

METRICALLY THIN SINGULARITIES OF INTEGRABLE CR FUNCTIONS

JOËL MERKER AND EGMONT PORTEN

ABSTRACT. In this article, we consider metrically thin singularities E of the solutions of the tangential Cauchy-Riemann operators on a smoothly embedded Cauchy-Riemann manifold M (CR functions on $M \setminus E$). Our main result establishes the removability of E within the space of locally integrable functions on M , which are CR on $M \setminus E$, under the hypothesis that the $(\dim M - 2)$ -dimensional Hausdorff volume of E is zero and that the CR-orbits of M and of $M \setminus E$ are comparable.

1. INTRODUCTION

Let M be a smooth CR submanifold of a complex manifold X . This paper is devoted to the study of the singularities of integrable CR-functions on M . A closed set $E \subset M$ is called L^p -removable ($p \geq 1$; cf. [HP], [JO], [CS]) if

$$L_{loc,CR}^p(M \setminus E) \cap L_{loc}^p(M) = L_{loc,CR}^p(M),$$

i.e. if any function $f \in L_{loc}^p(M)$ which is CR outside of E is automatically CR all over M . An analogous definition can be formulated for general linear partial differential operators instead of the tangential Cauchy-Riemann operators (see [HP]).

This notion appears probably for the first time in Riemann's removability theorem (1851), a classical theorem, which states that isolated points for the Cauchy-Riemann operator on the complex plane are L_{loc}^∞ -removable. Later on, the question was treated by Bochner, Carleson in the other L^p -spaces for the ordinary $\bar{\partial}$ and for the Laplace operator over \mathbb{C} and \mathbb{R}^n . In 1970, Harvey and Polking extended this circle of ideas to general linear partial differential operators. Their main theorem (Theorem 4.1. of [HP]), which is optimal in the general setting, implies for tangential Cauchy-Riemann operators (as a particular case) that E is $(L^p, \bar{\partial}_b)$ -removable if the Hausdorff measure $H^{\dim M - p'}(E) < \infty$, $p' = p/(p - 1)$. We stress that on this general level, no result at all in the L^1 spaces is possible (if $p = 1$, we have $p' = \infty$ and the above statement is meaningless): The obstruction

Date: June 19, 2001.

1991 Mathematics Subject Classification. 32D20, 32A20, 32D10, 32C16.

Key words and phrases. Integrable CR functions, Removable singularities for the tangential Cauchy-Riemann operator, Analytic discs.

is already shown by the rational function $z \mapsto 1/z$, which is locally integrable near the singularity $0 \in \mathbb{C}$.

Nevertheless, our main theorem in this article shows that essentially stronger removability phenomena hold true for the tangential Cauchy-Riemann operators on embedded CR-manifolds. Our approach relies on the close interplay between CR-functions and complex analysis by means of analytic extension. The rough idea is that singularities of CR-functions should behave in a way comparable to those of holomorphic functions.

For boundaries of domains in complex manifolds, this point of view was illustrated by contributions of many authors (we refer to the excellent survey [CS]; see also [LS], [JO], [ME], [PO], [DS]). In the present paper, a direct prolongation of [MP1,2,3], we focus our attention on CR-manifolds of arbitrary codimension.

We would like to mention that many problems in CR geometry require quite different and new methods when passing from the hypersurface case to the greater codimension case (for instance, the analogue of Radó's theorem is known only in codimension one, s. [RS], [C]).

As a first background, we need the notion of CR-orbits. For a point p of a CR-manifold M , its *CR-orbit* $\mathcal{O}(p, M)$ is defined to be the union of all points which are connected with p by a piecewise differentiable path $\gamma(t) \in M$, $t \in [0, 1]$, whose derivatives (both left-sided and right-sided at nonsmooth points) are always non-vanishing and contained in $T^c M$: $\frac{d}{dt}\gamma(t) \in T_{\gamma(t)}^c M$. We shall often denote CR orbits by \mathcal{O}_{CR} (independently of a reference point $p \in \mathcal{O}_{CR}$) or shortly by \mathcal{O} .

By a general result of Sussmann [SU], it is known that CR-orbits are smooth manifolds, injectively immersed in M . For information on their relation to analytic extension and further references, we refer the reader to contributions of Treves [TRV], Trépreau [TR], Tumanov [TU1], Jöricke [JO], and Merker [ME] (see also [MP1], Section 1, and [MP2], Section 4, for a summary of relevant notions).

Let us introduce some terminology and notation. A property P is said to be *true on almost every CR-orbit* if there is a union \mathcal{N} of CR-orbits which is of measure zero on M , such that P holds true on every orbit \mathcal{O} out of \mathcal{N} . Furthermore a CR-manifold M will be called *globally minimal* if it contains only one CR-orbit ([ME]). As above, let H^κ , $\kappa \in \mathbb{N}$, denote Hausdorff κ -dimensional measure on M equipped with some Riemannian metric.

The following theorem contains the essence of the article.

Theorem 1.1. *Let M be an embedded CR-manifold of class \mathcal{C}^3 , dimension $d \geq 3$, and CR-dimension $\dim_{CR} M = m \geq 1$. Then every closed subset E of M with $H^{d-2}(E) = 0$ such that for almost all CR orbits, $\mathcal{O}_{CR} \setminus E$ is globally minimal is L_{loc}^p -removable for $1 \leq p \leq \infty$.*

The above theorem is a refinement of results we have proved in [MP1,2], where either the hypothesis on Hausdorff measure was more restrictive (we suppose $H^{d-3}(E)$ locally finite in [MP1]), or M was assumed to be real analytic ([MP2]). This research was as a whole inspired by corresponding results of Lupaciuolu,

Stout, Chirka and Jöricke on analytic extension of continuous CR-functions defined on parts of boundaries of complex domains.

The organization of the proof is as follows: In a first reduction (section 2), we shall see that the problem restricts to solve the corresponding question on almost all CR-orbits. More concretely we shall be left with the hypothesis that both M and $M \setminus E$ are globally minimal.

Next we shall employ the technique of Bishop discs to extend our CR function f from $M \setminus E$ to a large portion of a wedge \mathcal{W} attached to M (section 3), except a singularity $E_{\mathcal{W}} \subset \mathcal{W}$. Then the final step should be to remove the singularity $E_{\mathcal{W}}$ in the wedge with good L^1 -estimates and to recover thereafter f globally as an L^1 -limit of a holomorphic functions in \mathcal{W} (as in the theory of Hardy spaces), thereby proving that f is CR everywhere. This scheme of proof is of completely different nature than the measure-theoretic arguments of [HP] (*cf.* [JO]) and the discussion in the introduction of [MP1]).

At the removal of the singularity $E_{\mathcal{W}}$ in \mathcal{W} , we encounter a special problem. This problem is overcome by the following theorem which seems to be new and of independent interest. Let $\mathcal{H}(\mathcal{U})$ denote the ring of holomorphic functions defined over the open set $\mathcal{U} \subset X$.

Theorem 1.2. *Let $D \subset \mathbb{C}^N$ be a domain equipped with a foliation \mathcal{F} by holomorphic curves, of class \mathcal{C}^2 . Further let $E \subset D$ be a closed union of leaves with $H^{2N}(E) = 0$. Then a function $F \in L^1(D) \cap \mathcal{H}(D \setminus E)$ extends holomorphically to D as soon as F extends to an open subset D' of D which intersects each leaf contained in E .*

Our proof is an extension of techniques used by Henkin and Tumanov in [HT] to prove a related statement about continuous CR functions on manifolds foliated by holomorphic curves. Thus, there is a serious difference in our proof of Theorem 1.2 above given in [MP2], where the foliation \mathcal{F} is supposed to be holomorphic or real analytic (Theorem 5.10 in [MP2]).

We believe that this result is true without the assumption $F \in L^1(D)$ (s. Remark 4.1 in this paper). This is an open problem. More generally, it seems to us that analogues of many classical theorems about partial analyticity or meromorphy (*e.g.* Hartogs', Rothstein's, Shiffman's theorems), mixed with removable singularity problems, remain completely open in case of smooth nonholomorphic foliations by complex curves.

Acknowledgement. We would like to thank B. Jöricke for fruitful discussions on the topic. Further we are grateful to the referee for valuable suggestions, which helped considerably to improve the overall quality of the paper.

2. REDUCTION TO CR-ORBITS

In this section, we shall reduce the proof of theorem 1.1 to the case where both M and $M \setminus E$ are globally minimal. This is a consequence of the following two statements.

Lemma 2.1. *Under the assumptions of Theorem 1.1, for almost every orbit \mathcal{O} , the intersection $E \cap \mathcal{O}$ satisfies $H_{\mathcal{O}}^{d_{\mathcal{O}}-2}(E \cap \mathcal{O}) = 0$, where $d_{\mathcal{O}} = \dim_{\mathbb{R}} \mathcal{O}$.*

Here, the index \mathcal{O} in $H_{\mathcal{O}}^{d_{\mathcal{O}}-2}(E \cap \mathcal{O})$ indicates that the Hausdorff measure is computed with respect to the manifold topology of \mathcal{O} and with respect to the pullback of some Riemannian metric of M to \mathcal{O} .

Proof. First, we recall the following local property of CR-orbits: Near every point $p \in M$, there exist a (topologically trivial) \mathcal{C}^2 -foliation of M by leaves of dimension $k = \dim \mathcal{O}(p, M)$ such that for any q near p , its orbit $\mathcal{O}(q, M)$ contains the leaf passing through q . This implies that the dimension of all the CR-orbits passing near p cannot be less than $\dim \mathcal{O}(p, M)$, and that in case of equality, the leaf is an open subset of the corresponding orbit. A well known property of Hausdorff measures assures that for almost every leaf L , the intersection $E \cap L$ satisfies $H_L^{\dim L-2}(E \cap L) = 0$.

To exploit this local information, we need the following simple covering lemma.

Lemma 2.2. *There is a countable covering of M by foliated open sets $U_j \subset\subset M$ as above, such that every point $p \in M$ appears in at least one U_j whose leaves are of dimension $\dim \mathcal{O}(p, M)$.*

Proof. We argue by induction on the dimension of orbits. By paracompactness of M , we easily find a countable covering $\{U_j^{(0)}\}$ as required for the open submanifold

$$M^{(0)} = \{p \in M : \dim \mathcal{O}(p, M) = \dim M\}.$$

In this case, the $U_j^{(0)}$ are just equipped with the trivial zero-codimensional foliation.

In the next, we look at

$$M^{(1)} = \{p \in M : \dim \mathcal{O}(p, M) \geq \dim M - 1\}.$$

By the above local property of CR-orbits, $M^{(1)}$ is also an open subset of M . We choose for any point $p \in M^{(1)} \setminus M^{(0)}$ an open set $V_p^{(1)}$ equipped with a convenient codimension one foliation and obtain thereby a covering $\{U_j^{(0)}, V_p^{(1)}\}$ of $M^{(1)}$. Again by paracompactness we extract a countable covering $\{U_j^{(1)}\} \supset \{U_j^{(0)}\}$ of $M^{(1)}$.

By induction we treat the open submanifolds

$$M^{(k)} = \{p \in M : \dim \mathcal{O}(p, M) \geq \dim M - k\}$$

in an analogous fashion. We thus get an increasing chain of systems $\{U_j^{(k)}\}$ and the desired covering after at most $\dim M - 2\dim_{CR} M$ steps \square

From Lemma 2.2 and the preceding remark we can deduce that the set of all CR-orbits \mathcal{O} of a given dimension k for which the intersection $\mathcal{O} \cap E$ with some leaf of some fixed U_j (with leaf-dimension k) has too large Hausdorff content is of zero measure. As countable unions of sets of measure zero are still of measure zero, Lemma 2.1 will be proved

Precisely, Let $k \in \mathbb{N}_*$, $2\dim_{CR}M \leq k \leq \dim M$. Let U_{j_k} denote all the foliated open sets U_{j_k} with leaf dimension k . By lemma 2.2, the union of the U_{j_k} is a countable covering of an open neighborhood in M of the set $\mathcal{O}_k := \{p \in M; \dim\mathcal{O}(p, M) = k\}$. Let $L_{j_k}(p)$ denote the leaf of U_{j_k} passing by p . By the remark preceding Lemma 2.2, the set

$$A_{j_k} := \{p \in U_{j_k}; p \in \mathcal{O}_k \text{ and } H^{k-2}(L_{j_k}(p) \cap E) > 0\}$$

is of zero d -dimensional measure in U_{j_k} . Finally, the countable union $\bigcup A_{j_k} := \Sigma_k$ remains of measure zero in M , where, by construction

$$\Sigma_k = \{p \in M; p \in \mathcal{O}_k \text{ and } H_{\mathcal{O}(p, M)}^{k-2}(E \cap \mathcal{O}(p, M)) > 0\},$$

which completes the proof of Lemma 2.1. \square

Next we need the following result from [PO].

Theorem 2.3. *If M is \mathcal{C}^3 , a function $f \in L_{loc}^1(M)$ is CR if and only if $f|_{\mathcal{O}}$ belongs to $L_{loc, CR}^1(\mathcal{O})$ for almost every CR-orbit \mathcal{O} .*

Lemma 2.1 obviously implies that the Hausdorff-theoretical assumptions of Theorem 1.1 are inherited by almost all orbits. Theorem 2.3 shows, first, that the restriction of an $f \in L_{loc, CR}^1(M \setminus E)$ is CR on almost every CR-orbit of $M \setminus E$. By assumption in Theorem 1, almost every CR orbit of $M \setminus E$ coincides with $\mathcal{O} \setminus E$ for an orbit \mathcal{O} of M , thus $f \in L_{loc, CR}^1(\mathcal{O} \setminus E)$ for almost all \mathcal{O} . Secondly, Theorem 2.3 shows that it is enough to remove the singularity orbit-wise, *i.e.* to show that $L_{loc, CR}^1(\mathcal{O} \setminus (E \cap \mathcal{O})) \cap L_{loc}^1(\mathcal{O}) = L_{loc, CR}^1(\mathcal{O})$ for such \mathcal{O} . Hence the desired reduction of Theorem 1.1 to globally minimal manifolds is complete. \square

3. REDUCTION TO SINGULARITIES OF HOLOMORPHIC FUNCTIONS

Now we shall use the technique of analytic discs to reduce the proof to a special problem concerning removable singularities of integrable holomorphic functions.

By the preceding section and by the fact that the CR-orbits of a manifold M of class $\mathcal{C}^{2, \alpha}$, $0 < \alpha < 1$, are of class $\mathcal{C}^{2, \beta}$, for any $\beta \in (0, \alpha)$ (s. [TU1,2], [MP1]), it is enough to prove the following theorem.

Theorem 3.1. *Let M be $\mathcal{C}^{2, \alpha}$, $0 < \alpha < 1$, $\dim_{CR}M = m \geq 1$, $\dim M \geq 3$, $d = \dim M$. Then every closed subset E of M such that M and $M \setminus E$ are globally minimal and such that $H^{d-2}(E) = 0$, is L_{loc}^1 -removable.*

Proof. As the proof is quite long, it will be subdivided in several steps. Until step three below, the scheme of proof here is similar to the one given in [MP1,2], modulo minor modifications. But we provide here a less formal and more summarized presentation, for completeness.

Step one: general setup. Fix a function $f \in L_{loc, CR}^1(M \setminus E) \cap L_{loc}^1(M)$. Define $\mathcal{A} = \{\Psi \subset E \text{ closed; } M \setminus \Psi \text{ globally minimal and } f \in L_{loc, CR}^1(M \setminus \Psi) \cap L_{loc}^1(M)\}$, and define $E_{nr} = \bigcap_{\Psi \in \mathcal{A}} \Psi$. Then $M \setminus E_{nr}$ is globally minimal, too. If $E_{nr} = \emptyset$, we are done.

To reach a contradiction, we assume $E_{\text{nr}} \neq \emptyset$ and we denote E_{nr} from now on again by E .

Notation. By $\mathcal{V}_X(\mathcal{F})$ or shortly $\mathcal{V}(\mathcal{F})$, we denote an open thin neighborhood in X of a subset \mathcal{F} of the ambient complex manifold X where M lives.

Firstly, as $M \setminus E$ is globally minimal and thanks to Proposition 1.16 in [MP1], we may (after a slight deformation of M fixing E) assume that $f \in L_{loc}^1(M) \cap \mathcal{H}(\mathcal{V}_X(M \setminus E))$, *i.e.* that f is holomorphic in a neighborhood of $M \setminus E$ in X .

Secondly, let $p \in E \neq \emptyset$ be an arbitrary point and let γ be a piecewise differentiable CR-curve linking p with a point $q \in M \setminus E$. Such a γ exists as M and $M \setminus E$ are globally minimal by assumption. After shortening γ , we may suppose that $\{p\} = E \cap \gamma$ and that γ is a smoothly embedded segment. Therefore γ can be described as the integral curve of some nonvanishing CR vector field Υ defined in a neighborhood of γ .

It suffices to show that f is CR near the endpoint p . Indeed, a standard argument using the dynamical flow of Υ proves the existence of arbitrarily small neighborhoods V of p for which $(M \setminus E) \cup V$ is globally minimal (see the details in [MP1], Lemma 2.3). If f is CR near p and V small enough, this will contradict the definition of $E = E_{\text{nr}}$.

The remainder of the proof is devoted to show that f is CR near p .

Step two: construction of analytic discs. First of all, let us briefly recall some basic terminology about analytic discs. An *analytic disc* A is a holomorphic mapping $A : \Delta \rightarrow X$ extending continuously to $\overline{\Delta}$. Sometimes, we will also denote by A , ∂A the ranges $A(\Delta)$, $A(\partial\Delta)$ and call them shortly the *disc* and its *boundary*. The disc A is said to be *attached* to M if $\partial A \subset M$.

In this step, as in [MP1], we shall construct a very small analytic disc A attached to M , whose boundary has nontrivial intersection with E : $\partial A \cap E \neq \emptyset$, but such that this boundary is not fully contained in E : $\partial A \not\subset E$.

In a rough sense, this disc will mimic an infinitesimal vector $\gamma'(t)dt \in T_{\gamma(t)}^c M$ at some point $p = \gamma(t)$ of E where there exists such a vector in $T_p^c M$ going “outside” E . Of course, the existence of such a point p could fail if M and $M \setminus E$ were not globally minimal, for instance if E would contain a CR orbit of M .

This is the intuitive explanation. Further we may achieve that A is *round* in the following sense: There is a point $p \in \partial A$ and holomorphic coordinates z, w around p such that $\{z = 0\}$ corresponds to $T_p^c M$ and such that the projection of A to $\{z = 0\}$ is a translated coordinate disc (*cf.* Lemma 2.4 in [MP1]). Thus we need to work in local coordinates.

Locally, we can identify X with \mathbb{C}^{m+n} , in holomorphic Euclidean coordinates $(z = x + iy, w)$ transferring a given point $p \in M$ to the origin, and such that M is given near p as a graph of a vector-valued function h

$$(1) \quad x = h(y, w),$$

satisfying $h(0, 0) = 0$ and $\nabla h(0, 0) = 0$. Here $w \in \mathbb{C}^m$, $z \in \mathbb{C}^n$, $m + n = N$, $m = \dim_{\mathbb{C}R} M$, $n = \text{codim } M$.

A small analytic disc A will be called *round* if, in appropriated local coordinates (z, w) as above, its w -components appear as a round disc in some complex line in \mathbb{C}^m . Roundness is only a technical quality which is required below in our use of the continuity principle to control monodromy of the holomorphic extensions.

By an elementary observation from [TU1], Proposition 2.1, we may approximate the CR-curve γ by a finite chain of round analytic discs A_1, \dots, A_k , with $A_1(1) = p$, $A_j(-1) = A_{j+1}(1)$, and with $A_k(-1)$ very close to q , where all A_j are of small diameter.

The logic of the remainder of the proof is now as follows: By an inductive argument we shall show that f is CR near the union of the boundaries $\partial A_1, \dots, \partial A_k$, hence in particular in some neighborhood of $p = A(1) \in E$. As explained in the previous step, this will give the desired contradiction to the assumption $E \neq \emptyset$ and therefore finish the proof. To this end it is clearly sufficient to prove the following general statement: If A is a small round disc attached to M with $\partial A \not\subset E$, then f is CR in a neighborhood of ∂A . By reparametrization, we only need to consider the situation where $A(-1) \notin E$ and $A(1) \in E$ and to prove that f is CR near $A(1)$.

Step three: partial analytic extension. For a small round disc A with $\partial A \not\subset E$ we have to prove that f is CR near ∂A . By symmetry it is enough to argue near $p = A(1) \in E$. We shall embed A into a family of analytic discs which sweeps out a wedge \mathcal{W} glued to M . Using the continuity principle, we shall in this step extend f holomorphically to a subset of \mathcal{W} of full measure and we shall prove that the extension F is integrable on \mathcal{W} .

We shall first develop A in a preliminary family $A_{\rho, s'}$ of analytic discs, $0 \leq \rho < \rho_2$, $\rho_2 > \rho_1 > 0$, $I_{\rho_2} = (0, \rho_2)$, $s' = (a_2, \dots, a_m, y_1^0, \dots, y_n^0)$ running in a neighborhood of 0 in $\mathbb{C}^{m-1} \times \mathbb{R}^n$, as follows. (This step is only a detailed summary of Proposition 2.6 of [MP1], to which we refer the reader.)

Set for the w -component

$$W_{\rho, s'}(\zeta) = (\rho\zeta - \rho_1, a_2, \dots, a_{m-1})$$

and take then $A_{\rho, s'}$ of the form

$$A_{\rho, s'}(\zeta) = (X_{\rho, s'}(\zeta) + iY_{\rho, s'}(\zeta), \rho\zeta - \rho_1, a_2, \dots, a_{m-1}),$$

where $Y_{\rho, s'}$ is the solution of Bishop's equation

$$Y_{\rho, s'} = T_1 h(Y_{\rho, s'}, W_{\rho, s'}) + y^0.$$

Here T_1 denotes the renormalized Hilbert transform which associates to a (vector-valued) real function on $\partial\Delta$ its harmonic conjugate vanishing at the point 1 $\in \partial\Delta$. Since the discs $W_{\rho, s'}$ are of small $\mathcal{C}^{2, \alpha}$ norm, some general strong regularity properties of the Bishop equation provide the existence of solutions depending $\mathcal{C}^{2, \beta}$ -smoothly on the data for any $\beta < \alpha$, if h is $\mathcal{C}^{2, \alpha}$, $\alpha > 0$ ([TU1, 2]).

After differentiating Bishop's equation as in [MP1, 2], Lemmas 2.7 and 2.8 (a computation mainly due to Tumanov in [TU1]), we see that there exist $\rho_2 > 0$, \mathcal{A} , a neighborhood of 0 in \mathbb{C}^{m-1} , \mathcal{V} , a neighborhood of 0 in \mathbb{C}^{m-1} and \mathcal{Y} , a

neighborhood of 0 in \mathbb{R}^n , such that the mapping

$$I_{\rho_2} \times \mathcal{A} \times \mathcal{Y} \times \partial\Delta \ni (\rho, a, y^0, \zeta) \mapsto A_{\rho, a, y^0}(\zeta) \in M$$

is an embedding.

This property is important. It shows that a neighborhood of $A_{0,0,0}(\overline{\Delta}) = p$ in M is foliated by \mathcal{C}^2 -smooth (non-holomorphic) two-dimensional discs $D_{a,y^0} = D_{s'} = \{A_{\rho, a, y^0}(\zeta) \in M : 0 \leq \rho < \rho_2, \zeta \in \partial\Delta\}$ (recall $s' := (a, y^0)$). Moreover, since $H^{d-2}(E) = 0$, the set $\mathcal{S}'_E = \{s' \in \mathcal{S}' : D_{s'} \cap E \neq \emptyset\}$ is a closed subset of $\mathcal{S}' := \mathcal{A} \times \mathcal{Y}$ of Lebesgue measure zero.

Now, recall (s. [ME] and [MP1], on the top of Proposition 1.6) that an analytic disc is said to be *analytically isotopic to a point in $M \setminus E$* , if it is isotopic to a constant mapping through analytic discs attached to $M \setminus E$ which are, moreover, embeddings of the unit disc into \mathbb{C}^{m+n} , except for the first (constant) map (for instance, $(t, \zeta) \mapsto t(\zeta - 1)$, $[0, 1] \times \Delta \rightarrow \mathbb{C}$).

Notice that all discs $A_{\rho, s'}$ for $\rho \neq 0$ are embeddings.

Thus, by choosing an evident path in the parameter space $I_{\rho_2} \times \mathcal{S}'$, we see that each disc $A_{\rho, s'}$ with $s' \notin \mathcal{S}'_E$ is therefore analytically isotopic to a point in $M \setminus E$ (cf. Section 3 in [MP1]). Indeed, for such s' , $D_{s'} \cap E = \emptyset$. Finally, we also get that *each* disc $A_{\rho, s'}$ with $A_{\rho, s'}(\partial\Delta) \cap E = \emptyset$ is analytically isotopic to a point in $M \setminus E$, since $\mathcal{S}' \setminus \mathcal{S}'_E$ is open and dense in \mathcal{S}' .

Since $A(1) = p \in E$ and $A(\partial\Delta) \not\subset E$, we can assume after a reparametrization of Δ that $A(-1) \notin E$. We can therefore push the disc, *i.e.* deform it, in a neighborhood of $A(-1)$, because f is already holomorphic in a neighborhood of $A(-1)$ ($f \in \mathcal{H}(\mathcal{V}_X(M \setminus E))$). Such deformations exist, thanks to the flexibility of Bishop's equation (see Tumanov, [TU1,2]). These deformations which we call *normal* in [MP1,2,3] (because they take place in the normal space to M at $A(-1)$) are appropriate to propagate wedge extendibility [TU1,2]. The rough idea here is that we will try to *propagate wedge extendibility, but taking into account at the same time the presence of singularities*.

Thus, by means of *normal deformations* of the family near $A(-1)$ as in [MP1], Proposition 2.6, we can develop A in a *regular family* $A_{\rho, s', v}$, for v running in an open set \mathcal{V} in \mathbb{R}^{n-1} containing the origin (see Definition 1.8 in [MP1] or Definition 3 in [MP3]). The main property of regular families is that the *direction of exit* $-\frac{\partial}{\partial \zeta} A_{\rho, s', v}(1) := \eta_v \notin T_{A(1)}M$ of the discs varies sufficiently as v varies to cover a *cone* in the normal bundle to M at $A(1)$, when we put $\rho = \rho_1$ and $s' = 0$: After projection to $T_p \mathbb{C}^{m+n} / T_p M$, the union of half-lines $\cup_v (\mathbb{R}^+ \eta_v)$ covers an open cone in the n -dimensional vector space $T_p \mathbb{C}^{m+n} / T_p M$. These deformations are nontrivial and appropriate for further holomorphic extension of f , thanks to $A(-1) \notin E$, $f \in \mathcal{H}(\mathcal{V}_X(M \setminus E))$.

Remark. That $-\frac{\partial}{\partial \zeta} A_{\rho, s', v}(1) := \eta_v \notin T_{A(1)}M$ and that the family $(\mathbb{R}^+ \eta_v)_{v \in \mathcal{V}}$ describes a cone in the normal bundle to M at p , are two conditions which are required to *cover a wedge of edge M at p* (see below). In the hypersurface case $n = 1$, $-\frac{\partial}{\partial \zeta} A_{\rho, s'}(1) \notin T_{A(1)}M$ would suffice and no extra parameter would

be needed. However, if from the beginning $-\frac{\partial}{\partial \zeta} A_{\rho, s'}(1) \in T_{A(1)}M$ (general case $n \geq 1$), it is really necessary for the sequel to proceed to a first normal deformation *depending on a bigger parameter space* $\mathcal{V}' \subset \mathbb{R}^n$, which will have the property that, for some $v'_0 \in \mathcal{V}'$ close to 0, then $-\frac{\partial}{\partial \zeta} A_{\rho, s', v'_0}(1) := \eta_{v'_0} \notin T_{A(1)}M$, and then we can replace \mathcal{V}' by $\mathcal{V}'_0 :=$ a neighborhood of v'_0 in \mathbb{R}^n , and replace \mathcal{V} by a suitable intersection of \mathcal{V}'_0 with an $(n-1)$ -dimensional real affine plane passing through v'_0 , thus getting a regular family. This point is unavoidable, because discs with $-\frac{\partial}{\partial \zeta} A_{\rho, s', v}(1) := \eta_v \in T_{A(1)}M$, $\forall \rho, s', v$, can fail to cover a wedge: for instance, this situation occurs necessarily if $n = 1$ and if there is a complex hypersurface H through M at p , because all discs $A_{\rho, s'}$ (no v) are necessarily attached to H . Thus, larger normal deformations are firstly needed. This crucial point is explained in [MP1] (s. Proposition 2.6 and the sequel). \square

As mentioned implicitly just above, any regular family has the fundamental property of providing a wedge of edge M at $A(1) = p$, which is covered by attached analytic discs. Such a wedge can be defined by $\mathcal{W}_{A,p} := \{A_{\rho, a, y^0, v}(\zeta) \in \mathbb{C}^{m+n} : \rho \in J_{\rho_1}, a \in \mathcal{A}, y^0 \in \mathcal{Y}, v \in \mathcal{V}, \zeta \in \Delta_1\}$, where J_{ρ_1} is a neighborhood of ρ_1 in I_{ρ_2} , \mathcal{A} is a neighborhood of 0 in \mathbb{C}^{m-1} , \mathcal{Y} is a neighborhood of 0 in \mathbb{R}^n and Δ_1 is a neighborhood of 1 in Δ . Such wedges are loci of holomorphic extension of nonsingular CR functions ([TU1], [TR], [ME]).

This regular family $A_{\rho, s', v}$ has also the property that, for each $v \in \mathcal{V}$, the set $\mathcal{S}'_{E,v} = \{s' \in \mathcal{S}' : D_{s', v} \cap E \neq \emptyset\}$ is again a closed subset of \mathcal{S}' of Lebesgue measure zero. Therefore, each disc $A_{\rho, s', v}$ with $A_{\rho, s', v}(\partial\Delta) \cap E = \emptyset$ is also analytically isotopic to a point in $M \setminus E$, since $\mathcal{S}' \setminus \mathcal{S}'_{E,v}$ is dense and open in \mathcal{S}' .

Then the isotopy property (s. [ME]) and a version of the continuity principle which controls the monodromy appropriately (s. [ME], [MP1,2]) imply that $\mathcal{H}(\mathcal{V}_X(M \setminus E))$ extends holomorphically into an open dense part $\mathcal{W} \setminus E_{\mathcal{W}}$ of a wedge \mathcal{W} with edge M . More precisely $\mathcal{W} = \mathcal{W}_{A,p}$ is defined by means of some (possibly) smaller open sets of parameters $J_{\rho_1} \subset I_{\rho_2}$, a neighborhood of ρ_1 in I_{ρ_2} , $\mathcal{A}_1 \subset \mathcal{A}$, $\mathcal{Y}_1 \subset \mathcal{Y}$, $\mathcal{V}_1 \subset \mathcal{V}$ and Δ_1 , a neighborhood of 1 in Δ :

$$\mathcal{W} = \{A_{\rho, a, y^0, v}(\zeta) \in \mathbb{C}^{m+n} : \rho \in J_{\rho_1}, a \in \mathcal{A}_1, y^0 \in \mathcal{Y}_1, v \in \mathcal{V}_1, \zeta \in \Delta_1\},$$

whereas $E_{\mathcal{W}}$ is the closed set

$$E_{\mathcal{W}} = \{A_{\rho, s', v}(\zeta) \in \mathbb{C}^{m+n} : A_{\rho, s', v}(\partial\Delta) \cap E \neq \emptyset\},$$

for which $H^{2m+2n-1}(E_{\mathcal{W}}) = 0$ (see below).

In other words, the usual process of extending holomorphically a CR function to a wedge is successful, except that the extension admits a singularity $E_{\mathcal{W}}$ inside \mathcal{W} which corresponds to the union of all traces in \mathcal{W} of all holomorphic curves $A_{\rho, s', v}(\Delta_1)$ which (unfortunately) meet the singularity E of f in M along their boundaries, $A_{\rho, s', v}(\partial\Delta) \cap E \neq \emptyset$.

Thus, we should study the new singularity $E_{\mathcal{W}}$, we should remove it, *i.e.* we should show that holomorphic functions in $\mathcal{W} \setminus E_{\mathcal{W}}$ extend holomorphically

through $E_{\mathcal{W}}$. This strategy appears to be successful, thanks to the fairly special structure of $E_{\mathcal{W}}$.

Of course we must estimate the size of $E_{\mathcal{W}}$. Recall that $M \subset \mathbb{C}^{m+n=N}$ has dimension $d = 2m + n$.

It is easy to see that the Hausdorff dimension $\text{H-dim}(E_{\mathcal{W}})$ of $E_{\mathcal{W}}$ is at worst less or equal to $n + 1 + \text{H-dim}(E)$ (s. [MP1]; [MP2], Lemma 5.9 and Proposition 5.7). Furthermore, we have $H^{\kappa+n+1}(E_{\mathcal{W}}) = 0$ if $H^{\kappa}(E) = 0$. Thus $H^{2m+2n-1}(E_{\mathcal{W}}) = 0$.

To summarize, the aim is now to establish that $E_{\mathcal{W}}$ is removable for the space $\mathcal{H}(\mathcal{W} \setminus E_{\mathcal{W}})$, namely that every $f \in \mathcal{H}(\mathcal{W} \setminus E_{\mathcal{W}})$ extends in a unique way to an $F \in \mathcal{H}(\mathcal{W})$. This will be accomplished in step four below, thanks to Theorem 1.2.

Let us first insist at this stage on heuristical aspects.

Because it is clear that not all singularities $\mathcal{F} \subset \mathcal{W}$ with $H^{2m+2n-1}(\mathcal{F}) = 0$ are removable for functions holomorphic in $\mathcal{W} \setminus \mathcal{F}$. Denote this ring by $\mathcal{H}(\mathcal{W} \setminus \mathcal{F})$.

Of course, if $E = \emptyset$ (nonsingular case), then $f \in \mathcal{H}(\mathcal{W})$ immediately ($E_{\mathcal{W}} = \emptyset$).

Next, if for instance, $H^{2m+2n-3}(E_{\mathcal{W}}) = 0$ (when $H^{2m+n-4}(E) < \infty$, this occurs automatically), then it is well known that $E_{\mathcal{W}}$ is removable for $\mathcal{H}(\mathcal{W} \setminus E_{\mathcal{W}})$ and we are done.

The special case $H^{2m+2n-2}(E_{\mathcal{W}}) < \infty$ (which occurs when $H^{2m+n-3}(E) < \infty$) is proved in [MP1], Theorem 3.1.

Warning: of course, not all subsets \mathcal{F} of \mathcal{W} with $H^{2m+2n-2}(\mathcal{F}) < \infty$ are removable for $\mathcal{H}(\mathcal{W} \setminus \mathcal{F})$ (just incorporate bad complex hypersurfaces to \mathcal{F} !). The main geometric point in the situation under consideration is intuitively that for a generic complex curve $A_{\rho,s',v}(\Delta_1) \subset E_{\mathcal{W}}$, one should expect that *not the whole of its interesting boundary* $A_{\rho,s',v}(\overline{\Delta_1} \cap \partial\Delta) \subset M$, *is contained in* E . Otherwise the Hausdorff dimensions of E and of $E_{\mathcal{W}}$ would satisfy, $\text{H-dim}(E_{\mathcal{W}}) = n + \text{H-dim}(E) \leq 2m + 2n - 2$ and $H^{2m+2n-2}(E_{\mathcal{W}}) = 0$ also. But then it is well known that closed sets $\mathcal{F} (:= E_{\mathcal{W}}$ here) of \mathcal{W} with $H^{2m+2n-2}(\mathcal{F}) = 0$ are (even locally) removable for $\mathcal{H}(\mathcal{W} \setminus \mathcal{F})$.

A particular situation confirms the above observation: supposing that E is a *manifold* (s. [ME], [PO], [MP1]: Theorem 4), then $A_{\rho,s',v}(\overline{\Delta_1} \cap \partial\Delta) \cap E$ is empty or equal to a single point.

That not all the boundary $A_{\rho,s',v}(\overline{\Delta_1} \cap \partial\Delta) \subset M$ is contained in the singularity E for most bad discs, will be crucial to show the removability of $E_{\mathcal{W}}$.

After these explanations, let us finally show that $F \in L^1(\mathcal{W})$. For each fixed v the boundaries of the discs $A_{\rho,s',v}$ foliate some open subset of M . By applying a standard estimate to almost all of these discs and integrating over ρ, s' we get an L^1 estimate for the restriction of F to the $(d+1)$ -dimensional submanifold of \mathcal{W} swept out by these discs corresponding to a fixed v . Integrating over v we finally get $F \in L^1(\mathcal{W})$.

Step four: extension of F . The singular set $E_{\mathcal{W}}$ contains the closed subset

$$E'_{\mathcal{W}} = \{A_{\rho,s',v}(\zeta) \in \mathbb{C}^{m+n} : A_{\rho,s',v}(\overline{\Delta_1} \cap \partial\Delta) \subset E\}.$$

The main problem consists first in removing $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ from $\mathcal{W} \setminus E'_{\mathcal{W}}$. This will be a consequence of Theorem 1.2. Indeed, the foliation of $\mathcal{W} \setminus E'_{\mathcal{W}}$ being given by the holomorphic discs $A_{\rho, s', v}$ we observe that $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ satisfies $H^{2m+2n-1}(E_{\mathcal{W}} \setminus E'_{\mathcal{W}}) = 0$, because $H^{2m+n-2}(E) = 0$ (cf. [MP2], Lemma 5.9). Furthermore, the set $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ satisfies the hypothesis of Theorem 1.2: it is *in part* foliated by the complex curves $A_{\rho, s', v}(\Delta_1)$ which are *not completely contained* in $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$, because $A_{\rho, s', v}(\overline{\Delta_1} \cap b\Delta) \not\subset E$ for such parameters (ρ, s', v) . Finally we have to verify that F extends holomorphically through any point of each leaf of $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$. But now, it is clear by the structure of $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ and by our initial assumption $f \in L^1_{loc}(M) \cap \mathcal{H}(\mathcal{V}_X(M \setminus E))$ that Theorem 1.2 applies.

Admitting for the moment that Theorem 1.2 is proved, we are thus left with $E'_{\mathcal{W}}$. Its definition and the hypothesis $H^{2m+n-2}(E) = 0$ yield $H^{2m+2n-2}(E'_{\mathcal{W}}) = 0$. In this situation, it is well known that we can extend F through $E'_{\mathcal{W}}$.

Step five: L^1 boundary values. Finally, we wish to recover f near p as the L^1 -limit of F . For $0 < \delta \ll 1, 0 \ll r < 1$, define the *approach manifolds*

$$M_{v,r} = \{A_{\rho, s', v}(re^{i\theta}) \in \mathbb{C}^{m+n} : |\rho_1 - \rho| < \delta, s' \in \mathcal{S}', |\theta| < \delta\}.$$

In step four above, we have proved $F \in L^1(\mathcal{W})$ by estimating along almost every disc. Similarly, we can prove a uniform L^1 bound for the restrictions of F to the approach manifolds (cf. [PO]; [MP1], from 1.11 to 1.14).

Namely, the Embedding Theorem of Carleson (s. [JO]) yields a uniform bound

$$\int_{-\delta}^{\delta} |F(A_{\rho, s', v}(re^{i\theta}))| d\theta < C \|f \circ A_{\rho, s', v}\|_{H^1},$$

valid for almost all discs $A_{\rho, s', v}$ with $\partial A_{\rho, s', v} \cap E = \emptyset$ (as usual $\|\cdot\|_{H^1}$ denotes the Hardy-space norm). Integrating over s' gives the desired estimate

$$\int_{M_{v,r}} |F| d\theta ds' < C \|f\|_{L^1(M')},$$

where M' is a sufficiently large neighborhood of p in M .

As in the usual theory of Hardy spaces (s. for the theory in several variables the standard reference [ST]), F attains near p a weak boundary value F^* , as $r \rightarrow 1, r < 1$, which is an integrable CR function on M . Obviously, F^* coincides (almost everywhere) with f , and the proof is completed. \square

4. L^1 -REMOVABILITY FOR HOLOMORPHIC FUNCTIONS

To complete the proof of Theorem 1.1, it remains to show Theorem 1.2.

Remark 4.1. As the reader may have noticed, we need in the application of Theorem 1.2 to the proof of Theorem 1.1, only a weaker version of Theorem 1.2: for this application, even $H^{2m+2n-1}(E) = 0$ holds true, instead of $H^{2m+2n}(E) = 0$ (this new E is what we called $E_{\mathcal{W}}$). Under this additional hypothesis and additional regularity, some special cases of Theorem 1.1 get much simpler.

In fact, if the foliation \mathcal{F} in Theorem 1.2 is a *holomorphic* or a *real analytic* foliation, we have provided an independent proof of the special case by means of the continuity principle ([MP2], Theorem 5.10). This result and its proof do not use the L^1 -integrability of F (s. the suggestions at the end of this introduction). For dimensional reasons, the same method applies to the \mathcal{C}^2 -smooth case, if $N = 2$, and more generally, if $N \geq 2$ but the \mathcal{C}^2 -smooth foliation \mathcal{F} is a foliation by complex hypersurfaces, thus having again codimension one.

For the general \mathcal{C}^2 -smooth situation and $N \geq 3$ (*i.e.* Theorem 1.2 without the assumption $F \in L^1(D)$), we do not know whether there exists a successful proof working only with holomorphic hulls, even after putting $H^{2N-1}(E) = 0$ or $H^{2N-2}(E) < \infty$. \square

Proof of Theorem 1.2. We take inspiration from an argument used by Henkin and Tumanov (s. [HT], Lemma 6) to treat a related question for continuous CR functions on CR manifolds foliated by complex curves.

Let E_{nr} be the complement of the maximal open subset of D to which F extends holomorphically. We assume $E_{\text{nr}} \neq \emptyset$ and have to deduce a contradiction. Our arguments are local and work near any point of $p_0 \in E_{\text{nr}}$ which is not an inner point of the set $L_{p_0} \cap E_{\text{nr}}$ with respect of the leaf-topology of the leaf L_{p_0} through p_0 .

Around p_0 we may choose a neighborhood Q with the following properties: Near \overline{Q} there are coordinates $w = u + iv$ and $r = (r_1, \dots, r_{2N-2})$, where w is holomorphic and the r_j of class \mathcal{C}^2 , such that Q is given as $\{0 < u < 1, 0 < v < 1, 0 < r_j < 1, j = 1, \dots, 2N-2\}$ and the foliation \mathcal{F} corresponds to the level sets of the mapping r . Contracting U around p_0 we may further assume that F is holomorphic near the bottom $\{v = 0, 0 \leq u \leq 1, 0 \leq r_j \leq 1, j = 1, \dots, 2N-2\}$. Further we may assume the existence of holomorphic coordinates $z = (z_1, \dots, z_N)$ near \overline{Q} . It shall be convenient to work with slightly smaller product domain $Q' = \{0 < v < 1\} \times B'$ where we get B' by smoothing the edges of the bottom. In the following we will tacitly suppose appropriate contractions of Q and Q' around p_0 which do not destroy the precedingly achieved properties.

We choose near \overline{Q} a basis $\overline{L}_1, \dots, \overline{L}_N$ of complex antilinear vector fields whose coefficients are \mathcal{C}^2 with respect to (w, r) such that L_1 is tangent to \mathcal{F} . As observed in [HT], the $\overline{L}_1, \dots, \overline{L}_N$ may be corrected such that all brackets of the form $[\overline{L}_1, \overline{L}_s]$ are tangent to \mathcal{F} .

Next we take a subdomain $G \subset Q'$ containing p_0 which is the region squeezed between the bottom B' and a smooth hypersurface M which cuts $\{v = 0\}$ transversely along bB' and is transverse to the leaves of \mathcal{F} . By Fubini's Theorem, after a slight deformation of M the restriction $F|_M$ may be supposed to be integrable with respect to $(2N-1)$ -dimensional volume. We shall show that (after some additional modifications) $F|_M$ is an integrable CR function. Afterwards, the usual Hartogs-Bochner Theorem gives a holomorphic extension of F to G , in contradiction to the choice of p_0 . Let us rename $B := B', Q := Q'$.

For technical reasons we have to fix in advance a special approximation of F . Let χ be a smooth nonnegative, compactly supported, rotation invariant function of the coordinates z with $\int \chi dm(z) = 1$. Take a smooth function $\eta \geq 0$ whose support is contained in a small neighborhood of \overline{Q} and which equals 1 near \overline{Q} . Setting $\chi_\varepsilon(z) = (1/\varepsilon)^{2N} \chi(z/\varepsilon)$, we define for sufficiently small $\varepsilon > 0$

$$F_\varepsilon = (\eta F) * \chi_\varepsilon,$$

where $*$ denotes convolution with respect to Lebesgue measure in z . It is standard that F_ε approximates F in $L^1(Q)$. From the mean value property of holomorphic functions and the rotation invariance of χ we further deduce that, for near any given point $z \in Q \setminus E_{\text{nr}}$, F_ε will coincide with F for ε sufficiently small. In particular, this is true near the bottom B .

We extract a subsequence $\varepsilon_k \searrow 0$ and claim that after a slight deformation of M we can assume that the restrictions of $F_k = F_{\varepsilon_k}$ to M tend in $L^1(M)$ to F . Indeed, this is a consequence of the following variant of Fubini's theorem: If we have a series of functions converging in L^1 on a product set, then their restriction to almost all slices will converge to the restriction of the limit.

Now we can return to the main part of the proof. Fix a point $q \in M \cap E_{\text{nr}}$. We have to show

$$(2) \quad \int_M F \wedge \overline{\partial} \phi = 0,$$

for any smooth $(N, N-2)$ -form ϕ such that $\text{supp } \phi \cap M$ is contained in some small neighborhood of q in M . By the very definition of the tangential CR complex, the right side of (2) is not changed if we add to ϕ an $(N, N-2)$ -form contained in the differential ideal generated by a defining function ρ of M and its derivative $\overline{\partial} \rho$ (s. [BO], 8.1). As M is transverse to \mathcal{F} , we may thus restrict our attention to more special ϕ : Let $\overline{\omega}_j$ be a complex antilinear dual basis of \overline{L}_j . As M is transverse to \mathcal{F} , it is enough to consider forms $\phi = \sum' \phi_J d\overline{\omega}_J$ such that all coefficients ϕ_J where J contains 1 are zero (here \sum' means summation over increasing indices, ϕ_J are $(N, 0)$ -forms).

Fixing such a ϕ , we wish to approximate its coefficients (with respect to the basis $\omega_j, \overline{\omega}_j$) by functions which are holomorphic along the leaves. In order to apply known techniques for approximation in the complex plane we use the holomorphic w -coordinate and argue fiber-wise. As the intersection of $(\text{supp } \phi) \cap M$ with a leaf L_w is contained in a short segment I_w , we may approximate the coefficient functions on L_w by taking convolution integrals over I_w with holomorphic kernels (as for instance in the proof of the Approximation Theorem of Baouendi and Treves). As the integrals depend smoothly on r we get that this sequence of approximating forms ϕ_j tends in \mathcal{C}^2 to ϕ . In particular, $\phi_j \rightarrow 0$ on ∂M .

The fact that $\text{supp } \phi_j$ can no longer be assumed to be of compact support seems to cause complications. Nevertheless, it shall be possible to establish

$$(3) \quad \int_M F \wedge \overline{\partial} \phi_j = \int_{\partial M} F \wedge \phi_j,$$

which implies (2) by going to the limit, since $\phi_j \rightarrow 0$ on ∂M .

Now we verify as in [HT] that $\bar{\partial}F_k \wedge \bar{\partial}\phi_j$ is of the form $\overline{L_1}F_k \wedge \alpha$ where α is a \mathcal{C}^2 form independent of k (but of course depending on j).

Indeed, for a selection $\overline{L_1}, \overline{L_{s_1}}, \dots, \overline{L_{s_{N-2}}}$ we apply Cartan's formula

$$\begin{aligned} (\overline{L_1}, \overline{L_{s_1}}, \dots, \overline{L_{s_{N-2}}}) \vdash \bar{\partial}\phi_j &= \\ \overline{L_1}(\overline{L_{s_1}}, \dots, \overline{L_{s_{N-2}}}) \vdash \phi_j + \sum (-1)^{k+1} \overline{L_{s_k}}(\overline{L_1}, \dots, \widehat{\overline{L_{s_k}}}, \dots) \vdash \phi_j \\ + \sum (-1)^{k+h+1} (\dots, [\overline{L_{s_k}}, \overline{L_{s_h}}], \dots) \vdash \phi_j + \sum (-1)^{k+1} ([\overline{L_1}, \overline{L_{s_k}}], \dots) \vdash \phi_j, \end{aligned}$$

where \vdash denotes interior multiplication of vectors and forms. By the choice of ϕ and of the $\overline{L_s}$ such that $[\overline{L_1}, \overline{L_s}] = a_s \overline{L_1}$, all the right-hand terms vanish. Consequently, $(\overline{L_1}, \overline{L_{s_1}}, \dots, \overline{L_{s_{N-2}}}) \vdash \bar{\partial}\phi_j = 0$, that is to say, if we write $\bar{\partial}\phi_j = \sum' \beta_J d\overline{\omega_J}$, all β_J with J containing 1 vanish identically. Therefore in the wedge product with $\bar{\partial}F_k = \sum_{j=1}^N \overline{L_i}F_k d\overline{\omega_i}$ only the term $\overline{L_1}F_k d\overline{\omega_1}$ survives.

Consequently we get

$$(4) \quad \int_M F_k \wedge \bar{\partial}\phi_j = \int_B F_k \wedge \bar{\partial}\phi_j + \int_G \overline{L_1}F_k \wedge \alpha.$$

So what remains to do is to prove the following.

Claim. *For fixed j and for $k \rightarrow \infty$, the integral $\int_G \overline{L_1}F_k \wedge \alpha$ tends to zero.*

Since by applying Stokes theorem afterwards, we will obtain (3):

$$\int_M F \wedge \bar{\partial}\phi_j = \int_B F \wedge \bar{\partial}\phi_j \stackrel{\partial M = -\partial B}{=} \int_{\partial M} F \wedge \phi_j \rightarrow 0, \quad j \rightarrow \infty.$$

Proof. Let R be the closed set of all $r^0 = (r_1^0, \dots, r_{2N-2}^0)$ such that the holomorphic curve $\{r = r^0\}$ has nonvoid intersection with E_{nr} . By a result of Whitney, there is a so called regularized distance function $\delta_R(r)$ which is comparable with the euclidean distance function $\text{dist}(r, R)$ (s. [ST]) and satisfies: (i) $(1/C)\text{dist}(r, R) \leq \delta_R(r) \leq C\text{dist}(r, R)$, $\forall r \in B$, $C \geq 1$; (ii) $\delta_R(r) \in \mathcal{C}^0(B) \cap \mathcal{C}^2(B \setminus R)$; (iii) $|\nabla \delta_R(r)| \leq C$ on $B \setminus R$.

Fix an arbitrary $\varepsilon > 0$. Further we take a small $\delta > 0$ whose precise choice shall be explained later. For the moment we only require δ to be a regular value of δ_R , i. e. to be generic in the sense of the theorem of Sard. Hence $Q \cap \{\delta_R = \delta\}$ is a \mathcal{C}^2 -smooth hypersurface (fibered by leaves of \mathcal{F} and with possibly, a number of connected components tending to ∞ as $\delta \rightarrow 0$).

Depending on δ we choose a large k such that, for any $p \in Q$ with $\delta_R(z) \geq \delta$, the distance of p to the singularity E_{nr} , measured with respect to the holomorphic coordinate z , is greater than ε_k (the scaling parameter appearing in the definition of F_k).

In the decomposition

$$\int_G \overline{L_1}F_k \wedge \alpha = \int_{G \cap \{\delta_R > \delta\}} \overline{L_1}F_k \wedge \alpha + \int_{G \cap \{\delta_R < \delta\}} \overline{L_1}F_k \wedge \alpha$$

the first term on the right is obviously zero since F is holomorphic on $G \cap \{\delta_R > \delta\}$. For the second term we integrate by parts

$$\int_{G \cap \{\delta_R < \delta\}} \bar{L}_1 F_k \wedge \alpha = \int_{G \cap \{\delta_R < \delta\}} F_k \wedge \bar{L}_1^t \alpha + \int_{\partial(G \cap \{\delta_R < \delta\})} \sigma(\bar{L}_1) F_k \wedge \alpha,$$

where \bar{L}_1^t denotes the formally adjoint operator of \bar{L}_1 and $\sigma(\bar{L}_1)$ the factor from the symbol of \bar{L}_1 appearing in the boundary integral.

We claim that the first summand on the right gets small if δ is small and k large. Indeed, as R is of $(2N - 2)$ -dimensional volume zero and F integrable, there is a δ such that

$$\int_{Q \cap \{\delta_R < \delta\}} |F| < \varepsilon.$$

By the theorem of Sard, we may suppose that δ is a regular value of δ_R , and therefore $Q \cap \{\delta_R = \delta\}$ a smooth hypersurface. Next we choose k so large that

$$\int_{Q \cap \{\delta_R < \delta\}} |F_k| < 2\varepsilon.$$

As $|\bar{L}_1^t \alpha| < C$ on \bar{Q} , we have $|\int_{G \cap \{\delta_R < \delta\}} F_k \wedge \bar{L}_1^t \alpha| \leq 2C\varepsilon$. The second term decomposes in three boundary integrals

$$\int_{\partial(G \cap \{\delta_R < \delta\})} = \int_{G \cap \{\delta_R = r\}} + \int_{M \cap \{\delta_R = r\}} + \int_{B \cap \{\delta_R = r\}}$$

For the first boundary term we remark that

$$\int_{G \cap \{\delta_R = \delta\}} \sigma(\bar{L}_1) F_k \wedge \alpha = 0,$$

as \bar{L}_1 is tangential to $G \cap \{\delta_R = \delta\}$, whence $\sigma(\bar{L}_1)$ vanishes on $G \cap \{\delta_R = r\}$. The rest of the boundary integral is estimated in analogous manner as the interior integral. We have only to use that $F_k|_M \rightarrow F|_M$ in L^1 and that F_k coincides with F near B . After summing up, the proof of the claim is finished. \square

The proof of Theorem 1.2 is complete. \square

REFERENCES

- [BO] A. BOGGESS, *CR manifolds and the tangential Cauchy-Riemann complex*, CRC Press, Boca Raton, 1991.
- [C] E. M. CHIRKA, *Radó's theorem for CR mappings of hypersurfaces*, Russian Acad. Sci. Sb. Math., **82** (1995), No1, 243-259.
- [CS] E. M. CHIRKA and E. L. STOUT, *Removable singularities in the boundary*, Contributions to Complex Analysis and Analytic Geometry, Aspects of Mathematics **E 26**, Vieweg 1994, 43-104.
- [DS] T.-C. DINH and F. SARKIS, *Wedge removability of metrically thin sets and application to the CR meromorphic extension*, Math. Z. (to appear).
- [HP] R. HARVEY and J.C. POLKING, *Removable singularities of solutions of linear partial differential equations*, Acta Math. **125** (1970), 39-76.
- [HT] G. M. HENKIN and A. E. TUMANOV, *Local characterization of holomorphic automorphisms of Siegel domains*, Funct. Anal. Appl. **15** (1983), 285-294.

- [JO] B. JÖRICKE, *Boundaries of singularity sets, removable singularities, and CR-invariant subsets of CR-manifolds*, J. Geom. Anal. **6** (1996), 555–611.
- [LS] G. LUPACCIOLU and E.L. STOUT, *Removable singularities for $\bar{\partial}_b$* , Proceedings of the Mittag Leffler Special Year in Complex Analysis, Princeton University Press, Princeton, NJ, 1993.
- [ME] J. MERKER, *On removable singularities for CR functions in higher codimension*, Int. Math. Res. Not. **1** (1997), 21–56.
- [MP1] J. MERKER and E. PORTEN, *On removable singularities for integrable CR functions*, Univ. Math. J. **48** (1999), 805–856.
- [MP2] J. MERKER and E. PORTEN, *On the local meromorphic extension of CR meromorphic functions*, Ann. Polon. Math. **70** (1998), 163–193.
- [MP3] J. MERKER and E. PORTEN, *Enveloppe d’holomorphie locale des variétés CR et élimination des singularités pour les fonctions CR intégrables*, C. R. Acad. Sci. Paris, t. **328**, Série I (1999), 853–858.
- [PO] E. PORTEN, *On removable singularities for L^p CR functions*. Preprint 1998.
- [RS] J.P. ROSAY and E.L. STOUT, *Radó’s theorem for CR functions*, Proc. Amer. Math. Soc. **106** (1989), 1017–1026.
- [ST] E.M. STEIN, *Boundary behaviour of holomorphic functions in several complex variables*. Mathematical Notes, Princeton University Press, Princeton, NJ, 1972.
- [SU] H. J. SUSSMANN, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.
- [TR] J.-M. TRÉPREAU, *Sur la propagation des singularités dans les variétés CR*, Bull. Soc. Math. Fr. **118** (1990), 403–450.
- [TRV] F. TREVES, *Hypoanalytic structures. Local theory*. Princeton Mathematical Series **40**, Princeton University Press, Princeton, NJ, 1992.
- [TU1] A. E. TUMANOV, *Connections and propagation of analyticity for CR functions*, Duke Math. J. **73** (1994), 1–24.
- [TU2] A. E. TUMANOV, *On the propagation of extendibility of CR functions*. Complex analysis and geometry (Trento, 1993), 479–498, Lecture Notes in Pure and Applied Mathematics, **173**, Dekker, New York, 1996.

LABORATOIRE D’ANALYSE, TOPOLOGIE ET PROBABILITÉS, CENTRE DE MATHÉMATIQUES ET D’INFORMATIQUE, UMR 6632, 39 RUE JOLIOT-CURIE, F-13453 MARSEILLE CEDEX 13, FRANCE

E-mail address: merker@cmi.univ-mrs.fr

MATHEMATISCHES INSTITUT DER HUMBOLDT-UNIVERSITÄT ZU BERLIN, JÄGERSTRASSE 10-11, D-10117 BERLIN, GERMANY

E-mail address: egmont@mathematik.hu-berlin.de