leq 0\}$, $\varphi(0) = 0$, $\varphi > 0$ on $\{u > 0\}$ has $\Phi = \{u = y = x = 0\}$, $S = \{y = x = 0, u < 0\}$, $M \setminus (S \cup \Phi)$ is a single orbit of $M \setminus \Phi$.

In the situation of Theorem 4 however, we will be able to realize our strategy by establishing the following sufficient technical condition in order that CR orbits of M are in one-to-one correspondence with those of $M \setminus N$.

THEOREM 6. *Let M be a locally embeddable \mathcal{C}^2 -smooth CR manifold with $\dim_{CR} M = m \geq 1$, let $N \subset M$ be a \mathcal{C}^2 -smooth submanifold with $\text{codim}_M N \geq 2$ and $T_p N \not\subset T_p^c M$, $\forall p \in N$, let $N^c = \{p \in N : \dim_{\mathbb{R}} T_p^c M \cap T_p N = 2m - 1\}$ and let Υ be the set of \mathcal{C}^1 sections of $T^c M$ such that $Y|_{N^c}$ is tangent to N . If for each $p \in N$, $\mathcal{O}_\Upsilon(M, p)$ is not contained in N , then every CR orbit of $M \setminus N$ is given by $\mathcal{O}_{CR} \setminus N$, for some CR orbit of M .*

This condition that M and $M \setminus \Phi$ too are globally minimal (*cf.* Theorems 2, 3, 4) cannot be dropped at least for the problem of wedge extension of $\mathcal{D}'_{CR}(M \setminus \Phi)$. A modification of Treves' construction [46] yields counterexamples as follows (we however do not know how to refine the example up to $L_{CR}^1(M \setminus \Phi)$ or $\mathcal{C}_{CR}^0(M \setminus \Phi)$).

The occurrence of non open CR orbits \mathcal{O} in $M \setminus N$ which are closed submanifolds of $M \setminus N$ (in a localized situation near $p \in M$ as in Example 2) can give rise to CR distributions with support on them. We assume $N = b\mathcal{O}$.

Indeed, since \mathcal{O} is a closed CR submanifold of $M \setminus N$ with the same CR dimension as M , the pullbacks to \mathcal{O} of $dw_1, \dots, dw_m, dz_1, \dots, dz_d$ span a subbundle $T'_\mathcal{O}$ of $\mathcal{C}T^*\mathcal{O}$, of rank $r := m + e$, where $e = \dim_{\mathbb{R}} \mathcal{O} - 2m$. Let $\omega_1, \dots, \omega_{m+e}$ be a basis of $T'_\mathcal{O}$, for some neighborhood U of p in M . Consider the linear functional u on $\mathcal{C}_c^\infty(U \setminus N)$ defined by

$$u(\varphi) = \int_{(U \setminus N) \cap \mathcal{O}} \varphi \omega_1 \wedge \dots \wedge \omega_r \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n.$$

This is a nonzero CR distribution in $U \setminus N$ with support in \mathcal{O} since the antiholomorphic sections $L_j, j = 1, \dots, m$ of $T^{0,1}M$ which form the dual basis to $d\bar{z}_1, \dots, d\bar{z}_n$, *i.e.* $L_j \bar{z}_k = \delta_{jk}$, $1 \leq j, k \leq m$, act as follows

$$(L_j \varphi) \omega_1 \wedge \dots \wedge \omega_r \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n = \pm d(\varphi \omega_1 \wedge \dots \wedge \omega_r \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \dots \wedge d\bar{z}_n),$$

and Stokes' theorem yields

$$\langle L_j u, \varphi \rangle = \pm \int_{(U \setminus N) \cap \mathcal{O}} d(\varphi \omega_1 \wedge \dots \wedge \omega_r \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \dots \wedge d\bar{z}_n) = 0.$$

Such a nonzero CR distribution u cannot be the boundary value of a holomorphic function in a wedge of edge M at p , because of the uniqueness principle and because u vanishes on the open set $U \setminus (N \cup \mathcal{O})$. In this example, u has the manifold $\overline{\mathcal{O}} \setminus \mathcal{O} = N$ as a singular set near p and $TN \cap T^c M|_N$ is a vector bundle of rank $2m - 1$.

Step 3: \mathcal{W} -removability of Φ . Here we use massively the theory of Bishop discs. It will turn out to be possible to construct sufficiently rich families of analytic discs which sweep out (almost all of) a wedge attached to M and whose boundaries are contained in $M \setminus \Phi$.

Next we would like to extend CR-functions along the discs. Most often this is done by an application of the Baouendi-Treves approximation theorem. Unfortunately, because of the presence of the singularity Φ , this tool is not valid.

To avoid confusion, let us mention that in the case that $m \geq 2$, there is the following device used in [20] to employ nevertheless the approximation theorem: Assume that $N \subset M$ is generic $\mathcal{C}^{2,\alpha}$ of codimension one (hence $\dim_{CR} N = m - 1 \geq 1$), and that N has already been minimalized at $p \in b\Phi$, where $\Phi \subset N$ is closed, $\Phi \neq N$ and $b\Phi$ is with respect to the topology of N . Therefore there is a family of Bishop discs sweeping out an open wedge \mathcal{W}_0 attached to N . By continuity of Bishop's equation we get perturbed families of discs attached to all nearby manifolds N_t , where $\bigcup_t N_t$ is a local real foliation of M with $N_0 = N$. Now we can apply the approximation theorem to each $N_t, t \neq 0$ to get extensions of continuous CR functions on $M \setminus N$ to the wedges $\mathcal{W}_t, t \neq 0$ glued to $N_t, t \neq 0$. As $p \in b\Phi$, all extensions will glue together in the wedge $\mathcal{W} = \bigcup_{t \neq 0} \mathcal{W}_t$ whose edge is a neighborhood of p in M .

But in case of general CR dimension $m \geq 1$, Φ cannot be included in a generic minimal N of positive CR-dimension. So we will follow an alternative strategy.

First we deform M in a very nearby manifold $M^d \supset \Phi$ inside the wedge $\mathcal{W}^0(M \setminus \Phi)$. Thereby, \mathcal{W}_0 becomes a neighborhood ω of $M^d \setminus \Phi$. Thus we are in position to use the *continuity principle* with discs with boundaries in $M^d \setminus \Phi \cup \omega$. Under the stated conditions on N, E , there are enough deformations of discs to get that $\mathcal{H}(\omega)$ extends to $\mathcal{H}(\mathcal{W}^d)$, where \mathcal{W}^d is a wedge attached to M^d . The main tool is Tumanov's theorem on deformations of analytic discs which has been successfully applied to propagation of CR extension in [49].

Finally, we realize that the construction depends smoothly on d and not on the size of ω , such that, letting $d \rightarrow 0$, *i.e.* $M^d \rightarrow M$, we get a wedge \mathcal{W} attached to M , *i.e.* $\Phi = N, E$ is \mathcal{W} -removable.

Step 4. L^p -removability. The general principle is that \mathcal{W} -removability (*i.e.* a hull result) yields L^p -removability provided the constructions of discs glued to $M \setminus \Phi$ are suitable to control the wedge extension in L^p norm, up to the edge M (*cf.* [1]). So that the link between hulls and L^p is clear, now (*cf.* [21], [33]).

In fact, to prove our results in L^p above is equivalent to do the constructions which prove \mathcal{W} -removability, with L^p control (however, in case N is generic of codimension ≥ 2 and $m \geq 2$, we give an argument like estimating $\int_{N^\varepsilon} f^\tau P(\varphi\chi_\varepsilon) \rightarrow 0, \varepsilon \rightarrow 0$, to remove N in L^p , see Proposition 1.17. This argument fails in case $m = 1$).

The paper is organized as follows. In Section 1, we will explain the main geometric approach and prepare a good deal of technical tools necessary for the sequel. This section is already sufficient to understand the proof of the following statement, which apparently was not known before: *isolated points in a real \mathcal{C}^2 -smooth hypersurface in \mathbb{C}^2 are L^1 -removable.*

In section 2, we prove Theorems 2, 4 and 5 by using the technique of deformation of discs and the continuity principle.

In Section 3, we prove Theorem 1, by combining our techniques studied before with a result of Shiffman about separate meromorphicity.

In Section 4, we derive the sufficient condition given in Theorem 3 insuring that CR orbits of M are in one-to-one correspondence with those of $M \setminus N$.

In Section 5 we prove Theorem 5 and in Section 6, we explain how these results extend to hypoanalytic structures (Treves [46]).

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1. Preliminaries. 1.0. NOTATIONS. Let M be generic in \mathbb{C}^{m+n} , $m = \dim_{\mathbb{C}R} M$, $n = \text{codim}_{\mathbb{C}^n} M$, $d = 2m + n = \dim_{\mathbb{R}} M$, $N \subset M$, a submanifold, $L_{loc,CR}^1 = L_{loc}^1 \cap \mathcal{D}'_{CR}$, \mathcal{O}_{CR} be a CR-orbit, $\mathcal{O}_{CR}(M, p)$ be the CR orbit of p and $\mathcal{H}(U)$ be the holomorphic functions in U . We will give a synthetic proof of the following.

THEOREM 1.1. (JÖRICKE, PORTEN, [20][21][33].) *Let M be a \mathcal{C}^3 -smooth locally embeddable abstract CR manifold. Then a function $f \in L_{loc}^p(M)$, $p \geq 1$, is in $L_{loc,CR}^p(M)$ if and only if $f|_{\mathcal{O}}$ is in $L_{loc,CR}^p(\mathcal{O})$ for almost all CR orbits \mathcal{O} in M .*

By almost all CR orbits, we mean except on a CR invariant set of zero d -dimensional measure, $d = \dim_{\mathbb{R}} M$. Since $L_{loc}^p(M)$ embeds in $L_{loc}^1(M)$, we set $p = 1$.

Remark. For integrable functions Theorem 1.1 is proved in [33] for M of class \mathcal{C}^3 . We give here a summary of proof, the complete proof appears in [33]. The smoothness \mathcal{C}^3 is essentially needed to assure that the CR orbits are of class \mathcal{C}^2 . On the other hand it is semi-known that for an embedded CR-manifold of class $\mathcal{C}^{2,\alpha}$ the local and global orbits are of class $\mathcal{C}^{2,\beta}$ for any $\beta < \alpha$. Careful examination shows that the arguments carry over to our situation assuming only that M is $\mathcal{C}^{2,\alpha}$.

A sharper result (\mathcal{C}^2 instead of $\mathcal{C}^{2,\alpha}$ or \mathcal{C}^3) for hypersurfaces relies on the fact that in this case orbits are either open or even analytic hypersurfaces ([21]).

First, we recall the following elementary facts about CR orbits. Let $T^c M$ be the \mathcal{C}^{k-1} complex tangent bundle to M of class \mathcal{C}^k , $k \geq 2$. Let $L \in \Gamma(U, T^c M)$ be a section over an open set $U \subset M$. Let $p \in U$. Denote by $t \mapsto L_t(p)$ the integral curve of L starting at the point p , $L_0(p) = p$, $\frac{d}{dt} L_t(p) = L(L_t(p))$. It is known that the global flow mapping $(p, t) \mapsto L_t(p)$ is of class \mathcal{C}^{k-1} (defined on some domain $\Omega_L \subset U \times \mathbb{R}$). Two elements $p, q \in M$ are *in the same CR orbit* if there exist $k \in \mathbb{N}_*$, L^1, \dots, L^k , $t_1, \dots, t_k \in \mathbb{R}$ such that $q = L_{t_k}^k \circ \dots \circ L_{t_1}^1(p)$. We write $q = L_T(p)$. An *immersed submanifold* S of M is an abstract manifold S together with a smooth immersion $S \rightarrow M$. S is *CR-integral* if $TS \supset T^c M|_S$.

THEOREM. (SUSSMANN, [41].) *Let M be $\mathcal{C}^{k,\alpha}$, $k \geq 2$, $0 \leq \alpha < 1$. Then every CR orbit \mathcal{O}_{CR} of M admits a unique differentiable structure such that \mathcal{O}_{CR} is an immersed $\mathcal{C}^{k-1,\alpha}$ submanifold of M .*

Remark. In our case that M is locally embeddable CR, a better regularity result holds (Tumanov): orbits are of class $\mathcal{C}^{k,\alpha-0} = \bigcup_{\varepsilon > 0} \mathcal{C}^{k,\alpha-\varepsilon}$. Indeed, for local CR orbits, this follows from [48]. For global CR orbits, this follows from [51]. Furthermore, the estimates for Bishop's equation in [51] *also* yield that the *transversal structure to orbits* is $\mathcal{C}^{k,\alpha-0}$, chains of analytic discs replacing Sussmann's curves. Therefore, in Proposition below, the map ψ can be assumed

to be $\mathcal{C}^{2,\beta}$ if ψ is constructed from chains of attached analytic discs instead of coming from Sussmann's chains. As we would only need such ψ to be \mathcal{C}^2 in our proof, this entails that our Theorem 1.1 is true in $\mathcal{C}^{2,\alpha}$, $0 < \alpha < 1$. \square

Let us denote the mapping $T \mapsto L_T(p)$ by $\gamma_{L,p}$.

We now formulate Sussmann's main lemma on orbits.

LEMMA. (SUSSMANN, [41].) *Let \mathcal{O}_{CR} be an orbit of M . Let $p \in \mathcal{O}_{CR}$. Then there exist $k \in \mathbb{N}_*$, $L^1, \dots, L^k \in \Gamma(T^c M)$, $T^0 = (t_1^0, \dots, t_k^0) \in \mathbb{R}^k$ such that $p = L_T(p)$ is defined and $(d\gamma_{L,p})_{T^0}(\Omega_{L,p}) = T_p \mathcal{O}_{CR}$.*

In other words, the differential at $T = T^0$ of the multitime parameter variation $T \mapsto L_T(p)$ covers the tangent space to \mathcal{O}_{CR} at p .

COROLLARY. *Let $e = \dim_{\mathbb{R}} \mathcal{O}_{CR}$. Then there exists an e -dimensional affine subspace $H \subset \mathbb{R}^k$ through T^0 and a ball B of center T^0 such that $B \cap H \ni T \mapsto L_T(p) \in \mathcal{O}_{CR} \cap \mathcal{V}_M(p)$ is a $\mathcal{C}^{k-1,\alpha}$ embedding.*

Now, choosing a small manifold Λ through p transversal to \mathcal{O}_{CR} at p , i.e. $T_p \Lambda \oplus T_p \mathcal{O}_{CR} = T_p M$, we recall that the local transversal structure of orbits at P can be described by applied the multitime map $\gamma_{L,p}$ to points $q \in \Lambda$ close to p to derive :

PROPOSITION. *Let $p \in M$ and $e = \dim_{\mathbb{R}} \mathcal{O}_{CR}(M, p)$. Then there exist $\mathcal{C}^{k-1,\alpha}$ coordinates $\psi = (s, t) : V \rightarrow I^e \times I^{d-e}$ ($I = (-1, 1)$) on a neighborhood V of p in M such that*

- 1) $\psi^{-1}(I^e \times \{0\}) = \mathcal{O}_{CR}(M, p) \cap V$;
- 2) $\forall t \in I^{d-e}$, the set $\psi^{-1}(I^e \times \{t\})$ is contained in a single CR orbit of M .

As a corollary, the mapping $q \mapsto \dim_{\mathbb{R}} \mathcal{O}_{CR}(M, q)$ is lower semicontinuous (all of this is classical). We use this proposition in 2) just below.

Proposition above provides a local slice structure for global orbits. The main idea in proving Theorem 1.1 is to

Proof of Theorem 1.1. Assume f is L^1 -locally CR on M . There exists $\bigcup_{j \in J} U_j = \mathcal{U}$ a countable open covering of M by relatively compact $U_j \subset M$ with the properties that

- 1) there exist embeddable open sets $U'_j \supset U_j$ such that $L^1_{CR}(U'_j)$ is approximable, in the L^1 sense, by holomorphic polynomials on \overline{U}_j ;
- 2) each U_j is equipped with \mathcal{C}^2 coordinates $(s, t) \in U_j = S_j \times T_j$ such that for every $t \in T_j$, $S_j \times \{t\}$ is contained a single CR orbit of M ;
- 3) for every point $p \in M$, there is a $j \in J$ such that $p = (s_p, t_p) \in U_j$ and such that the leaf $S_j \times \{t_p\}$ is an open subset of $\mathcal{O}_{CR}(M, p)$;

In fact, U_j is locally described by full pieces of the piecewise smooth flow of L , each of which is contained in a single CR orbit of M [41] [20]. Regularity of the flow is \mathcal{C}^2 , so coordinates (s, t) are \mathcal{C}^2 . According to 1), for each $f \in L^1_{loc, CR}(M)$, there exists a sequence $(P_{j,l})_{l \in \mathbb{N}}$ of holomorphic polynomials such that $\lim_{l \rightarrow \infty} |f - P_{j,l}|_{L^1(U_j)} = 0$. Choose coordinates (s, t) on $U_j = S_j \times T_j$ and let $\dim_{\mathbb{R}} S_j = 2m + k$, i.e. $p_j \in \mathcal{A}_k := \{p \in M : \dim_{\mathbb{R}} \mathcal{O}_{CR}(M, p) = 2m + k\}$. Since $\mathcal{A}_k \cap U$ is a measurable subset of $S_j \times T_j$ of the form $S_j \times T_{j(k)}$, with $T_{j(k)} = (\{0\} \times T_j) \cap \mathcal{A}_k$ closed in T_j , and since $f|_{\mathcal{A}_k \cap U_j} \in L^1(\mathcal{A}_k \cap U_j)$, Fubini's theorem implies (passing to a subsequence if necessary) that $P_{j,l}$ converges in $L^1(S_j \times \{t\})$ to $f|_{S_j \times \{t\}}$ for almost all $t \in T_{j(k)}$. Therefore, $f|_{S_j \times \{t\}}$ is CR, for almost all $t \in T_{j(k)}$. Fix k . If $U_j \cap U_i \neq \emptyset$, $f|_{S_{i,j} \times \{t_{i,j}\}}$ is in $L^1_{CR}(S_{i,j} \times \{t_{i,j}\})$ for almost all $t_{i,j} \in T_{i,j(k)}$. Then the set of orbits \mathcal{O}_k contained in \mathcal{A}_k such that $f|_{\mathcal{O}_k \cap U_j} \in L^1_{CR}(\mathcal{O}_k \cap U_j)$ for each $j \in J$ consists of almost all of \mathcal{A}_k , since J is countable. For such orbits, $f \in L^1_{loc, CR}(\mathcal{O}_k)$.

Conversely, assume $f \in L^1_{loc}(M)$ is CR on almost all CR orbits of M . The sets $\mathcal{B}_k = \{p \in M : \dim_{\mathbb{R}} \mathcal{O}_{CR}(M, p) \geq 2m + k\}$ are open in M (cf. 2) above), $0 \leq k \leq n$. Then $f|_{\mathcal{B}_n}$ is CR on \mathcal{B}_n , since $\mathcal{B}_n = \mathcal{A}_n$. By induction and localization, it is sufficient to prove that $f \in L^1_{CR}(U_j)$, given that f is in $L^1_{CR}(\mathcal{B}_{k+1})$, where $U_j = U$ is an open subset of M as in 2), $p_j \in \mathcal{A}_k$, $0 \leq k \leq n-1$. Drop j , set $F = (\{0\} \times T) \cap \mathcal{A}_k \subset T$ and let (s, t) be \mathcal{C}^2 coordinates on $U = S \times T$.

Let $\delta(t)$ be a so-called regularized distance function associated with F , i.e. a real function $\delta \in \mathcal{C}^0(T) \cap \mathcal{C}^2(T \setminus F)$ satisfying $1/A \operatorname{dist}(t, F) \leq \delta(t) \leq A \operatorname{dist}(t, F)$ and $|\nabla \delta| \leq A$ on $T \setminus F$, for some constant $A > 0$ ([39]). Let $\psi(x)$ be a \mathcal{C}^2 real-valued function with $\psi(x) \equiv 1$ on $\{x \leq 1/2\}$, $\psi(x) \equiv 0$ on $\{x \geq 1\}$, set $\psi_l(x) = \psi(lx)$ for $l \in \mathbb{N} \setminus \{0\}$ and put $\chi_l(t) = \psi_l(\delta(t))$. Then $\chi_l \equiv 1$ on F , χ_l is \mathcal{C}^2 -smooth on T , $\operatorname{supp} \chi_l$ is of order $1/l$ close to F and $|\nabla \chi_l| \leq Cl$, for some constant $C > 0$.

Let L be a \mathcal{C}^1 CR vector field on U and let ${}^{\tau}L$ denote the transpose of L with respect to coordinates (s, t) on U . Notice that L has \mathcal{C}^1 -smooth coefficients, since (s, t) are \mathcal{C}^2 , so ${}^{\tau}L$ also is \mathcal{C}^1 . Let $\varphi \in \mathcal{C}^2_c(U)$. We have to prove that $\int_U f {}^{\tau}L(\varphi) dsdt = 0$. Write ${}^{\tau}L\varphi = {}^{\tau}L((1 - \chi_l)\varphi) + {}^{\tau}L(\chi_l\varphi)$. Since f is CR on the open set $U \setminus (\mathcal{A}_k \cap U) \subset \mathcal{B}_{k+1}$, we have

$$\int_U f {}^{\tau}L((1 - \chi_l)\varphi) dsdt = 0.$$

Now, write ${}^{\tau}L = D_t + D_s$, with D_t involving only derivatives with respect to t and no zero order terms. Then

$$\left| \int_U f (D_t + D_s)(\chi_l\varphi) dsdt \right| \leq \left| \int_U f D_t(\chi_l\varphi) dsdt \right| + \left| \int_U f \chi_l D_s(\varphi) dsdt \right|.$$

Letting $l \rightarrow \infty$, we get that the last integral tends to $\int_F dt (\int_{S \times \{t\}} f {}^{\tau}L\varphi ds)$, which is zero, since $L|_{S \times \{t\}}$ is a CR vector field tangent to $S \times \{t\}$ for each $t \in F$, and since we assumed that f is CR on almost all orbits in \mathcal{A}_k .

On the other hand, for each $\varepsilon > 0$, there exists $l \in \mathbb{N}$ such that $|\int_U f D_t(\chi_l\varphi) dsdt| < \varepsilon$. Indeed, the \mathcal{C}^1 -smooth coefficients of D_t are bounded by $C_1 \operatorname{dist}(\cdot, F)$, since $L|_{S \times \{t\}}$ is tangent to $S \times \{t\}$ for each $t \in F$. So $|D_t(\chi_l\varphi)| \leq C_2$ on U . Moreover, we can write

$$\left| \int_U f D_t(\chi_l\varphi) dsdt \right| \leq \left| \int_S ds \int_{F^{(1/2l)}} f D_t(\chi_l\varphi) dt \right| + \left| \int_S ds \int_{F^{(1/l)} \setminus F^{(1/2l)}} f D_t(\chi_l\varphi) dt \right|,$$

where $F^{(r)} = \{t \in T : \delta(t) < r\}$. The first term tends to $\int_{S \times F} f D_t(\varphi) dsdt$, which is zero, as $D_t = 0$ on $S \times F$. As the \mathcal{C}^1 -smooth coefficients of D_t are bounded by $C_1 \operatorname{dist}(\cdot, S \times F)$, we may estimate the second term uniformly by $C \int_S ds \int_{F^{(1/l)} \setminus F^{(1/2l)}} |f| dsdt$. For $l \rightarrow \infty$, the volume of the domain of integration goes to zero.

The proof of Theorem 1.1 is complete. \square

1.1.' COROLLARY. *A function $f \in \mathcal{C}^0(M)$ is continuous CR on M if and only if, for each CR orbit \mathcal{O} of M , $f|_{\mathcal{O}}$ is continuous CR on \mathcal{O} .*

In the rest of Section 1, we shall prove Theorem 4 (i) in a special case which simplifies the geometric treatment of the singularity, namely $\dim_{CR} M = 1$ and $T_p N \cap T_p^c M = \{0\}$, $\forall p \in N$. This special case is much easier to handle and we will see an elementary construction of discs providing removability. But we need already in this case to apply the *continuity principle*, so a deformation argument will be necessary. Therefore, during the course of the proof of

Theorem 1.2 below, we will prove our main deformation Proposition 1.16 which works for Theorems 2, 3 and 4. It states, roughly (see 1.7), that \mathcal{C}^0 -removability or \mathcal{W} -removability implies L^p -removability after some supplementary work, *i.e.* that after it is known that a set Φ is \mathcal{W} -removable, one can easily deduce that Φ is L^p removable under circumstances which are all satisfied in our Theorems 2, 3 ([29]) and 4.

THEOREM 1.2. *Let M be a $\mathcal{C}^{2,\alpha}$ -smooth ($0 < \alpha < 1$) generic manifold in \mathbb{C}^{1+n} with $\dim_{CR}M = 1$ and let $N \subset M$ be a \mathcal{C}^1 submanifold such that $N \cap \mathcal{O}$ is a \mathcal{C}^1 submanifold of \mathcal{O} with $\text{codim}_{\mathcal{O}}(N \cap \mathcal{O}) = \text{codim}_M N$ for every CR orbit \mathcal{O} and $T_p N \cap T_p^c M = \{0\}, \forall p \in N$. Then N is L^1 -removable, *i.e.* $L^1_{loc,CR}(M \setminus N) \cap L^1(M) = L^1_{loc,CR}(M)$.*

Example. We could choose a local piece M of a generic manifold with $\dim_{\mathbb{R}} \mathcal{O}_{CR}^{loc}(M, 0) = 3$ in \mathbb{C}^{1+n} passing by 0 and as $N \ni 0$ a submanifold transversal to \mathcal{O}_{CR}^{loc} with $\text{codim}_M N = 3$ (so $T_0^c M \cap T_0 N = \{0\}$).

Proof. After replacing M by an orbit \mathcal{O} of M ($\mathcal{O} := M$), we have that $\mathcal{O} = M$ and $\mathcal{O} \setminus (N \cap \mathcal{O})$ are globally minimal by our lemma 1.3 below, that $N \subset M$ has $\text{codim}_M N \geq 3$ and that $T_p N \cap T_p^c M = \{0\} \forall p \in N$. Hence we can assume thanks to Theorem 1.1 that M is a single orbit. It is clear that we can also include N in a bigger manifold N_1 of codimension exactly three and remove N_1 instead of N . So we treat the case $\text{codim}_M N = 3$.

Now, we prove a capital lemma.

LEMMA 1.3. *For each $p_0 \in M \setminus N$, $\mathcal{O}_{CR}(M \setminus N, p_0) = \mathcal{O}_{CR}(M, p_0) \setminus N$.*

Proof. Let $p_1 \in N$, let L be a \mathcal{C}^1 -smooth section of $T^c M$ in the neighborhood U of p_1 , and let $(t, p) \mapsto L_t(p)$ be the flow of L . If $\varepsilon > 0$ is small enough, one has $p_\varepsilon = L_\varepsilon(p_1) \in U \setminus N$. Take ω_ε a \mathcal{C}^1 piece of $\mathcal{O}_{CR}(U \setminus N, p_\varepsilon)$ through p_ε . Then $L_{-\varepsilon}(\omega_\varepsilon) \setminus \cup_{t \leq 0} L_t(N)$ is contained in $L_{-\varepsilon}(\omega_\varepsilon)$. Observe that $\omega_0 = L_{-\varepsilon}(\omega_\varepsilon)$ is outside of $L_{-\varepsilon}(\omega_\varepsilon)$ a CR-submanifold because $\omega_0 \setminus L_{-\varepsilon}(\omega_\varepsilon)$ is open in $\mathcal{O}_{CR}(U \setminus N, p_\varepsilon)$. By continuity, ω_0 is a CR-submanifold and contains some neighborhood ω of p_1 in $\mathcal{O}_{CR}^{loc}(p_1)$.

So it is enough to show $\omega_0 \setminus N \subset \mathcal{O}_{CR}(U \setminus N, p_\varepsilon)$, which is done by introducing a second CR vector field R' as a small rotation of R . We remark that R' is tangent to ω_0 and that $\cup_{t \leq 0} L_t(N) \cap \cup_{t \leq 0} L_t'(N) = N$ in a neighborhood of p_1 in M by the fact that $\dim_{\mathbb{R}}(\mathbb{R}L(p_1) + \mathbb{R}L'(p_1)) = 2$. \square

1.4. SPECIAL CASE. We will first remove functions in $L^1(M)$ which extend holomorphically into a neighborhood of $M \setminus N$ in \mathbb{C}^{1+n} and then prove that a deformation argument reduces to that situation (see 1.7 and 1.16 below). First, we glue families of analytic discs to M which shrink as a slice pressing N . Considering functions in $L^1_{CR}(M \setminus N)$ which are holomorphic in a neighborhood of $M \setminus N$, we can apply a continuity principle to discs with their boundaries in $M \setminus N$ to *modify* M in a new manifold where singularities disappear. Here is the program.

There exist holomorphic coordinates (z, w) in \mathbb{C}^{n+1} such that $p_0 = 0$ in these coordinates, $w \in \mathbb{C}$, $z = x + iy \in \mathbb{C}^n$, $T_0 M = \{x = 0\}$, $T_0 N = \{x = 0, w = 0, y_1 = 0\}$ and, if we set $y' = (y_2, \dots, y_n)$, M, N are given by

$$(2) \quad M : x = h(y, w), \quad N : x = h(y, w), y_1 = k(y'), w = g(y'),$$

for a $\mathcal{C}^{2,\alpha}$ vector valued function h with $h(0) = 0, \nabla h(0) = 0$, and \mathcal{C}^1 functions g, k with $g(0) = 0, \nabla g(0) = 0, k(0) = 0, \nabla k(0) = 0$. First, we construct a family of analytic discs such that their boundaries parametrize a neighborhood of 0 in M . Let $A_\rho^y(\zeta) = (Z_\rho^y(\zeta), W_\rho^y(\zeta))$ be the family of analytic discs parametrized by ρ, y with w -component $W_\rho^y(\zeta) = \rho\zeta + g(y')$ and

let Y_ρ^y be the harmonic extension of the solution on $b\Delta$ of Bishop's equation (see after Lemma 1.10 the theorem of Tumanov for existence)

$$(3) \quad Y_\rho^y = T(h(Y_\rho^y, \rho\zeta + g(y'))) + (k(y') + y_1, y').$$

(T denotes the usual Hilbert transform on the unit circle, *i.e.* $T : L^2(b\Delta) \rightarrow L^2(b\Delta)$ is the unique operator such that $u + iTu$ extends holomorphically into Δ and the harmonic extension $PH(Tu)(0) = 0$ [22].) Because of the differentiability of the solution, there exists an open neighborhood \mathcal{Y} of 0 in the y -space and $\rho_0 > 0$, $I_{\rho_0} = [0, \rho_0)$, such that the mapping $\mathcal{Y} \times I_{\rho_0} \times (\mathbb{R}/2\pi\mathbb{Z}) \ni (y, \rho, \theta) \mapsto A_\rho^y(e^{i\theta}) \in M$ is a smooth embedding. This gives polar coordinates on M and the volume form $\rho d\rho d\theta dy$ is proportional to $d\text{Vol}_M$. Fix small $\eta > 0$, write $|y| = \max_{1 \leq j \leq n-1} |y_j|$ and set, for $0 < \varepsilon \ll \eta$,

$$M_{\eta, \varepsilon} = \{p \in M : |y(p)| < \eta, \rho(p) < \varepsilon\}.$$

Define also

$$(4) \quad M_\varepsilon = (M_{\eta, \eta} \setminus M_{\eta, \varepsilon}) \cup \bigcup_{|y| < \eta} A_\varepsilon^y(\overline{\Delta}) = (M_{\eta, \eta} \setminus M_{\eta, \varepsilon}) \cup \widetilde{M}_\varepsilon.$$

M_ε is a piecewise smooth Lipschitz CR graph, a weakly differentiable manifold on which the existence of CR functions still makes sense (*cf.* [10], [11]). $\mathcal{V}(E)$ will denote a small open neighborhood of a set $E \subset \mathbb{C}^{1+n}$.

PROPOSITION 1.5. *If $f \in L_{loc, CR}^1(M) \cap \mathcal{H}(\mathcal{V}(M \setminus N))$, there exists $f_\varepsilon \in \mathcal{C}_{CR}^0(M_\varepsilon)$ such that $f_\varepsilon \equiv f$ on $M_\varepsilon \setminus \widetilde{M}_\varepsilon (\subset M)$. Moreover, there exists a sequence $\varepsilon_j \rightarrow 0$ such that $\int_{\widetilde{M}_{\varepsilon_j}} |f_{\varepsilon_j}| \rightarrow 0$ as $j \rightarrow \infty$.*

We claim that Proposition 1.5 implies the theorem in the special case where our function $f \in L_{loc, CR}^1(M \setminus N)$ is holomorphic in a neighborhood of $M \setminus N$ in \mathbb{C}^{1+n} . Indeed, take an arbitrary \mathcal{C}^2 -smooth $(1+n, 0)$ -form φ with support in a small ball centred at 0. Then $f_{\varepsilon_j} \rightarrow f$ in L^1 norm according to Proposition 1.5 and

$$\int_{M_{\eta, \eta}} f \bar{\partial} \varphi = \lim_{j \rightarrow \infty} \int_{M_{\eta, \eta} \setminus M_{\eta, \varepsilon_j}} f \bar{\partial} \varphi = \lim_{j \rightarrow \infty} \int_{M_{\eta, \eta} \setminus M_{\eta, \varepsilon_j}} f_{\varepsilon_j} \bar{\partial} \varphi = - \lim_{j \rightarrow \infty} \int_{\widetilde{M}_{\varepsilon_j}} f_{\varepsilon_j} \bar{\partial} \varphi \stackrel{1.5}{=} 0,$$

since, because f is L^1 on $M_{\eta, \eta}$, we have $\lim_{j \rightarrow \infty} \int_{M_{\eta, \varepsilon_j}} f \bar{\partial} \varphi = 0$, since $f \equiv f_{\varepsilon_j}$ on $M_{\eta, \eta} - M_{\eta, \varepsilon_j}$ and since $f_{\varepsilon_j} \in \mathcal{C}_{CR}^0(M_{\varepsilon_j})$. \square

Proof of Proposition 1.5. We recall facts about analytic isotopies of discs [28]. Let M be generic. An embedded analytic disc A attached to M is said to be *analytically isotopic* to a point in M if there exists a \mathcal{C}^1 -smooth mapping $(s, \zeta) \mapsto A_s(\zeta)$, $0 \leq s \leq 1$, $\zeta \in \overline{\Delta}$, such that $A_0 = A$, each A_s is an embedded analytic disc attached to M for $0 \leq s < 1$ and A_1 is a constant mapping $\overline{\Delta} \rightarrow \{pt\} \in M$. We shall use the following continuity principle:

PROPOSITION 1.6. ([28]) *Let M be generic, $\mathcal{C}^{2, \alpha}$, let Φ be a proper closed subset of M and let ω be a neighborhood of $M \setminus \Phi$ in \mathbb{C}^{m+n} . If an embedded disc A attached to $M \setminus \Phi$ is analytically isotopic to a point in $M \setminus \Phi$, then, there exists a neighborhood $\mathcal{V}(A(\overline{\Delta}))$ in \mathbb{C}^{m+n} such that, for each function $f \in \mathcal{H}(\omega)$, there exists a function $F \in \mathcal{H}(\mathcal{V}(A(\overline{\Delta})))$ such that $F = f$ in a neighborhood of $A(b\Delta)$. \square*

We then have to isotope the analytic discs to points with boundaries in $\omega = \mathcal{V}(M \setminus N)$ to show that $\zeta \mapsto f \circ A_\varepsilon^y(\zeta)$ extends holomorphically into Δ for each ε, y . One explicit mean to get the isotopy is the following: we extend the family A_ρ^y for the parameter $\eta \leq y_1 \leq 2\eta$ (and similarly also for $-2\eta \leq y_1 \leq -\eta$) by setting $A_\rho^y(\zeta) = (Z_\rho^y(\zeta), W_\rho^y(\zeta))$, where $W_\rho^y(\zeta) = \rho/\eta|2\eta - y_1|\zeta + g(y')$ and Y_ρ^y is the solution of Bishop's equation

$$(5) \quad Y_\rho^y = T(h(Y_\rho^y, \rho/\eta|2\eta - y_1|\zeta + g(y'))) + (k(y') + y_1, y').$$

Varying y_1 in $(0, 2\eta)$, we see that each disc A_ε^y is analytically isotopic in ω to disc $A_{\tilde{\varepsilon}}^{\tilde{y}}$ with $\tilde{y}_1 = 2\eta$ and $\tilde{y}' = y'$, which is a point with (y, w) -coordinates on M equal to $(k(y') + 2\eta, y', g(y'))$ (due to the fact that the solution of (5) is a constant, because the term behind T is independent of ζ). Therefore, in view of Proposition 1.6, each $f \in \mathcal{H}(\omega)$ extends to be a *continuous* CR function f_ε on $\widetilde{M}_\varepsilon$ as $f_\varepsilon \equiv f$ on $M_\varepsilon \setminus \widetilde{M}_\varepsilon$ and

$$f_\varepsilon(p) = \frac{1}{2i\pi} \int_{b\Delta} \frac{f \circ A_\varepsilon^y(\xi)}{\xi - \zeta_p} d\xi,$$

for each point $p = A_\varepsilon^y(\zeta_p)$. Put $f \circ A_\varepsilon^y(e^{i\theta}) = f(\rho, \theta, y)$. For each small $\varepsilon > 0$, there exists $\varepsilon' < \varepsilon$ such that

$$(6) \quad \varepsilon' \int_{|\theta| \leq \pi, |y| < \eta} |f(\varepsilon', \theta, y)| d\theta dy \leq \frac{1}{\varepsilon} \int_{\rho < \varepsilon, |\theta| \leq \pi, |y| < \eta} |f(\rho, \theta, y)| \rho d\rho dy = I_\varepsilon/\varepsilon.$$

On the other hand, there exists a constant $C > 0$, depending only on the \mathcal{C}^1 -norm of the embedding $\mathcal{Y} \times I_{\rho_0} \times I_{2\pi} \ni (y, \rho, \theta) \mapsto A_\rho^y(e^{i\theta}) \in M$ with

$$J_{\varepsilon'} = \int_{\widetilde{M}_{\varepsilon'}} |f_{\varepsilon'}| d\text{Vol}_{\widetilde{M}_{\varepsilon'}} \leq C(\varepsilon')^2 \int_{\rho < 1, |\theta| \leq \pi, |y| < \eta} |f \circ A_{\varepsilon'}^y(\rho e^{i\theta})| \rho d\rho d\theta dy.$$

Recall that, for $u \in \mathcal{C}^0(\overline{\Delta}) \cap \mathcal{H}(\Delta)$, the subharmonicity of $|u|$ implies $\int_0^{2\pi} |u(re^{i\theta})| d\theta \leq \int_0^{2\pi} |u(e^{i\theta})| d\theta$, for every $0 \leq r \leq 1$. Then

$$\begin{aligned} J_{\varepsilon'} &\leq C(\varepsilon')^2 \int_{|y| < \eta} dy \int_{\rho < 1} \rho d\rho \int_{|\theta| \leq \pi} |f \circ A_{\varepsilon'}^y(\rho e^{i\theta})| d\theta \leq \\ &\leq C(\varepsilon')^2 \int_0^1 \rho d\rho \int_{|\theta| \leq \pi, |y| < \eta} |f \circ A_{\varepsilon'}^y(e^{i\theta})| d\theta dy \end{aligned}$$

and this is $\leq CI_\varepsilon \varepsilon'/2\varepsilon \leq CI_\varepsilon/2$, in view of (6). But $I_\varepsilon \rightarrow 0$, since f is L^1 .

The proof of Proposition 1.5 is complete. \square

1.7. GENERAL CASE. Let M be generic, let $p \in M$ and suppose that $\Phi \subset M$ is a given closed singularity set for locally integrable CR functions which are $L_{loc, CR}^1$ on $M \setminus \Phi$. Our aim is to *reduce the problem of removing Φ to the problem of removing Φ for the space of functions holomorphic in some neighborhood of $M \setminus \Phi$ and locally integrable on M , $\mathcal{H}(\mathcal{V}(M \setminus \Phi)) \cap L_{loc}^1(M)$* . Informally, this can be performed by suitable very small deformations M^d of M over a wedge attached to $M \setminus \Phi$, so small that we only very slightly change a fixed function $f \in L_{loc}^1(M) \cap L_{loc, CR}^1(M \setminus \Phi)$.

To be precise, we localize first M near one of its points $p \in b\Phi$. Suppose that we can prove (see Proposition 1.16) that, given $f \in L^1(M) \cap L_{loc, CR}^1(M \setminus \Phi)$, for each $\varepsilon > 0$, there exists a

$\mathcal{C}^{2,\alpha}$ -smooth compactly supported deformation M^d of M , $M^d \supset \Phi$, with $\|M^d - M\|_{\mathcal{C}^{2,\alpha}} < \varepsilon$ (the $\mathcal{C}^{2,\alpha}$ distance $\|M^d - M\|_{\mathcal{C}^{2,\alpha}}$ between M and M^d can be measured by $\|h^d - h\|_{\mathcal{C}^{2,\alpha}}$ for graphing functions h and h^d like in (2)) such that there exists $f^d \in L^1(M^d)$ which is holomorphic in a neighborhood of $(M^d \setminus \Phi)$ in \mathbb{C}^{m+n} and such that $\|f^d - f\|_{L^1(M)} < \varepsilon$. If, for each M^d ,

$$L^1(M^d) \cap \mathcal{H}(\mathcal{V}(M^d \setminus \Phi)) = L^1_{loc,CR}(M^d),$$

then $f \in L^1_{loc,CR}(M)$ near p . Indeed, fix $\varepsilon > 0$ and choose d , M^d , $f^d \in L^1(M^d)$ as above. Let φ be a \mathcal{C}^2 -smooth $(m+n, m-1)$ -form, compactly supported near p . Then $|\int_M (f^d - f)\bar{\partial}\varphi| \leq C_\varphi \varepsilon$, since $\|f^d - f\|_{L^1(M)} < \varepsilon$ and we can identify abusively the volume form on M and on M^d . Therefore, $\int_{M^d} f^d \bar{\partial}\varphi = 0$ implies that $\int_M f \bar{\partial}\varphi = 0$, since ε was arbitrary.

To fulfill this program, it is convenient to introduce families of discs which enjoy better regularity properties than the general discs of minimal defect [47]. These discs are suitable to apply measure theoretic arguments in L^p , *e.g.* Fubini's theorem or convergence almost everywhere and they in a certain sense reduce the L^p analysis on wedges to the L^p analysis on bundles of analytic discs.

DEFINITION 1.8. By a *regular family of analytic discs attached to M at p* , we mean a $\mathcal{C}^{2,\beta}$ -smooth mapping $A : \mathcal{S} \times \mathcal{V} \times \overline{\Delta} \rightarrow \mathbb{C}^{m+n}$, $(s, v, \zeta) \mapsto A_{s,v}(\zeta)$, $A_{0,v}(1) = p$, holomorphic with respect to ζ , the parameters s, v run over 0-neighborhoods $\mathcal{S} \subset \mathbb{R}^{2m+n-1}$, $\mathcal{V} \subset \mathbb{R}^{n-1}$, respectively, such that

- 1) The mapping $\mathcal{S} \times b\Delta \rightarrow M$, $(s, \zeta) \mapsto A_{s,v}(\zeta)$ is an embedding, $\forall v \in \mathcal{V}$ uniformly;
- 2) The vector $Y = -\frac{\partial}{\partial \zeta} A_{0,v}(1) \notin T_p M$ and the rank of the mapping $v \mapsto \text{pr}_{T_p \mathbb{C}^{m+n} / (T_p M \oplus \mathbb{R}Y)} \left(-\frac{\partial}{\partial \zeta} A_{0,v}(1) \right)$ is equal to $n - 1$;
- 3) There exists a neighborhood V of p such that $\{A_{s,v}(\zeta) : \zeta \in b\Delta\} \subset V$ and $L^1_{loc,CR}$ functions are approximable by holomorphic polynomials in $L^1(V)$.

Here, we require that one set of parameters \mathcal{V} describes the total amount of necessary outer directions $-\frac{\partial}{\partial \zeta} A_{0,v}(1)$ to cover a cone in $T_p \mathbb{C}^{m+n} / T_p M$ of dimension n and that the set of parameters \mathcal{S} describes, together with the arc-length on $b\Delta$ near 1, a set which gives coordinates on M near p . This is crucial to use Fubini's theorem in order to exhibit L^1 traces of f on almost every circle $A_{s,v}(b\Delta)$ if $f \in L^1(M)$ (*cf.* [24], [21]).

Remark 1. We stress that existence of regular families of analytic discs does not hold automatically at minimal points, because a disc of minimal defect need not be an embedding [47], [3] and that on a globally minimal M which is nowhere locally minimal, condition 2) above cannot at all be satisfied. Fortunately, the existence of regular families of analytic discs attached to *deformations* of M follows from the work of Tumanov on propagation [49], [51] (but is nontrivial) and is *equivalent to global minimality*.

Furthermore, it follows from [49] in the construction of such $A_{s,v}$ that one deforms a small round disc whose projection on the $T_p^c M$ space is a trivial disc, like $\zeta \mapsto (c(1 - \zeta), 0) \in \mathbb{C}^m$, implying that each disc $A_{s,v}$ is an embedding (*cf.* also Lemma 2.4).

We shall use L^p wedge extension of L^p_{CR} functions on a globally minimal M , which is proved in [32], [33] using the minimalization theorem in [20].

Remark 2. According to conditions 1) and 2), there exist $\mathcal{S}_1 \subset \mathcal{S}$ a subneighborhood of 0, $\mathcal{V}_1 \subset \mathcal{V}$ a subneighborhood of 0, $\Delta_1 \subset \overline{\Delta}$ a neighborhood of 1, such that the mapping

$$\mathcal{S}_1 \times \mathcal{V}_1 \times \overset{\circ}{\Delta}_1 \ni (s, v, \zeta) \mapsto A_{s,v}(\zeta) \in \mathbb{C}^{m+n} \setminus M$$

is an embedding, whose image

$$\mathcal{W} = \{A_{s,v}(\zeta) \in \mathbb{C}^{m+n}; (s, v, \zeta) \in \mathcal{S}_1 \times \mathcal{V}_1 \times \overset{\circ}{\Delta}_1\}$$

constitutes a wedge open set with edge M at p . \square

Remark 3. For sufficiently small families of discs, the approximation property is automatic, according to a version of a theorem due to Baouendi and Treves proved by Jöricke [21].

THEOREM 1.9. [4] [21]. *Let M be a \mathcal{C}^2 -smooth locally embeddable CR manifold, let p , $1 \leq p < \infty$. Then, for every neighborhood U of p in M , there exists another neighborhood $V \subset\subset U$ of p such that, for each $f \in L^p_{loc, CR}(U)$, each $\varepsilon > 0$, there exists a polynomial P in z_1, \dots, z_{m+n} with*

$$\int_V |f - P| dVol_M < \varepsilon. \square$$

Remark 4. In general, we shall deal below with regular families such that $\mathcal{S} \times b\Delta \rightarrow M$, $(s, \zeta) \mapsto A_{s,v}(\zeta)$ is an embedding, $\forall v \in \mathcal{V}$. But the weakened condition 1) that the disc mapping with parameters should only be an immersion along the boundary can be reached directly by suitable modifications of a given disc as follows. If E is any set in a metric space B , we shall denote by $\mathcal{V}(E, r)$ the set $\{p \in E : \text{dist}(E, p) < r\}$ and $\mathcal{V}(E)$ any small open neighborhood of E in B . For instance the space B of $\mathcal{C}^{2,\alpha}$ discs is a Banach space and we write $A' \in \mathcal{V}(A, \delta)$.

LEMMA 1.10. [24]. *Let M be generic, $\mathcal{C}^{2,\alpha}$ -smooth, and let A be a sufficiently small $\mathcal{C}^{2,\beta}$ -smooth analytic disc attached to M , $A(1) = p_0$. Then, for each $\delta > 0$, there exists a family of analytic discs A_s attached to M with $A_0(1) = p_0$, the parameter s runs over a 0-neighborhood $\mathcal{S} \subset \mathbb{R}^{2m+n-1}$, such that $A_s \in \mathcal{V}(A, \delta)$, for $s \in \mathcal{S}$ and the rank of the mapping $\mathcal{S} \times \mathbb{R} \ni (s, \theta) \mapsto A_s(e^{i\theta}) \in M$ is equal to $2m + n = \dim_{\mathbb{R}} M$.*

To begin with, let T denote the Hilbert transform *i.e.* the harmonic conjugation operator) on the unit circle $b\Delta$. Recall that for a function ϕ on $b\Delta$, $T\phi$ is the unique function on $b\Delta$ such that $\phi + iT\phi$ extends holomorphically into Δ and $PH(T\phi)(0) = 0$, where PH denotes the harmonic extension operator. We also denote by T_1 the Hilbert transform vanishing at 1, $T_1\phi = T\phi - T\phi(1)$. It is known that T and T_1 are bounded operators $\mathcal{C}^{k,\alpha} \rightarrow \mathcal{C}^{k,\alpha}$ ($k \geq 0$, $0 < \alpha < 1$) and in L^p ($1 < p \leq \infty$).

The following equation, called Bishop's equation with parameters, arises in constructing analytic discs with boundaries in a given generic manifold

$$(7) \quad y = T_1 H(y, \cdot, t) + y_0,$$

where H is a given \mathbb{R}^n -valued function depending on $y \in \mathbb{R}^n$, $\zeta \in b\Delta$ and a parameter $t \in \mathbb{R}^l$. The solution $y = y(\zeta, t, y_0)$ is a function of $\zeta \in b\Delta$, the variables $t \in \mathbb{R}^l$ and $y_0 \in \mathbb{R}^n$ being parameters. We shall repeatedly use the best result below on solvability and regularity of Bishop's equation. Let B_r^n denote the ball of radius r centered at the origin.

THEOREM. (TUMANOV, [51]). *Let $H \in \mathcal{C}^{k,\alpha}(B_1^n \times b\Delta \times B_1^l, \mathbb{R}^n)$, $k \geq 1$, $0 < \alpha < 1$. For every constant $C > 0$, there exists $c > 0$ such that, if $\|H\|_{\mathcal{C}^{k,\alpha}} < C$, $\|H\|_{\mathcal{C}^0} < c$, $\|H_y\|_{\mathcal{C}^0} < c$ and $\|H_\zeta\|_{\mathcal{C}^0} < c$, then*

- (i) (6) has a unique solution $\zeta \mapsto y(\zeta, t, y_0)$ in $L^2(b\Delta)$;
- (ii) $y \in \mathcal{C}^{k,\alpha}(b\Delta)$ and $\|y\|_{\mathcal{C}^{k,\alpha}}$ is uniformly bounded with respect to parameters;

(iii) $y \in \mathcal{C}^{k,\beta}(b\Delta \times B_1^l \times B_c^n, \mathbb{R}^n)$ for all $0 < \beta < \alpha$. \square

Proof of Lemma 1.10. Assume that M is given by (2) and write $A(\zeta) = (Z(\zeta), W(\zeta))$. We can slightly perturb the component $W_m(\zeta)$ to insure that $W'_m(\zeta) \neq 0$ for every $\zeta \in b\Delta$ and that this property holds for every disc in $\mathcal{V}(A, \delta')$, $\delta' > 0$, $\delta' \ll \delta$ small enough. Still denote by A this disc, where $y(\zeta)$ is obtained by taking the harmonic extension of the solution of Bishop's equation $Y = T_1 h(Y, W)$ on $b\Delta$. Using the notation $W^*(\zeta) = (W_1(\zeta), \dots, W_{m-1}(\zeta))$, we set, for small $|t| < \varepsilon$, $\varepsilon \ll \delta$, $\varepsilon > 0$,

$$W^t(\zeta) = (W^*(\zeta), W_m(\zeta) + t\zeta W'_m(\lambda\zeta)),$$

where $\lambda < 1$ is close enough to 1 to insure that the jacobian of the map $(t, \theta) \mapsto W_m^t(e^{i\theta})$ is nowhere vanishing on $\{|t| < \varepsilon\} \times \mathbb{R}$. Then we consider the family of analytic discs

$$A_{t,w^*0,y^0}(\zeta) = (Z_{t,w^*0,y^0}(\zeta), W^*(\zeta) + w^*0, W_m^t(\zeta) - W_m^t(1)),$$

where Y_{t,w^*0,y^0} satisfies

$$Y_{t,w^*0,y^0} = T_1 h(Y_{t,w^*0,y^0}, W^*(\cdot) + w^*0, W_m^t(\cdot) - W_m^t(1)) + y^0$$

on $b\Delta$. It is $\mathcal{C}^{2,\beta}$ -smooth with respect to all the variables by the Theorem of Tumanov. Then the jacobian matrix $E = D_{y^0} Y_{t,w^*0,y^0}$ has non vanishing determinant, if the disc A_{t,w^*0,y^0} is small enough, since it satisfies $E = T_1 h(E, W_{t,w^*0,y^0}) + I$, and $dh(0) = 0$. Since the jacobian matrix $D_{(w^*0,t,\theta)} W_{t,w^*0,y^0}$ has non vanishing determinant too, we get the lemma by letting $s = (t, w^*0, y^0)$ vary in a sufficiently small neighborhood \mathcal{S} of 0 in \mathbb{R}^{2m+n-1} .

The proof of Lemma 1.10 is complete. \square

From now on, since it will be more convenient and not restrictive for our purpose, we shall assume that regular families of discs embed $\mathcal{S} \times b\Delta$ in M as in Definition 1.18. We check:

LEMMA 1.11. *Let $f \in L^1_{loc,CR}(M)$ and let $v \in \mathcal{V}$ be given. Then, for almost all $s \in \mathcal{S}$, the mapping $b\Delta \ni \zeta \mapsto f \circ A_{s,v}(\zeta) \in \mathbb{C}$ is well-defined and belongs to $L^1(b\Delta)$.*

Proof. The mapping $\mathcal{S} \times b\Delta \ni (s, \theta) \mapsto A_{s,v}(e^{i\theta})$ is a smooth embedding of $\mathcal{S} \times b\Delta$ onto a tubular open connected neighborhood \mathcal{T} of $A(b\Delta)$ in M and this gives coordinates (s, θ) on \mathcal{T} together with a volume element $dsd\theta$ on \mathcal{T} which is proportional to $d\text{Vol}_M$. Since $f \in L^1(\mathcal{T})$, Fubini's theorem written in the form

$$\int_{\mathcal{S} \times b\Delta} f \circ A_{s,v}(e^{i\theta}) dsd\theta = \int_{\mathcal{S}} ds \left(\int_0^{2\pi} f \circ A_{s,v}(e^{i\theta}) d\theta \right) < \infty,$$

implies that $b\Delta \ni \zeta \mapsto f \circ A_{s,v}(\zeta)$ belongs to $L^1(b\Delta)$ for almost all $s \in \mathcal{S}$, which completes the proof. \square

A $L^1_{loc,CR}$ function f on M is said to be holomorphically extendable to a wedge $\mathcal{W} = \mathcal{W}(U, C)$ if there exists $F \in \mathcal{H}(\mathcal{W})$ such that $F|_{U_\eta} \rightarrow f$ in the L^1 sense, as $\eta \rightarrow 0$, $U_\eta = U + \eta$, uniformly for $\eta \in C$. Recall that this implies that, given a subcone $C_1 \subset\subset C$, given $U_1 \subset\subset U$, for every one-parameter \mathcal{C}^2 -smooth family of open sets $U_{1\varepsilon}$ with $U_{1\varepsilon} \rightarrow U_1$ in \mathcal{C}^2 norm and such that $U_{1\varepsilon} \subset \mathcal{W}(U_1, C_1)$, then $F|_{U_{1\varepsilon}} \rightarrow f|_{U_1}$ in L^1 norm, and that, conversely, if $F|_{U_{1\varepsilon}} \rightarrow f|_{U_1}$ in L^1 , then $F|_{\mathcal{W}(U_2, C_2)}$ is a holomorphic extension of the $L^1_{loc,CR}$ function f in the first sense, for possibly smaller $U_2 \subset\subset U_1$, $C_2 \subset\subset C_1$. Therefore, the sizes of U and C are not essential.

Let $H_a^1(\Delta)$ denote Hardy space in Δ , $H_a^1(\Delta) = \{u \in \mathcal{H}(\Delta) : \sup_{r < 1} \int_0^{2\pi} |u(re^{i\theta})| d\theta < \infty\}$. Accordingly, we denote by $H_a^1(\mathcal{W})$ the Hardy space of holomorphic functions in \mathcal{W} with L^1 boundary values on M , i.e. $F|_{U_{1\varepsilon}} \rightarrow f|_{U_1}$ in L^1 norm.

The existence of regular families of discs is well suited to get holomorphic extension of measurable CR functions. This is the content of the next proposition.

PROPOSITION 1.12. *Let M be generic, $\mathcal{C}^{2,\alpha}$, let $p \in M$ and assume that there exists a regular family $A : \mathcal{S} \times \mathcal{V} \times \overset{\circ}{\Delta}_1 \rightarrow \mathbb{C}^{m+n}$ of analytic discs attached to M at p . Then there exist open neighborhoods $0 \in \mathcal{S}_1 \subset \mathcal{S}$, $0 \in \mathcal{V}_1 \subset \mathcal{V}$, $1 \in \Delta_1 \subset \overline{\Delta}$, such that every $L_{loc,CR}^1$ function on M extends holomorphically as $F \in H_a^1(\mathcal{W})$ into the wedge open set*

$$\mathcal{W} = \{A_{s,v}(\zeta) \in \mathbb{C}^{m+n} : (s, v, \zeta) \in \mathcal{S}_1 \times \mathcal{V}_1 \times \overset{\circ}{\Delta}_1\}.$$

Remark. As \mathcal{W} is foliated by pieces of discs $A_{s,v}$, the overall scheme of our proof will be a reduction to Hardy spaces of discs $A_{s,v}$. Hence $H_a^1(\mathcal{W})$ appears as a *bundle of Hardy spaces* on $\overset{\circ}{\Delta}_1$. Our proof is a repetition of [21] in higher codimension.

Proof. Choose $\mathcal{S}_1 \times \mathcal{V}_1 \times \Delta_1 \subset \mathcal{S} \times \mathcal{V} \times \overline{\Delta}$ such that Δ_1 is in the form $\Delta_1 = \{\zeta = re^{i\theta} \in \overline{\Delta} : r_1 < r \leq 1, |\theta| < \theta_1\}$, $\theta_1 > 0$ and $\mathcal{S}_1 \times \mathcal{V}_1 \times \overset{\circ}{\Delta}_1 \ni (s, v, \zeta) \mapsto A_{s,v}(\zeta) \in \mathbb{C}^{m+n}$ is an embedding. Set $I_1 = (-\theta_1, \theta_1)$. For $r_1 < r \leq 1$, $v \in \mathcal{V}_1$, define a partial copy of M contained in \mathcal{W}

$$M_r(v) = \{A_{s,v}(re^{i\theta}); \theta \in I_1, s \in \mathcal{S}_1\}.$$

There exists a uniform constant $C > 0$ such that $(1/C)d\text{Vol}_M \leq d\text{Vol}_{M_r(v)} \leq Cd\text{Vol}_M$, where $d\text{Vol}_M(p) = g_M(s, \theta) ds d\theta$, $d\text{Vol}_{M_r(v)}(p) = g_{M_r(v)}(s, \theta) ds d\theta$ are computed with respect to the \mathcal{C}^2 -smooth parametrization $(s, \theta) \mapsto A_{s,v}(re^{i\theta})$ with respect to which both $d\text{Vol}_M$ and $d\text{Vol}_{M_r(v)}$ can be expressed, since $M_r(v)$ is a small deformation of M . In the rest of the paper, we shall denote by C an unspecified constant > 0 , depending on the context.

LEMMA 1.13. *There exists a constant $C > 0$ such that the following estimate holds*

$$\int_{M_r(v)} |P| d\text{Vol}_{M_r(v)} \leq C \int_V |P| d\text{Vol}_M,$$

for every holomorphic polynomial P , every $r \in (r_1, 1]$ and every $v \in \mathcal{V}_1$.

Proof. Recall $g_{M_r(v)}(p)/g_M(p) \leq C$. By plurisubharmonicity, for every polynomial P ,

$$\int_{\mathcal{S}_1} \int_{-\theta_1}^{\theta_1} |P \circ A_{s,v}(re^{i\theta})| g_{M_r(v)}(s, \theta) ds d\theta \leq C_1 C \int_{\mathcal{S}_1} \int_{-\pi}^{\pi} |P \circ A_{s,v}(e^{i\theta})| g_M(s, \theta) ds d\theta,$$

which yields, since the disc mapping $(s, \theta) \mapsto A_{s,v}(e^{i\theta})$ is an embedding (uniform in v),

$$\int_{\mathcal{S}_1} \int_{-\theta_1}^{\theta_1} |P \circ A_{s,v}(re^{i\theta})| g_{M_r(v)}(s, \theta) ds d\theta \leq C |P|_{L^1(V)}.$$

The proof of Lemma 1.13 is complete. \square

Integrate now 1.13 with respect to $v \in \mathcal{V}_1$ and to $r \in (r_1, 1]$. We get

$$(8) \quad \int_{\mathcal{W}} |P| d\text{Vol}_{\mathbb{C}^{m+n}} \leq C \int_V |P| d\text{Vol}_M.$$

Let f be a CR function on M of class L^1_{loc} . According to the approximation theorem, there exists a sequence $\{P_\mu\}_{\mu=1}^\infty$ of polynomials in z_1, \dots, z_{m+n} such that $f = \lim_{\mu \rightarrow \infty} P_\mu$ in $L^1(V)$. Then, for $P_\mu - P_\nu$, we have the estimate (8), so $|P_\mu - P_\nu|_{L^1(\mathcal{W})}$ tends to 0 as μ, ν go to infinity. Hence P_μ converges to a holomorphic function F in $\mathcal{H}(\mathcal{W})$ (Cauchy's Theorem).

To exhibit f as the L^1 boundary value of F , take M' as a neighborhood of p contained in

$$\bigcap_{v \in \mathcal{V}} \{A_{s,v}(e^{i\theta}) : s \in \mathcal{S}_1, \theta \in I_1\}.$$

Define $M'_r(v) = \{A_{s,v}(re^{i\theta}) : s \in \mathcal{S}_1, \theta \in I_1, A_{s,v}(e^{i\theta}) \in M'\}$. We have to show the uniform L^1 convergence $F|_{M'_r(v)} \rightarrow f|_{M'}$, as r tends to 1. For $\varepsilon > 0$, 1.9 shows that there is a polynomial P with $|P|_V - f|_V|_{L^1(V)} < \varepsilon$. We notice that $f = \lim_{\mu \rightarrow \infty} P_\mu$ in $L^1(U)$ also implies that $f \circ A_{s,v}(e^{i\theta})$ extends holomorphically to Δ for almost all $s \in \mathcal{S}$, i.e. $f \circ A_{s,v} \in H^1_a(\Delta)$. By 1.13, applied to $F - P$, which is possible, because $f \circ A_{s,v} \in H^1_a(\Delta)$, we get that for all r, v , the estimate $|P|_{M'_r(v)} - F|_{M'_r(v)}|_{L^1(M'_r(v))} < C\varepsilon$ holds true. By continuity of P , we can now choose $r_0 > 0$ such that, for $0 < r < r_0$, $|P|_{M'_r(v)} - P|_{M'}|_{L^1(M')} < \varepsilon$. This implies that $|F|_{M'_r(v)} - f|_{M'}|_{L^1(M')}$ can be made arbitrarily small (with the identifications of volume forms on M' and on $M'_r(v)$).

The proof of Proposition 1.12 is complete. \square

PROPOSITION 1.14. *Under the hypotheses of 1.12, let $W_2 = \{A_{s,0}(e^{i\theta}) \in M; 2s \in \mathcal{S}_1, 2\theta \in I_1\}$. Then, for each $\varepsilon > 0$, there exists a $\mathcal{C}^{2,\beta}$ -smooth deformation $M^d \subset M \cup \mathcal{W}$ of M with $W_2 \subset \text{supp } d \subset\subset V$ such that $\|M^d - M\|_{\mathcal{C}^{2,\beta}} < \varepsilon$ and there exists a function $f^d \in L^1_{loc,CR}(M^d) \cap \mathcal{H}(\mathcal{V}(W_2^d))$ such that $|f^d - f|_{L^1(V)} < \varepsilon$ and $f^d \equiv f$ in $M \setminus \text{supp } d$.*

Proof. Fix $\varepsilon > 0$ and define $W_1 = \{A_{s,0}(e^{i\theta}) \in M; s \in \mathcal{S}_1, \theta \in I_1\} \subset V$, which is a neighborhood of 0 in M . Let $r : \mathcal{S}_1 \times I_1 \rightarrow (r_1, 1]$ be a $\mathcal{C}^{2,\beta}$ -smooth function with $\mathcal{S}_1 \times I_1 \supset \text{supp } (1 - r) \supset \frac{1}{2}\mathcal{S}_1 \times \frac{1}{2}I_1$ and consider the deformation M^d of M with support in W_1 for which

$$W_1^d = \{A_{s,0}(r(s, \theta)e^{i\theta}); s \in \mathcal{S}_1, \theta \in I_1\} \subset \mathcal{W} \cup M.$$

Let $f \in L^1_{loc,CR}(M)$. Since f extends holomorphically into the wedge \mathcal{W} in 1.12, one can define f^d to be equal to f in $M \setminus \text{supp } d$ and to be the restriction $F|_{W_1^d}$ of the holomorphic extension F of f to \mathcal{W} . By a similar reasoning as in 1.12, assume $f = \lim_{\nu \rightarrow \infty} P_\nu$ in $L^1(V)$ as in 1.9. Then $F|_{W_1^d} = f^d = \lim_{\nu \rightarrow \infty} P_\nu|_{W_1^d}$ in $L^1(W_1^d)$, since $\int_{W_1^d} |P_\nu - P_\mu| \leq C \int_{W_1} |P_\nu - P_\mu|$. Now, we have

$$|f^d - f|_{L^1(W_1)} < |f - P_\nu|_{L^1(W_1)} + |P_\nu|_{W_1^d} - P_\nu|_{W_1}|_{L^1} + |P_\nu - f^d|_{L^1(W_1^d)}.$$

The first term is estimated in terms of the approximation theorem. The third term is estimated by Carleson's embedding theorem. According to Carleson's embedding theorem (see [22], p. 181), given a function $r(\theta) \in \mathcal{C}^1$ such that $r \equiv r_1$ in $(-\theta_1, \theta_1)$, $0 < r_1 < 1$ and $\text{supp } (1 - r) \subset (-\theta_0, \theta_0)$, $0 < \theta_1 < \theta_0 < \pi$, there exists a constant $C_1 > 0$ such that

$$\forall f \in H^1_a(\Delta), \quad \int_{-\theta_0}^{\theta_0} f(re^{i\theta})d\theta \leq C_1|f|_{L^1(b\Delta)}$$

and C_1 depends only on r . Applying therefore this inequality to each disc and choosing d sufficiently small in order that $|P_\nu|_{W_1^d} - P_\nu|_{W_1}|_{L^1} < \varepsilon$, we get $|f^d - f|_{L^1(W_1)}$ arbitrarily small.

The proof of Proposition 1.14 is complete. \square

Remark. In 1.12, we had $M'_r(v) \subset \mathcal{W}$ being small approach manifolds *not glued* to M and we had that $f|_{M'_r(v)} \rightarrow f|_{M'}$ in L^1 followed by a trivial subharmonic inequality, whereas Carleson's estimate was needed in 1.14 for the smoothing and gluing of $M'_r(v)$ to M \square

The following is a consequence of the minimalization process given by successive deformations of discs along the orbits as in [49] or [20].

PROPOSITION 1.15. [32]. *Let M be generic, let $p \in M$ and assume that M is globally minimal at p . Then, for each $\varepsilon > 0$, each $0 < \beta < \alpha$, there exists a compactly supported $\mathcal{C}^{2,\beta}$ deformation M^d of M with $M^d = M$ in a neighborhood of p in M , $\|M^d - M\|_{\mathcal{C}^{2,\beta}} < \varepsilon$, such that*

- 1) *There exists a $\mathcal{C}^{2,\beta}$ regular family of analytic discs $A_{s,v}$ attached to M^d at p ;*
- 2) *There exists a bounded linear extension operator*

$$L^1_{loc,CR}(M) \rightarrow L^1_{loc,CR}(M^d), \quad f \mapsto f^d, \quad f^d = f \text{ on } M \cap M^d;$$

- 3) *Moreover, given $f \in L^1_{loc,CR}(M)$, there exists such a deformation with $|f^d - f|_{L^1} < \varepsilon$.*

Then Propositions 1.14 and 1.12 show that the regular family $A_{s,v}$ given in 1.15 enables one to extend f with L^1 control 2ε at p into the wedge of edge M at p defined by $A_{s,v}$. By means of the above result on the existence of regular families of discs, and Proposition 1.14, we now can prove the main result of Section 1. We recall that Proposition 1.16 below will be suitable to apply the continuity principle and that, as explained in 1.7, it reduces the problem to considering $L^1 M_{loc} \cap \mathcal{H}(\mathcal{V}(M \setminus N))$ instead of simply $L^1_{loc}(M) \cap L^1_{loc,CR}(M \setminus N)$. The proof is technical and consists in a great number of deformations of M into wedges defined by regular families of analytic discs attached to M and attached to its subsequent deformations and in applying the estimates given by Propositions 1.14 and 1.15.

PROPOSITION 1.16. *Let M be generic, $\mathcal{C}^{2,\alpha}$ -smooth, let $\Phi \subset M$ be a proper closed subset of M , assume that $M \setminus \Phi$ is globally minimal, let $f \in L^1_{loc,CR}(M \setminus \Phi) \cap L^1_{loc}(M)$ and let U be an open subset of $M \setminus \Phi$ whose closure is compact in M . Then, for each $\varepsilon > 0$, there exists a $\mathcal{C}^{2,\beta}$ deformation d , $M^d \supset \Phi$, of M with $\text{supp } d = \bar{U}$, $\|M^d - M\|_{\mathcal{C}^{2,\beta}} < \varepsilon$, such that there exists a function*

$$f^d \in L^1_{loc}(M^d) \cap L^1_{loc,CR}(M^d \setminus \Phi) \cap \mathcal{H}(\mathcal{V}(U^d)),$$

coinciding with f on $M \setminus (U \cup \Phi)$ and such that $|f^d - f|_{L^1(U)} < \varepsilon$.

Proof. Consider an exhaustion of U by compact sets

$$U = \bigcup_{\nu=0}^{\infty} K_{\nu}, \quad K_{\nu} \subset \overset{\circ}{K}_{\nu+1}.$$

Since global minimality is a stable property under smooth deformations vanishing sufficiently fast at infinity, there exists a decreasing sequence $(\delta_{\nu})_{\nu \in \mathbb{N}}$ of positive numbers, $0 < \delta_{\nu+1} \leq \delta_{\nu}$, such that for every $\mathcal{C}^{2,\beta}$ deformation d of M with $\|M^d - M\|_{\mathcal{C}^{2,\beta}(K_{\nu} \setminus \overset{\circ}{K}_{\nu-1})} < \delta_{\nu}$, $\nu = 0, 1, \dots$, $K_{-1} = \emptyset$, $M^d \setminus \Phi$ is globally minimal.

We shall assume by induction on ν that, for each sequence $0 < \varepsilon_{\nu+1} \leq \varepsilon_{\nu} \leq \dots \leq \varepsilon_1 \leq \varepsilon_0 \leq \varepsilon$, there exists a deformation d_{ν} of M , $M^{d_{\nu}}$, such that $K_{\nu} \subset \text{supp } d_{\nu} \subset \overset{\circ}{K}_{\nu+1}$, $\|M^{d_{\nu}} - M\|_{\mathcal{C}^{2,\beta}(K_j \setminus \overset{\circ}{K}_{j-1})} < \varepsilon_j$, $j = 0, \dots, \nu + 1$ and there exists a function

$$f^{d_{\nu}} \in L^1_{loc,CR}(M^{d_{\nu}}) \cap \mathcal{H}(\mathcal{V}(K_{\nu}^{d_{\nu}}))$$

such that $|f^{d\nu} - f|_{L^1(U)} < \varepsilon$.

For $\nu = 0$, this is a consequence of 1.14, if one chooses $K_0 = \overline{W}_2^0$ and $K_1 = \overline{W}_1^0$, changing only the first two compact sets in the exhaustion if necessary.

Fix $\varepsilon > 0$ and $0 < \varepsilon_{\nu+2} \leq \varepsilon_{\nu+1} \leq \varepsilon_\nu \leq \dots \leq \varepsilon_1 \leq \varepsilon_0 \leq \varepsilon/2$ arbitrary, with $\varepsilon_j \leq \delta_j$, $j = 0, \dots, \nu + 2$. Let $d_\nu, M^{d\nu}, f^{d\nu}$ be as in the induction hypothesis, with $\varepsilon/2$ in place of ε and

$$\|M^{d\nu} - M\|_{\mathcal{C}^{2,\beta}(K_{\nu+1} \setminus \overset{\circ}{K}_\nu)} < \varepsilon_{\nu+1}/2, \quad \|M^{d\nu} - M\|_{\mathcal{C}^{2,\beta}(K_\nu \setminus \overset{\circ}{K}_{\nu-1})} < \varepsilon_\nu/2,$$

and $\|M^{d\nu} - M\|_{\mathcal{C}^{2,\beta}(K_j \setminus \overset{\circ}{K}_{j-1})} < \varepsilon_j$, for $j = \nu - 1, \dots, 0$. Then $|f^{d\nu} - f|_{L^1(U)} \leq \varepsilon/2$.

$M^{d\nu} \setminus \Phi$ is globally minimal. By compactness of $K_{\nu+1}^{d\nu} \setminus (\overset{\circ}{K}_\nu)^{d\nu}$, there exists a finite number $\mu = \mu(\nu)$ of points $p_1, \dots, p_\mu \in K_{\nu+1}^{d\nu} \setminus (\overset{\circ}{K}_\nu)^{d\nu}$ and there exist regular families $A_{s_1, v_1} : \mathcal{S}^1 \times \mathcal{V}^1 \times \overline{\Delta} \rightarrow \mathbb{C}^{m+n}, \dots, A_{s_\mu, v_\mu} : \mathcal{S}^\mu \times \mathcal{V}^\mu \times \overline{\Delta} \rightarrow \mathbb{C}^{m+n}$ of analytic discs attached to suited deformations of $M^{d\nu}$ as in 1.15, with $A_{s_j, v_j}(1) = p_j$, such that $A_{s_j, 0}(e^{i\theta})$ embeds $\mathcal{S}_1^j \times I_1^j$ onto $W_1^j = \{A_{s_j, 0}(e^{i\theta}) \in M^{d\nu}; s_j \in \mathcal{S}_1^j, \theta \in I_1^j\}$ and such that

$$K_{\nu+2}^{d\nu} \setminus (\overset{\circ}{K}_{\nu-1})^{d\nu} \supset W_3^1 \cup \dots \cup W_3^\mu \supset K_{\nu+1}^{d\nu} \setminus (\overset{\circ}{K}_\nu)^{d\nu},$$

for some $S_1^j \subset \subset S^j, I_1^j \subset \subset I^j, W_2^j = \{A_{s_j, 0}(e^{i\theta}) \in M^{d\nu}; 2s_j \in \mathcal{S}_1^j, 2\theta \in I_1^j\}, W_3^j = \{A_{s_j, 0}(e^{i\theta}) \in M^{d\nu}; 3s_j \in \mathcal{S}_1^j, 3\theta \in I_1^j\}$. Set also $\mathcal{S}_2^j = \frac{1}{2}\mathcal{S}_1^j, I_2^j = \frac{1}{2}I_1^j, \mathcal{S}_3^j = \frac{1}{3}\mathcal{S}_1^j, I_3^j = \frac{1}{3}I_1^j$.

Moreover, we can assume that there exists $\gamma_\nu > 0$ such that for every $\mathcal{C}^{2,\beta}$ deformation $(M^{d\nu})^d$ of $M^{d\nu}$ with $\text{supp } d \subset K_{\nu+2}^{d\nu} \setminus K_{\nu-1}^{d\nu}$, $\|(M^{d\nu})^d - M^{d\nu}\|_{\mathcal{C}^{2,\beta}(K_{\nu+2}^{d\nu} \setminus (\overset{\circ}{K}_{\nu-1})^{d\nu})} < \gamma_\nu$, and the support of d not meeting the part where A_{s_j, v_j}^d is not attached to $M^{d\nu}$ (as in 1.15), then the deformed discs A_{s_j, v_j}^d exist and give regular families. Indeed, Bishop's equation allows small perturbations.

Let φ_3^1 be a smooth function on $\mathcal{S}_1^1 \times I_1^1$ such that $\text{supp}(1 - \varphi_3^1) \subset \subset \mathcal{S}_1^2 \times I_1^2, \varphi_3^1 \equiv r_1$ near $\mathcal{S}_3^1 \times I_3^1$, with $r_1 < 1$ very close to 1 and φ_3^1 very close to the constant r_1 in \mathcal{C}^∞ norm. We set a deformation of $M^{d\nu}$ to be $(M^{d\nu})^{d_\nu^1} = M^{d\nu}$ outside W_1^1 and

$$(M^{d\nu})^{d_\nu^1} = \{A_{s_1, 0}(\varphi_3^1(s_1, \theta_1)e^{i\theta_1}) : s_1 \in \mathcal{S}_1^1, \theta_1 \in I_1^1\},$$

We can assume $\varepsilon_{\nu+2} \leq \gamma_\nu$. Proposition 1.14 with $\varepsilon = \varepsilon_{\nu+2}/4$, implies that we can choose r_1 and φ_3^1 close to 1 in order that

$$\|(M^{d\nu})^{d_\nu^1} - M\|_{\mathcal{C}^{2,\beta}(K_{\nu+1} \setminus \overset{\circ}{K}_\nu)} \leq \varepsilon_{\nu+1}/2 + \varepsilon_{\nu+2}/4 \leq \varepsilon_{\nu+1}/2 + \varepsilon_{\nu+1}/4,$$

$$\|(M^{d\nu})^{d_\nu^1} - M\|_{\mathcal{C}^{2,\beta}(K_\nu \setminus \overset{\circ}{K}_{\nu-1})} \leq \varepsilon_\nu/2 + \varepsilon_{\nu+2}/4 \leq \varepsilon_\nu/2 + \varepsilon_\nu/4,$$

and such that there exists $(f^{d\nu})^{d_\nu^1} \in L_{loc, CR}^1((M^{d\nu})^{d_\nu^1})$, such that

$$|f - (f^{d\nu})^{d_\nu^1}|_{L^1(U)} < |f - f^{d\nu}|_{L^1(U)} + |f^{d\nu} - (f^{d\nu})^{d_\nu^1}|_{L^1(U)} < \varepsilon/2 + \varepsilon_{\nu+2}/4 < \varepsilon/2 + \varepsilon/4.$$

To make a second (and decisive) step, let A_{s_2, v_2} and W_1^2, W_2^2, W_3^2 be as above. We denote by $\psi_1 : (s_1, \theta_1) \mapsto A_{s_1, 0}(e^{i\theta_1}), \psi_2 : (s_2, \theta_2) \mapsto A_{s_2, 0}(e^{i\theta_2})$. Choose a smooth function χ on $\mathcal{S}_1^1 \times I_1^1$ such that $\text{supp}(1 - \chi) \subset \subset \mathcal{S}_1^1 \times I_1^1, \chi \equiv \varphi_3^1$ near $\psi_1^{-1}(W_2^1 \cap W_2^2)$ and $\chi \equiv 1$ outside $\psi_1^{-1}(W_1^1 \cap W_1^2)$. Define

$$(M^{d\nu})^{d_\nu^{1/2}} = \{A_{s_1, 0}(\chi(s_1, \theta_1)e^{i\theta_1}) : (s_1, \theta_1) \in \mathcal{S}_1^1 \times I_1^1\},$$

and $(M^{d\nu})^{d\nu^{1/2}} = M^{d\nu}$ outside W_1^1 . By Proposition 1.14, we can choose χ in order that

$$\|(M^{d\nu})^{d\nu^{1/2}} - M\|_{\mathcal{C}^{2,\beta}(K_{\nu+1} \setminus \overset{\circ}{K}_\nu)} \leq \varepsilon_{\nu+1}/2 + \varepsilon_{\nu+2}/4 \leq \varepsilon_{\nu+1}/2 + \varepsilon_{\nu+1}/4,$$

$$\|(M^{d\nu})^{d\nu^{1/2}} - M\|_{\mathcal{C}^{2,\beta}(K_\nu \setminus \overset{\circ}{K}_{\nu-1})} \leq \varepsilon_\nu/2 + \varepsilon_{\nu+2}/4 \leq \varepsilon_\nu/2 + \varepsilon_\nu/4,$$

and such that there exists $(f^{d\nu})^{d\nu^{1/2}} \in L_{loc,CR}^1((M^{d\nu})^{d\nu^{1/2}})$ such that

$$|f - (f^{d\nu})^{d\nu^{1/2}}|_{L^1(U)} \leq \varepsilon/2 + \varepsilon/4.$$

Now, we consider the deformed regular family of discs $(A_{s_2, v_2})^{d\nu^{1/2}}$ attached to $(M^{d\nu})^{d\nu^{1/2}}$. The crucial fact is that $(M^{d\nu})^{d\nu^{1/2}}$ is not deformed where the discs A_{s_2, v_2} is not attached to $M^{d\nu}$, as stated in 1.15. Moreover, since $\varepsilon_{\nu+2}/4 \leq \gamma_\nu$, $(A_{s_2, v_2})^{d\nu^{1/2}}$ exists and gives a regular family. Choose a smooth function φ_3^2 such that $\text{supp}(1 - \varphi_3^2) \subset \mathcal{S}_2^2 \times I_2^2$, $\varphi_3^2 \equiv r_2 < 1$ near $\mathcal{S}_3^2 \times I_3^2$, and set

$$(M^{d\nu})^{d\nu^2} = \{(A_{s_2, v_2})^{d\nu^{1/2}}(\varphi_3^2(s_2, \theta_2)e^{i\theta_2}) : (s_2, \theta_2) \in \mathcal{S}_2^2 \times I_2^2\},$$

and $(M^{d\nu})^{d\nu^2} = M^{d\nu}$ outside $(W_2^2)^{d\nu^{1/2}}$. If φ_3^2 is sufficiently close to 1, we have

$$\|(M^{d\nu})^{d\nu^2} - (M^{d\nu})^{d\nu}\|_{\mathcal{C}^{2,\beta}(K_{\nu+1} \setminus \overset{\circ}{K}_\nu)} \leq \varepsilon_{\nu+2}/8 \leq \varepsilon_{\nu+1}/8,$$

$$\|(M^{d\nu})^{d\nu^2} - (M^{d\nu})^{d\nu}\|_{\mathcal{C}^{2,\beta}(K_\nu \setminus \overset{\circ}{K}_{\nu-1})} \leq \varepsilon_{\nu+2}/8 \leq \varepsilon_\nu/8,$$

and there exists $(f^{d\nu})^{d\nu^2} \in L_{loc,CR}^1((M^{d\nu})^{d\nu^2})$ such that

$$|(f^{d\nu})^{d\nu^2} - (f^{d\nu})^{d\nu}|_{L^1(U)} \leq \varepsilon/8.$$

It suffices to repeat the deformations with $j = 3, \dots, \mu$.

The proof of Proposition 1.16 is complete. \square

We end this section with an elementary proof of the following.

PROPOSITION 1.17. *Let M be generic, \mathcal{C}^2 -smooth, in \mathbb{C}^{m+n} , $\dim_{CR} M = m$, let $N \subset M$ be a generic two-codimensional \mathcal{C}^2 -smooth submanifold. If $m \geq 2$, N is L^1 -removable.*

Proof. Let $p \in N$, let $U \ni p$ be a neighborhood of p in M and choose two one-codimensional \mathcal{C}^2 -smooth submanifolds M_1, M_2 of M containing N near p such that $T_p M_1 + T_p M_2 = T_p M$. Let Υ_j , $j = 1, 2$, denote the set of \mathcal{C}^1 sections $L_j \in \Gamma(T^{0,1}U)$ such that $L_j|_{M_j} \in \Gamma(T^{0,1}M_j)$, $j = 1, 2$. Any $L \in \Gamma(T^{0,1}U)$ is a linear combination of a section L_1 of Υ_1 and a section L_2 of Υ_2 . For this, we use $m \geq 2$. Therefore, it suffices to show that, given $f \in L_{CR}^1(U \setminus N)$, then $L_j(f) = 0$ in the weak sense, for any $L_j \in \Upsilon_j$, $j = 1, 2$. Let $\varphi \in \mathcal{C}_c^\infty(U)$, let $\chi_{j,l} \in \mathcal{C}_c^\infty(U)$, $l \in \mathbb{N}$, $|\nabla \chi_{j,l}| \leq Cl$, with $\chi_{j,l} \equiv 1$ in a neighborhood of $M_j \cap \text{supp } \varphi$, and $\lim \chi_{j,l} = 0$ in $L^1(U)$, as $l \rightarrow \infty$. Then

$$\int_U {}^\tau L_j(\varphi)f = \int_{U \setminus M_j} {}^\tau L_j((1 - \chi_{j,l})\varphi)f + \int_{\mathcal{V}(M_j \cap U)} {}^\tau L_j(\chi_{j,l}\varphi)f = \int_{\mathcal{V}(M_j \cap U)} {}^\tau L_j(\chi_{j,l}\varphi)f,$$

since f is CR outside M_j , where the transposition is relative to a fixed measure on U . This last term tends to zero as l tends to infinity, since $L_j|_{M_j}$ is tangent to M_j , exactly as in the proof of Theorem 1.1. For this we need M_1, M_2 \mathcal{C}^2 .

The proof of Proposition 1.17 is complete. \square

2. Proofs of Theorem 4 (i), (ii) and (iii). Now, we come to the the general case. We plan to show first that these three theorems reduce to a single statement, namely Proposition 2.2 below. This is done as follows. Recall that a point $p \in \Phi$ is called *removable in $M \setminus \Phi$* (*removable*, for short) if there exists a small neighborhood V of p in M such that L^1_{CR} or \mathcal{W} functions over $M \setminus \Phi$ extend in the L^1_{CR} or \mathcal{W} sense over V .

LEMMA 2.1. (i) and (iii) in Theorem 4 reduce to (ii).

Proof. We have $m \geq 2$. Since $M \setminus N$ is globally minimal, Proposition 1.16 with 1.7 show that it suffices to prove that $L^1_{loc,CR}(M \setminus N) \cap \mathcal{H}(\mathcal{V}(M \setminus N)) \subset L^1_{loc,CR}(M)$. Let $p \in M$, $\dim_{CR} T_p N = m - 2$. By Proposition 1.17, p is L^1 -removable. An alternative argument is as follows. By the remark after Lemma 2.10 or by Theorem 5.A.1 in [28], p is \mathcal{W} -removable. By Proposition 2.11, this implies that p is L^1 -removable. This proves that it suffices to show that the proper closed subset $\Phi = \{p \in N : \dim_{CR} T_p N \geq m - 1\}$ is L^1 -removable. Clearly also, Theorem 4 (i) is implied by (ii).

The proof that (i) and (iii) reduce to (ii) in Theorem 4 is complete. \square

To prove Theorem 4 (ii), proceed as follows. Set

$$\mathcal{A} = \{\Psi \subset \Phi \text{ closed} : M \setminus \Psi \text{ is globally minimal}$$

$$\text{and } L^1_{loc,CR}(M \setminus \Phi) \cap L^1_{loc}(M) = L^1_{loc,CR}(M \setminus \Psi) \cap L^1_{loc}(M)\},$$

and

$$\mathcal{B} = \{\Psi \subset \Phi \text{ closed} ; M \setminus \Psi \text{ globally minimal and } \Phi \setminus \Psi \text{ is } \mathcal{W}\text{-removable in } M \setminus \Phi\}.$$

As \mathcal{A} (*resp.* \mathcal{B}) is closed under arbitrary intersections (obvious), it contains especially

$$\Psi_{nr} = \bigcap_{\Psi \in \mathcal{A}} \Psi \in \mathcal{A}, \quad (\text{resp. } \Psi_{nr} = \bigcap_{\Psi \in \mathcal{B}} \Psi \in \mathcal{B}).$$

Assume that $\Psi_{nr} \neq \emptyset$. We shall reach to a contradiction by proving that there is some point p_1 in Ψ_{nr} such that

1. There exists a section $L \in \Gamma(T^c M)$ near p_1 with $L_t(p_1) \in M \setminus \Psi_{nr}$, $0 < t < \delta$, for some $\delta > 0$ and

2. The point p_1 is L^1 - (*resp.* \mathcal{W} -) removable in $M \setminus \Psi_{nr}$.

Then by 1., if V denotes a small neighborhood of p_1 in M which is removable as in 2., we have that $(M \setminus \Psi_{nr}) \cup V$ is globally minimal and that $\Phi \setminus (V \cup (M \setminus \Psi_{nr}))$ is removable in $M \setminus \Phi$, contradicting the definition of \mathcal{A} or \mathcal{B} .

Denote from now on Ψ_{nr} again by $\Phi \neq \emptyset$.

Since $M \setminus \Phi$ is globally minimal, $L^1_{loc,CR}(M \setminus \Phi)$ extends holomorphically into a wedge at every point of $M \setminus \Phi$. Notice that we have derived in Proposition 1.16 (see also 1.7) that we can slightly deform M over $M \setminus \Phi$ in a manifold M^d in order that we are given the space $L^1_{loc,CR}(M^d \setminus \Phi) \cap \mathcal{H}(\mathcal{V}(M^d \setminus \Phi))$ for L^1 -removability, with L^1 -control, and $\mathcal{H}(\mathcal{V}(M^d \setminus \Phi))$ for \mathcal{W} -removability, which is a crucial reduction to apply the continuity principle. Since we can let d tend to 0 and since the construction of any wedge \mathcal{W}^d attached to M^d by means of discs will depend smoothly on d (see also Section 5 in [28]), it is sufficient to prove the following statement, a single proposition which implies therefore \mathcal{W} -removability in Theorem 4 (ii). L^1 -removability in Theorem 4 (ii) will be derived thereafter in Proposition 2.11 from Proposition 2.2. We rename such M^d as M .

PROPOSITION 2.2. *Let M be generic in \mathbb{C}^{m+n} , $\mathcal{C}^{2,\alpha}$, $\dim_{CR}M = m \geq 1$, let $N \subset M$ be a connected \mathcal{C}^2 -smooth submanifold with $\text{codim}_M N = 2$, let $\Phi \subset N$ be a proper closed subset, $\Phi \neq \emptyset$, assume that M and $M \setminus \Phi$ are globally minimal and let ω be a neighborhood of $M \setminus \Phi$ in \mathbb{C}^{m+n} . Then there exists a point $p_1 \in b\Phi$ such that there exists a section $L \in \Gamma(T^c M)$ near p_1 with $L_t(p_1) \in M \setminus \Phi$, $0 < t < \delta$, for some $\delta > 0$, and there exists a wedge \mathcal{W}_{p_1} at p_1 such that, for every function $f \in \mathcal{H}(\omega)$, there exists a function $F \in \mathcal{H}(\mathcal{W}_{p_1})$ with $F = f$ in the intersection of \mathcal{W}_{p_1} with a neighborhood of $M \setminus \Phi$ in \mathbb{C}^{m+n} .*

Proof. The proof will be given in four steps. Let us begin with the following. We consider the interior and the boundary of Φ as a subset of the manifold N .

LEMMA 2.3. *There exists a point $p_1 \in b\Phi$ such that there exists a section $L \in \Gamma(T^c M)$ near p_1 with $L_t(p_1) \in M \setminus \Phi$, $0 < t < \delta$, some $\delta > 0$, and such that there exists a generic one codimensional \mathcal{C}^2 -smooth manifold M_1 through p_1 with Φ contained in one closed side M_1^- of M_1 at p_1 , i.e. a side V^- of a small neighborhood V of p_1 in M divided by M_1 in two closed parts V^+ and V^- with $V^+ \cap V^- = V \cap M$.*

Proof. Assume first that there exists a point $p_0 \in b\Phi$ such that $T_{p_0}N \not\supset T_{p_0}^c M$. Then we easily construct a generic hypersurface $M_1 \subset M$ touching N exactly in p_0 and therefore containing $N \supset \Phi$ in one of its sides. Existence of L with $L_t(p_1) \in M \setminus \Phi$, $0 < t < \delta$ is trivial.

Assume now that $T_q N \supset T_q^c M$, for each point q in the closed subset $b\Phi \subset N$. By the *local CR orbit* $\mathcal{O}_{CR}^{loc}(p)$ of a point $p \in M$, we mean a representative of $\lim_{U \ni p} \mathcal{O}_{CR}(U, p)$ ([27]). Recall N is \mathcal{C}^2 -smooth. Choose a point $p_0 \in b\Phi$ such that $\mathcal{O}_{CR}^{loc}(p_0) \not\subset b\Phi$ and let U be a neighborhood of p_0 in M . (Otherwise, if $\mathcal{O}_{CR}^{loc}(q) \subset b\Phi$, $\forall q \in b\Phi$, $b\Phi$ contains a CR orbit, which contradicts the assumption that M and $M \setminus N$ are globally minimal). We have $T_{p_0}N \supset T_{p_0}^c M$. Let S be a two-dimensional submanifold of M through p_0 with $T_{p_0}S + T_{p_0}N = T_{p_0}M$. Let $\pi : M \rightarrow N$ be a submersion parallel to S in smooth (linear) coordinates on M such that S and N are coordinate spaces. For any vector field $L \in \Gamma(U, T^c M)$, one can define a vector field L_N on $N \cap U$ by taking $L_N(q) = \pi_*(L(q))$, $q \in N \cap U$. This defines a vector bundle $K \subset TN$ of rank $2m$ (this holds since $T_{p_0}N \supset T_{p_0}^c M$). Assume that $\mathcal{O}_K^{loc}(p_0) \subset b\Phi$. Since, by assumption, $K(q) = T_q^c M$, for each $q \in b\Phi$, this implies that $\mathcal{O}_{CR}^{loc}(p_0) \subset b\Phi$, which is not true. Therefore, $\mathcal{O}_K^{loc}(p_0) \not\subset b\Phi$ near p_0 .

Following Bony [9], we shall say that a vector field $L_N \in \Gamma(K)$ is tangent to the closed set $b\Phi$ near p_0 if, for any point $q \in U \cap N$, any open ball B with center q such that $B \subset N \setminus b\Phi$, then, for every point $p_1 \in bB \cap b\Phi$, $L_N(p_1)$ is tangent to bB at p_1 . We shall use the following theorem of Bony [9]. *If a Lipschitz real vector field X on \mathbb{R}^k is tangent in the above sense to a closed subset $F \subset \mathbb{R}^k$, then every integral curve meeting F stays in F .* By this theorem, the condition that $\mathcal{O}_K^{loc}(p_0) \not\subset b\Phi$ implies that there exists an open ball $B \subset N \setminus b\Phi$ such that $bB \cap b\Phi = \{p_1\}$ (is a single point, which holds true after a homothety with center p_1) and there exists a section $L \in \Gamma(U, K)$ such that $L(p_1) \notin T_{p_1}bB$. Choose \mathcal{C}^1 coordinates s_1, \dots, s_{2m+n} on M such that lines $(s_2, \dots, s_{2m+n}) = \text{const}$ correspond to integral curves of L . Since $p_1 \in b\Phi$, we can choose a point $q \in N \setminus \Phi$ close to p_1 . The set $\{L_{N,t}(q) : t \in I\}$, I an open interval with origin 0 in \mathbb{R} , uniform in q , i.e. the integral curve of L_N with origin q , meets Φ in $U \cap N$ or does not meet Φ . If it does meet Φ , it meets $b\Phi$ for a smallest $|t|$, say $t_0 < 0$ (changing L_N in $-L_N$ if necessary), so we get the existence of a new $p_0 = L_{N,t_0}(q) \in b\Phi$ such that $L_{N,t}(p_0) \in N \setminus \Phi$, $0 < t < \delta$, some $\delta > 0$. Since π is parallel to S and since $T_{p_0}^c M \subset T_{p_0}N$, this implies that $L_t(p_0) \in M \setminus \Phi$, $0 < t < \delta$. Assume on the contrary that, for each $q \in N \setminus \Phi$, the set $\{L_{N,t}(q) : t \in I\}$ does not meet Φ in $U \cap N$. As a consequence, $\{L_{N,t}(p_1) : t \in I\} \subset \Phi$. But since $p_1 \in b\Phi$, there exist

points $q \in N \setminus \Phi$ arbitrarily close to p_1 . Therefore, in fact, $\{L_{N,t}(p_1) : t \in I\} \subset b\Phi$, which is contrary to the fact that $B \subset N \setminus b\Phi$, $p_1 \in b\Phi$ and $L_N(p_1) \notin T_{p_1}bB$.

Fix a point $p_0 \in b\Phi$ as above with $L_{N,t}(p_0) \in N \setminus \Phi$, $0 < t < \delta$. Let $q = L_{N,t}(p_0) \in N \setminus \Phi$, some $t > 0$. We can assume that $s_1, s_2, \dots, s_{2m+n-2}$ give coordinates which are \mathcal{C}^1 on N (via a graphing function for N) and that the lines $s_2, \dots, s_{2m+n-2} = \text{const}$ correspond to integral curves of L_N . Consider, for $\varepsilon \geq 1$, the increasing family of domains with \mathcal{C}^1 -smooth boundary contained in N

$$D_N(q, \varepsilon) = \{(s_1 - s_1(q))^2/\varepsilon^2 + s_2^2 + \dots + s_{2m+n-2}^2 < r^2\},$$

where $r > 0$ is small enough to insure that $D_N(q, 1) \subset\subset N \setminus \Phi$. Then, for each $\varepsilon \geq 1$, a point $r \in bD_N(q, \varepsilon)$ satisfies $L_N(r)$ tangent to $D_N(q, \varepsilon)$ if and only if $s_1(r) = s_1(q)$, and this implies that $r \in N \setminus \Phi$. Therefore, there exists the unique least $\varepsilon_0 > 1$ such that $bD_N(q, \varepsilon_0) \cap \Phi = bD_N(q, \varepsilon_0) \cap b\Phi \neq \emptyset$ and for each p_1 (new p_1) in the intersection, one has $L(p_1)$ nontangential to $bD_N(q, \varepsilon_0)$. For \mathcal{C}^2 coordinates close in \mathcal{C}^1 norm to the coordinates (s_1, \dots, s_{2m+n}) , in this new geometric situation, we can repeat the argument of increasing balls and get the same result with $bD_N(q, \varepsilon_0)$ \mathcal{C}^2 . Now a part of the \mathcal{C}^2 -smooth boundary of the ball $bD_N(q, \varepsilon_0)$ can be included in a \mathcal{C}^2 -smooth manifold $M_1 \subset M$ of codimension one in M which is generic in \mathbb{C}^{m+n} and such that there exists $p_1 \in b\Phi \cap M_1$ and $\Phi \subset M_1^-$.

The proof of Lemma 2.3 is complete. \square

Step one: existence of a disc. Choose holomorphic coordinates (w, z) such that $p_1 = 0$, M_1 is a germ through p_1 of a generic one codimensional \mathcal{C}^2 -smooth submanifold of M with $T_0M_1 = \{x = 0, u_1 = 0\}$, $M = \{x = h(y, w)\}$ and M_1 is given in M by the supplementary equation $u_1 = k(v_1, w_2, \dots, w_m, y)$, $w_1 = u_1 + iv_1$, for a \mathcal{C}^2 -smooth function k with $k(0) = 0$ and $dk(0) = 0$. We note $M_1^- = \{u_1 \leq k(v_1, w_2, \dots, w_m, y)\}$.

LEMMA 2.4. *There exists an embedded analytic disc $A \in C^{2,\beta}(\overline{\Delta})$ with $A(1) = p_1$, $A(b\Delta) \setminus \{1\} \subset M \setminus M_1^-$ and $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) = v_0 \in T_{p_1}M_1$.*

Proof. For small $c > 0$, take $W_c(\zeta) = (c(1 - \zeta), 0, \dots, 0)$ and consider the analytic disc $A_c(\zeta) = (Z_c(\zeta), W_c(\zeta))$, where Z_c is the $\mathcal{C}^{2,\beta}$ solution of Bishop's equation $Y_c = T_1h(Y_c, W_c)$ on $b\Delta$. Then, for $\zeta \in b\Delta$,

$$U_1(\zeta) = \frac{c}{2}|1 - \zeta|^2, \quad |Z_c(\zeta)| \leq O(c|1 - \zeta|(c^\beta + |1 - \zeta|^\beta)),$$

and there exists $c_0 > 0$ and a constant $C > 0$, depending on the second derivatives of k in a ball containing 0 in $\mathbb{R} \times \mathbb{C}^{m-1} \times \mathbb{R}^n$, such that, for each $c \leq c_0$, $|k \circ A(\zeta)| < C(c^2|1 - \zeta|^2)$. Recall $M_1^+ = \{u_1 \geq k(v_1, w_2, \dots, w_m, y)\}$. Choose now c with $cC < \frac{1}{2}$.

The proof of Lemma 2.4 is complete. \square

Therefore, \mathcal{W} -removability of p_1 in $M \setminus \Phi$ is a consequence of the following.

PROPOSITION 2.5. *Let M be generic, $\mathcal{C}^{2,\alpha}$ -smooth, let $p_1 \in M$, let $N \ni p_1$ be a \mathcal{C}^2 submanifold with $\text{codim}_M N = 2$, let $\Phi \subset N$ be closed, let $p_1 \in b\Phi$ and assume that there exists a one codimensional generic \mathcal{C}^2 -smooth manifold $M_1 \subset M$ such that $\Phi \subset M_1^-$ and let ω be a neighborhood of $M \setminus N$ in \mathbb{C}^{m+n} . Let, as in 2.4, a sufficiently small embedded analytic disc $A \in C^{2,\beta}(\overline{\Delta})$ be attached to M , $A(1) = p_1$, $\frac{d}{d\theta}|_{\theta=0}A(e^{i\theta}) = v_0 \in T_{p_1}M_1$, with $A(b\Delta \setminus \{1\}) \subset M \setminus M_1^-$. Then for each $\varepsilon > 0$, there exist $v_{00} \in T_{p_1}M_1$ with $|v_{00} - v_0| < \varepsilon$, $v_{00} \notin T_{p_1}N$, $v_{00} \notin T_{p_1}^c M$ and a wedge \mathcal{W} of edge M at (p_1, Jv_{00}) such that for every holomorphic*

function $f \in \mathcal{H}(\omega)$ there exists a function $F \in \mathcal{H}(\mathcal{W})$ with $F = f$ in the intersection of \mathcal{W} with a neighborhood of $M \setminus N$ in \mathbb{C}^{m+n} .

Proof. Fix a function $f \in \mathcal{H}(\omega)$. The goal will be to construct deformations of our given original disc A as in [28] with boundaries in $M \cup \omega$ to show that the envelope of holomorphy of ω contains a (very thin) wedge of edge M at $A(1)$ (instead of appealing to a Baouendi-Treves approximation theorem, a version of which *by no means* being *a priori* valuable here, which is explained in the remarks before step five below), the natural tool being the so-called *continuity principle* 1.6.

We can assume that $A(1) = 0$ and that M is given in a coordinate system as in Lemma 1.12. Set $A(\zeta) = (X(\zeta) + iY(\zeta), W(\zeta))$. Since $v_0 \in T_0 M_1$, $\frac{d}{d\theta}|_{\theta=0} w_1(e^{i\theta})$ is purely imaginary.

The proof of Proposition 2.5 will be divided in four more steps. During the second one, we will introduce a large family of normal deformations of analytic discs, and during the third one, we will check and use the isotopy properties of this family. This is the most important step in our approach.

Step two: normal deformations. The following result shows that any disc can be included in a regular family.

PROPOSITION 2.6. *Let M be generic, $\mathcal{C}^{2,\alpha}$ -smooth, let $A \in \mathcal{C}^{2,\beta}(\overline{\Delta})$ be a sufficiently small analytic disc attached to M , $A(1) = p_0$, with $v_0 = \frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) \neq 0$ and let ω be a neighborhood of $A(-1)$ in \mathbb{C}^{m+n} . Then there exists a $\mathcal{C}^{2,\beta}$ -smooth family of analytic discs attached to $M \cup \omega$, $A_{t,\tau,a,p}(\zeta)$, with t in a neighborhood \mathcal{T} of 0 in \mathbb{R}^n , $\tau \in I_{\tau_0} = (-\tau_0, \tau_0)$, $\tau_0 > 0$, $a \in \mathcal{A} \subset \mathbb{C}^{m-1}$, \mathcal{A} a neighborhood of 0 in \mathbb{C}^{m-1} , with p in a neighborhood \mathcal{M} of p_0 in M , $A_{t,\tau,a,p}(1) = p$, such that the rank of the mapping $t \mapsto -\partial A_{t,0,0,p_0} / \partial \zeta(1)$ is equal to n and such that the set $\Gamma_0 = \{s \frac{d}{d\theta}|_{\theta=0} A_{t,\tau,a,p_0}(e^{i\theta}) : s > 0, t \in \mathcal{T}, \tau \in I_{\tau_0}, a \in \mathcal{A}\}$ is a $(2m+n)$ -dimensional cone with vertex p_0 in $T_{p_0} M$.*

Proof. We include a proof of this result which is crucial for the proof of Theorem 1. The main argument Lemma 2.7 below will be recalled here for completeness ([49], [28]): it is this availability of variation of the outer direction of discs as to describe a cone in $T_p \mathbb{C}^{m+n} / T_p M$ which underlies our propagation of removability process.

We can assume that $A(1) = 0$ in a coordinate system as (1), and that the projection of $v_0 = \frac{d}{d\theta}|_{\theta=0} A(e^{i\theta})$ on the v_1 -axis is non zero.

Let $\mu = \mu(y, w)$ be a \mathcal{C}^∞ , \mathbb{R} -valued function with support near the point $(y(-1), w(-1))$ that equals 1 there and let $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a \mathcal{C}^∞ function with $\kappa(0) = 0$ and $\kappa'(0) = Id$. We can assume that the supports of μ and κ are sufficiently concentrated in order that every manifold M_t with equation

$$x = H(y, w, t) = h(y, w) + \kappa(t)\mu(y, w)$$

is contained in ω and the deformation is localized in a neighborhood of $A(-1)$ in \mathbb{C}^{m+n} . Let $\chi = \chi(\zeta)$ be a smooth function on the unit circle supported in a small neighborhood of $\zeta = -1$ that will be chosen later. Set $A(\zeta) = (Z(\zeta), W(\zeta))$. For every small t , $t \in \mathcal{T}$, every small τ , $\tau \in I_{\tau_0}$, every $a \in \mathcal{A}$, every $p \in \mathcal{M}$, after denoting by $(y^0, w_1^0, \dots, w_m^0)$ the coordinates of p on M , we consider the disc

$$A_{t,\tau,a,p}(\zeta) = (X_{t,\tau,a,p}(\zeta) + iY_{t,\tau,a,p}(\zeta), e^{i\tau} W_1(\zeta) + w_1^0, W^*(\zeta) + a(\zeta - 1) + w^{*0}),$$

where $Y_{t,\tau,a,p}$ is the solution of Bishop's equation with parameters

$$Y_{t,\tau,a,p}(\zeta) = T_1 H(Y_{t,\tau,a,p}, e^{i\tau} W_1 + w_1^0, W^* + a(\cdot - 1) + w^{*0}, t\chi) + y^0,$$

which exists and depends in a $\mathcal{C}^{2,\beta}$ -smooth fashion on (t, τ, a, p, ζ) . Then $A_{t,\tau,a,p}(1) = p$. When $\tau = 0$, $a = 0$ and $p = p_0 = 0$, simply denote $A_{t,0,0,0}$ by A_t . We prove the crucial lemma 2.7 below, due to Tumanov, on normal deformations of discs: by pushing A into ω near $A(-1)$ along every direction given by t in the normal bundle to M at $A(-1)$, the inner tangential direction $-\frac{\partial A_t}{\partial \zeta}(1)$ will describe a *whole* open cone in the normal bundle to M at $A(1)$.

let Π denote the canonical bundle epimorphism $\Pi : T\mathbb{C}^{m+n}|_M \rightarrow T\mathbb{C}^{m+n}|_M/TM$ and consider the $\mathcal{C}^{1,\beta}$ mapping

$$(9) \quad D : \quad \mathbb{R}^n \ni t \longmapsto \Pi \left(-\frac{\partial A_t}{\partial \zeta}(1) \right) \in T_0\mathbb{C}^{m+n}/T_0M \simeq \mathbb{R}^n.$$

LEMMA 2.7. (TUMANOV [49]). χ can be chosen in order that $\text{rk } D'(0) = n$.

Proof. Set $t = (t_1, \dots, t_n)$. Differentiating the equation $X_t(\zeta) = H(Y_t(\zeta), W(\zeta), t\chi(\zeta))$, $\zeta \in b\Delta$, with respect to $t_j, j = 1, \dots, n$, we obtain that the holomorphic disc $\frac{\partial}{\partial t_j}|_{t=0}A_t(\zeta) = \dot{A}(\zeta) = (\dot{X}(\zeta) + i\dot{Y}(\zeta), 0)$ satisfies the following equation on the unit circle

$$(10) \quad \dot{X} = H_y \circ A\dot{Y} + \chi H_{t_j} \circ A.$$

We also introduce some notations. For a $\mathcal{C}^{1,\beta}$ -smooth function $g(\zeta)$ on the unit circle with $g(1) = 0$, we write

$$\mathcal{J}(g) = \frac{1}{\pi} \int_0^{2\pi} \frac{g(e^{i\theta})}{|e^{i\theta} - 1|^2} d\theta,$$

where the integral is understood in the sense of principal value. Then, if $g \in \mathcal{C}^{1,\beta}(\overline{\Delta})$ is holomorphic in Δ and vanishes at 1, we have

$$(11) \quad \mathcal{J}(g) = -\frac{\partial g}{\partial \zeta}(1) = i \frac{d}{d\theta}|_{\theta=0} g(e^{i\theta}).$$

Notice also that for $\mathcal{C}^{1,\beta}$ real-valued functions g, g' with $g(1) = g'(1) = 0$, applying (11) to the holomorphic function $(g + iT_1g)(g' + iT_1g')$ vanishing to second order at 1, we obtain

$$(12) \quad \mathcal{J}(gg' - T_1gT_1g') = 0.$$

Associate with A and H a $n \times n$ matrix-valued function $G(\zeta)$ on the unit circle as a solution to the equation

$$(13) \quad G = I - T_1(GH_y \circ A).$$

The definition of G implies that $G(1) = I$ and $T_1G = GH_y \circ A - H_y \circ A(1) = GH_y \circ A$, since $A(1) = 0$ and $h(0) = 0, dh(0) = 0$. Using (14) and G , we can write on the unit circle

$$\begin{aligned} G\chi H_{t_j} \circ A &= G(\dot{X} - H_y \circ A\dot{Y}) \\ &= G\dot{X} - (T_1G)(T_1\dot{X}) \\ &= \dot{X} + (G - I)\dot{X} - T_1(G - I)T_1\dot{X}. \end{aligned}$$

By virtue of (12),

$$(14) \quad \mathcal{J}(G\chi H_{t_j} \circ A) = \mathcal{J}(\dot{X}).$$

On the other hand, according to (11) and the fact that $\dot{Y} = T_1\dot{X}$,

$$(15) \quad -i\mathcal{J}(\dot{X}) + \mathcal{J}(T_1\dot{X}) = i\frac{\partial\dot{Z}}{\partial\zeta}(1) = \frac{d}{d\theta}|_{\theta=0}(\dot{X} + i\dot{Y}).$$

Identifying the real part of the two extreme terms and taking (10) into account, we have

$$(16) \quad \mathcal{J}(T_1\dot{X}) = \frac{d}{d\theta}|_{\theta=0}\dot{X} = \frac{d}{d\theta}|_{\theta=0}(H_y \circ A\dot{Y} + \chi H_{t_j} \circ A) = 0,$$

if we choose χ in order that χ is equal to zero near $\zeta = 1$ and since $\dot{Y}(1) = 0$, $dH(0) = 0$. (14), (15) and (16) therefore yield

$$-\frac{\partial\dot{Z}}{\partial\zeta}(1) = \mathcal{J}(G\chi H_{t_j} \circ A).$$

Natural coordinates on $T_0\mathbb{C}^{m+n}/T_0M$ being given by x_1, \dots, x_n , we obtain in these coordinates

$$(17) \quad \frac{\partial D}{\partial t_j}(0) = \Pi \left(-\frac{\partial\dot{Z}}{\partial\zeta}(1) \right) = \mathcal{J}(G\chi H_{t_j} \circ A).$$

Furthermore, choose χ in order that $\mathcal{J}(\chi) = 1$ and the support of χ is concentrated near $\zeta = -1$ so that the vectors $\mathcal{J}(G\chi H_{t_j} \circ A)$ are close to the vectors $G(-1)H_{t_j} \circ A(-1)$ and linearly independent, for $j = 1, \dots, n$ respectively. This is possible, since G is non singular at every point on the unit circle and the $H_{t_j} \circ A(-1)$, $j = 1, \dots, n$ are linearly independent by the choice of κ .

The proof of Lemma 2.7 is complete. \square

When $p = 0$, simply denote $A_{t,\tau,a,0}$ by $A_{t,\tau,a}$.

LEMMA 2.8. χ can be chosen in order that the following holds: there exist $\tau_0 > 0$, \mathcal{T} a neighborhood of 0 in \mathbb{R}^q and \mathcal{A} a neighborhood of 0 in \mathbb{C}^{p-1} such that the set

$$(18) \quad \Gamma_0 = \left\{ s \frac{dA_{t,\tau,a}}{d\theta}(1); s > 0, t \in \mathcal{T}, \tau \in I_{\tau_0}, a \in \mathcal{A} \right\}$$

is a $(2m+n)$ -dimensional open connected cone with vertex 0 in T_0M .

Proof. Indeed, the smooth mapping

$$E : \mathcal{T} \times I_{\tau_0} \times \mathcal{A} \ni (t, \tau, a) \mapsto \frac{d}{d\theta}|_{\theta=0} A_{t,\tau,a}(e^{i\theta}) \in T_0M$$

satisfies $\text{rk } E'(0) = 2m + n - 1$, for a choice of the function χ as in Lemma 1.11, since we defined $W_{t,\tau,a}(\zeta)$ not depending on t , and the partial rank of E with respect to (τ, a) is equal to $(2m - 1)$. Then the rank of $(s, t, \tau, a) \mapsto sE(t, \tau, a)$ is equal to $2m + n$.

The proof of Proposition 2.6 is complete. \square

It easy to see that if one chooses the disc A as in Proposition 2.6, then a subfamily of a slight modification of $A_{t,\tau,a,p}$ gives a regular family for $(0, M)$ in the sense of Definition 1.8.

Step three: isotopies. Choose a disc A_{t_1,τ_1,a_1} such that $v_{00} = \frac{d}{d\theta}|_{\theta=0} A_{t_1,\tau_1,a_1}(e^{i\theta}) \in T_0M_1$, $v_{00} \notin T_0N$, $v_{00} \notin T_0^cM$ and $|v_{00} - v_0| < \varepsilon$. This is possible, since the conification of the set

of tangential directions to discs in the family $A_{t,\tau,a}$ covers an open cone Γ_0 in T_0M . We now prove that isotopy properties are satisfied.

LEMMA 2.9. *There exist an open cone $\Gamma_1 \subset\subset \Gamma_0$, $v_{00} \in \Gamma_1$, a generic \mathcal{C}^2 -smooth one-codimensional submanifold K of M with $K \supset N$ near 0 and $\Gamma_1 \cap T_0K = \emptyset$ and there exists \mathcal{K} a neighborhood of 0 in K such that, if $\mathcal{P}_1 = \{(t, \tau, a) \in \mathcal{T} \times I_{\tau_0} \times \mathcal{A} : \frac{d}{d\theta} A_{t,\tau,a}(1) \in \Gamma_1\}$,*

- (i) *for each $p \in \mathcal{K}$ and $(t, \tau, a) \in \mathcal{P}_1$, $A_{t,\tau,a,p}(b\Delta) \cap \Phi \neq \emptyset$ if and only if $p \in \mathcal{K} \cap \Phi$,*
- (ii) *each $A_{t,\tau,a,p}$ for $p \in \mathcal{K} \setminus \Phi$ and $(t, \tau, a) \in \mathcal{P}_1$ is analytically isotopic to a point in ω .*

Proof. By shrinking all the open sets in the parameter space, (i) follows, since $v_{00} \notin T_0K$ and the embedded disc A_{t_1,τ_1,a_1} satisfies $A_{t_1,\tau_1,a_1}(b\Delta \setminus \{1\}) \subset M \setminus M_1^-$, and $\Phi \subset M_1^-$.

The second part is as follows.

(a) Each disc $A_{t,\tau,a,p}$ with $p \in \mathcal{K} \setminus \Phi$ is analytically isotopic in ω to $A_{t_1,\tau_1,a_1,p}$. Indeed, the discs $A_{t,\tau,a,p}$ can meet Φ only if $p \in \Phi$, by (i).

(b) The discs $A_{t_1,\tau_1,a_1,p}$ for different $p \in \mathcal{K} \setminus \Phi$ are analytically isotopic between each other in ω . Indeed, Φ does not divide \mathcal{K} near 0, since $\text{codim}_K N = 1$ and $0 \in b\Phi$. Then one can take a curve p_s , $0 \leq s \leq 1$, in $\mathcal{K} \setminus \Phi$ between two p_1, p_2 and all the discs A_{t_1,τ_1,a_1,p_s} do not meet Φ along their boundaries.

(c) For $u_1^0 > 0$, set $M_{1,u_1^0} = \{u_1 = u_1^0 + k(v_1, w_2, \dots, w_p, y)\}$ viewed in M . For small enough $u_1^0 > 0$, take $p_2 \in \mathcal{K} \cap M_{1,u_1^0}$. By (a), we can assume that after an isotopy, we have a disc A_{t_2,τ_2,a_2,p_2} such that $\frac{d}{d\theta} A_{t_2,\tau_2,a_2,p_2}(1) \in T_{p_2}M_{1,u_1^0}$. Now, push the disc away from M_{1,u_1^0} in the direction of the positive u_1 -axis by letting it be attached to M in the region $\{u_1 \geq u_1^0 + k(v_1, w_2, \dots, w_p, y)\} \cap M$. This can be done by using a single parameter family through Bishop's equation. When the disc is far enough from N , it is analytically isotopic to a point in ω .

The proof of Lemma 2.9 is complete. \square

Step four: holomorphic extension. Let $v_{00} \in C$ be a n -dimensional proper linear cone in the $(2m+n)$ -dimensional space T_0M and contained in Γ_1 such that the projection $T_0C \rightarrow T_0M/T_0^cM$ is surjective and $\overline{C} \cap T_0^cM = \{0\}$. Let \mathcal{P} denote the set of parameters

$$\mathcal{P} = \{(t, \tau, a) \in \mathcal{T} \times I_{\tau_0} \times \mathcal{V}; \frac{d}{d\theta} A_{t,\tau,a}(1) \in C\},$$

which is a \mathcal{C}^1 -smooth $(q-1)$ -dimensional submanifold of $\mathcal{T} \times I_{\tau_0} \times \mathcal{V}$. We choose a similar germ of a manifold, still denoted by \mathcal{P} , with same tangent space at 0 which is \mathcal{C}^2 . As in [28], one observes that a consequence of the isotopy property 1.16 and of the fact that the mapping

$$\mathcal{P} \times \mathcal{K} \times \overset{\circ}{\Delta}_1 \ni (t, \tau, a, p, \zeta) \mapsto A_{t,\tau,a,p}(\zeta) \in \mathbb{C}^{m+n} \setminus M$$

is a smooth embedding is that $\mathcal{H}(\omega)$ extends holomorphically into the open wedge set

$$\mathcal{W} = \{A_{t,\tau,a,p}(\zeta); (t, \tau, a) \in \mathcal{P}, p \in \mathcal{K}, \zeta \in \overset{\circ}{\Delta}_1\}$$

minus the set

$$\Phi_{\mathcal{P}} = \{A_{t,\tau,a,p}(\zeta) : (t, \tau, a) \in \mathcal{P}, p \in \mathcal{K} \cap \Phi, \zeta \in \overset{\circ}{\Delta}_1\}.$$

Indeed, since the mapping remains injective on $\mathcal{P} \times \mathcal{K} \setminus \Phi \times \overset{\circ}{\Delta}_1$, we can set unambiguously

$$F(z) := \frac{1}{2i\pi} \int_{b\Delta} \frac{f \circ A_{t,\tau,a,p}(\eta)}{\eta - \zeta} d\eta$$

as a value at points $z = A_{t,\tau,a,p}(\zeta)$ for an extension of $f|_{M\setminus\Phi}$, $p \in \mathcal{K}\setminus\Phi$, $(t, \tau, a) \in \mathcal{P}$, $\zeta \in \overset{\circ}{\Delta}_1$. Since f extends holomorphically to the interior of these discs, we get a continuous extension F on each $A_{t,\tau,a,p}(\Delta_1)$, $p \in \mathcal{K}\setminus\Phi$. Thus, the extension F of $f|_{M\setminus\Phi}$ also becomes continuous on

$$(\mathcal{W}\setminus\Phi_{\mathcal{P}}) \cup (M\setminus\Phi),$$

where $\Phi_{\mathcal{P}}$ is a proper closed subset of the one generic closed one codimensional submanifold of \mathcal{W}

$$N_{\mathcal{P}} = \{A_{t,\tau,a,p}(\zeta) : (t, \tau, a) \in \mathcal{P}, p \in \mathcal{K} \cap N, \zeta \in \overset{\circ}{\Delta}_1\}.$$

Since $f|_{M\setminus\Phi}$ extends analytically to a neighborhood of $A_{t,\tau,a,p}(\overline{\Delta})$, F is holomorphic in $\mathcal{W}\setminus\Phi_{\mathcal{P}}$. Indeed, fix a point $(\tilde{t}, \tilde{\tau}, \tilde{a}, \tilde{p}_0) \in \mathcal{P} \times (\mathcal{K}\setminus\Phi)$ and let $\tilde{\mathcal{P}} \times \tilde{\mathcal{K}}$ be a neighborhood of $(\tilde{t}, \tilde{\tau}, \tilde{a}, \tilde{p}_0)$ in $\mathcal{P} \times (\mathcal{K}\setminus\Phi)$ such that for each $(t, \tau, a, p) \in \tilde{\mathcal{P}} \times \tilde{\mathcal{K}}$, $A_{t,\tau,a,p}(\overline{\Delta})$ is contained in some neighborhood $\tilde{\omega}$ of $A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(\overline{\Delta})$ in \mathbb{C}^{m+n} such that there exists a holomorphic function $\tilde{f} \in \mathcal{H}(\tilde{\omega})$ with \tilde{f} equal to f near $A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(b\Delta)$. Let $\tilde{\zeta} \in \overset{\circ}{\Delta}_1$ and $\tilde{z} = A_{\tilde{t},\tilde{\tau},\tilde{a},\tilde{p}_0}(\tilde{\zeta})$. To check that the previously defined function F is holomorphic in a neighborhood of \tilde{z} , we note that for $z = A_{t,\tau,a,p}(\zeta)$, $(t, \tau, a, p) \in \tilde{\mathcal{P}} \times \tilde{\mathcal{K}}$, ζ in some neighborhood $\tilde{\Delta}_1$ of $\tilde{\zeta}$ in $\overset{\circ}{\Delta}_1$, $\tilde{f}(z)$ is given by the Cauchy integral formula

$$\tilde{f}(z) = \frac{1}{2i\pi} \int_{b\Delta} \frac{\tilde{f} \circ A_{t,\tau,a,p}(\eta)}{\eta - \zeta} d\eta = \frac{1}{2i\pi} \int_{b\Delta} \frac{f \circ A_{t,\tau,a,p}(\eta)}{\eta - \zeta} d\eta = F(z).$$

As a consequence, $\tilde{f}(z) = F(z)$ for z in a small neighborhood of \tilde{z} in \mathbb{C}^{m+n} , since the mapping $(t, \tau, a, p, \zeta) \mapsto A_{t,\tau,a,p}(\zeta)$ from $\tilde{\mathcal{P}} \times \tilde{\mathcal{K}} \times \tilde{\Delta}_1$ to \mathbb{C}^{m+n} has rank $2n$ at $(\tilde{t}, \tilde{\tau}, \tilde{a}, \tilde{p}_0, \tilde{\zeta})$.

This proves that F is holomorphic into $\mathcal{W}\setminus\Phi_{\mathcal{P}}$.

By shrinking ω near 0, which does not modify the possible disc deformations, we can insure that $\omega \cap \mathcal{W}$ is connected, since $\overline{\mathcal{C}} \cap T_0^c M = \{0\}$ and then also $\omega \cap (\mathcal{W}\setminus\Phi_{\mathcal{P}})$, since $N_{\mathcal{P}}$ is a closed one-codimensional submanifold of \mathcal{W} . Therefore $f \in \mathcal{H}(\omega)$ and $F \in \mathcal{H}(\mathcal{W}\setminus\Phi_{\mathcal{P}})$ stick together in a single holomorphic function in $\omega \cup (\mathcal{W}\setminus\Phi_{\mathcal{P}})$, since both are continuous up to $M\setminus\Phi$, which is a uniqueness set, and coincide there.

Recall the following result of Jöricke [21].

LEMMA 2.10. *Let $U \subset \mathbb{C}^{m+n}$ be an open set, $m+n \geq 2$, let $M \subset U$ be a connected \mathcal{C}^2 -smooth hypersurface and let $\Phi \subset M$ be a proper closed subset. If Φ does not contain any CR orbit of M , then, for every function $f \in \mathcal{H}(U\setminus\Phi)$, there exists a function $F \in \mathcal{H}(U)$ with $F|_{U\setminus\Phi} = f$.*

Proof. Let Φ_r denote the set of removable points of Φ and assume that $\Phi_{\text{nr}} = \Phi \setminus \Phi_r$ is nonempty. Replace Φ_{nr} by $\Phi \neq \emptyset$. By Lemma 2.3, which applies also in that situation, there exists a point $p_1 \in M\setminus\Phi$ such that half of M near p_1 is contained in $U\setminus\Phi$ and the half is divided by a complex tangential direction to M at p_1 . If M is not minimal at p_1 , M contains a germ through p_1 of a complex hypersurface $H \subset M$, with a half H^+ of H contained in $U\setminus\Phi$. Slightly deform M on that side to drop H , in order that M becomes minimal at p_1 . We can therefore assume that M is minimal at p_1 . By Trépreau's extension theorem, a side, say M^- , of M , has the holomorphic extension property at p_1 to the other side M^+ . This proves that f extends through Φ at p_1 . Indeed, the univalence is easy to see.

The proof of Lemma 2.10 is complete. \square

End of proof of Proposition 2.2. Recall that the set

$$N_{\mathcal{P}} = \{A_{t,\tau,a,p}(\zeta) : (t, \tau, a) \in \mathcal{P}, p \in \mathcal{K}, \zeta \in \overset{\circ}{\Delta}_1\}$$

is a closed \mathcal{C}^2 -smooth one-codimensional submanifold of \mathcal{W} and that $\Phi_{\mathcal{P}}$ is a proper closed subset of $N_{\mathcal{P}}$. Furthermore, the closed subset $N_{\mathcal{P}} \setminus (N_{\mathcal{P}} \cap \omega)$ of the generic \mathcal{C}^2 -smooth manifold $N_{\mathcal{P}}$ cannot contain any CR invariant subset. Indeed, each piece of complex curve $A_{t,\tau,a,p}(\overset{\circ}{\Delta}_1)$ meets $\omega \setminus N_{\mathcal{P}}$, since ω is a neighborhood of $M \setminus N$ in \mathbb{C}^{m+n} and $\frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) \notin T_{p_1} N$. Therefore, Lemma 2.10 yields that $\mathcal{H}((\mathcal{W} \setminus N_{\mathcal{P}}) \cup \omega)$ extends holomorphically into \mathcal{W} .

The proof of Proposition 2.2 is complete. \square

Remark. Assume that $T_{p_1} N$ is generic in $T_{p_1} \mathbb{C}^{m+n}$ and that $m \geq 2$. By a geometric adaptation of the approximation theorem given in [4], it is easy to prove that there exist two neighborhoods $U, V \subset \subset U$ of p_1 in M such that each function $f \in \mathcal{C}_{CR}^0(U \setminus N)$ can be uniformly approximated by holomorphic polynomials on compact subsets of $V \setminus N$ ([28], Proposition 5.B; the L_{CR}^1 case as in [21]). This shows that the continuity principle is not needed. Therefore, Proposition 2.2 holds, with $\Phi = N$ near a point where N is generic, if $\dim_{CR} M \geq 2$.

Remark. An adaptation of Baouendi-Treves approximation fails necessarily in many cases, e.g. when $m = 1$, $\Phi = \{p\}$. Indeed, let $U \ni p$ be a small neighborhood, let $V \subset \subset U$, let $q_1, q_2 \in V \setminus \{p\}$. One should find maximally real manifolds $L_{q_1} \subset U$, $L_{q_2} \subset U$ with $L_{q_1} \cap (U \setminus V) \equiv L_{q_2} \cap (U \setminus V)$ and a manifold $\Sigma \subset V$ with $b\Sigma = L_{q_1} - L_{q_2}$ such that $\Sigma \subset V \setminus \{p\}$ and this is clearly impossible for arbitrary choices of q_1, q_2 , since $\dim_{\mathbb{R}} L_{q_1} = \dim_{\mathbb{R}} M - 1 = 1 + n$, $\dim_{\mathbb{R}} \Sigma = \dim_{\mathbb{R}} M = 2 + n$. \square

The proof of \mathcal{W} -removability in Theorem 4 is complete now.

Step five: End of proof of Theorem 4, L^1 -removability. We got analytic extension of $L_{CR,loc}^1$ functions on $M \setminus N$ over M and to get L_{CR}^1 extension over M we need Hardy space like estimates for the extension as in the nonsingular case Proposition 1.12.

We assume the situation given as in the end of the proof of Proposition 2.2. We first include the disc given in 2.4 in a partial regular family by choosing parameters $s = (\rho, w^{*0}, y^0)$ running in a neighborhood \mathcal{S} of 0 in \mathbb{R}^{2m+n-1} , $w^{*0} = (w_2^0, \dots, w_m^0)$, $y^0 = (y_1^0, \dots, y_n^0)$, and by setting

$$A_s(\zeta) = (X_s(\zeta) + iY_s(\zeta), c(1 - (1 + \rho)\zeta), w^{*0}),$$

where Y_s is the solution of the equation

$$Y_s = T_1 h(Y_s, c(1 - (1 + \rho)\zeta), w^{*0}) + y^0.$$

This family satisfies the requirements of the following Proposition.

PROPOSITION 2.11. *Let M be generic, $\mathcal{C}^{2,\alpha}$ -smooth, let $p_1 \in M$, let N be a two codimensional submanifold of M with $p_1 \in N$ and let ω be a neighborhood of $M \setminus N$ in \mathbb{C}^{m+n} . Assume that there exists a family A_s , $s \in \mathcal{S}$, \mathcal{S} a neighborhood of 0 in \mathbb{R}^{2m+n-1} , of discs such that the mapping $\mathcal{S} \times b\Delta \ni (s, \zeta) \mapsto A_s(\zeta) \in M$ is an embedding and $A(b\Delta) \not\subset N$, $A_0 = A$, $A(1) = p_1$ and each A_s is analytically isotopic to a point. If $\mathcal{H}(\omega)$ extends holomorphically into a wedge \mathcal{W} of edge a small ball in M around p_1 containing $A(b\Delta)$, then p_1 is L^1 removable.*

Proof. Extend first A_s in a regular family $A_{s,v}$ by deforming A_s near a point $A(\zeta) \notin N$ (Proposition 2.6). Let

$$\mathcal{W}_A = \{A_{s,v}(\zeta) \in \mathbb{C}^{m+n} : (s, v, \zeta) \in \mathcal{S}_1 \times \mathcal{V}_1 \times \overset{\circ}{\Delta}_1\}$$

denote a wedge defined by $A_{s,v}$ at p_1 (Proposition 1.12). Introduce a one parameter family M^d , $d \geq 0$, of small smooth deformations of M into \mathcal{W} , construct the deformed family of analytic discs $A_{s,v}^d$ and isotope each of these discs to a point on M^d , which is possible, since the wedge has a big simply connected and round edge containing $A_{s,v}(b\Delta)$.

This family gives holomorphic extension of $\mathcal{H}(\omega \cup \mathcal{W})$ to \mathcal{W}_{A^d} for each d . By letting d tend to zero, the deformed wedge tends in a smooth fashion to \mathcal{W}_A , so we can control the connectedness of the intersections of the sets where the extensions are defined. By taking a smaller set of v parameters, we see that $\mathcal{H}(\omega \cup \mathcal{W})$ extends holomorphically into \mathcal{W}_A . (Notice that two wedges of edge M that are in general position have *empty* intersection.)

By Lemma A.6.4 in [10], the closed set $\mathcal{S}_N = \{s \in \mathcal{S} : A_s(b\Delta) \cap N \neq \emptyset\}$ is of Hausdorff dimension $2m + n - 2$.

Take a sequence of open neighborhoods $U_n \supset \mathcal{S}_N$ with $\text{Vol}(U_n) \rightarrow 0$ as $n \rightarrow \infty$ and a sequence of small nonnegative functions $\chi_n \in \mathcal{C}^2(\mathcal{S}_N)$, $\text{supp } \chi_n \subset U_n$, positive on \mathcal{S}_N , such that $\chi_n \rightarrow 0$ in $\mathcal{C}^2(\mathcal{S})$. Define

$$M_n = \bigcup_{\zeta \in b\Delta, s \in \mathcal{S}} A_s((1 - \chi_n(s))\zeta) \subset \mathcal{W}_A \cup \omega$$

and define $f_n \in CR(M_n)$ by restriction of the extension $f \in \mathcal{H}(\mathcal{W}_A \cup \omega)$. By subharmonicity of almost all functions $\zeta \mapsto A_s(\zeta)$ and integrating over $s \in U_n$ one gets

$$\int_{s \in U_n} \int_{-\pi}^{\pi} |f_n(A_s(1 - \chi_n(s))\zeta)| \, ds d\theta < \int_{s \in U_n} \int_{-\pi}^{\pi} |f \circ A_s(\zeta)| \, ds d\theta \rightarrow_{n \rightarrow \infty} 0.$$

So we conclude that $\lim_{n \rightarrow \infty} f_n = f$ in L^1 , hence that f is CR near p_1 , since each $f_n \in \mathcal{H}(\mathcal{V}(M_n))$.

The proof of Proposition 2.11 is complete. \square

The proof of Theorem 4 is complete now.

COROLLARY 2.12. *Let M be a real analytic generic manifold in \mathbb{C}^{m+n} of finite type at every point with $\dim_{CR} M = m \geq 2$. Then every connected real analytic submanifold $N \subset M$ with $\text{codim}_M N = 2$ is \mathcal{W} - and L^1 -removable if it does not consist of a CR manifold with $\dim_{CR} N = (m - 1)$.*

Proof. Since M is everywhere minimal, N cannot contain any (open) CR orbit of M , so the hypotheses of Theorem 4 (iii) are satisfied. \square

3. Metrically thin singularities. This paragraph is devoted to prove Theorem 2.

THEOREM 3.1. *Let M be a locally embeddable $\mathcal{C}^{2,\alpha}$ -smooth CR manifold, of dimension $d = 2m + n$, $\dim_{CR} M = m \geq 1$, let $1 \leq p \leq \infty$, and let E be a closed subset of M such that the Hausdorff measure $H_{d-3}(K) < \infty$ for each compact set $K \subset E$. Assume that for almost all CR orbits, $\mathcal{O}_{CR} \setminus E$ is globally minimal. Then E is L^p -removable.*

Proof. Fix a function $f \in L^1_{loc}(M) \cap L^1_{loc,CR}(M \setminus E)$. Call M_f the union of all CR orbits \mathcal{O}_{CR} of M such that $f|_{\mathcal{O}_{CR}}$ is locally integrable on \mathcal{O}_{CR} and CR on $\mathcal{O}_{CR} \setminus (\mathcal{O}_{CR} \cap E)$, such that $\mathcal{O}_{CR} \setminus (\mathcal{O}_{CR} \cap E)$ is globally minimal and such that $\mathcal{O}_{CR} \cap E$ is locally of Hausdorff codimension at least three in \mathcal{O}_{CR} . By a measure-theoretic lemma [10], combined with arguments in the proof of Theorem 1.1, M_f is of full measure. Therefore it is now sufficient to prove the theorem in the case that M and $M \setminus E$ are globally minimal.

Define

$$\mathcal{A} = \{\Psi \subset E \text{ closed; } M \setminus \Psi \text{ is globally minimal and } f \in L^1_{loc,CR}(M \setminus \Psi) \cap L^1_{loc}(M)\}$$

and define $E_{\text{nr}} = \cap_{\Psi \in \mathcal{A}} \Psi$. Then $M \setminus E_{\text{nr}}$ is globally minimal too. Use for E_{nr} the previous notation E . According to Proposition 1.16, we can assume that $f \in L^1_{\text{loc}}(M) \cap \mathcal{H}(\mathcal{V}(M \setminus E))$. Assume $E \neq \emptyset$. We shall reach a contradiction.

According to the idea of the proof of lemma 2.3, there exists a generic manifold M_1 of codimension one in M through a point $p_1 \in E$ such that $T_{p_1} M_1 \not\supset T_{p_1}^c M$ and $E \subset M_1^-$ locally. By the definition of \mathcal{A} and genericity of M_1 , it suffices to show that p_1 is L^1 -removable. Lemma 2.4 then provides an embedded analytic disc $A \in \mathcal{C}^{2,\alpha}(\overline{\Delta})$ with $A(1) = p_1$, $A(b\Delta \setminus \{1\}) \subset M \setminus M_1^-$ and $\frac{d}{d\theta}|_{\theta=0} A(e^{i\theta}) \in T_{p_1} M_1$. In a coordinate system as in (2) and as in 2.4, we can assume that $A(\zeta) = (Z(\zeta), W(\zeta))$, with $W(\zeta) = (\rho_1(\zeta - \rho_1), 0, \dots, 0)$, $\rho_1 > 0$. For the rest of the proof below, we only need a disc A with $A(1) \in E$, but $A(b\Delta) \not\subset E$.

We shall first developpe A in a partial regular family $A_{\rho,s'}$ of analytic discs, $0 \leq \rho < \rho_2$, $\rho_2 > \rho_1$, $I_{\rho_2} = (0, \rho_2)$, $s' = (a_2, \dots, a_m, y_1^0, \dots, y_n^0)$ running in a neighborhood of 0 in $\mathbb{C}^{m-1} \times \mathbb{R}^m$, as follows: $W_{\rho,a}(\zeta) = (\rho(\zeta - \rho_1), a_2, \dots, a_{m-1})$ and $A_{\rho,s'}(\zeta) = X_{\rho,s'}(\zeta) + iY_{\rho,s'}(\zeta), \rho(\zeta - \rho_1), a_2, \dots, a_{m-1})$, where $Y_{\rho,s'} = T_1 h(Y_{\rho,s'}, W_{\rho,s'}) + y^0$. Then there exists \mathcal{V} , a neighborhood of 0 in \mathbb{C}^{m-1} , \mathcal{Y} , a neighborhood of 0 in \mathbb{R}^n , such that the mapping

$$I_{\rho_2} \times \mathcal{A} \times \mathcal{Y} \times b\Delta \ni (\rho, a, y^0, \zeta) \mapsto A_{\rho,a,y^0}(\zeta) \in M$$

is an embedding. This shows that a neighborhood in M of $A_{0,0,0}(\overline{\Delta}) = p_1$ is foliated by \mathcal{C}^2 -smooth real discs $D_{a,y^0} = D_{s'} = \{A_{\rho,a,y^0}(\zeta) \in M : 0 \leq \rho < \rho_2, \zeta \in b\Delta\}$. Moreover, since $H_{d-3}(E) < \infty$, the set $\mathcal{S}'_E = \{s' \in \mathcal{S}' : D_{s'} \cap E \neq \emptyset\}$ is a closed subset of $\mathcal{S}' = \mathcal{A} \times \mathcal{Y}$ of Hausdorff codimension ≥ 1 . By construction, each disc $A_{\rho,s'}$ with $s' \notin \mathcal{S}'_E$ is therefore analytically isotopic to a point in $M \setminus E$.

Furthermore, by means of normal deformations of the family near $A(-1)$ as in Proposition 2.6, we can developpe A in a regular family $A_{\rho,s',v}$ which has the property that, for each $v \in \mathcal{V}$, the set $\mathcal{S}'_{E,v} = \{s' \in \mathcal{S}' : D_{s',v} \cap E \neq \emptyset\}$ is a closed subset of \mathcal{S}' of Hausdorff codimension ≥ 1 . Therefore, each disc $A_{\rho,s',v}$ with $A_{\rho,s',v}(b\Delta) \cap E = \emptyset$ is analytically isotopic to a point in $M \setminus E$, since $\mathcal{S}' \setminus \mathcal{S}'_{E,v}$ is dense and open in \mathcal{S}' .

Then the isotopy property and Proposition 1.6 imply that $\mathcal{H}(\mathcal{V}(M \setminus E))$ extends holomorphically into

$$\mathcal{W} = \{A_{\rho,a,y^0,v}(\zeta) \in \mathbb{C}^n : \rho \in I_{\rho_1}, a \in \mathcal{A}_1, y^0 \in \mathcal{Y}_1, v \in \mathcal{V}_1, \zeta \in \Delta_1\}$$

minus the set

$$E_{\mathcal{W}} = \{A_{\rho,s',v}(\zeta) \in \mathbb{C}^n : A_{\rho,s',v}(b\Delta) \cap E \neq \emptyset\}$$

for which $H_{2m+2n-2}(E_{\mathcal{W}}) < \infty$. If even $H_{2m+2n-3}(E_{\mathcal{W}}) < \infty$, then $E_{\mathcal{W}}$ is removable and we are done. Let $f \in \mathcal{H}(\omega)$ and let F denote its extension to $\mathcal{W} \setminus E_{\mathcal{W}}$.

By using a subharmonic estimate as in 1.12 and 1.13, we get $F \in L^1(\mathcal{W})$. If P is a polydisc into some \mathbb{C}^n , we call D a *coordinate disc* if D is the intersection of P with a coordinate line in \mathbb{C}^n .

LEMMA 3.2. *Let $p_0 \in \mathcal{W}$ and let $p_0 \in P \subset \mathcal{W}$ be a polydisc. Then, for almost every coordinate disc $D \subset P$, $D \cap E_{\mathcal{W}}$ is a union of isolated points and $F|_D$ is meromorphic with poles of order at most one.*

Proof. By Lemma A.6.4 from [10], for almost every coordinate disc D , $D \cap E_{\mathcal{W}}$ is of Hausdorff dimension zero, *i.e.* is a union of isolated points. Fubini's theorem implies that $F|_D \in L^1(D)$ for almost all D . To show meromorphicity, if $g \in \mathcal{O}(\Delta \setminus \{0\}) \cap L^1(\Delta)$, then $\bar{\partial}(zg) = 0$ in the distribution sense, which completes the proof.

We can then apply the following theorem of Shiffmann on separate meromorphicity to get that $F \in L^1(\mathcal{W}) \cap \mathcal{H}(\mathcal{W} \setminus E_{\mathcal{W}})$ is *meromorphic* on \mathcal{W} . A subset Q of a polydisc P is said to be a *full subset* of P if $Q \cap D$ is a set of full measure in D for almost every coordinate disc $D \subset P$.

THEOREM 3.3. (SHIFFMAN, [36]). *Let $P \subset \subset \mathbb{C}$ be a polydisc and let $Q \subset P$ be a full subset of P . Then a function $F : Q \rightarrow \mathbb{P}^1(\mathbb{C})$ has a meromorphic extension in P if and only if, for almost every coordinate disc $D \subset P$, $F|_{D \cap Q}$ extends meromorphically in D .*

Assume that F is not holomorphic in \mathcal{W} , *i.e.* the polar variety $P_F = \{p \in \mathcal{W}; F(p) \notin \mathbb{C}\}$, of pure dimension $(n-1)$, is nonempty. We shall prove that either $A_{s,v}(\overset{\circ}{\Delta}_1)$ does not intersect P_F or is contained in it.

LEMMA 3.4. *If $A_{s,v}(\overset{\circ}{\Delta}_1) \cap P_F \neq \emptyset$, then $A_{s,v}(\overset{\circ}{\Delta}_1) \subset P_F$.*

Proof. Indeed, if $A_{s,v}(\zeta_1) \in P_F$ and the intersection number of P_F with $A_{s,v}(\overset{\circ}{\Delta}_1)$ at $A_{s,v}(\zeta_1)$ is a finite positive number, P_F will intersect every nearby disc $A_{\tilde{s},\tilde{v}}(\overset{\circ}{\Delta}_1)$, contradicting the fact that $F \in \mathcal{H}(\mathcal{V}(A_{\tilde{s},\tilde{v}}(\overset{\circ}{\Delta}_1)))$ for almost all $(\tilde{s}, \tilde{v}) \in \mathcal{S} \times \mathcal{V}$.

End of proof of Theorem 3.1. Consider the manifolds with boundary near p_0 which foliate \mathcal{W} , $M_v = \{A_{s,v}(\zeta) \in \mathbb{C}^n : s \in \mathcal{S}_1, \zeta \in \overset{\circ}{\Delta}_1\}$, for $v \in \mathcal{V}_1$. Since the $(2m+2n-2)$ -dimensional measure of P_F is positive, for almost all $v \in \mathcal{V}_1$, the $(2m+n-1)$ -dimensional measure of $P_F \cap M_v$ is strictly positive. Fix $v_1 \in \mathcal{V}_1$ with that property. Then the set \mathcal{S}_2 of $s \in \mathcal{S}_1$ such that $A_{s,v_1}(\overset{\circ}{\Delta}_1) \subset P_F \cap M_{v_1}$ is of positive $(2m+n-3)$ -dimensional Hausdorff measure, because of Lemma 3.4. Since E is of Hausdorff codimension at least 3 in M and $\{A_{s_1,v_1}(b\Delta \cap \Delta_1) \in M; s_2 \in \mathcal{S}_2\}$ is of positive $(2m+n-2)$ -dimensional measure, there exists $s_2 \in \mathcal{S}_2$ such that $A_{s_2,v_1}(b\Delta \cap \Delta_1) \not\subset E$, say $A_{s_2,v_1}(\zeta_0) \notin E$. But $A_{s_2,v_1}(\overset{\circ}{\Delta}_1) \subset P_F$ and F holomorphic in a neighborhood of $A_{s_2,v_1}(\zeta_0)$ in \mathbb{C}^n gives the desired contradiction. Finally, the proof of Proposition 2.11, applies verbally to that situation and yields the desired L^1 -removability of E near p_1 .

The proof of Theorem 3.1 is complete. \square

4. Orbit decomposition. In this section, we shall prove the technical Theorem 6 about orbits. Since the result is a real differential geometric lemma, let us start as follows.

Let M be a real \mathcal{C}^2 -smooth manifold of dimension $m+n$, $n \geq 1$, $m \geq 2$ and let $K \subset TM$ be a \mathcal{C}^1 -smooth real subbundle of rank m . Given a section L of K , we denote by $\mathbb{R} \ni t \mapsto L_t(p)$ an integral curve of L with origin p , and call it a K -curve. According to the analysis of Sussmann [41], orbits $\mathcal{O}_K(M, p)$ under K of points $p \in M$ can be naturally equipped with a structure of a \mathcal{C}^1 -smooth manifold making the canonical inclusion $i : \mathcal{O}_K(M, p) \rightarrow M$ an injective immersion of class \mathcal{C}^1 .

By a *leaf* ω_p of $\mathcal{O}_K(M, p)$ through p , we shall mean an open small neighborhood of p in $\mathcal{O}_K(M, p)$ for the topology of the orbit, such that ω_p is a \mathcal{C}^1 -smooth submanifold of M too. By a K -integral manifold S , we shall mean a submanifold with the property that $T_q S \supset K(q)$, $\forall q \in S$. Orbits are immersed K -integral submanifolds.

Local K -orbits are defined to be the inductive limit, as U ranges through the set of open neighborhoods of p in M , of the K -orbit of p in U . Since the dimension of orbits is a well-defined integer ≥ 2 , these stabilize and define the unique germ through p of a K -integral submanifold of M with minimal possible dimension. We call M *K -minimal at p* if $\mathcal{O}_K^{loc}(p)$ is

an open neighborhood of p in M , *i.e.* has maximal possible dimension. We shall prove the following.

THEOREM 4.1. *Let M be a real \mathcal{C}^2 -smooth manifold, $\dim_{\mathbb{R}} M = m + n$, $m \geq 2$, $n \geq 1$, let $K \subset TM$ be a \mathcal{C}^1 subbundle of rank m , let $N \subset M$ be a \mathcal{C}^2 -smooth submanifold, $\text{codim}_M N \geq 2$, $T_p N \not\supset K(p)$, $\forall p \in N$, let $N^c = \{p \in M : \dim_{\mathbb{R}} K(p) \cap T_p N = m - 1\}$ and let Υ denote the set of \mathcal{C}^1 sections of K such that $Y|_{N^c}$ is tangent to N . If, for each $p \in N$, $\mathcal{O}_{\Upsilon}(M, p)$ is not contained in N , then every K -orbit of $M \setminus N$ is given by $\mathcal{O}_K \setminus N$, for some K -orbit of M .*

Proof. Theorem 4.1 can be reduced to two lemmas, 4.2 and 4.4 below.

LEMMA 4.2. *Let M, N be as in Theorem 4.1. Then, for each $q \in N$, there exists ω_q a K -integral manifold through q such that $T_q \omega_q \not\subset T_q N$, $\omega_q \cap N \subset H_q$, H_q a one codimensional closed submanifold of ω_q , and $\omega_q \setminus (\omega_q \cap N)$ is contained in a single K -orbit of $M \setminus N$.*

Proof. Remark that $\omega_q \cap N$ is thin in ω_q . Remark also $\mathcal{O}_K(M \setminus N, p) \subset \mathcal{O}_K(M, p) \setminus N$, if $p \in M \setminus N$ (obvious). Without loss of generality, we shall denote by $[0, t_1] \ni t \mapsto L_t(p)$ any piecewise smooth K -integral curve, as if there were a single smooth piece.

Since $\mathcal{O}_{\Upsilon}(M, q) \not\subset N$, there exists a point $q_0 \in \mathcal{O}_{\Upsilon}(M, q)$, $q_0 = L_{s_0}(q)$, such that $L_s(q) \in N$, $\forall s, 0 \leq s \leq s_0$ and $\mathcal{O}_{\Upsilon}^{loc}(M, q_0) \not\subset N$. Choose a germ of a \mathcal{C}^2 -smooth generic one codimensional manifold $M_1 \subset M$ through q_0 such that M_1 contains N . Then $K|_{M_1} \cap TM_1$ defines a \mathcal{C}^1 vector bundle K_1 of rank $m - 1$ and we have that $\mathcal{O}_{K_1}^{loc}(M_1, q_0) \not\subset N$. Indeed, otherwise, $\mathcal{O}_{K_1}^{loc}(M, q_0) = S$ satisfies $S \subset N$, $\dim_{\mathbb{R}}(T_q S \cap K(q)) = m - 1$, $\forall q \in S$, therefore $\mathcal{O}_{\Upsilon}^{loc}(M, q_0) \subset S \subset N$.

Choose a section L' of K near q_0 with $L'(q_0) \notin T_{q_0} N$. Let $q_1 = L_{t_1}(q_0) \in M_1 \setminus N$ be the endpoint of a piecewise smooth integral curve of K_1 -tangent vector fields and extend L in a neighborhood of M_1 in M . Choose a leaf ω_1 of $\mathcal{O}_K(M \setminus N, q_1)$ through q_1 . Set $\omega_t = L_{t-t_1}(\omega_1)$, $0 \leq t \leq t_1$, $\omega_{t_1} = \omega_1$. Then we have the following. Notice that since $T_{q_1} M_1 \not\supset K(q_1)$ and since $L'(q_1) \in T_{q_1} \omega_1$, then $T_{q_1} \omega_1 \not\subset T_{q_1} M_1$.

LEMMA 4.3. *For every t , $0 \leq t \leq t_1$, $T_{q_t} \omega_t \not\subset T_{q_t} N$, ω_t is a K -integral manifold and $\omega_t \setminus (\omega_t \cap N)$ is contained in a single K -orbit of $M \setminus N$.*

Proof. Since $L|_{M_1}$ is tangent to M_1 , the flow of L stabilizes M_1 . Therefore $T_{q_t} M_1 + T_{q_t} \omega_t = T_{q_t} M$, for every $0 \leq t \leq t_1$. This implies also that for any point $r \in \omega_1 \setminus N$, $L_{t-t_1}(r)$ does not meet N also. So $\omega_t \setminus M_1$ is a K -integral manifold, since everything flows in $M \setminus N$ and since we choosed ω_1 as a leaf of $\mathcal{O}_K(M \setminus N, q_1)$ through q_1 . Moreover, since $H_t = \omega_t \cap M_1$ is a one codimensional submanifold, the closure ω_t of $\omega_t \setminus H_t$ still is a K integral manifold of M . Since $\omega_t \setminus M_1$ is contained in a single K -orbit of $M \setminus N$, all points of $M_1 \cap \omega_t$ not in $N \cap \omega_t$ can be reached by means of L' , which completes the proof. \square

We have $q_0 = L_{s_0}(q)$. A repetition of 4.3 along $L_{s-s_0}(q_0)$, $0 \leq s \leq s_0$, gives 4.2.

The proof of Lemma 4.2 is complete. \square

Let $p \in M \setminus N$ and let $p_1 \in \mathcal{O}_K(M, p) \setminus N$. The following lemma shows $p_1 \in \mathcal{O}_K(M \setminus N, p)$.

LEMMA 4.4. *Let $p \in M \setminus N$, let $t \mapsto L_t(p)$ be piecewise smooth integral curve of K , set $p_t = L_t(p)$, $t \in I = [0, t_1]$. Then, for every $t \in I$, there exists a K -integral submanifold χ_t through p_t such that $\chi_t \setminus (\chi_t \cap N) \subset \mathcal{O}_K(M \setminus N, p)$.*

Proof. Let E denote the set of $t \in I$ such that, for every $s \leq t$, there exists a K -integral manifold χ_s through p_s with $\chi_s \setminus (\chi_s \cap N) \subset \mathcal{O}_K(M \setminus N, p)$. Then E contains $[0, \delta)$, for small $\delta > 0$, since $p \in M \setminus N$. E is open, by definition. Indeed, let $t \in E$ with $0 < t < t_1$, let χ_t be a K -integral manifold through p_t . For every small $\delta > 0$, $L_{\delta}(p_t) \in \chi_t$, so a neighborhood $\chi_{t+\delta}$ of p_t in χ_t satisfies $\chi_{t+\delta} \setminus (\chi_{t+\delta} \cap N) \subset \mathcal{O}_K(M \setminus N, p)$.

To prove closedness, let $t \leq t_1$ such that $s \in E$, for every $s < t$. Let $J = \{t \in I : p_t \in N\}$ and let $\overset{\circ}{J}$ denote the interior of J . Assume first that $(t - \delta, t)$ is contained in $\overset{\circ}{J}$, for small $\delta > 0$. Let ω_s be as in Lemma 4.2. We can replace χ_s by ω_s , for each $s \in (t - \delta, t)$. Indeed, notice that $\mathcal{O}_K^{loc}(p_s) \subset \chi_s \cap \omega_s$, since ω_s and χ_s are K -integral manifolds through p_s . But $\mathcal{O}_K^{loc}(p_s) \not\subset N$, since $T_{p_s}N \not\subset T_{p_s}^cM$. Since both $\chi_s \setminus (\chi_s \cap N)$ and $\omega_s \setminus (\omega_s \cap N)$ are contained in a single K orbit of $M \setminus N$, this yields that $\omega_s \setminus (\omega_s \cap N) \subset \mathcal{O}_K(M \setminus N, p)$. Then $\omega_t \setminus (\omega_t \cap N) \subset \mathcal{O}_K(M \setminus N, p)$ also, since $\omega_t = L_{\delta'}(\omega_{t-\delta'})$, $0 < \delta' < \delta$. Assume now that there exists $\delta > 0$ such that $L_{-\delta}(p_t) = p_s \notin N$. Since ω_t is a K -integral manifold, $L_{-\delta}(p_t) \in \omega_t$. But $\chi_s \subset \mathcal{O}_K(M \setminus N, p)$. So a neighborhood of p_s in ω_t is also contained in a single K orbit of $M \setminus N$, namely $\mathcal{O}_K(M \setminus N, p)$. It suffices to take $\chi_t = \omega_t$.

The proof of Theorem 4.1 is complete. \square

5. \mathcal{C}^λ peak sets. Here, we the constructions are simpler than those about global minimality and continuity principle, since they do not involve considerations of envelope of holomorphy. Let $S \subset M$ be a \mathcal{C}^λ peak set, $0 < \lambda < 1$, i.e. there exists a *nonconstant* function $\varpi \in \mathcal{C}_{CR}^\lambda(M)$ such that $S = \{\varpi = 1\}$ and $|\varpi| \leq 1$ on M . We use three lemmas from [24], after which Theorem 5 is an easy consequence. Recall that a regular family of analytic discs $A_{s,v}$ generates a wedge $\mathcal{W} = \mathcal{W}_{A,p}$.

LEMMA 5.1. ([24].) *Let M be $\mathcal{C}^{2,\alpha}$ globally minimal and let $p \in M$. If $A_{s,v}$ is a regular family of analytic discs at p , then for each \mathcal{C}^λ peak function $\varpi : M \rightarrow \mathbb{C}$:*

- 1) $|\varpi| < 1$ in $\mathcal{W}_{A,p}$ and $\varpi \circ A_{s,v} < 1$ on Δ ; and
- 2) $\varpi \circ A_{s,v} < 1$ almost everywhere on $b\Delta$.

Proof. Of course, we can assume that $S \neq \emptyset$. By the *maximum principle on discs*, $|\varpi \circ A_{s,v}(\zeta)| \leq 1$ on $\overline{\Delta}$, since $|\varpi \circ A_{s,v}|_{b\Delta} \leq |\varpi|_M \leq 1$. Hence $|\varpi| \leq 1$ in $\mathcal{W}_{A,p} = \{A_{s,v}(\zeta) : (s, v, \zeta) \in \mathcal{S}_1 \times \mathcal{V}_1 \times \overset{\circ}{\Delta}_1\}$. But $|\varpi|_S \equiv 1$, $S \neq \emptyset$, so, if $|\varpi| = 1$ at a point of $\mathcal{W}_{A,p}$ (or if $|\varpi \circ A_{s,v}(\zeta_0)| = 1$ at some $\zeta_0 \in \Delta$), then $\varpi \equiv 1$ in $\mathcal{W}_{A,p}$, again because of the maximum principle, so $\varpi|_M \equiv 1$, contradiction. \square

LEMMA 5.2. ([24], Lemma 3.) *Let $L = \sum_{j=1}^d a_j(x) \partial / \partial x_j$, $a_j(x) \in \mathcal{C}^1(\Omega, \mathbb{C})$, $\Omega \subset \mathbb{R}^r$ open, let $u \in \mathcal{C}^\lambda(\Omega)$, $0 < \lambda < 1$, be a solution of $Lu = 0$. Then for every $f \in L_{loc}^1(\Omega)$ such that $Lf = 0$ on $\Omega \setminus \{u = 0\}$, one has $L(u^{2/\lambda}f) = 0$ in Ω . \square*

LEMMA 5.3. ([24], Lemma 4.) *If $g \in H_a^1(\Delta)$, $u \in \mathcal{H}(\Delta)$, $\operatorname{Re} u^{\lambda/2} > 0$ in Δ and $g = uf$ on $b\Delta$ with $f \in L^1(b\Delta)$, then $g/u \in H_a^1(\Delta)$. \square*

Now, let $f \in L_{loc}^1(M) \cap L_{loc,CR}^1(M \setminus S)$ where $S = \{\varpi = 1\}$ is \mathcal{C}^λ peak. Let $p \in M$, $A_{s,v}$ regular at p . Then $g := (1 - \varpi)^{2/\lambda} f \in L_{loc,CR}^1(M)$ by Lemma 5.2, hence admits a holomorphic extension in $H_a^1(\mathcal{W}_{A,p})$, thanks to Proposition 2. But $u := (1 - \varpi)^{2/\lambda} \neq 0$ in $\mathcal{W}_{A,p}$ and $\operatorname{Re} u^{\lambda/2} = \operatorname{Re} (1 - \varpi) > 0$ and also $\operatorname{Re}(u \circ A_{s,v})^{\lambda/2} > 0$ in Δ thanks to Lemma 5, hence $(g/u) \circ A_{s,v} \in H_a^1(\Delta)$ by Lemma 7 and finally $g/u \in H_a^1(\mathcal{W}_{A,p})$. \square

6. Hypoanalytic structures. In this section, M will be a d -dimensional manifold, of class $\mathcal{C}^{k,\alpha}$, $k \geq 2$, $0 < \alpha < 1$, countable at infinity. A *hypoanalytic structure* on M ([45], [46]) means the data of an open covering $\{U^j\}_{j \in J}$ of M and, for each $j \in J$, the data of n complex-valued $\mathcal{C}^{k,\alpha}$ functions Z_1^j, \dots, Z_n^j in U^j with linearly independent differentials over U^j , with $n \geq 1$ independent of j , such that the following is satisfied: whenever $U^j \cap U^k \neq \emptyset$, there is a holomorphic mapping F_k^j from a neighborhood of $Z^j(U^j \cap U^k)$ in \mathbb{C}^n such that $Z^k = F_k^j \circ Z^j$ in $U^j \cap U^k$. The integer $m = d - n$ is called the *codimension* of the hypoanalytic structure.

Smooth *hypoanalytic functions* are complex-valued functions which are locally pullbacks by Z^j of holomorphic functions on open neighborhoods of $Z^j(U^j)$ in \mathbb{C}^n . We shall say that f is a *RC function* if $df \in \Gamma(T', U)$, where T' is defined as follows.

If (U, Z_1, \dots, Z_n) is a hypoanalytic local chart in M , the differentials dZ_1, \dots, dZ_n span a complex vector subbundle T'_U of $\mathbb{C}T^*U$. According to the compatibility condition, this defines a complex vector subbundle T' of $\mathbb{C}T^*M$ of rank n , called the *structure bundle*. Under the duality between tangent and cotangent vectors, it is equivalent to give the formally integrable complex vector subbundle $\mathcal{V} = (T')^\perp$ of $\mathbb{C}TM$ of rank m .

Let (U, Z_1, \dots, Z_n) be a hypoanalytic chart in M . Possibly after contracting U about one of its points p_0 , we can find m $\mathcal{C}^{k,\alpha}$ real-valued functions (y_1, \dots, y_m) in U such that $(dZ_1, \dots, dZ_n, dy_1, \dots, dy_m)$ span the whole cotangent space $\mathbb{C}T_p^*M$ at every point of U . We then define $m+n$ $\mathcal{C}^{k,\alpha}$ complex vector fields in U , L_1, \dots, L_m and M_1, \dots, M_n by $M_j Z_k = \delta_j^k$, $L_i y_l = \delta_l^i$, $M_j y_l = 0$, $L_i Z_k = 0$, $i, l = 1, \dots, m$, $j, k = 1, \dots, n$. Then the commutation relations $[L_i, L_l] = [L_i, M_j] = [M_j, M_k] = 0$ follow at once. Put $x_1 = \operatorname{Re} Z_1, \dots, x_n = \operatorname{Re} Z_n$. Then $(x_1, \dots, x_n, y_1, \dots, y_m)$ is a system of local real coordinates for M in the neighborhood U of p_0 . We can assume that the hypoanalytic functions Z_1, \dots, Z_n vanish at p_0 .

Let $r = n - \dim T^*M \cap T'[p_0]$. We can make a \mathbb{C} -linear transformation on the functions Z_1, \dots, Z_n and choose coordinates $(x_1, \dots, x_n, y_1, \dots, y_m)$ on a neighborhood U of p_0 with $x_j = \operatorname{Re} Z_j$, $j = 1, \dots, n$ and $y_k = \operatorname{Im} Z_k$, $k = 1, \dots, r$, such that

$$Z_k(x, y) = x_k + iy_k, \quad k = 1, \dots, r, \quad Z_{r+k}(x, y) = x_{r+k} + ih_{r+k}(x, y), \quad k = 1, \dots, n-r,$$

with $h_k(0, 0) = 0$, $dh_k(0, 0) = 0$, $k = r+1, \dots, n$.

Therefore, a basis of $(T')^\perp = \mathcal{V}$ is given by

$$(19) \quad L_j = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) + \sum_{k=r+1}^n a_{jk}(x, y) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, r,$$

$$L_{r+j} = \frac{i}{2} \frac{\partial}{\partial y_{r+j}} + \sum_{k=r+1}^n a_{r+j, k}(x, y) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, m-r,$$

with $a_{jk}(0) = 0$, the a_{jk} being determined by $L_j Z_k = 0$, $j = 1, \dots, m$, $k = 1, \dots, n$, and we have $L_j y_l = \frac{i}{2} \delta_l^j$. We are now prepared to speak of the local CR lifting of a hypoanalytic manifold.

The intersection $T^0 = T' \cap T^*M$, called the *characteristic set* of the differential operators in $\mathcal{V} = (T')^\perp$ is *not*, in general, a vector bundle, nor is $\operatorname{Re} \mathcal{V} = (T' \cap T^*M)^\perp$. However, it has the property that the integer-valued function $M \ni p \mapsto r(p) = n - \dim T^*M \cap T'[p]$ is lower semicontinuous.

Let $r = n - \dim T^*M \cap T'[p_0]$. Equivalently, $r = r(p_0)$ is the integer which satisfies $\dim \operatorname{Re} T'[p_0] = n + r(p_0)$.

6.1. DEFINITION. By a *regular hypoanalytic chart* (Z, y, U) at p_0 , we mean the data of a hypoanalytic chart $Z : U \rightarrow \mathbb{C}^n$, $U \ni p_0$, together with $m-r$ real-valued functions $y_{r+1}, \dots, y_m : U \rightarrow \mathbb{R}$, $r = \dim \operatorname{Re} T'[p_0] - n$, such that $(d\operatorname{Re} Z_1, \dots, d\operatorname{Re} Z_n, d\operatorname{Im} Z_1, \dots, d\operatorname{Im} Z_n, dy_1, \dots, dy_m)$ generate T_p^*M at every point $p \in U$. To a regular hypoanalytic chart (Z, y, U) is associated a local embedding of U in $\mathbb{C}^n \times \mathbb{R}^{m-r}$, $p \mapsto (Z(p), y(p))$, which is also a real chart on U .

If (Z, y, U) and (Z, \tilde{y}, U) are two regular hypoanalytic charts at p_0 , there exists a local diffeomorphism G at 0 in $\mathbb{C}^n \times \mathbb{R}^{m-r}$ of the form $G(z, y) = (z, G_y(z, y))$ such that $(Z(p), \tilde{y}(p)) =$

$(Z(p), G_y(Z(p), y(p)))$, so $(Z, \tilde{y})_* \mathcal{V}|_U = G_*(Z, y)_* \mathcal{V}|_U$. Therefore, the subbundle $(Z, y)_* \mathcal{V}|_U = \mathcal{V}'|_{U'} \subset \mathbb{C}T'$ does not depend on the choice of (y_1, \dots, y_{m-r}) .

Then the push forward by $(Z, y) = Z'$ of the vector fields given by (20) are the restriction to the manifold U' of the vector fields on $\mathbb{C}^n \times \mathbb{R}^{m-r}$, equipped with coordinates $(z_1, \dots, z_n, y_{r+1}, \dots, y_m)$, given by

$$(20) \quad L'_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=r+1}^n b_{jk}(z, \bar{z}, y) \frac{\partial}{\partial \bar{z}_k}, \quad j = 1, \dots, r,$$

$$L'_{r+j} = \frac{i}{2} \frac{\partial}{\partial y_{r+j}} + \sum_{k=r+1}^n b_{r+j \ k}(z, \bar{z}, y) \frac{\partial}{\partial \bar{z}_k}, \quad j = 1, \dots, m-r,$$

that are $L'_j|_{U'} = Z'_* L_j$, $j = 1, \dots, m$, with b_{jk} of class $\mathcal{C}^{k-1, \alpha}$ near 0 satisfying $b_{jk}(0) = 0$.

For every completion of the coordinate system (y_{r+1}, \dots, y_m) at a point $p_0 \in M$, $r = r(p_0)$, for which $(d\operatorname{Re} Z_1, \dots, d\operatorname{Re} Z_n, d\operatorname{Im} Z_1, \dots, d\operatorname{Im} Z_n, dy_{r+1}, \dots, dy_m)$ span T^*M in a neighborhood of p_0 , we introduce extra real variables $(x_{n+1}, \dots, x_{n+m-r})$ in \mathbb{R}^{m-r} and a mapping $\tilde{Z} : U \times V = \tilde{U} \ni (p, x) \mapsto \tilde{Z}(p, x) \in \mathbb{C}^{n+m-r}$, V a neighborhood of 0 in \mathbb{R}^{m-r} , defined by

$$\tilde{Z}_k(p, x) = Z_k(p), \quad k = 1, \dots, n, \quad \tilde{Z}_{n+k}(p, x) = x_{n+k} + iy_{r+k}(p), \quad k = 1, \dots, m-r.$$

Then the image $\tilde{Z}(\tilde{U}) = \tilde{M}$ is a germ of a CR generic submanifold in \mathbb{C}^{n+m-r} , equipped with coordinates $(z_1, \dots, z_n, w_{n+1}, \dots, w_{n+m-r})$, given by the equation

$$\operatorname{Im} z_k = h_k(z_1, \dots, z_r, w_{n+1}, \dots, w_{n+m-r}, \operatorname{Re} z_{r+1}, \dots, \operatorname{Re} z_m), \quad k = r+1, \dots, m.$$

We have a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{id} & U & \xrightarrow{i} & U \times V^{m-r} \\ z \downarrow & & \downarrow Z' & & \downarrow \tilde{Z} \\ \mathbb{C}^n & \xleftarrow{\pi_1} & \mathbb{C}^n \times \mathbb{R}^{m-r} & \xleftarrow{\pi_2} & \mathbb{C}^{n+m-r} \end{array}$$

Now, the vector fields $i_* L_1, \dots, i_* L_r, \frac{1}{2} \frac{\partial}{\partial x_{r+1}} + i_* L_{r+1}, \dots, \frac{1}{2} \frac{\partial}{\partial x_m} + i_* L_m$ on $U \times V^{m-r}$ span a formally integrable complex subbundle $\tilde{\mathcal{V}}$ of $\mathbb{C}T\tilde{U}$ such that the push-forward $\tilde{Z}_* \tilde{\mathcal{V}}$ becomes the bundle $T^{0,1} \tilde{M}$ of antiholomorphic tangent vector fields to \tilde{M} which does not depend on the choice of (y_1, \dots, y_{m-r}) . These vector fields are the restriction to \tilde{M} of the vector fields on $\mathbb{C}^n \times \mathbb{C}^{m-r}$ given by

$$(21) \quad \tilde{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=r+1}^n b_{r+j \ k}(z, \bar{z}, \operatorname{Im} w) \frac{\partial}{\partial \bar{z}_k}, \quad j = 1, \dots, r,$$

$$\tilde{L}_{r+j} = \frac{\partial}{\partial \bar{w}_{n+j}} + \sum_{k=r+1}^n b_{j \ k}(z, \bar{z}, \operatorname{Im} w) \frac{\partial}{\partial \bar{z}_k}, \quad j = 1, \dots, m-r.$$

6.2. PROPOSITION. *Let M be a smooth hypoanalytic manifold. Then, for every $p_0 \in M$, there exists (Z, y, U) a regular hypoanalytic chart at p_0 , there exists a neighborhood \mathcal{X} of 0 in \mathbb{R}^{m-r} , $r = n - \dim T' \cap T^*M[p_0]$, such that the mapping $U \times \mathcal{X} \rightarrow \mathbb{C}^{n+m-r}$ given by*

$(p, x) \mapsto (Z(p), x + iy(p))$ realizes $U \times \mathcal{X}$ as a generic submanifold of \mathbb{C}^{n+m-r} . Moreover, a function f is a RC function in some regularity class \mathcal{D} on U if and only if there exists a CR function $\tilde{F} \in \mathcal{D}_{CR}(\tilde{M})$ which is independent of $(\operatorname{Re} w_{n+1}, \dots, \operatorname{Re} w_{n+m-r})$ and $\tilde{F} \circ \tilde{Z} \circ i = f$.

$$\begin{array}{ccc}
 U \times \mathcal{X} & \xrightarrow{\tilde{Z}} & \mathbb{C}^n \times \mathbb{C}^{m-r} \\
 \uparrow i & & \downarrow \pi_2 \\
 U & \xrightarrow{Z'} & \mathbb{C}^n \times \mathbb{R}^{m-r} \\
 \downarrow Z & & \downarrow F' \\
 \mathbb{C}^n & \xrightarrow{F} & \mathbb{C}
 \end{array}$$

Proof. Prove the statement for the weakest class of functions. Namely, a distribution f is a RC distribution on U if and only if f is annihilated by $\mathcal{V}' = Z'_* \mathcal{V}$. Since the function $\tilde{F} = f \circ (\tilde{Z} \circ i)^{-1} = F' \circ \pi_2$ is by definition independent of the extra variables $(x_{n+1}, \dots, x_{n+m-r})$, \tilde{F} is a solution of \mathcal{V} , so is CR on \tilde{M} . The proof of Proposition 6.2 is complete. \square

6.3. DEFINITION. By a *local CR lifting* of a hypoanalytic structure, we mean datas like 6.2.

Let $p_0 \in M$. According to Sussmann, the $\operatorname{Re} \mathcal{V}$ -orbit of p_0 can be equipped with a structure of a $\mathcal{C}^{k-1, \alpha}$ manifold, possibly with a finer topology than that induced by M . If $\mathcal{O}_{\operatorname{Re} \mathcal{V}}(M, p_0)$ contains a neighborhood of p_0 in M , the propagation of analyticity method yields that $\tilde{F} = f \circ (\tilde{Z} \circ i)^{-1}$ extends holomorphically into a wedge of edge M at 0 in \mathbb{C}^{n+m-r} .

Indeed, the crucial fact which reduces this last statement to the generalized Tumanov's theorem ([20], [27], [43], [49]), is that changes of local CR liftings at p_0 behave correctly and always are possible, even if $r(p)$ changes, since all the extensions successively obtained will *never* depend on the extra variables. Therefore, extension of RC functions in the CR lifting sense, support propagation, propagation of the wave front set, are straightforward consequences of those results which hold true for CR structures. This idea goes back to Treves [45]. In conclusion, we can restate theorems 1 and 2 for hypoanalytic structures.

6.4. THEOREM. *Theorems 1 and 2 hold for $\mathcal{C}^{2, \alpha}$ hypoanalytic manifolds and RC functions.* \square

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