

**CHARACTERIZATION OF THE NEWTONIAN
FREE PARTICLE SYSTEM
IN $m \geq 2$ DEPENDENT VARIABLES**

JOËL MERKER

ABSTRACT. We treat the problem of linearizability of a system of second order ordinary differential equations. The criterion we provide has applications to nonlinear Newtonian mechanics, especially in three-dimensional space.

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , let $x \in \mathbb{K}$, let $m \geq 2$, let $y := (y^1, \dots, y^m) \in \mathbb{K}^m$ and let

$$y_{xx}^1 = F^1(x, y, y_x), \dots, y_{xx}^m = F^m(x, y, y_x),$$

be a collection of m analytic second order ordinary differential equations, in general nonlinear. We obtain a new and applicable necessary and sufficient condition in order that this system is equivalent, under a point transformation

$$(x, y^1, \dots, y^m) \mapsto (X(x, y), Y^1(x, y), \dots, Y^m(x, y)),$$

to the Newtonian free particle system $Y_{XX}^1 = \dots = Y_{XX}^m = 0$.

Strikingly, the explicit differential system that we obtain is of first order in the case $m \geq 2$, whereas according to a classical result due to Lie, it is of second order the case of a single equation ($m = 1$).

CNRS, Université de Provence, LATP, UMR 6632, CMI, 39 rue Joliot-Curie, F-13453 Marseille Cedex 13, France.

merker@cmi.univ-mrs.fr

<http://www.cmi.univ-mrs.fr/~merker/index.html>

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§1. INTRODUCTION

Several physically meaningful systems of ordinary differential equations are of second order, as for instance the free particle in m -dimensional space, the damped or undamped harmonic oscillator, coupled or not, having constant time-dependent frequency or not, *etc.* Such systems are ubiquitous in Newtonian Mechanics, in Hamiltonian Dynamics and in General Relativity.

Two classical major problems are to classify these systems up to point or contact equivalence (Lie's Grail) and to recognize when they coincide with the Euler equations associated to a Lagrangian (inverse variational problem). In small dimensions, complete results hold (Lie, Tresse, Cartan; Darboux, Douglas). However, in arbitrary dimension, both tasks quickly exceed the human as well as the digital computer scale,

Date: 2006-4-6.

2000 Mathematics Subject Classification. Primary: 58F36 Secondary: 34A05, 58A15, 58A20, 58F36, 34C14, 32V40.

due to the intrinsic complexity of the underlying symbolic computations (explosion, swelling) and to the exponentially increasing number of cases to be treated. We refer to Olver’s monograph [OL1995] for a panorama of problems, methods and results.

At least, as a first step in classification, with respect to applications, there are both a mathematical and a physical interest in determining concrete, explicit and applicable (“ready-made”) criteria for a system of ordinary differential equation to be equivalent, via a local point transformation, to a linear equation.

1.1. Scalar equation. In this respect, we remind the celebrated linearizability criterion for a single equation, due to Lie. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $x \in \mathbb{K}$ and $y \in \mathbb{K}$. Consider a local second order ordinary differential equation $y_{xx} = F(x, y, y_x)$, possibly nonlinear, with a locally \mathbb{K} -analytic right-hand side¹.

Theorem 1.2. ([Lie1883], pp. 362–365; [GTW1989]; [OL1995], p. 406) *The following four conditions are equivalent:*

- (1) $y_{xx} = F(x, y, y_x)$ is equivalent under a local point transformation $(x, y) \mapsto (X, Y)$ to the free particle equation $Y_{XX} = 0$;
- (2) $y_{xx} = F(x, y, y_x)$ is equivalent to some linear equation $Y_{XX} = G_0(X) + G_1(X)Y + H(X)Y_X$;
- (3) the local Lie symmetry group of $y_{xx} = F(x, y, y_x)$ is eight-dimensional, locally isomorphic to the group $\mathrm{PGL}(3, \mathbb{K})$ of all projective transformations of $P_2(\mathbb{K})$.
- (4) $F_{y_x y_x y_x} = 0$, or equivalently $F = G + y_x H + (y_x)^2 L + (y_x)^3 M$, where G, H, L, M are functions of (x, y) that satisfy:

$$0 = -2G_{yy} + \frac{4}{3}H_{xy} - \frac{1}{3}L_{xx} + 2(GL)y - 2G_x M - 4GM_x + \frac{2}{3}HL_x - \frac{4}{3}HH_y,$$

$$0 = -\frac{2}{3}H_{yy} + \frac{4}{3}L_{xy} - 2M_{xx} + 2GM_y + 4G_y M - 2(HM)_x - \frac{2}{3}H_y L + \frac{4}{3}LL_x.$$

Section 2 of this paper is devoted to a detailed exposition of the original proof of the equivalence between (1) and (4), following [Lie1883]. In the contemporary literature, to the author’s knowledge, there is no modern restitution of Lie’s elegant proof, whereas the description of an alternative proof of Theorem 1.2 as a byproduct of É. Cartan’s equivalence algorithm appears in the references [TR1896], [Ca1924], [GTW1989], [OL1995], [NS2003]².

Lie’s Grail would comprise:

- a complete classification of all Lie algebras of local vector fields; Lie achieved this task in dimension 2 over \mathbb{C} (real case: [GKO1992]); however, as soon

¹By borrowing techniques developed in [Ma2003], this theorem as well as the next both hold under weaker smoothness assumptions, namely with a \mathcal{C}^2 or a $W_{\mathrm{loc}}^{1, \infty}$ right-hand side.

²We note that in these references, the already substantial computations are stopped just after the reduction to an $\{e\}$ -structure on an eight-dimensional (local) principal bundle over the three-dimensional first order jet space. The vanishing of two (among four) fundamental tensors in the structure equations of the obtained $\{e\}$ -structure yields two partial differential equations satisfied by the right-hand side $F(x, y, y_x)$, which are equivalent to (4) of Theorem 1.2. We mention that with the help of Maple programming, the complete reduction to an $\{e\}$ -structure on the base (not only on the principal bundle) is achieved in [HK1989], in the simpler case of so-called fiber-preserving transformations, namely point transformations leaving invariant the “vertical” foliation $\{x = \mathrm{ct.}\}$. To the author’s knowledge, the complete confirmation of Tresse’s results by means of É. Cartan’s method has never been achieved.

as the dimension is ≥ 3 , the complete classification is unknown, due to the intrinsic richness of imprimitive Lie algebras of vector fields;

- a list of all the possible Lie algebras that can be realized as infinitesimal Lie symmetry algebras of partial differential equations, together with their Levi-Malčev decomposition;
- an explicit Gröbner basis of the (noncommutative) algebra of all differential invariants of each equation in the list.

However, complete results hold only for the scalar second order ordinary differential equation. Some tables extracted from Lie's *Gesammelte Abhandlungen* and from Tresse's prized thesis [TR1896] may be found in [OL1995].

1.3. Systems. Let $x \in \mathbb{K}$, let $m \geq 2$, let $y := (y^1, \dots, y^m) \in \mathbb{K}^m$ and let

$$(1.4) \quad y_{xx}^1(x) = F^1(x, y(x), y_x(x)), \dots, y_{xx}^m(x) = F^m(x, y(x), y_x(x))$$

be a collection of m analytic second order ordinary differential equations, possibly nonlinear, of the most general form.

Motivated by physical applications and by geometrical questions, some authors (Leach [Le1980], Grissom-Thompson-Wilkens [GTW1989], González-López [GL1988], Fels [Fe1995], Crampin-Martínez-Sarlet [CMS1996], Doubrov [Do2000], Grossman [Gr2000], Mahomed-Soh [MS2001], and others) have been interested in a least characterizing those that have the Lie symmetry group of maximal dimension. In [Le1980], based on the belief that the equivalence between (1), (2) and (3) of Theorem 1.2 would persist in the case $m \geq 2$, it was conjectured that the symmetry algebra of *every* linear system

$$(1.5) \quad y_{xx}^j = G_0^j(x) + \sum_{l=1}^m y^l G_{1,l}^j(x) + \sum_{l_1=1}^m y_{x}^{l_1} H_{l_1}^j(x), \quad j = 1, \dots, m,$$

of $m \geq 2$ second order differential equations has a Lie symmetry group locally isomorphic to the full transformation group $\text{PGL}(m+1, \mathbb{K})$ of the projective space $P_{m+1}(\mathbb{K})$. However, González-López [GL1988] infirmed this expectation, and produced a necessary and sufficient condition (*see* Corollary 1.8 below) for local equivalence of such linear systems to the free particle system

$$(1.6) \quad Y_{XX}^j = 0, \quad j = 1, \dots, m.$$

Applying Lie's algorithm it is easily seen ([GG1983]) that the free particle system has a local symmetry algebra of dimension $\leq m^2 + 4m + 3$, the bound being attained by $Y_{XX}^j = 0, j = 1, \dots, m$, with group $\text{PGL}(m+1, \mathbb{K})$.

In 1939, for fiber-preserving transformations only, Chern [Ch1939] conducted the É. Cartan algorithm through absorptions of torsion, normalizations and prolongations up to the reduction to an $\{e\}$ -structure. Remarkably, in 1995, Fels [Fe1995] conducted the É. Cartan algorithm for general systems $y_{xx}^j = F^j(x, y, y_x)$ and for general point transformations. As a byproduct of the uniqueness of the obtained $\{e\}$ -structure for which all invariant tensors vanish, Fels deduced in [Fe1995] that the flat system $Y_{XX}^j = 0, j = 1, \dots, m$, is, up to equivalence, the only system of second order possessing a symmetry group of maximal dimension. Alas, the $\{e\}$ -structures obtained by Chern and by Fels are not parametric, so that the counterpart to (4) of Theorem 1.2 was

lacking as soon as $m \geq 2$. The extraordinary heaviness of É. Cartan's method is a well known obstacle.

Recently, Neut [N2003] (*see also* [DNP2005]) wrote a Maple program to compute (among other $\{e\}$ -structures) Fels' tensors in a parametric, explicit way. The digital computations succeeded in the case $m = 2$, yielding Theorem 3 of [DNP2005], a statement that may be checked to be equivalent to Theorem 1.7 (3) just below in the case $m = 2$. In the physically most meaningful case $m = 3$, a present-day computer is stuck ([N2003], [DNP2005]).

Extending Lie's less heavy computations³, we present here a complete solution to the characterization of $Y_{XX}^j = 0$ for arbitrary $m \geq 2$.

Theorem 1.7. Suppose $m \geq 2$. The following three conditions are equivalent:

- (1) *the system $y_{xx}^j = F^j(x, y, y_x)$, $j = 1, \dots, m$, is equivalent, under a local point transformation $(x, y^j) \mapsto (X, Y^j)$ to the free particle equation $Y_{XX}^j = 0$;*
- (2) *the local Lie symmetry group of $y_{xx}^j = F^j(x, y, y_x)$ is $(m^2 + 4m + 3)$ -dimensional and locally isomorphic to $\mathrm{PGL}(m + 1, \mathbb{K})$;*
- (3) *the right hand sides $F^j(x, y, y_x)$ are of a special form, described as follows.*

- (i) *There exist local \mathbb{K} -analytic functions G^j , $H_{l_1}^j$, L_{l_1, l_2}^j and M_{l_1, l_2} , where $j, l_1, l_2 = 1, \dots, m$, enjoying the symmetries $L_{l_1, l_2}^j = L_{l_2, l_1}^j$ and $M_{l_1, l_2} = M_{l_2, l_1}$ and depending only on (x, y) such that $F^j(x, y, y_x)$ may be written as the following specific cubic polynomial with respect to y_x :*

$$y_{xx}^j = G^j + \sum_{l_1=1}^m y_x^{l_1} H_{l_1}^j + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} L_{l_1, l_2}^j + y_x^j \cdot \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} M_{l_1, l_2}.$$

- (ii) *The functions G^j , $H_{l_1}^j$, L_{l_1, l_2}^j and M_{l_1, l_2} satisfy the following explicit system of four families of first order partial differential equations:*

$$(I) \quad \left\{ \begin{array}{l} 0 = -2 G_{y^{l_1}}^j + 2 \delta_{l_1}^j G_{y^{l_2}}^{l_2} + H_{l_1, x}^j - \delta_{l_1}^j H_{l_2, x}^{l_2} + \\ \quad + 2 \sum_{k=1}^m G^k L_{l_1, k}^j - 2 \delta_{l_1}^j \sum_{k=1}^m G^k L_{l_2, k}^{l_2} + \\ \quad + \frac{1}{2} \delta_{l_1}^j \sum_{k=1}^m H_{l_2}^k H_k^{l_2} - \frac{1}{2} \sum_{k=1}^m H_{l_1}^k H_k^j, \end{array} \right.$$

³Throughout the article, we do not adopt the summation convention, because in several subsequent equations, some repeated indices shall appear that will not be summed. Also, we always put commas between the indices. For instance $L_{l_1, l_3, y^{l_2}}^j$ denotes $\partial L_{l_1, l_3}^j / \partial y^{l_2}$ shortly. As usual, δ_i^j is the Kronecker symbol.

where the indices j, l_1 vary in $\{1, 2, \dots, m\}$;

$$(II) \quad \left\{ \begin{aligned} 0 = & -\frac{1}{2} H_{l_1, y^{l_2}}^j + \frac{1}{6} \delta_{l_1}^j H_{l_2, y^{l_2}}^{l_2} + \frac{1}{3} \delta_{l_2}^j H_{l_1, y^{l_1}}^{l_1} + \\ & + L_{l_1, l_2, x}^j - \frac{1}{3} \delta_{l_1}^j L_{l_2, l_2, x}^{l_2} - \frac{2}{3} \delta_{l_2}^j L_{l_1, l_1, x}^{l_1} + \\ & + G^j M_{l_1, l_2} - \frac{1}{3} \delta_{l_1}^j G^{l_2} M_{l_2, l_2} - \frac{2}{3} \delta_{l_2}^j G^{l_1} M_{l_1, l_1} + \\ & + \frac{1}{3} \delta_{l_1}^j \sum_{k=1}^m G^k M_{l_2, k} - \frac{1}{3} \delta_{l_2}^j \sum_{k=1}^m G^k M_{l_1, k} - \\ & - \frac{1}{2} \sum_{k=1}^m H_k^j L_{l_1, l_2}^k + \frac{1}{2} \sum_{k=1}^m H_{l_1}^k L_{l_2, k}^j + \\ & + \delta_{l_1}^j \left(\frac{1}{6} \sum_{k=1}^m H_k^{l_2} L_{l_2, l_2}^k - \frac{1}{6} \sum_{k=1}^m H_{l_2}^k L_{l_2, k}^{l_2} \right) + \\ & + \delta_{l_2}^j \left(\frac{1}{3} \sum_{k=1}^m H_k^{l_1} L_{l_1, l_1}^k - \frac{1}{3} \sum_{k=1}^m H_{l_1}^k L_{l_1, k}^{l_1} \right), \end{aligned} \right.$$

where the indices j, l_1, l_2 vary in $\{1, 2, \dots, m\}$;

$$(III) \quad \left\{ \begin{aligned} 0 = & L_{l_1, l_2, y^{l_3}}^j - L_{l_1, l_3, y^{l_2}}^j + \delta_{l_3}^j M_{l_1, l_2, x} - \delta_{l_2}^j M_{l_1, l_3, x} + \\ & + \frac{1}{2} H_{l_3}^j M_{l_1, l_2} - \frac{1}{2} H_{l_2}^j M_{l_1, l_3} + \\ & + \frac{1}{2} \delta_{l_1}^j \sum_{k=1}^m H_{l_3}^k M_{l_2, k} - \frac{1}{2} \delta_{l_1}^j \sum_{k=1}^m H_{l_2}^k M_{l_3, k} + \\ & + \frac{1}{2} \delta_{l_3}^j \sum_{k=1}^m H_{l_1}^k M_{l_2, k} - \frac{1}{2} \delta_{l_2}^j \sum_{k=1}^m H_{l_1}^k M_{l_3, k} + \\ & + \sum_{k=1}^m L_{l_1, l_3}^k L_{l_2, k}^j - \sum_{k=1}^m L_{l_1, l_2}^k L_{l_3, k}^j, \end{aligned} \right.$$

where the indices j, l_1, l_2, l_3 vary in $\{1, \dots, m\}$; and

$$(IV) \quad \left\{ \begin{aligned} 0 = & M_{l_1, l_2, y^{l_3}} - M_{l_1, l_3, y^{l_2}} - \sum_{k=1}^m L_{l_1, l_2}^k M_{l_3, k} + \sum_{k=1}^m L_{l_1, l_3}^k M_{l_2, k}, \end{aligned} \right.$$

where the indices l_1, l_2, l_3 vary in $\{1, \dots, m\}$.

Let us provide commentaries and explanations. The form of the right-hand side of **(3)(i)** is the analog of the form of the right-hand side F in **(4)** of Lie's Theorem 1.2. However, we notice that the right-hand side of **(3)(i)** is not the most general degree three polynomial in the variables y_x^j , $j = 1, \dots, m$: some coefficients of the cubic terms vanish.

Very strikingly, the differential system **(I)**, **(II)**, **(III)**, **(IV)** satisfied by the functions $G^j, H_{l_1}^j, L_{l_1, l_2}^j, M_{l_1, l_2}$ is of *first order* for $m \geq 2$, whereas the system **(4)** satisfied by G, H, L, M in Lie's Theorem 1.2 is of *second order* for $m = 1$. This confirms the

main theorem of González-López [GL1988], that is recovered here in the special linear case (1.5).

Corollary 1.8. ([GL1988]) *For $m \geq 2$, a linear (nonhomogeneous) system $y_{xx}^j = G_0^j(x) + \sum_{l=1}^m y^l G_{1,l}^j(x) + \sum_{l_1=1}^m y_{x}^{l_1} H_{l_1}^j(x)$ is equivalent to $Y_{XX}^j = 0$ if and only if there exists a function $B(x)$ such that the $m \times m$ matrix G_1 may be written under the specific form:*

$$(1.9) \quad G_{1,l}^j = \frac{1}{2} H_l^j - \frac{1}{4} \sum_{k=1}^m H_k^j H_l^k + \delta_l^j B.$$

We also mention that Fels obtained a characterization of equivalence to the free particle $Y_{XX}^j = 0$, $j = 1, \dots, m$, by the vanishing of two (nonexplicit) tensors \tilde{S}_{ikl}^j and \tilde{P}_i^j (Corollary 5.1 in [Fe1995]). In this reference, some parametric computations are achieved after restricting the initial G -structure, together with its subsequent prolongations, to the identity element of the group; explicit expressions of $\tilde{S}_{ikl}^j|_{\text{Id}}$ and of $\tilde{P}_i^j|_{\text{Id}}$ are then obtained, through already hard computations. The vanishing of the two tensors \tilde{P}_i^j and \tilde{S}_{ikl}^j at the identity of the structure group, namely, as computed in Lemma 4.1 of [Fe1995], yields (translating into our notation)

$$(1.10) \quad \left\{ \begin{array}{l} 0 = (\tilde{S}_{ikl}^j)|_{\text{Id}} = F_{y_x^i y_x^k y_x^l}^j - \frac{1}{n+2} \sum_{l_1=1}^m \sum_{\sigma \in \mathfrak{S}_3} \delta_{\sigma(l)}^j F_{y_x^{l_1} y_x^{\sigma(i)} y_x^{\sigma(k)}}^{l_1}, \\ 0 = (\tilde{P}_i^j)|_{\text{Id}} = \frac{1}{2} D(F_{y_x^i}^j) - F_{y^i}^j - \frac{1}{4} \sum_{k=1}^m F_{y_x^k}^j F_{y_x^i}^k - \\ \quad - \frac{1}{m} \delta_i^j \left[\frac{1}{2} D\left(\sum_{k=1}^m F_{y_x^k}^k\right) - \sum_{k=1}^m F_{y^k}^k - \frac{1}{4} \sum_{k=1}^m \sum_{l=1}^m F_{y_x^k}^l F_{y_x^l}^k \right], \end{array} \right.$$

where D is the total differentiation operator $\frac{\partial}{\partial x} + \sum_{l=1}^m y_x^l \frac{\partial}{\partial y^l} + \sum_{l=1}^m F^l \frac{\partial}{\partial y_x^l}$, and where $i, j, k, l = 1, \dots, m$. Strikingly, one may check that the first equation is equivalent to **(3)(i)** of Theorem 1.2 and then that the second equation yields the (complicated) four families of first order partial differential equations (I), (II), (III) and (IV). So, in Corollary 5.1 of [Fe1995], one may replace the vanishing of \tilde{S}_{ikl}^j and of \tilde{P}_i^j by the vanishing of $\tilde{S}_{ikl}^j|_{\text{Id}}$ and of $\tilde{P}_i^j|_{\text{Id}}$, which were *explicitly computed* there.

This phenomenon could be explained as follows: as soon as the tensors \tilde{S}_{ikl}^j vanish, the system enjoys a projective connection (appendix of [Fe1995]); with such a connection, the tensors \tilde{P}_i^j then transform according to a specific rule via tensorial rotation formulas and their general expression may be deduced from their expression at the identity⁴. We have checked this, but as we try to avoid the method of equivalence, details will not be reproduced here. Similar observations appear in Hachtroudi [Ha1937].

Even if the expressions (1.10) are more compact than the (equivalent) conditions in Theorem 1.7 **(3)**, we prefer the complete expressions of Theorem 1.7 **(3)**, since they are more explicit and ready-made for checking whether a physically given nonlinear

⁴Similar rotation formulas are known in the much simpler case of (pseudo-) Riemannian metrics, see Chapter 12 of Olver [OL1995]. It would be interesting to write a program, finer and more efficient than [N2003], which would systematically recognize such rotation formulas in any application of É. Cartan's equivalence method.

system is equivalent to a free particle. If the reader prefers compact expressions and “short” theorems, (s)he may replace the conditions of Theorem 1.7 (3) by (1.10).

Open Problem 1.11. *Characterize explicitly the linearizability of a Newtonian system in $m \geq 2$ degrees of freedom, i.e. local equivalence to :*

$$(1.12) \quad Y_{XX}^j = G_0^j(X) + \sum_{l=1}^m Y^l G_{1,l}^j(X) + \sum_{l_1=1}^m Y_X^{l_1} H_{l_1}^j(X).$$

1.13. Organization, avertissement and acknowledgment. Section 2 is devoted to a thorough restitution of Lie’s original proof of the equivalence between (1) and (4) in Theorem 1.2. Section 3 is devoted to the formulation of combinatorial formulas yielding the general form of a system equivalent to $Y_{XX}^j = 0$, $j = 1, \dots, m$, under a local \mathbb{K} -analytic point transformation $(x, y^j) \mapsto (X, Y^j)$, for general $m \geq 2$; the proof of the main technical Lemma 3.32 is exposed in Section 5. Section 4 is devoted to the final proof of the equivalence between (1) and (3) in Theorem 1.7, the equivalence between (1) and (2) being already proved by Fels [Fe1995].

Some word about style and intentions. We wanted the proof of the equivalence between (1) and (3) be totally complete, every tiny detail being rigorously and patiently checked. This is why we decided to *carefully detail each intermediate computational step*, seeking first the combinatorics of the formal calculations in the case $m = 1$ and devising then the underlying combinatorics for the case $m \geq 2$. Actually, the size of differential expressions is relatively impressive, as will become soon evident. Thus, no intermediate symbolic computation will be hidden, hence essentially no checking work is left to the reader, as would have been the case if we did not have typed all the computations.

We also would like to point out that except in every specific standard situations, as with the much studied Riemann and Ricci tensors, present-day computer programs are not yet powerful enough to apply the É. Cartan equivalence method when the number of some collection of variables is a general integer. All the formulas obtained in Sections 2, 3 and 4 were first treated completely by hand and then, some of them were confirmed afterwards with the help of MAPLE release 6 in the cases $m = 2$ and $m = 3$. The author is indebted to Sylvain Neut and to Michel Petitot, from the University of Lille 1, for their help in computer machine confirmations.

§2. PROOF OF LIE’S THEOREM

2.1. Argument. This preliminary section contains a detailed exposition of Lie’s original proof of the equivalence between (1) and (4) in Theorem 1.2. Since our goal is to guess the combinatorics of computations in several variables, it will be a crucial point for us to explain thoroughly and patiently each step of Lie’s computation. Without such an intuitive control, it would be hopeless to conduct any generalization to several variables. Hence we shall respect a fundamental principle: always explain clearly and completely what sort of computation is achieved at each step. Also, we shall many times introduce some appropriate new notation.

2.2. Combinatorics of the second order prolongation of a point transformation. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $(x, y) \mapsto (X(x, y), Y(x, y))$ be a local \mathbb{K} -analytic invertible transformation, defined in a neighborhood of the origin in \mathbb{K}^2 , which maps the second order differential equation $y_{xx} = F(x, y, y_x)$ to the flat equation $Y_{XX} = 0$. By

assumption, the Jacobian determinant

$$(2.3) \quad \Delta(x|y) := \begin{vmatrix} X_x & X_y \\ Y_x & Y_y \end{vmatrix}$$

is nowhere vanishing. Since the equation $Y_{XX} = 0$ is left unchanged by any affine transformation in the (X, Y) space, we can (and we shall) assume that the transformation is tangent to the identity at the origin, namely the above Jacobian matrix equals the identity matrix at $(x, y) = (0, 0)$.

The computation how the differential equation in the (X, Y) coordinates is related to the differential equation in the (x, y) -coordinates is classical, *cf.* [Lie1883], [TR1896], [BK1989], [IB1992]: let us remind it. A local graph $\{y = y(x)\}$ being transformed to a local graph $\{Y = Y(X)\}$, we have a direct formula for the first derivative Y_X :

$$(2.4) \quad Y_X := \frac{dY}{dX} = \frac{dx \cdot \partial Y(x, y(x)) / \partial x}{dx \cdot \partial X(x, y(x)) / \partial x} = \frac{Y_x + y_x Y_y}{X_x + y_x X_y}.$$

This yields the prolongation of the transformation to the first order jet space. For the second order prolongation, introducing the second order total differentiation operator (which geometrically corresponds to differentiation along graphs $\{(x, y(x))\}$) defined by

$$(2.5) \quad D := \frac{\partial}{\partial x} + y_x \frac{\partial}{\partial y} + y_{xx} \frac{\partial}{\partial y_x},$$

we may compute, simplify and reorder the expression of the second order derivative in the (X, Y) -coordinates:

$$(2.6) \quad \left\{ \begin{aligned} Y_{XX} &:= \frac{d^2 Y}{dX^2} \equiv \frac{DY_X}{DX} = \frac{D[(Y_x + y_x Y_y)(X_x + y_x X_x)^{-1}]}{X_x + y_x X_y} = \\ &= \frac{1}{[X_x + y_x X_y]^3} \{ y_{xx} [X_x Y_y - Y_x X_y] + X_x Y_{xx} - Y_x X_{xx} + \\ &+ y_x [2(X_x Y_{xy} - Y_x X_{xy}) - (X_{xx} Y_y - Y_{xx} X_y)] + \\ &+ y_x y_x [X_x Y_{yy} - Y_x X_{yy} - 2(X_{xy} Y_y - Y_{xy} X_y)] + \\ &+ y_x y_x y_x [-(X_{yy} Y_y - Y_{yy} X_y)] \}. \end{aligned} \right.$$

Even if not too complicated, the internal combinatorics of this expression has to be analyzed and expressed thoroughly. First of all, as $Y_{XX} = 0$ by assumption, we may erase the cubic factor $[X_x + y_x X_y]^{-3}$. Next, as the factor of y_{xx} in the right-hand side of (2.6), we just recognize the Jacobian $\Delta(x|y)$ expressed in (2.3) above. Also, all the other factors are modifications of the Jacobian $\Delta(x|y)$, whose combinatorics may be understood as follows.

There exist exactly three possible distinct second order derivatives: xx , xy and yy . There are also exactly two columns in (2.3). By replacing each of the two columns of first order derivative in $\Delta(x|y)$ by any column of second order derivative (leaving X and Y unchanged), we may build exactly six new determinants

$$(2.7) \quad \begin{cases} \Delta(xx|y) & \Delta(xy|y) & \Delta(yy|y) \\ \Delta(x|xx) & \Delta(x|xy) & \Delta(x|yy) \end{cases}$$

where for instance

$$(2.8) \quad \left\{ \Delta(\underline{xx}|y) := \begin{vmatrix} X_{xx} & X_y \\ Y_{xx} & Y_y \end{vmatrix} \quad \text{and} \quad \Delta(x|\underline{xy}) := \begin{vmatrix} X_x & X_{xy} \\ Y_x & Y_{xy} \end{vmatrix} \right\}.$$

Hence, by rewriting (2.6), we see that the equation $y_{xx} = F(x, y, y_x)$ equivalent to $Y_{XX} = 0$ may be written under the general explicit form, involving determinants

$$(2.9) \quad \left\{ \begin{aligned} 0 = & y_{xx} \cdot \begin{vmatrix} X_x & X_y \\ Y_x & Y_y \end{vmatrix} + \begin{vmatrix} X_x & X_{xx} \\ Y_x & Y_{xx} \end{vmatrix} + y_x \cdot \left\{ 2 \begin{vmatrix} X_x & X_{xy} \\ Y_x & Y_{xy} \end{vmatrix} - \begin{vmatrix} X_{xx} & X_y \\ Y_{xx} & Y_y \end{vmatrix} \right\} + \\ & + y_x y_x \cdot \left\{ \begin{vmatrix} X_x & X_{yy} \\ Y_x & Y_{yy} \end{vmatrix} - 2 \begin{vmatrix} X_{xy} & X_y \\ Y_{xy} & Y_y \end{vmatrix} \right\} + y_x y_x y_x \cdot \left\{ - \begin{vmatrix} X_{yy} & X_y \\ Y_{yy} & Y_y \end{vmatrix} \right\} \end{aligned} \right\} +$$

or equivalently, after solving in y_{xx} , *i.e.* after dividing by the Jacobian $\Delta(x|y)$:

$$(2.10) \quad \left\{ \begin{aligned} y_{xx} = & - \frac{\Delta(x|xx)}{\Delta(x|y)} + y_x \cdot \left\{ -2 \frac{\Delta(x|xy)}{\Delta(x|y)} + \frac{\Delta(xx|y)}{\Delta(x|y)} \right\} + \\ & + (y_x)^2 \cdot \left\{ - \frac{\Delta(x|yy)}{\Delta(x|y)} + 2 \frac{\Delta(xy|y)}{\Delta(x|y)} \right\} + (y_x)^3 \cdot \left\{ \frac{\Delta(yy|y)}{\Delta(x|y)} \right\}. \end{aligned} \right\}.$$

At this point, it will be convenient to slightly contract the notation by introducing a new family of *square functions* as follows. We first index the coordinates (x, y) as (y^0, y^1) , namely we introduce the two notational equivalences

$$(2.11) \quad \boxed{y^0 \equiv x, \quad y^1 \equiv y},$$

which will be very convenient in the sequel, especially to write down general combinatorial formulas anticipating our treatment of the case of $m \geq 2$ dependent variables (y^1, \dots, y^m) , to be achieved in Sections 3, 4 and 5 below. With this convention at hand, our six square functions $\square_{y^{j_1} y^{j_2}}^{k_1}$, symmetric with respect to the lower indices, where $0 \leq j_1, j_2, k_1 \leq 1$, are defined by

$$(2.12) \quad \left\{ \begin{aligned} \square_{xx}^0 &:= \frac{\Delta(xx|y)}{\Delta(x|y)}, & \square_{xy}^0 &:= \frac{\Delta(xy|y)}{\Delta(x|y)}, & \square_{yy}^0 &:= \frac{\Delta(yy|y)}{\Delta(x|y)}, \\ \square_{xx}^1 &:= \frac{\Delta(x|xx)}{\Delta(x|y)}, & \square_{xy}^1 &:= \frac{\Delta(x|xy)}{\Delta(x|y)}, & \square_{yy}^1 &:= \frac{\Delta(x|yy)}{\Delta(x|y)}. \end{aligned} \right.$$

Here of course, the upper index designates the column upon which the second order derivative appears, itself being encoded by the two lower indices. Even if this is hidden in the notation, we shall remember that the square functions are explicit rational expressions in terms of the second order jet of the transformation $(x, y) \mapsto (X(x, y), Y(x, y))$. However, we shall be aware of not confusing the index in the square functions with a second order partial derivative of some function “ \square^j ”, denoted by the square symbol: indeed, the partial derivatives are hidden in some determinant.

At this point, we may summarize what we have established so far.

Lemma 2.13. *The equation $y_{xx} = F(x, y, y_x)$ is equivalent to the flat equation $Y_{XX} = 0$ if and only if there exist two local \mathbb{K} -analytic functions $X(x, y)$ and $Y(x, y)$ such that it may be written under the form*

$$(2.14) \quad y_{xx} = -\square_{xx}^1 + y_x \cdot (-2\square_{xy}^1 + \square_{xx}^0) + (y_x)^2 \cdot (-\square_{yy}^1 + 2\square_{xy}^0) + (y_x)^3 \cdot \square_{yy}^0.$$

At this point, for heuristic reasons, it may be useful to compare the right-hand side of (2.14) with the classical expression of the prolongation to the second order jet space of a general vector field of the form $L := X(x, y) \frac{\partial}{\partial x} + Y(x, y) \frac{\partial}{\partial y}$, which is given, according to [Lie1883], [Ol1986], [BK1989], by

$$(2.15) \quad \left\{ \begin{array}{l} L^{(2)} = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + [Y_x + y_x \cdot (Y_y - X_x) + (y_x)^2 \cdot (-X_y)] \frac{\partial}{\partial y_x} + \\ \quad + [Y_{xx} + y_x \cdot (2Y_{xy} - X_{xx}) + (y_x)^2 \cdot (Y_{yy} - 2X_{xy}) + (y_x)^3 \cdot (-X_{yy}) + \\ \quad + y_{xx} \cdot (Y_y - 2X_x) + y_x y_{xx} \cdot (-3X_y)] \frac{\partial}{\partial y_{xx}}. \end{array} \right.$$

We immediately see that (up to an overall minus sign) the right-hand side of (2.14) is formally analogous to the second line of (2.15) : the letter X corresponds to the symbol \square^0 and the letter Y corresponds to the symbol \square^1 . This analogy is no mystery, just because the formula for $L^{(2)}$ is classically obtained by differentiating at $\varepsilon = 0$ the second order prolongation $[\exp(\varepsilon L)(\cdot)]^{(2)}$ of the flow of L !

In fact, as we assumed that the transformation $(x, y) \mapsto (X(x, y), Y(x, y))$ is tangent to the identity at the origin, we may think that $X_x \cong 1$, $X_y \cong 0$, $Y_x \cong 0$ and $Y_y \cong 1$, whence the Jacobian $\Delta(x|y) \cong 1$ and moreover

$$(2.16) \quad \left\{ \begin{array}{lll} \square_{xx}^0 \cong X_{xx}, & \square_{xy}^0 \cong X_{xy}, & \square_{yy}^0 \cong X_{yy}, \\ \square_{xx}^1 \cong Y_{xx}, & \square_{xy}^1 \cong Y_{xy}, & \square_{yy}^1 \cong Y_{yy}. \end{array} \right.$$

By means of this (abusive) notational correspondence, we see that, up to an overall minus sign, the right-hand side of (2.14) transforms precisely to the second line of (2.15). This analogy will be useful in devising combinatorial formulas for the generalization of Lemma 2.13 to the case of $m \geq 2$ variables (y^1, \dots, y^m) , see Lemmas 3.22 and 3.32 below.

2.17. Continuation. Clearly, since the right-hand side of (2.14) is a polynomial of degree three in y_x , the first condition of Theorem 1.2 (4) immediately holds. We are therefore led to establish that the second condition is necessary and sufficient in order that there exist two local \mathbb{K} -analytic functions $X(x, y)$ and $Y(x, y)$ which solve the following system of nonlinear second order partial differential equations (remind that the second order jet of (X, Y) is hidden in the square functions):

$$(2.18) \quad \left\{ \begin{array}{l} G = -\square_{xx}^1, \\ H = -2\square_{xy}^1 + \square_{xx}^0, \\ L = -\square_{yy}^1 + 2\square_{xy}^0, \\ M = \square_{yy}^0. \end{array} \right.$$

In the remainder of this section, following [Lie1883], p. 364, we shall study this second order system by introducing two auxiliary systems of partial differential equations which are *complete*, and we shall see in §2.38 below that the compatibility conditions (insuring involutivity, hence complete integrability) of the second auxiliary system exactly provide the two partial differential equations appearing in Theorem 1.2 (4).

2.19. First auxiliary system. We notice that in (2.18), there are two more square functions $\square_{xx}^0, \square_{xy}^0, \square_{yy}^0, \square_{xx}^1, \square_{xy}^1, \square_{yy}^1$, than functions G, H, L and M . Hence, as a trick, let us introduce six new independent functions $\Pi_{j_1, j_2}^{k_1}$ of (x, y) , symmetric with respect to the lower indices, for $0 \leq j_1, j_2, k_1 \leq 1$ and let us seek necessary and sufficient conditions in order that there exist solutions (X, Y) to the *first auxiliary system*:

$$(2.20) \quad \begin{cases} \square_{xx}^0 = \Pi_{0,0}^0, & \square_{xy}^0 = \Pi_{0,1}^0, & \square_{yy}^0 = \Pi_{1,1}^0, \\ \square_{xx}^1 = \Pi_{0,0}^1, & \square_{xy}^1 = \Pi_{0,1}^1, & \square_{yy}^1 = \Pi_{1,1}^1. \end{cases}$$

According to the (aproximate) identities (2.16), this system looks like a complete second order system of partial differential equations in two variables (x, y) and in two unknowns (X, Y) . More rigorously, by means of elementary algebraic operations, taking account of the fact that $X_x \cong 1, X_y \cong 0, Y_x \cong 0$ and $Y_y \cong 1$, one may transform this sytem in a true second order *complete* system, solved with respect to the top order derivatives, namely of the form

$$(2.21) \quad \begin{cases} X_{xx} = \Lambda_{0,0}^0, & X_{xy} = \Lambda_{0,1}^0, & X_{yy} = \Lambda_{1,1}^0, \\ Y_{xx} = \Lambda_{0,0}^1, & Y_{xy} = \Lambda_{0,1}^1, & Y_{yy} = \Lambda_{1,1}^1, \end{cases}$$

where the $\Lambda_{j_1, j_2}^{k_1}$ are local \mathbb{K} -analytic functions of $(x, y, X, Y, X_x, X_y, Y_x, Y_y)$. For such a system, the compatibility conditions [which are necessary and sufficient for the existence of a solution (X, Y)] are easily formulated:

$$(2.22) \quad \begin{cases} (\Lambda_{0,0}^0)_y = (\Lambda_{0,1}^0)_x, & (\Lambda_{0,1}^0)_y = (\Lambda_{1,1}^0)_x, \\ (\Lambda_{0,0}^1)_y = (\Lambda_{0,1}^1)_x, & (\Lambda_{0,1}^1)_y = (\Lambda_{1,1}^1)_x. \end{cases}$$

Equivalently, we may express the compatibility conditions directly with the system (2.20), without transforming it to the form (2.21). This direct strategy will be more appropriate.

2.23. Compatibility conditions for the first auxiliary system. Indeed, to begin with, let us remind that the $\Delta(\cdot|\cdot)$ are determinant, hence we have the skew-symmetry relation $\Delta(x^a y^b | x^c y^d) = -\Delta(x^c y^d | x^a y^b)$ and the following two formulas for partial differentiation

$$(2.24) \quad \begin{cases} [\Delta(x^a y^b | x^c y^d)]_x = \Delta(x^{a+1} y^b | x^c y^d) + \Delta(x^a y^b | x^{c+1} y^d), \\ [\Delta(x^a y^b | x^c y^d)]_y = \Delta(x^a y^{b+1} | x^c y^d) + \Delta(x^a y^b | x^c y^{d+1}). \end{cases}$$

With these formal rules at hand, as an exercise, let us compute for instance the following cross differentiation (remember that the lower index in the square functions is *not*

a partial derivative):

$$(2.25) \quad \left\{ \begin{aligned} & (\square_{xx}^0)_y - (\square_{xy}^0)_x = \frac{\partial}{\partial y} \left(\frac{\Delta(xx|y)}{\Delta(x|y)} \right) - \frac{\partial}{\partial x} \left(\frac{\Delta(xy|y)}{\Delta(x|y)} \right) = \\ & = \frac{1}{[\Delta(x|y)]^2} \left\{ \frac{\Delta(xxy|y) \cdot \Delta(x|y)}{\textcircled{a}} + \Delta(xx|yy) \cdot \Delta(x|y) - \right. \\ & \quad \left. - \frac{\Delta(xy|y) \cdot \Delta(xx|y)}{\textcircled{b}} - \Delta(x|yy) \cdot \Delta(xx|y) - \right. \\ & \quad \left. - \frac{\Delta(xxy|y) \cdot \Delta(x|y)}{\textcircled{a}} - \frac{\Delta(xy|xy) \cdot \Delta(x|y)}{\textcircled{c}} + \right. \\ & \quad \left. + \frac{\Delta(xy|y) \cdot \Delta(xx|y)}{\textcircled{b}} + \Delta(xy|y) \cdot \Delta(x|xy) \right\} = \\ & = \frac{1}{[\Delta(x|y)]^2} \left\{ \Delta(xx|yy) \cdot \Delta(x|y) - \Delta(x|yy) \cdot \Delta(xx|y) + \right. \\ & \quad \left. + \Delta(xy|y) \cdot \Delta(x|xy) \right\}. \end{aligned} \right.$$

Crucially, we observe that the third order derivatives kill each other and disappear, *see* the underlined terms with \textcircled{a} appended. Also, two products of two determinants $\Delta(\cdot|\cdot)$ involving a second order derivative upon one column of each determinant kill each other: they are underlined with \textcircled{b} appended. Finally, by antisymmetry of determinants, the term $\Delta(xy|xy) \cdot \Delta(x|y)$ vanishes gratuitously: it is underlined with \textcircled{c} appended.

However, there still remains one term involving second order derivatives upon the *two* columns of a determinant: it is $\Delta(xx|yy)$.

We must transform this unpleasant term $\Delta(xx|yy) \cdot \Delta(x|y)$ and express it as a product of two determinants, each involving a second order derivative only in one column. To this aim, we have:

Lemma 2.26. *The following three relations between the differential determinants $\Delta(\cdot|\cdot)$ hold true:*

$$(2.27) \quad \left\{ \begin{aligned} & \Delta(xx|xy) \cdot \Delta(x|y) = \Delta(xx|y) \cdot \Delta(x|xy) - \Delta(xy|y) \cdot \Delta(x|xx), \\ & \Delta(xx|yy) \cdot \Delta(x|y) = \Delta(xx|y) \cdot \Delta(x|yy) - \Delta(yy|y) \cdot \Delta(x|xx), \\ & \Delta(xy|yy) \cdot \Delta(x|y) = \Delta(xy|y) \cdot \Delta(x|yy) - \Delta(yy|y) \cdot \Delta(x|xy). \end{aligned} \right.$$

Proof. Each of these three formal identities is an immediate direct consequence of the following Plücker type identity, easily verified by developing all the determinants :

$$(2.28) \quad \left| \begin{array}{cc} A_1 & B_1 \\ A_2 & B_2 \end{array} \right| \cdot \left| \begin{array}{cc} C_1 & D_1 \\ C_2 & D_2 \end{array} \right| = \left| \begin{array}{cc} A_1 & D_1 \\ A_2 & D_2 \end{array} \right| \cdot \left| \begin{array}{cc} C_1 & B_1 \\ C_2 & B_2 \end{array} \right| - \left| \begin{array}{cc} B_1 & D_1 \\ B_2 & D_2 \end{array} \right| \cdot \left| \begin{array}{cc} C_1 & A_1 \\ C_2 & A_2 \end{array} \right|,$$

where the variables $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 \in \mathbb{K}$ are arbitrary. \square

Thanks to the second identity (2.27), we may therefore transform the result left above in the last two lines of (2.25); as desired, it will remain determinants having only one second order derivative per column, so that after division by $[\Delta(x|y)]^2$, we

discover a quadratic expression involving only the square functions themselves:

$$(2.29) \quad \left\{ \begin{aligned} (\square_{xx}^0)_y - (\square_{xy}^0)_x &= \frac{1}{[\Delta(x|y)]^2} \{ \Delta(xx|y) \cdot \Delta(x|yy) - \Delta(yy|y) \cdot \Delta(x|xx) - \\ &\quad - \Delta(x|yy) \cdot \Delta(xx|y) + \Delta(xy|y) \cdot \Delta(x|xy) \} \\ &= \frac{1}{[\Delta(x|y)]^2} \{ -\Delta(yy|y) \cdot \Delta(x|xx) + \Delta(xy|y) \cdot \Delta(x|xy) \} \\ &= -\square_{yy}^0 \cdot \square_{xx}^1 + \square_{xy}^0 \cdot \square_{xy}^1. \end{aligned} \right.$$

In sum, the result of the cross differentiation $(\square_{xx}^0)_y - (\square_{xy}^0)_x$ is a quadratic expression in terms of the square functions themselves! Following the same recipe (with no surprise), one may establish the following relations, listing all the compatibility conditions (the first one is nothing else than (2.29)):

$$(2.30) \quad \left\{ \begin{aligned} (\square_{xx}^0)_y - (\square_{xy}^0)_x &= -\square_{xx}^1 \cdot \square_{yy}^0 + \square_{xy}^1 \cdot \square_{xy}^0, \\ (\square_{xy}^0)_y - (\square_{yy}^0)_x &= -\square_{xy}^0 \cdot \square_{yx}^0 - \square_{xy}^1 \cdot \square_{yy}^0 + \square_{yy}^0 \cdot \square_{xx}^0 + \square_{yy}^1 \cdot \square_{xy}^0, \\ (\square_{xx}^1)_y - (\square_{xy}^1)_x &= -\square_{xx}^0 \cdot \square_{yx}^1 - \square_{xy}^1 \cdot \square_{yy}^1 + \square_{xy}^0 \cdot \square_{xx}^1 + \square_{xy}^1 \cdot \square_{xy}^1, \\ (\square_{xy}^1)_y - (\square_{yy}^1)_x &= -\square_{xy}^0 \cdot \square_{yx}^1 + \square_{yy}^0 \cdot \square_{xx}^1. \end{aligned} \right.$$

Instead of checking patiently each of the remaining three cross above cross differentiation identities, it is better to establish directly the following general relation.

Lemma 2.31. *Remind from (2.11) that we identify y^0 with x and y^1 with y and let $0 \leq j_1, j_2, j_3, k_1 \leq 1$. Then*

$$(2.32) \quad \left(\square_{y^{j_1} y^{j_2}}^{k_1} \right)_{y^{j_3}} - \left(\square_{y^{j_1} y^{j_3}}^{k_1} \right)_{y^{j_2}} = - \sum_{k_2=0}^1 \square_{y^{j_1} y^{j_2}}^{k_2} \cdot \square_{y^{j_3} y^{k_2}}^{k_1} + \sum_{k_2=0}^1 \square_{y^{j_1} y^{j_3}}^{k_2} \cdot \square_{y^{j_2} y^{k_2}}^{k_1}.$$

This lemma is left to the reader; anyway, we shall complete the proof of a generalization of Lemma 2.31 to the case of $m \geq 1$ dependent variables (y^1, \dots, y^m) in Section 2 below (Lemma 3.40).

Coming back to the first auxiliary system (2.20), we therefore have obtained a necessary and sufficient condition for the existence of (X, Y) : the functions $\Pi_{j^1, j^2}^{k_1}$ should satisfy the following system of first order partial differential equations, just obtained from (2.30) by replacing the square functions by the Pi functions:

$$(2.33) \quad \left\{ \begin{aligned} (\Pi_{0,0}^0)_y - (\Pi_{0,1}^0)_x &= -\Pi_{0,0}^1 \cdot \Pi_{1,1}^0 + \Pi_{0,1}^1 \cdot \Pi_{0,1}^0, \\ (\Pi_{0,1}^0)_y - (\Pi_{1,1}^0)_x &= -\Pi_{0,1}^0 \cdot \Pi_{0,1}^0 - \Pi_{0,1}^1 \cdot \Pi_{1,1}^0 + \Pi_{1,1}^0 \cdot \Pi_{0,0}^0 + \Pi_{1,1}^1 \cdot \Pi_{0,1}^0, \\ (\Pi_{0,0}^1)_y - (\Pi_{0,1}^1)_x &= -\Pi_{0,0}^0 \cdot \Pi_{0,1}^1 - \Pi_{0,0}^1 \cdot \Pi_{1,1}^1 + \Pi_{0,1}^0 \cdot \Pi_{1,1}^1 + \Pi_{0,1}^1 \cdot \Pi_{0,1}^1, \\ (\Pi_{0,1}^1)_y - (\Pi_{1,1}^1)_x &= -\Pi_{0,1}^0 \cdot \Pi_{0,1}^1 + \Pi_{1,1}^0 \cdot \Pi_{0,0}^1. \end{aligned} \right.$$

2.34. Second auxiliary system. It is now time to come back to the functions G, H, L and M and to get rid of the auxiliary ‘‘Pi’’ functions. Unfortunately, we cannot invert directly the linear system (2.18), hence we must choose two specific square functions as *principal unknowns*, and the best, from a combinatorial point of view, is to choose \square_{xx}^0 and \square_{yy}^1 . Remind that by (2.20), we have $\square_{xx}^0 = \Pi_{0,0}^0$ and $\square_{yy}^1 = \Pi_{1,1}^1$. For clarity,

it will be useful to adopt the notational equivalences

$$(2.35) \quad \Theta^0 \equiv \Pi_{0,0}^0 \quad \text{and} \quad \Theta^1 \equiv \Pi_{1,1}^1.$$

We may therefore quasi-inverse the linear system (2.18), obtaining that the four functions $\Pi_{0,0}^1$, $\Pi_{0,1}^1$, $\Pi_{0,1}^0$ and $\Pi_{1,1}^0$ may be expressed in terms of the functions G , H , L and M and in terms of the remaining two principal unknowns (2.35), which yields:

$$(2.36) \quad \begin{cases} \Pi_{0,0}^1 = \square_{xx}^1 = -G \\ \Pi_{0,1}^1 = \square_{xy}^1 = -\frac{1}{2}H + \frac{1}{2}\Theta^0, \\ \Pi_{0,1}^0 = \square_{xy}^0 = \frac{1}{2}L + \frac{1}{2}\Theta^1, \\ \Pi_{1,1}^0 = \square_{yy}^0 = M. \end{cases}$$

Replacing now each of these four expressions in the compatibility conditions of the first auxiliary system (2.33), solving the four equations with respect to Θ_y^1 , Θ_x^0 , Θ_x^1 and Θ_y^0 , we get after hygienic simplifications what we shall call the *second auxiliary system*, which is a complete system of first order partial derivatives in the remaining two principal unknowns Θ^0 and Θ^1 :

$$(2.37) \quad \begin{cases} \Theta_y^1 = -L_y + 2M_x + HM - \frac{1}{2}L^2 + M\Theta^0 + \frac{1}{2}(\Theta^1)^2, \\ \Theta_x^0 = -2G_y + H_x + GL - \frac{1}{2}H^2 - G\Theta^1 + \frac{1}{2}(\Theta^0)^2, \\ \Theta_x^1 = -\frac{2}{3}H_y + \frac{1}{3}L_x + 2GM - \frac{1}{2}HL - \frac{1}{2}H\Theta^1 + \frac{1}{2}L\Theta^0 + \frac{1}{2}\Theta^0\Theta^1, \\ \Theta_y^0 = -\frac{1}{3}H_y + \frac{2}{3}L_x + 2GM - \frac{1}{2}HL - \frac{1}{2}H\Theta^1 + \frac{1}{2}L\Theta^0 + \frac{1}{2}\Theta^0\Theta^1. \end{cases}$$

We do not comment the intermediate computations, since they offer no new combinatorial discovery.

2.38. Precise lexicographic rules. We group first order derivatives before zeroth order derivatives; in each group, we respect the lexicographic order of appearance given by the sequence G , H , L , M , Θ^0 , Θ^1 ; we always put rational coefficient of every differential monomial in its left; consequently, we accept a minus sign just after an equality sign, as for instance in (2.36)₁ and in (2.37)₂; for clarity, we prefer to write a complicated differential equation as $0 = \Phi$, with 0 on the left, instead of $\Phi = 0$, since Φ may incorporate 10, 20 and up to 150 monomials, as will happen for instance in the next sections below.

2.39. Compatibility conditions for the second auxiliary system. Clearly, the necessary and sufficient condition for the existence of solutions (Θ^0, Θ^1) to the second auxiliary system (2.37) is that the two cross differentiations vanish:

$$(2.40) \quad \begin{cases} 0 = (\Theta_x^0)_y - (\Theta_y^0)_x, \\ 0 = (\Theta_x^1)_y - (\Theta_y^1)_x. \end{cases}$$

Using (2.37), we shall see that we exactly obtain the two second order partial differential equations written in Theorem 1.2 (4). For completeness, we shall perform completely the computation of the first compatibility condition (2.40) and leave the second as an (easy) exercise.

First of all, inserting (2.37) and using the rule of Leibniz for the differentiation of a product, let us write the crude result, performing neither any simplification nor any reordering:

$$(2.41) \quad \left\{ \begin{array}{l} 0 = (\Theta_x^0)_y - (\Theta_y^0)_x \\ = -2G_{yy} + H_{xy} + G_y L + G L_y - H H_y - G_y \Theta^1 - G \Theta_y^1 + \Theta^0 \Theta_y^0 + \\ + \frac{1}{3} H_{xy} - \frac{2}{3} L_{xx} - 2G_x M - 2G M_x + \frac{1}{2} H_x L + \frac{1}{2} H L_x + \\ + \frac{1}{2} H_x \Theta^1 + \frac{1}{2} H \Theta_x^1 - \frac{1}{2} L_x \Theta^0 - \frac{1}{2} L \Theta_x^0 - \frac{1}{2} \Theta_x^0 \Theta^1 - \frac{1}{2} \Theta^0 \Theta_x^1. \end{array} \right.$$

Next, replacing each first order derivative Θ_x^0 , Θ_y^0 , Θ_x^1 and Θ_y^1 occuring in (2.41) by its expression given in (2.37), we obtain (suffering a little) as a brute result, before any simplification (except that we put all second order derivatives in the beginning):

$$(2.42) \quad \left\{ \begin{array}{l} 0 = -2G_{yy} + \frac{4}{3}H_{xy} - \frac{2}{3}L_{xx} + \\ + G_y L + G L_y - H H_y - G_y \Theta^1 + G L_y - 2G M_x - G H M + \\ + \frac{1}{2}G(L)^2 - G M \Theta^0 - \frac{1}{2}G(\Theta^1)^2 - \frac{1}{3}H_y \Theta^0 + \frac{2}{3}L_x \Theta^0 + \\ + 2G M \Theta^0 - \frac{1}{2}H L \Theta^0 - \frac{1}{2}H \Theta^0 \Theta^1 + \frac{1}{2}L(\Theta^0)^2 + \frac{1}{2}(\Theta^0)^2 \Theta^1 - \\ - 2G_x M - 2G M_x + \frac{1}{2}H_x L + \frac{1}{2}H L_x + \frac{1}{2}H_x \Theta^1 + \\ + \frac{1}{6}H L_x - \frac{1}{3}H H_y + G H M - \frac{1}{4}(H)^2 L - \frac{1}{4}(H)^2 \Theta^1 + \\ + \frac{1}{4}H L \Theta^0 + \frac{1}{4}H \Theta^0 \Theta^1 - \frac{1}{2}L_x \Theta^0 + G_y L - \frac{1}{2}H_x L - \frac{1}{2}G(L)^2 + \\ + \frac{1}{4}H^2 L + \frac{1}{2}G L \Theta^1 - \frac{1}{4}L(\Theta^0)^2 + G_y \Theta^1 - \frac{1}{2}H_x \Theta^1 - \\ - \frac{1}{2}G L \Theta^1 + \frac{1}{4}H^2 \Theta^1 + \frac{1}{2}G(\Theta^1)^2 - \frac{1}{4}(\Theta^0)^2 \Theta^1 - \frac{1}{6}L_x \Theta^0 + \\ + \frac{1}{3}H_y \Theta^0 - G M \Theta^0 + \frac{1}{4}H L \Theta^0 + \frac{1}{4}H \Theta^0 \Theta^1 - \frac{1}{4}L(\Theta^0)^2 - \\ - \frac{1}{4}(\Theta^0)^2 \Theta^1. \end{array} \right.$$

Now, we can simplify this brute expression by chasing every couple (or triple, or quadruple) of terms killing each other. After (patient) simplification and lexicographic ordering, we obtain the equation

$$(2.43) \quad \left\{ \begin{array}{l} 0 = -2G_{yy} + \frac{4}{3}H_{xy} - \frac{2}{3}L_{xx} + \\ + 2(G L)_y - 2G_x M - 4G M_x + \frac{2}{3}H L_x - \frac{4}{3}H H_y, \end{array} \right.$$

which is exactly the first equation of **(4)** of Theorem 1.2. The treatment of the second one is totally similar. This completes the proof of the equivalence between **(1)** and **(4)** in Theorem 1.2. \square

2.44. Interlude: about hand-computed formulas. In Section 4 below, when dealing with several dependent variables y^1, \dots, y^m , many simplifications of identities which are much more massive than (2.42) will occur several times. It is therefore welcome to explain how we manage to achieve such computations, without mistakes at the end and strictly by hand. One of the trick is to use colors, which, unfortunately, cannot be restituted in this printed document. Another trick is to *underline and to number the terms which disappear together*, by pair, by triple, by quadruple, *etc.* This trick is illustrated in the detailed identity (2.45) below, extracted from our manuscript, which is a copy of (2.42) together with the designation of all the terms which vanish together. Hence, *we keep a written track of each intermediate step of every computation and of every simplification.* Checking the correctness of a computation simply by reading is then the easiest way, both for the writer and for the reader, although of course, it takes time, anyway.

On the contrary, when relying upon a digital computer, most intermediate steps are invisible; the chase of mistakes is by reading the program and by testing it on several instances. Alas, all the finest intuitions which may awake in the extreme inside of a long computation are essentially absent, the mind believing that the machine is stronger for such tasks. This last belief is in part true, in the case where straightforward known computations are concerned, and in part untrue, in the case where some new hidden mathematical reality is concerned.

For us, *the challenge is to control everything in a sea of signs.* Computations are to be organized like a living giant coral tree, all part of which should be clearly visible in a transparent fluid of thought, and permanently subject to corrections.

Indeed, it often happens that going through a problem involving massive formal computations, some disharmony or some incoherency is discovered. Then one has to inspect every living atom in the preceding branches of the growing coral tree of computations until some very tiny or ridiculous mistake is found. In addition to making easy the reading, *a perfectly rigorous way of writing the formal identities which respects a large amount of virtual conventions facilitates to reorganize rapidly the coral tree after a mistake has been found.* The accumulation of new virtual conventions, all of which we cannot speak, constitutes another coral meta-tree and another profound collection of trick. Finally, we use a blank fluid corrector to avoid copying to much.

Extracted from our manuscript, here is the identity (2.42) with the underlining-numbering of all the vanishing terms (without the original colours) until we get the final equation (2.43):

$$\begin{aligned}
0 = & -2G_{yy} + \frac{4}{3}H_{xy} - \frac{2}{3}L_{xx} + \\
& + G_y L + G L_y - H H_y - \underline{G_y \Theta^1}_{\textcircled{a}} + G L_y - 2G M_x - \underline{G H M}_{\textcircled{b}} + \\
& + \frac{1}{2} \underline{G(L)^2}_{\textcircled{c}} - \underline{G M \Theta^0}_{\textcircled{d}} - \frac{1}{2} \underline{G(\Theta^1)^2}_{\textcircled{e}} - \frac{1}{3} \underline{H_y \Theta^0}_{\textcircled{f}} + \frac{2}{3} \underline{L_x \Theta^0}_{\textcircled{g}} + \\
& + \underline{2G M \Theta^0}_{\textcircled{h}} - \frac{1}{2} \underline{H L \Theta^0}_{\textcircled{i}} - \frac{1}{2} \underline{H \Theta^0 \Theta^1}_{\textcircled{j}} + \frac{1}{2} \underline{L(\Theta^0)^2}_{\textcircled{k}} + \frac{1}{2} \underline{(\Theta^0)^2 \Theta^1}_{\textcircled{l}} - \\
& - 2G_x M - 2G M_x + \frac{1}{2} H_x L + \frac{1}{2} H L_x + \frac{1}{2} \underline{H_x \Theta^1}_{\textcircled{m}} + \\
& + \frac{1}{6} H L_x - \frac{1}{3} H H_y + \underline{G H M}_{\textcircled{n}} - \frac{1}{4} \underline{(H)^2 L}_{\textcircled{o}} - \frac{1}{4} \underline{(H)^2 \Theta^1}_{\textcircled{p}} +
\end{aligned}$$

$$\begin{aligned}
(2.45) \quad & + \frac{1}{4} \underline{H L \Theta^0}_{\textcircled{f}} + \frac{1}{4} \underline{H \Theta^0 \Theta^1}_{\textcircled{g}} - \frac{1}{2} \underline{L_x \Theta^0}_{\textcircled{e}} + G_y L - \frac{1}{2} H_x L - \frac{1}{2} \underline{G (L)^2}_{\textcircled{a}} + \\
& + \frac{1}{4} \underline{H^2 L}_{\textcircled{l}} + \frac{1}{2} \underline{G L \Theta^1}_{\textcircled{n}} - \frac{1}{4} \underline{L (\Theta^0)^2}_{\textcircled{h}} + G_y \Theta^1 - \frac{1}{2} \underline{H_x \Theta^1}_{\textcircled{j}} - \\
& - \frac{1}{2} \underline{G L \Theta^1}_{\textcircled{n}} + \frac{1}{4} \underline{H^2 \Theta^1}_{\textcircled{m}} + \frac{1}{2} \underline{G (\Theta^1)^2}_{\textcircled{c}} - \frac{1}{4} \underline{(\Theta^0)^2 \Theta^1}_{\textcircled{i}} - \frac{1}{6} \underline{L_x \Theta^0}_{\textcircled{e}} + \\
& + \frac{1}{3} \underline{H_y \Theta^0}_{\textcircled{d}} - \underline{G M \Theta^0}_{\textcircled{b}} + \frac{1}{4} \underline{H L \Theta^0}_{\textcircled{f}} + \frac{1}{4} \underline{H \Theta^0 \Theta^1}_{\textcircled{g}} - \frac{1}{4} \underline{L (\Theta^0)^2}_{\textcircled{h}} - \\
& - \frac{1}{4} \underline{(\Theta^0)^2 \Theta^1}_{\textcircled{i}}.
\end{aligned}$$

As may be observed, the order in which we discover the terms which vanish is governed by chance. After some terms are underlined, they are automatically disregarded by the eyes, which lightens the chasing of other terms to be simplified. To collect the remaining terms in order to obtain the final expression (2.43), our method is similar: we underline the terms which may be summed together. However, whereas we use the red pencil to underline the vanishing terms, we use the green pencil to underline the remaining terms. This small trick is to avoid as much as possible to copy several times some long formal expressions. Finally, we reorder everything lexicographically, so as to get the conclusion (2.43). In order to obtain the final equation (2.43) as efficiently as possible, we read the remaining terms, picking them directly in lexicographic order. If, by lack of luck, one or two terms are forgotten by the eyes and not written in the right place, we copy once more the very final result in the right order, or we use the blank corrector.

Of course, such a refined methodology could seem to be essentially superfluous for such relatively accessible computations. However, when passing to several dependent variables, the current expressions will be approximatively five times more massive. We may really ascertain that a clever methodology of hand computations is helpful in this category.

§3. SYSTEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS EQUIVALENT TO FREE PARTICLES

3.1. Combinatorics of the second order prolongation of a point transformation. In this section, we endeavour to explain how Lie's theorem and proof may be generalized to the case of several dependent variables. As in the statement of Theorem 1.7 (1), let us assume that the system $y_{xx}^j = F^j(x, y, y_x)$, $j = 1, \dots, m$, is equivalent under an invertible point transformation $(x, y) \mapsto (X(x, y), Y(x, y))$ to the free particle system $Y_{XX}^j = 0$, $j = 1, \dots, m$. By assumption, the Jacobian determinant

$$(3.2) \quad \Delta(x|y^1 | \dots | y^m) := \begin{vmatrix} X_x & X_{y^1} & \dots & X_{y^m} \\ Y_x^1 & Y_{y^1}^1 & \dots & Y_{y^m}^1 \\ \dots & \dots & \dots & \dots \\ Y_x^m & Y_{y^1}^m & \dots & Y_{y^m}^m \end{vmatrix}$$

does not vanish at the origin. As in the case $m = 1$, since the flat system $Y_{XX}^j = 0$ is left unchanged by any affine transformation, we can (and we shall) assume that the

transformation is tangent to the identity at the origin, so that the above Jacobian matrix equals the identity matrix at $(x, y) = (0, 0)$, whence in a neighborhood of the origin it is close to the identity matrix, namely

$$(3.3) \quad X_x \cong 1, \quad X_{y^j} \cong 0, \quad Y_x^j \cong 0, \quad Y_{y^{j_1}}^{j_2} \cong \delta_{j_1}^{j_2}.$$

Inductive formulas for the computation how the differential equation in the (X, Y) coordinates is related to the differential equation in the (x, y) coordinates may be found in [BK1989], [OL1995]; the explicit formulas are not achieved in these references. Let us recall the inductive formulas, just on the computational level (differential-geometric conceptual background about graph transformations may be found in [OI1986], Ch. 2).

First of all, we seek how the $Y_X^j := \frac{dY^j}{dX}$ are explicitly related to the y_x^l . It suffices to replace, in the identity

$$(3.4) \quad Y_X^j \cdot \left(X_x dx + \sum_{l=1}^m X_{y^l} dy^l \right) = Y_X^j dX = dY^j = Y_x^j dx + \sum_{l=1}^m Y_{y^l}^j dy^l$$

the differentials dy^l by $y_x^l dx$ and then to identify the coefficient of dx on both sides, which rapidly yields the formulas

$$(3.5) \quad Y_X^j = \frac{Y_x^j + \sum_{l=1}^m y_x^l Y_{y^l}^j}{X_x + \sum_{l=1}^m y_x^l X_{y^l}},$$

for $j = 1, \dots, m$.

Next, we seek how the $Y_{XX}^j := \frac{d^2 Y^j}{dX^2} = \frac{dY_X^j}{dX}$ are related to the $y_x^{l_1}, y_{xx}^{l_2}$. It suffices to again replace each dy_l by $y_x^l dx$ and each dy_x^l by $y_{xx}^l dx$ in the identity

$$(3.6) \quad \left\{ \begin{aligned} Y_{XX}^j \cdot \left(X_x dx + \sum_{l=1}^m X_{y^l} dy^l \right) &= Y_{XX}^j \cdot dX = dY_X^j \\ &= \frac{\partial Y_X^j}{\partial x} dx + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y^l} dy^l + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y_x^l} dy_x^l \\ &= \left(\frac{\partial Y_X^j}{\partial x} + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y^l} y_x^l + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y_x^l} y_{xx}^l \right) \cdot dx. \end{aligned} \right.$$

Before entering the precise combinatorics of the explicit expression of Y_{XX}^j , let us observe that the last term of (3.6) simply writes $D(Y_X^j) dx$, where D denotes the *total differentiation operator* (of order two) defined by

$$(3.7) \quad D := \frac{\partial}{\partial x} + \sum_{l=1}^m y_x^l \frac{\partial}{\partial y^l} + \sum_{l=1}^m y_{xx}^l \frac{\partial}{\partial y_x^l}.$$

Since $dX \equiv DX$ after replacing each dy^l by $y_x^l dx$, it follows that we may compactly rewrite (3.6) as

$$(3.8) \quad Y_{XX}^j DX \cdot dx = D(Y_X^j) \cdot dx$$

Consequently, the expressions of Y_X^j (obtained in (3.5)) and of Y_{XX}^j are

$$(3.9) \quad Y_X^j = \frac{DY^j}{DX} \quad \text{and} \quad Y_{XX}^j = \frac{D(Y_X^j)}{DX} = \frac{DDY^j \cdot DX - DDX \cdot DY^j}{[DX]^3}.$$

As, by assumption, the system $y_{xx}^j = F^j(x, y, y_x)$ transforms to the flat system $Y_{XX}^j = 0$, after erasing the denominator of (3.7), we come to the equations

$$(3.10) \quad 0 = DDY^j \cdot DX - DDX \cdot DY^j,$$

for $j = 1, \dots, m$. However, this too simple and too compact expression of the system $y_{xx}^j = F^j(x, y, y_x)$ is of no use and we must develop (patiently!) the explicit expressions of DDY^j , of DX , of DDX and of DY^j , using the complete expression of D defined in (3.6).

At this point, we would like to stress that *it constitutes already a nontrivial computational and combinatorial task to obtain a complete explicit formula for the system $y_{xx}^j = F^j(x, y, y_x)$ hidden in the compact form (3.10)*, which would be the generalization of the nice formula (2.9) involving modifications of the Jacobian determinant. For general $m \geq 2$, the complete proofs are postponed to Section 5 below.

Since it would be intuitively unsatisfactory to provide directly the final simplified expression of the development of (3.10) in the general case $m \geq 2$, let us firstly describe step by step how one may guess what is the generalization of (2.9).

For instance, in the case $m = 2$, by a direct and relatively short computation which consists in developing plainly (3.10), we obtain for $j = 1, 2$:

$$(3.11) \quad \left\{ \begin{array}{l} 0 = -X_{xx} Y_x^j + Y_{xx}^j X_x + \\ + y_x^1 \cdot \left[-X_{xx} Y_{y^1}^j + Y_{xx}^j X_{y^1} - 2X_{xy^1} Y_x^j + 2Y_{xy^1}^j X_x \right] + \\ + y_x^2 \cdot \left[-X_{xx} Y_{y^2}^j + Y_{xx}^j X_{y^2} - 2X_{xy^2} Y_x^j + 2Y_{xy^2}^j X_x \right] + \\ + y_x^1 y_x^1 \cdot \left[-2X_{xy^1} Y_{y^1}^j + 2Y_{xy^1}^j X_{y^1} - X_{y^1 y^1} Y_x^j + Y_{y^1 y^1}^j X_x \right] + \\ + y_x^1 y_x^2 \cdot \left[-2X_{xy^1} Y_{y^2}^j + 2Y_{xy^1}^j X_{y^2} - 2X_{xy^2} Y_{y^1}^j + 2Y_{xy^2}^j X_{y^1} - \right. \\ \qquad \qquad \qquad \left. - 2X_{y^1 y^2} Y_x^j + 2Y_{y^1 y^2}^j X_x \right] + \\ + y_x^2 y_x^2 \cdot \left[-2X_{xy^2} Y_{y^2}^j + 2Y_{xy^2}^j X_{y^2} - X_{y^2 y^2} Y_x^j + Y_{y^2 y^2}^j X_x \right] + \\ + y_x^1 y_x^1 y_x^1 \cdot \left[-X_{y^1 y^1} Y_{y^1}^j + Y_{y^1 y^1}^j X_{y^1} \right] + \\ + y_x^1 y_x^1 y_x^2 \cdot \left[-X_{y^1 y^1} Y_{y^2}^j + Y_{y^1 y^1}^j X_{y^2} - 2X_{y^1 y^2} Y_{y^1}^j + 2Y_{y^1 y^2}^j X_{y^1} \right] + \\ + y_x^1 y_x^2 y_x^2 \cdot \left[-X_{y^2 y^2} Y_{y^1}^j + Y_{y^2 y^2}^j X_{y^1} - 2X_{y^1 y^2} Y_{y^2}^j + 2Y_{y^1 y^2}^j X_{y^2} \right] + \\ + y_x^2 y_x^2 y_x^2 \cdot \left[-X_{y^2 y^2} Y_{y^2}^j + Y_{y^2 y^2}^j X_{y^2} \right] + \\ + y_{xx}^1 \cdot \left[-X_{y^1} Y_x^j + Y_{y^1}^j X_x + y_x^2 \cdot \left\{ -X_{y^1} Y_{y^2}^j + Y_{y^1}^j X_{y^2} \right\} \right] + \\ + y_{xx}^2 \cdot \left[-X_{y^2} Y_x^j + Y_{y^2}^j X_x + y_x^1 \cdot \left\{ -X_{y^2} Y_{y^1}^j + Y_{y^2}^j X_{y^1} \right\} \right]. \end{array} \right.$$

Unfortunately, *the above two equations are not solved with respect to y_{xx}^1 and to y_{xx}^2* . Consequently, if we abbreviate them as a linear system of the form

$$(3.12) \quad \begin{cases} 0 = A^1 + y_{xx}^1 \cdot B_1^1 + y_{xx}^2 \cdot B_2^1, \\ 0 = A^2 + y_{xx}^1 \cdot B_1^2 + y_{xx}^2 \cdot B_2^2, \end{cases}$$

we have to solve for y_{xx}^1 and for y_{xx}^2 by means of the classical rule of Cramer. Here, it is rather quick to check manually that the determinant of this system has the following nice expression:

$$(3.13) \quad \begin{cases} \left| \begin{array}{cc} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{array} \right| = \Delta(x|y^1|y^2) \cdot \{X_x + y_x^1 X_{y^1} + y_x^2 X_{y^2}\} \\ \left| \begin{array}{cc} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{array} \right| = \Delta(x|y^1|y^2) \cdot DX. \end{cases}$$

However, the complete solving for y_{xx}^1 and for y_{xx}^2 requires some more time. After a direct and rather long hand computation (or alternately, using Maple or Mathematica) one obtains formulas involving hidden 3×3 determinants, which have to be guessed by the intuition; the first equation that we obtain, namely for y_{xx}^1 is as follows:

$$(3.14) \quad \begin{aligned} 0 = & y_{xx}^1 \cdot \left(\begin{vmatrix} X_x & X_{y^1} & X_{y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} + \begin{vmatrix} X_x & X_{xx} & X_{y^2} \\ Y_x^1 & Y_{xx}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{xx}^2 & Y_{y^2}^2 \end{vmatrix} + \right. \\ & + y_x^1 \cdot \left. \left\{ 2 \begin{vmatrix} X_x & X_{xy^1} & X_{y^2} \\ Y_x^1 & Y_{xy^1}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{xy^1}^2 & Y_{y^2}^2 \end{vmatrix} - \begin{vmatrix} X_{xx} & X_{y^1} & X_{y^2} \\ Y_{xx}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xx}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \right\} + \right. \\ & + y_x^2 \cdot \left. \left\{ 2 \begin{vmatrix} X_x & X_{xy^2} & X_{y^2} \\ Y_x^1 & Y_{xy^2}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{xy^2}^2 & Y_{y^2}^2 \end{vmatrix} \right\} + \right. \\ & + y_x^1 y_x^1 \cdot \left(\begin{vmatrix} X_x & X_{y^1 y^1} & X_{y^2} \\ Y_x^1 & Y_{y^1 y^1}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{y^1 y^1}^2 & Y_{y^2}^2 \end{vmatrix} - 2 \begin{vmatrix} X_{xy^1} & X_{y^1} & X_{y^2} \\ Y_{xy^1}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xy^1}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \right) + \\ & + y_x^1 y_x^2 \cdot \left\{ 2 \begin{vmatrix} X_x & X_{y^1 y^2} & X_{y^2} \\ Y_x^1 & Y_{y^1 y^2}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{y^1 y^2}^2 & Y_{y^2}^2 \end{vmatrix} - 2 \begin{vmatrix} X_{xy^2} & X_{y^1} & X_{y^2} \\ Y_{xy^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xy^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \right\} + \\ & + y_x^2 y_x^2 \cdot \left(\begin{vmatrix} X_x & X_{y^2 y^2} & X_{y^2} \\ Y_x^1 & Y_{y^2 y^2}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{y^2 y^2}^2 & Y_{y^2}^2 \end{vmatrix} \right) + \\ & + y_x^1 y_x^1 y_x^1 \cdot \left(- \begin{vmatrix} X_{y^1 y^1} & X_{y^1} & X_{y^2} \\ Y_{y^1 y^1}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1 y^1}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \right) + y_x^1 y_x^1 y_x^2 \cdot \left(-2 \begin{vmatrix} X_{y^1 y^2} & X_{y^1} & X_{y^2} \\ Y_{y^1 y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1 y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \right) + \\ & + y_x^1 y_x^2 y_x^2 \cdot \left(- \begin{vmatrix} X_{y^2 y^2} & X_{y^1} & X_{y^2} \\ Y_{y^2 y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^2 y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \right). \end{aligned}$$

This formula and the next have been checked by Sylvain Neut and Michel Petitot with the help of Maple. We notice that the size is not negligible, but fortunately, there

appears some combinatorics, much more visible than in (3.11). The second equation that we obtain, namely for y_{xx}^2 , is as follows:

$$\begin{aligned}
 (3.15) \quad 0 = & y_{xx}^2 \cdot \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| + \left| \begin{array}{ccc} X_x & X_{y^1} & X_{xx} \\ Y_x^1 & Y_{y^1}^1 & Y_{xx}^1 \\ Y_x^2 & Y_{y^2}^2 & Y_{xx}^2 \end{array} \right| + \\
 & + y_x^1 \cdot \left\{ 2 \left| \begin{array}{ccc} X_x & X_{y^1} & X_{xy^1} \\ Y_x^1 & Y_{y^1}^1 & Y_{xy^1}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{xy^1}^2 \end{array} \right| \right\} + \\
 & + y_x^2 \cdot \left\{ 2 \left| \begin{array}{ccc} X_x & X_{y^1} & X_{xy^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{xy^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{xy^2}^2 \end{array} \right| - \left| \begin{array}{ccc} X_{xx} & X_{y^1} & X_{y^2} \\ Y_{xx}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xx}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} + \\
 & + y_x^1 y_x^1 \cdot \left\{ \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^1 y^1} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^1 y^1}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^1 y^1}^2 \end{array} \right| \right\} + \\
 & + y_x^1 y_x^2 \cdot \left\{ 2 \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^1 y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^1 y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^1 y^2}^2 \end{array} \right| - 2 \left| \begin{array}{ccc} X_{xy^1} & X_{y^1} & X_{y^2} \\ Y_{xy^1}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xy^1}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} + \\
 & + y_x^2 y_x^2 \cdot \left\{ \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^2 y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^2 y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^2 y^2}^2 \end{array} \right| - 2 \left| \begin{array}{ccc} X_{xy^2} & X_{y^1} & X_{y^2} \\ Y_{xy^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xy^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} + \\
 & + y_x^1 y_x^1 y_x^2 \cdot \left\{ - \left| \begin{array}{ccc} X_{y^1 y^1} & X_{y^1} & X_{y^2} \\ Y_{y^1 y^1}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1 y^1}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} + y_x^1 y_x^2 y_x^2 \cdot \left\{ -2 \left| \begin{array}{ccc} X_{y^1 y^2} & X_{y^1} & X_{y^2} \\ Y_{y^1 y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1 y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} + \\
 & + y_x^2 y_x^2 y_x^2 \cdot \left\{ - \left| \begin{array}{ccc} X_{y^2 y^2} & X_{y^1} & X_{y^2} \\ Y_{y^2 y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^2 y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\}.
 \end{aligned}$$

Importantly, the obtained formulas seem to be analogous to the formula (2.9), since we observe that the coefficients of the degree three polynomial in the y_x^l are modifications of the Jacobian determinant $\Delta(x|y^1|y^2)$.

To describe the underlying combinatorics, let us observe that there exist exactly six possible distinct second order derivatives: xx , xy^1 , xy^2 , $y^1 y^1$, $y^1 y^2$ and $y^2 y^2$. There are also exactly three columns in the Jacobian determinant (3.2). By replacing each of the three columns of first order derivatives by a column of second order derivatives (leaving X , Y^1 and Y^2 unchanged), we may build exactly eighteen new determinants

$$(3.16) \quad \left\{ \begin{array}{ccc} \Delta(xx|y^1|y^2) & \Delta(x|xx|y^2) & \Delta(x|y^1|xx) \\ \Delta(xy^1|y^1|y^2) & \Delta(x|xy^1|y^2) & \Delta(x|y^1|xy^1) \\ \Delta(xy^2|y^1|y^2) & \Delta(x|xy^2|y^2) & \Delta(x|y^1|xy^2) \\ \Delta(y^1 y^1|y^1|y^2) & \Delta(x|y^1 y^1|y^2) & \Delta(x|y^1|y^1 y^1) \\ \Delta(y^1 y^2|y^1|y^2) & \Delta(x|y^1 y^2|y^2) & \Delta(x|y^1|y^1 y^2) \\ \Delta(y^2 y^2|y^1|y^2) & \Delta(x|y^2 y^2|y^2) & \Delta(x|y^1|y^2 y^2), \end{array} \right.$$

where for instance

$$(3.17) \quad \left\{ \begin{array}{l} \Delta(\underline{y^1 y^2} | y^1 | y^2) := \begin{vmatrix} X_{y^1 y^2} & X_{y^1} & X_{y^2} \\ Y_{y^1 y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1 y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{vmatrix} \\ \Delta(x | y^1 | \underline{x y^2}) := \begin{vmatrix} X_x & X_{y^1} & X_{x y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{x y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{x y^2}^2 \end{vmatrix} \end{array} \right. \quad \text{and}$$

Hence, using the Δ -notation, we may rewrite the two equation (3.14) and (3.15) under a more compact form; after division by the Jacobian determinant $\Delta(x|y^1|y^2)$, the first equation becomes:

$$(3.18) \quad \left\{ \begin{array}{l} 0 = y_{xx}^1 + \frac{\Delta(x|xx|y^2)}{\Delta(x|y^1|y^2)} + y_x^1 \cdot \left\{ 2 \frac{\Delta(x|xy^1|y^2)}{\Delta(x|y^1|y^2)} - \frac{\Delta(xx|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^2 \cdot \left\{ 2 \frac{\Delta(x|xy^2|y^2)}{\Delta(x|y^1|y^2)} \right\} + y_x^1 y_x^1 \cdot \left\{ \frac{\Delta(x|y^1 y^1|y^2)}{\Delta(x|y^1|y^2)} - 2 \frac{\Delta(xy^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^1 y_x^2 \cdot \left\{ 2 \frac{\Delta(x|y^1 y^2|y^2)}{\Delta(x|y^1|y^2)} - 2 \frac{\Delta(xy^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y_x^2 y_x^2 \cdot \left\{ \frac{\Delta(x|y^2 y^2|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^1 y_x^1 y_x^1 \cdot \left\{ -\frac{\Delta(y^1 y^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y_x^1 y_x^1 y_x^2 \cdot \left\{ -2 \frac{\Delta(y^1 y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^1 y_x^2 y_x^2 \cdot \left\{ -\frac{\Delta(y^2 y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\}. \end{array} \right.$$

Similarly, the second equation takes the form:

$$(3.19) \quad \left\{ \begin{array}{l} 0 = y_{xx}^2 + \frac{\Delta(x|y^1|xx)}{\Delta(x|y^1|y^2)} + y_x^1 \cdot \left\{ 2 \frac{\Delta(x|y^1|xy^1)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^2 \cdot \left\{ 2 \frac{\Delta(x|y^1|xy^2)}{\Delta(x|y^1|y^2)} - \frac{\Delta(xx|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y_x^1 y_x^1 \cdot \left\{ \frac{\Delta(x|y^1|y^1 y^1)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^1 y_x^2 \cdot \left\{ 2 \frac{\Delta(x|y^1|y^1 y^2)}{\Delta(x|y^1|y^2)} - 2 \frac{\Delta(xy^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^2 y_x^2 \cdot \left\{ \frac{\Delta(x|y^2|y^2 y^2)}{\Delta(x|y^1|y^2)} - 2 \frac{\Delta(xy^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^1 y_x^1 y_x^2 \cdot \left\{ -\frac{\Delta(y^1 y^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y_x^1 y_x^2 y_x^2 \cdot \left\{ -2 \frac{\Delta(y^1 y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ + y_x^2 y_x^2 y_x^2 \cdot \left\{ -\frac{\Delta(y^2 y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\}. \end{array} \right.$$

Since the formulas are still of a consequent size, analogously to what was achieved in Section 2, we shall introduce a new family of *square functions* as follows. We first index the coordinates (x, y^1, \dots, y^m) as (y^0, y^1, \dots, y^m) , namely we introduce the notational equivalence

$$(3.20) \quad \boxed{y^0 \equiv x},$$

which will be very convenient in the sequel, especially in order to write general formulas. With this convention at hand, our eighteen square functions $\square_{y^1 y^2}^{k_1}$, defined for $0 \leq j_1, j_2, k_1 \leq 2$ are defined by

$$(3.21) \quad \left\{ \begin{array}{lll} \square_{xx}^0 := \frac{\Delta(xx|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^1}^0 := \frac{\Delta(xy^1|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^2}^0 := \frac{\Delta(xy^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \\ \square_{y^1 y^1}^0 := \frac{\Delta(y^1 y^1|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^1 y^2}^0 := \frac{\Delta(y^1 y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^2 y^2}^0 := \frac{\Delta(y^2 y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \\ \square_{xx}^1 := \frac{\Delta(x|xx|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^1}^1 := \frac{\Delta(x|xy^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^2}^1 := \frac{\Delta(x|xy^2|y^2)}{\Delta(x|y^1|y^2)} \\ \square_{y^1 y^1}^1 := \frac{\Delta(x|y^1 y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^1 y^2}^1 := \frac{\Delta(x|y^1 y^2|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^2 y^2}^1 := \frac{\Delta(x|y^2 y^2|y^2)}{\Delta(x|y^1|y^2)} \\ \square_{xx}^2 := \frac{\Delta(x|y^1|xx)}{\Delta(x|y^1|y^2)} & \square_{xy^1}^2 := \frac{\Delta(x|y^1|xy^1)}{\Delta(x|y^1|y^2)} & \square_{xy^2}^2 := \frac{\Delta(x|y^1|xy^2)}{\Delta(x|y^1|y^2)} \\ \square_{y^1 y^1}^2 := \frac{\Delta(x|y^1|y^1 y^1)}{\Delta(x|y^1|y^2)} & \square_{y^1 y^2}^2 := \frac{\Delta(x|y^1|y^1 y^2)}{\Delta(x|y^1|y^2)} & \square_{y^2 y^2}^2 := \frac{\Delta(x|y^1|y^2 y^2)}{\Delta(x|y^1|y^2)} \end{array} \right.$$

Obviously, the square functions are symmetric with respect to the lower indices: $\square_{y^1 y^2}^{k_1} = \square_{y^2 y^1}^{k_1}$. Here, the upper index designates the column upon which the second order derivative appears, itself being encoded by the two lower indices. Even if this is hidden in the notation, we shall remember that *the square functions are explicit rational expressions in terms of the second order jet of the transformation* $(x, y) \mapsto (X(x, y), Y(x, y))$. At this point, we may summarize what we have established so far.

Lemma 3.22. *The system of two second order ordinary differential equations $y_{xx}^1 = F^1(x, y, y_x)$ and $y_{xx}^2 = F^2(x, y, y_x)$ is equivalent, under a point transformation, to the flat system $Y_{XX}^1 = 0$ and $Y_{XX}^2 = 0$ if and only if there exist three local \mathbb{K} -analytic functions $X(x, y)$, $Y^1(x, y)$ and $Y^2(x, y)$ such that it may be written under the form*

$$(3.23) \quad \left\{ \begin{array}{l} 0 = y_{xx}^1 + \square_{xx}^1 + y_x^1 \cdot (2\square_{xy^1}^1 - \square_{xx}^0) + y_x^2 \cdot (2\square_{xy^2}^1) + \\ \quad + y_x^1 y_x^1 \cdot (\square_{y^1 y^1}^1 - 2\square_{xy^1}^0) + y_x^1 y_x^2 \cdot (2\square_{y^1 y^2}^1 - 2\square_{xy^2}^0) + y_x^2 y_x^2 \cdot (\square_{y^2 y^2}^1) + \\ \quad + y_x^1 y_x^1 y_x^1 \cdot (-\square_{y^1 y^1}^0) + y_x^1 y_x^1 y_x^2 \cdot (-2\square_{y^1 y^2}^0) + y_x^1 y_x^2 y_x^2 \cdot (-\square_{y^2 y^2}^0), \\ 0 = y_{xx}^2 + \square_{xx}^2 + y_x^1 \cdot (2\square_{xy^1}^2) + y_x^2 \cdot (2\square_{xy^2}^2 - \square_{xx}^0) + \\ \quad + y_x^1 y_x^1 \cdot (\square_{y^1 y^1}^2) + y_x^1 y_x^2 \cdot (2\square_{y^1 y^2}^2 - 2\square_{xy^1}^0) + y_x^2 y_x^2 \cdot (\square_{y^2 y^2}^2 - 2\square_{xy^2}^0) + \\ \quad + y_x^1 y_x^1 y_x^2 \cdot (-\square_{y^1 y^1}^0) + y_x^1 y_x^2 y_x^2 \cdot (-2\square_{y^1 y^2}^0) + y_x^2 y_x^2 y_x^2 \cdot (-\square_{y^2 y^2}^0). \end{array} \right.$$

3.24. Second Lie prolongation of a vector field. At this point, instead of proceeding further with the case $m = 2$, it is now time to pass to the general case $m \geq 2$. First of all, we would like to remind from [GM2003] the complete explicit expression of the point prolongation to the second order jet space of a general vector field of the form $L = X \frac{\partial}{\partial x} + \sum_{j=1}^m Y^j \frac{\partial}{\partial y^j}$: it is a vector field of the form

$$(3.25) \quad L^{(2)} = X \frac{\partial}{\partial x} + \sum_{j=1}^m Y^j \frac{\partial}{\partial y^j} + \sum_{j=1}^m \mathbf{R}_1^j \frac{\partial}{\partial y_x^j} + \sum_{j=1}^m \mathbf{R}_2^j \frac{\partial}{\partial y_{xx}^j},$$

where the coefficients \mathbf{R}_1^j and \mathbf{R}_2^j are polynomials in the jet space variables having as coefficients certain specific linear combinations of first and second order derivatives of X and of the Y^j :

$$(3.26) \quad \left\{ \begin{array}{l} \mathbf{R}_1^j = Y_x^j + \sum_{l_1=1}^m y_x^{l_1} \cdot \left[Y_{y^{l_1}}^j - \delta_{l_1}^j X_x \right] + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[-\delta_{l_1}^j X_{y^{l_2}} \right], \\ \mathbf{R}_2^j = Y_{xx}^j + \sum_{l_1=1}^m y_x^{l_1} \cdot \left[2 Y_{xy^{l_1}}^j - \delta_{l_1}^j X_{xx} \right] + \\ + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[Y_{y^{l_1} y^{l_2}}^j - \delta_{l_1}^j X_{xy^{l_2}} - \delta_{l_2}^j X_{xy^{l_1}} \right] + \\ + \sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l_3=1}^m y_x^{l_1} y_x^{l_2} y_x^{l_3} \cdot \left[-\delta_{l_1}^j X_{y^{l_2} y^{l_3}} \right] + \\ + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot \left[Y_{y^{l_1}}^j - 2 \delta_{l_1}^j X_x \right] + \\ + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_{xx}^{l_2} \cdot \left[-\delta_{l_1}^j X_{y^{l_2}} - 2 \delta_{l_2}^j X_{y^{l_1}} \right]. \end{array} \right.$$

However, since the notations in [GM2003] are different and since the general case of $n \geq 1$ independent variables and $m \geq 1$ dependent variables is considered there, it is certainly easier to reconstitute formulas (3.26) directly by means of the inductive formulas described in [Ol1986], [BK1989]).

Analogously to the observation made in Section 2, we guess that there exists a formal correspondence between the terms of \mathbf{R}_2^j not involving y_{xx}^l and the explicit form of the equation $y_{xx}^j = F^j(x, y, y_x)$ equivalent to $Y_{XX}^j = 0$. In the case $m = 2$, we claim that this formal correspondence also holds true. Indeed, it suffices to write formula (3.26) for \mathbf{R}_2^j modulo the y_{xx}^l , which yields two expressions in total analogy with the two explicit polynomials appearing in the right-hand side of (3.23):

$$(3.27) \quad \left\{ \begin{array}{l} \mathbf{R}_2^1 \pmod{y_{xx}^l} \equiv Y_{xx}^1 + y_x^1 \cdot \{2 Y_{xy^1}^1 - X_{xx}\} + y_x^2 \cdot \{2 Y_{xy^2}^1\} + y_x^1 y_x^1 \cdot \{Y_{y^1 y^1}^1 - 2 X_{xy^1}\} + \\ + y_x^1 y_x^2 \cdot \{2 Y_{y^1 y^2}^1 - 2 X_{xy^2}\} + y_x^2 y_x^2 \cdot \{Y_{y^2 y^2}^1\} + \\ + y_x^1 y_x^1 y_x^1 \cdot \{-X_{y^1 y^1}\} + y_x^1 y_x^1 y_x^2 \cdot \{-2 X_{y^1 y^2}\} + y_x^1 y_x^2 y_x^2 \cdot \{-X_{y^2 y^2}\}, \\ \mathbf{R}_2^2 \pmod{y_{xx}^l} \equiv Y_{xx}^2 + y_x^1 \cdot \{2 Y_{xy^1}^2\} + y_x^2 \cdot \{2 Y_{xy^2}^2 - X_{xx}\} + y_x^1 y_x^1 \cdot \{Y_{y^1 y^1}^2\} + \\ + y_x^1 y_x^2 \cdot \{2 Y_{y^1 y^2}^2 - 2 X_{xy^1}\} + y_x^2 y_x^2 \cdot \{Y_{y^2 y^2}^2 - 2 X_{xy^2}\} + \\ + y_x^1 y_x^1 y_x^2 \cdot \{-X_{y^1 y^1}\} + y_x^1 y_x^2 y_x^2 \cdot \{-2 X_{y^1 y^2}\} + y_x^2 y_x^2 y_x^2 \cdot \{-X_{y^2 y^2}\}, \end{array} \right.$$

Except for inductive inspiration (*see* the formulation of Lemma 3.32 below), this observation will not be used further. At this stage, it helps at least to maintain a strong intuitive control of the correctness of the underlying combinatorics.

3.28. System equivalent to the flat system. By induction, we therefore guess that the analogy holds for general $m \geq 2$, namely we guess the following combinatorics, which requires some preliminaries.

As in the beginning of §3.1, let $x \in \mathbb{K}$, let $y = (y^1, \dots, y^m) \in \mathbb{K}^m$, let $(x, y) \mapsto (X(x, y), Y(x, y))$ be a local \mathbb{K} -analytic transformation defined in a neighborhood of the origin in \mathbb{K}^{m+1} and assume that the system $y_{xx}^j = F^j(x, y, y_x)$, $j = 1, \dots, m$, is equivalent to the flat system $Y_{XX}^j = 0$, $j = 1, \dots, m$. By assumption, the Jacobian matrix of the equivalence equals the identity matrix at the origin. Remind that we identify x with y^0 . For all $k_1, l_1, l_2 = 0, \dots, m$, we define a modification

$$(3.29) \quad \Delta(x | \dots |^{k_1} y^{l_1} y^{l_2} | \dots | y^m)$$

of the Jacobian determinant as follows. We replace the k_1 -th column of the determinant (3.2), which consists of first order derivatives $\cdot y^{k_1}$, by a column which consists of second order derivatives $\cdot y^{l_1} y^{l_2}$. In (3.29), the notation $|^{k_1}$ designates the k_1 -th column, the first one being labelled by $k_1 = 0$ and the last one by $k_1 = m$. With this notation at hand, we may define the square functions

$$(3.30) \quad \square_{y^{l_1} y^{l_2}}^{k_1} := \frac{\Delta(x | \dots |^{k_1} y^{l_1} y^{l_2} | \dots | y^m)}{\Delta(x | \dots |^{k_1} y^{k_1} | \dots | y^m)},$$

which are rational expression in the second order jet of the transformation $(x, y) \mapsto (X(x, y), Y(x, y))$. As before, the denominator is the Jacobian determinant of the change of coordinates.

Since, according to (3.26), the expression of $\mathbf{R}_2^j \pmod{y_{xx}^l}$ is

$$(3.31) \quad \left\{ \begin{aligned} \mathbf{R}_2^j \pmod{y_{xx}^l} &= Y_{xx}^j + \sum_{l_1=1}^m y_x^{l_1} \cdot \left[2 Y_{xy^{l_1}}^j - \delta_{l_1}^j X_{xx} \right] + \\ &+ \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[Y_{y^{l_1} y^{l_2}}^j - \delta_{l_1}^j X_{xy^{l_2}} - \delta_{l_2}^j X_{xy^{l_1}} \right] + \\ &+ \sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l_3=1}^m y_x^{l_1} y_x^{l_2} y_x^{l_3} \cdot \left[-\delta_{l_1}^j X_{y^{l_2} y^{l_3}} \right], \end{aligned} \right.$$

and since, in the cases $m = 1$ and $m = 2$, we have already observed strong analogies between (3.31) and the complete explicit expression of the system $y_{xx}^j = F^j(x, y, y_x)$ equivalent to the flat system $Y_{XX}^j = 0$, we guess that the following lemma is formally true.

Lemma 3.32. *The system $y_{xx}^j = F^j(x, y, y_x)$, $j = 1, \dots, m$, is equivalent to the flat system $Y_{XX}^j = 0$, $j = 1, \dots, m$, if and only if there exist local \mathbb{K} -analytic functions $X(x, y)$ and $Y^j(x, y)$, $j = 1, \dots, m$, such that it may be written under the specific form*

$$(3.33) \quad \left\{ \begin{aligned} 0 &= y_{xx}^j + \square_{xx}^j + \sum_{l_1=1}^m y_x^{l_1} \cdot \left[2 \square_{xy^{l_1}}^j - \delta_{l_1}^j \square_{xx}^0 \right] + \\ &+ \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[\square_{y^{l_1} y^{l_2}}^j - \delta_{l_1}^j \square_{xy^{l_2}}^0 - \delta_{l_2}^j \square_{xy^{l_1}}^0 \right] + \\ &+ y_x^j \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[-\square_{y^{l_1} y^{l_2}}^0 \right]. \end{aligned} \right.$$

The complete proof of this lemma involves only linear algebra considerations, although with rather massive terms. This makes it rather lengthy. Consequently, we postpone it to the final Section 5 below.

3.34. First auxiliary system. Clearly, if we set

$$(3.35) \quad \begin{cases} G^j := -\square_{xx}^j, \\ H_{l_1}^j := -2\square_{xy^{l_1}}^j + \delta_{l_1}^j \square_{xx}^0, \\ L_{l_1, l_2}^j := -\square_{y^{l_1} y^{l_2}}^j + \delta_{l_1}^j \square_{xy^{l_2}}^0 + \delta_{l_2}^j \square_{xy^{l_1}}^0, \\ M_{l_1, l_2} := \square_{y^{l_1} y^{l_2}}^0, \end{cases}$$

we immediately see that the first condition of Theorem 1.7 holds true. Moreover, we claim that there are $m + 1$ more square functions than functions G^j , $H_{l_1}^j$, L_{l_1, l_2}^j and M_{l_1, l_2} . Indeed, taking account of the symmetries, we enumerate:

$$(3.36) \quad \begin{cases} \#\{\square_{xx}^j\} = m, & \#\{\square_{xx}^0\} = 1, \\ \#\{\square_{xy^{l_1}}^j\} = m^2, & \#\{\square_{xy^{l_1}}^0\} = m, \\ \#\{\square_{y^{l_1} y^{l_2}}^j\} = \frac{m^2(m+1)}{2}, & \#\{\square_{y^{l_1} y^{l_2}}^0\} = \frac{m(m+1)}{2}, \end{cases}$$

whereas

$$(3.37) \quad \begin{cases} \#\{G^j\} = m, & \#\{H_{l_1}^j\} = m^2, \\ \#\{L_{l_1, l_2}^j\} = \frac{m^2(m+1)}{2}, & \#\{M_{l_1, l_2}\} = \frac{m(m+1)}{2}. \end{cases}$$

Similarly as in Section 2, for $j, l_1, l_2 = 0, 1, \dots, m$, let us introduce functions Π_{l_1, l_2}^j of (x, y^1, \dots, y^m) , symmetric with respect to the lower indices, and let us seek necessary and sufficient conditions in order that there exist solutions (X, Y) to the *first auxiliary system* defined precisely by:

$$(3.38) \quad \begin{cases} \square_{xx}^0 = \Pi_{0,0}^0, & \square_{xy^{l_1}}^0 = \Pi_{0, l_1}^0, & \square_{y^{l_1} y^{l_2}}^0 = \Pi_{l_1, l_2}^0, \\ \square_{xx}^j = \Pi_{0,0}^j, & \square_{xy^{l_1}}^j = \Pi_{0, l_1}^j, & \square_{y^{l_1} y^{l_2}}^j = \Pi_{l_1, l_2}^j. \end{cases}$$

3.39. Compatibility conditions for the first auxiliary system. As in Section 2, the compatibility conditions for this system will simply be obtained by computing the cross differentiations. The following statement generalizes Lemma 2.31 and also provides a proof of it, in the case $m = 1$.

Lemma 3.40. *For all $j, l_1, l_2, l_3 = 0, 1, \dots, m$, we have the cross differentiation relations*

$$(3.41) \quad \left(\square_{y^{l_1} y^{l_2}}^j \right)_{y^{l_3}} - \left(\square_{y^{l_1} y^{l_3}}^j \right)_{y^{l_2}} = - \sum_{k=0}^m \square_{y^{l_1} y^{l_2}}^k \cdot \square_{y^{l_3} y^k}^j + \sum_{k=0}^m \square_{y^{l_1} y^{l_3}}^k \cdot \square_{y^{l_2} y^k}^j.$$

Proof. To begin with, as a preliminary, let us generalize the Plücker identity (2.28). Let $C_1, C_2, \dots, C_m, D, E$ be $(m + 2)$ column vectors in \mathbb{K}^m and introduce the following notation for the $m \times (m + 2)$ matrix consisting of these vectors:

$$(3.42) \quad [C_1 | C_2 | \dots | C_m | D | E].$$

Extracting columns from this matrix, we shall construct $m \times m$ determinants which are modification of the following “fundamental” determinant

$$(3.43) \quad \|C_1 | \cdots | C_m\| \equiv \|C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} C_{j_2} | \cdots | C_m\|.$$

Here and in the sequel, we use a double vertical line in the beginning and in the end to denote a determinant. Also, we emphasize two distinct columns, the j_1 -th and the j_2 -th, where $j_2 > j_1$, since we will modify them. For instance in this matrix, let us replace these two columns by the column D and by the column E , which yields the determinant

$$(3.44) \quad \|C_1 | \cdots |^{j_1} D | \cdots |^{j_2} E | \cdots | C_m\|.$$

In this notation, one should understand that *only* the j_1 -th and the j_2 -th columns are distinct from the columns of the fundamental $m \times m$ determinant (3.43). With this notation at hand, we can now formulate and prove a preliminary lemma that will be useful later.

Lemma 3.45. *The following quadratic identity between determinants holds true:*

$$(3.46) \quad \begin{cases} \|C_1 | \cdots |^{j_1} D | \cdots |^{j_2} E | \cdots | C_n\| \cdot \|C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} C_{j_2} | \cdots | C_n\| = \\ = \|C_1 | \cdots |^{j_1} D | \cdots |^{j_2} C_{j_2} | \cdots | C_n\| \cdot \|C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} E | \cdots | C_n\| - \\ - \|C_1 | \cdots |^{j_1} E | \cdots |^{j_2} C_{j_2} | \cdots | C_n\| \cdot \|C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} D | \cdots | C_n\|. \end{cases}$$

Proof. After some permutations of columns, this identity amounts to

$$(3.47) \quad \begin{cases} \|C_1 | \cdots | C_{m-2} | D | E\| \cdot \|C_1 | \cdots | C_{m-2} | C_{m-1} | C_m\| = \\ = \|C_1 | \cdots | C_{m-2} | D | C_m\| \cdot \|C_1 | \cdots | C_{m-2} | C_{m-1} | E\| - \\ - \|C_1 | \cdots | C_{m-2} | E | C_m\| \cdot \|C_1 | \cdots | C_{m-2} | C_{m-1} | D\|. \end{cases}$$

To establish this identity, we introduce some notation. If A and B are vertical vectors in \mathbb{K}^m and if $i_1, i_2 = 1, \dots, m$ with $i_1 < i_2$, we denote

$$(3.48) \quad \Delta_{i_1, i_2}^2(A|B) := \begin{vmatrix} A_{i_1} & B_{i_1} \\ A_{i_2} & B_{i_2} \end{vmatrix}.$$

If $\|A_1 | A_2 | A_3 | \cdots | A_m\|$ is a $m \times m$ determinant, and if $i_1, i_2 = 1, \dots, m$ with $i_1 < i_2$, we denote by $M_{i_1, i_2}^{m-2}(A_3 | \cdots | A_m)$ the $(m-2) \times (m-2)$ determinant obtained from the matrix $[A_3 | \cdots | A_m]$ by erasing the i_1 -th line and the i_2 -th line. Without proof, we recall an elementary classical formula

$$(3.49) \quad \begin{cases} \|A_1 | A_2 | A_3 | \cdots | A_m\| = \\ = \sum_{1 \leq i_1 < i_2 \leq m} (-1)^{i_1+i_2-1} \Delta_{i_1, i_2}^2(A_1 | A_2) \cdot M_{i_1, i_2}^{m-2}(A_3 | \cdots | A_m), \end{cases}$$

which may be established by developing the determinant $\|A_1 | A_2 | A_3 | \cdots | A_m\|$ with respect to its first column, and then re-developing all the obtained $(m-1) \times (m-1)$ determinants with respect to their first columns. To establish (3.47), we start with an equivalent version of the identity (2.28):

$$(3.50) \quad \begin{cases} \Delta_{i_1, i_2}^2(D|E) \cdot \Delta_{i_3, i_4}^2(C_1|C_2) = \Delta_{i_1, i_2}^2(D|C_2) \cdot \Delta_{i_3, i_4}^2(C_1|E) - \\ - \Delta_{i_1, i_2}^2(E|C_2) \cdot \Delta_{i_3, i_4}^2(C_1|D), \end{cases}$$

where $1 \leq i_1 < i_2 \leq m$ and $1 \leq i_3 < i_4 \leq m$. Multiplying by $(-1)^{i_1+i_2+i_3+i_4-2}$, multiplying by $M_{i_1, i_2}^{m-2}(C_3 | \cdots | C_m)$, and multiplying by $M_{i_3, i_4}^{m-2}(C_3 | \cdots | C_m)$ applying the double summation $\sum_{1 \leq i_1 < i_2 \leq m} \sum_{1 \leq i_3 < i_4 \leq m}$, we get

$$(3.51) \quad \left\{ \begin{array}{l} \sum_{1 \leq i_1 < i_2 \leq m} \sum_{1 \leq i_3 < i_4 \leq m} (-1)^{i_1+i_2+i_3+i_4-2} \Delta_{i_1, i_2}^2(D|E) \cdot \Delta_{i_3, i_4}^2(C_1|C_2) \cdot \\ \quad \cdot M_{i_1, i_2}^{m-2}(C_3 | \cdots | C_m) \cdot M_{i_3, i_4}^{m-2}(C_3 | \cdots | C_m) = \\ = \sum_{1 \leq i_1 < i_2 \leq m} \sum_{1 \leq i_3 < i_4 \leq m} (-1)^{i_1+i_2-1} (-1)^{i_3+i_4-1} [\Delta_{i_1, i_2}^2(D|C_2) \cdot \Delta_{i_3, i_4}^2(C_1|E) - \\ \quad - \Delta_{i_1, i_2}^2(E|C_2) \cdot \Delta_{i_3, i_4}^2(C_1|D)] \cdot M_{i_1, i_2}^{m-2}(C_3 | \cdots | C_m) \cdot M_{i_3, i_4}^{m-2}(C_3 | \cdots | C_m). \end{array} \right.$$

Thanks to the relation (3.49), this last identity coincides exactly with the desired identity (3.47). The proof is complete. \square

We can now establish Lemma 3.40. As a preliminary observation, by the Leibniz rule for the differentiation of a determinant, we must differentiate every column:

$$(3.52) \quad \left\{ \begin{array}{l} [\Delta(y^{l_1}y^{k_1} | \cdots | y^{l_m}y^{k_m})]_{y^j} = \Delta(y^j y^{l_1} y^{k_1} | \cdots | y^{l_m} y^{k_m}) + \cdots + \\ \quad + \Delta(y^{l_1} y^{k_1} | \cdots | y^j y^{l_m} y^{k_m}). \end{array} \right.$$

Using also the rule for the differentiation of a quotient, we may endeavour to compute the cross differentiations $\left(\square_{y^{l_1}y^{l_2}}^j\right)_{y^{l_3}} - \left(\square_{y^{l_1}y^{l_3}}^j\right)_{y^{l_2}}$ of the left-hand side of (3.41). This will generalize (2.25). Sometimes in the computation, we shall abbreviate the Jacobian determinant $\Delta(y^0 | \cdots | y^m)$ using the shorter notation Δ ; as before, a product between two elements of \mathbb{K} will often be denoted by the sign “ \cdot ”, for clarity. Here is the computation:

$$(3.53) \quad \left\{ \begin{array}{l} \left(\square_{y^{l_1}y^{l_2}}^j\right)_{y^{l_3}} - \left(\square_{y^{l_1}y^{l_3}}^j\right)_{y^{l_2}} = \\ = \frac{\partial}{\partial y^{l_3}} \left(\frac{\Delta(y^0 | \cdots | y^j y^{l_1} y^{l_2} | \cdots | y^m)}{\Delta(y^0 | \cdots | y^m)} \right) - \frac{\partial}{\partial y^{l_2}} \left(\frac{\Delta(y^0 | \cdots | y^j y^{l_1} y^{l_3} | \cdots | y^m)}{\Delta(y^0 | \cdots | y^m)} \right) \\ = \frac{1}{[\Delta]^2} \left[\begin{array}{l} \Delta(y^0 y^{l_3} | \cdots | y^j y^{l_1} y^{l_2} | \cdots | y^m) \cdot \Delta + \cdots + \\ + \Delta(y^0 | \cdots | y^j y^{l_3} y^{l_1} y^{l_2} | \cdots | y^m) \cdot \Delta_{\text{a}} + \cdots + \\ + \Delta(y^0 | \cdots | y^j y^{l_1} y^{l_2} | \cdots | y^{l_3} y^m) \cdot \Delta - \\ - \Delta(y^0 | \cdots | y^j y^{l_1} y^{l_2} | \cdots | y^m) \cdot [\Delta(y^0 y^{l_3} | \cdots | y^m) + \cdots + \\ \quad + \Delta(y^0 | \cdots | y^{l_3} y^m)] \end{array} \right] - \\ - \frac{1}{[\Delta]^2} \left[\begin{array}{l} \Delta(y^0 y^{l_2} | \cdots | y^j y^{l_1} y^{l_3} | \cdots | y^m) \cdot \Delta + \cdots + \\ + \Delta(y^0 | \cdots | y^j y^{l_2} y^{l_1} y^{l_3} | \cdots | y^m) \cdot \Delta_{\text{a}} + \cdots + \\ + \Delta(y^0 | \cdots | y^j y^{l_1} y^{l_3} | \cdots | y^{l_2} y^m) \cdot \Delta - \\ - \Delta(y^0 | \cdots | y^j y^{l_1} y^{l_3} | \cdots | y^m) \cdot [\Delta(y^0 y^{l_2} | \cdots | y^m) + \cdots + \\ \quad + \Delta(y^0 | \cdots | y^{l_2} y^m)] \end{array} \right]. \end{array} \right.$$

Crucially, we observe that all the determinants involving a third order derivative upon one of their columns kill each other and disappear: we have underlined them with \textcircled{a} appended. However, it still remains plenty of determinants involving a second order derivative upon two different columns. We must transform all of them and express them in terms of determinants involving a second order derivative upon only one column. To this aim, as an application of our preliminary Lemma 3.45, we have the following relations, valid for $j_1, j_2, l_1, l_2, l_3, l_4 = 0, \dots, m$ and $j_1 < j_2$:

$$(3.54) \quad \left\{ \begin{array}{l} \Delta(y^0 | \dots | \underline{j_1 y^{l_1} y^{l_2}} | \dots | \underline{j_2 y^{l_3} y^{l_4}} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j_1 y^{j_1}} | \dots | \underline{j_2 y^{j_2}} | \dots | y^m) = \\ = \Delta(y^0 | \dots | \underline{j_1 y^{l_1} y^{l_2}} | \dots | \underline{j_2 y^{j_2}} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j_1 y^{j_1}} | \dots | \underline{j_2 y^{l_3} y^{l_4}} | \dots | y^m) \\ - \Delta(y^0 | \dots | \underline{j_1 y^{l_3} y^{l_4}} | \dots | \underline{j_2 y^{j_2}} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j_1 y^{j_1}} | \dots | \underline{j_2 y^{l_1} y^{l_2}} | \dots | y^m). \end{array} \right.$$

With these formulas, we may transform the lines number 3, 4, 5 and 8, 9, 10 of (3.53). Also, we observe that the lines 6, 7 and 11, 12 of (3.53) involve determinants having a single second order derivative. Taking account of the $\frac{1}{[\Delta]^2}$ factor, we deduce that the lines 6, 7 and 11, 12 of (3.53) may already be expressed as sums of square functions. Achieving all these transformations, we may rewrite (3.53) as follows

$$(3.55) \quad \left\{ \begin{array}{l} \left(\square_{y^{l_1} y^{l_2}}^j \right)_{y^{l_3}} - \left(\square_{y^{l_1} y^{l_3}}^j \right)_{y^{l_2}} = \\ = \frac{1}{[\Delta]^2} \left[\begin{array}{l} \Delta(y^0 y^{l_3} | \dots | \underline{j y^j} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^{l_1} y^{l_2}} | \dots | y^m) - \\ - \Delta(y^{l_1} y^{l_2} | \dots | \underline{j y^j} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^0 y^{l_3}} | \dots | y^m) + \\ + \dots + \\ + \Delta(y^0 | \dots | \underline{j y^{l_1} y^{l_2}} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^j} | \dots | y^{l_3} y^m) - \\ - \Delta(y^0 | \dots | \underline{j y^{l_3} y^m} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^j} | \dots | y^{l_1} y^{l_2}) \end{array} \right] - \\ - \sum_{k=0}^m \square_{y^{l_1} y^{l_2}}^j \square_{y^{l_3} y^k}^k - \\ - \frac{1}{[\Delta]^2} \left[\begin{array}{l} \Delta(y^0 y^{l_2} | \dots | \underline{j y^j} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^{l_1} y^{l_3}} | \dots | y^m) - \\ - \Delta(y^{l_1} y^{l_3} | \dots | \underline{j y^j} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^0 y^{l_2}} | \dots | y^m) + \\ + \dots + \\ + \Delta(y^0 | \dots | \underline{j y^{l_1} y^{l_3}} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^j} | \dots | y^{l_2} y^m) - \\ - \Delta(y^0 | \dots | \underline{j y^{l_2} y^m} | \dots | y^m) \cdot \Delta(y^0 | \dots | \underline{j y^j} | \dots | y^{l_1} y^{l_3}) \end{array} \right] + \\ + \sum_{k=0}^m \square_{y^{l_1} y^{l_3}}^j \square_{y^{l_2} y^k}^k. \end{array} \right.$$

Notice that two pairs of “cdots” terms $+\dots+$ appearing in the lines 3, 4 and 8, 9 of (3.53) are replaced by a single “cdots” term $+\dots+$ in the lines 4 and 9 of (3.55). Importantly, we point that in the “middle” of the two “cdots” terms $+\dots+$ appearing in the lines 4 and 10 of (3.55) just above, there are two terms which do not occur: they simply correspond to the two underlined terms having \textcircled{a} appended appearing in the lines 4 and 9 (3.53).

Now, taking account of the factor $\frac{1}{[\Delta]^2}$, we can re-express all the terms of (3.55) as sums of square functions:

$$(3.56) \quad \left\{ \begin{aligned} & \left(\square_{y^{l_1} y^{l_2}}^j \right)_{y^{l_3}} - \left(\square_{y^{l_1} y^{l_3}}^j \right)_{y^{l_2}} = \\ & = \sum_{k=0; k \neq j}^m \square_{y^{l_3} y^k}^k \square_{y^{l_1} y^{l_2}}^j - \sum_{k=0; k \neq j}^m \square_{y^{l_1} y^{l_2}}^k \square_{y^{l_3} y^k}^j - \\ & - \sum_{k=0}^m \square_{y^{l_1} y^{l_2}}^j \square_{y^{l_3} y^k}^k - \\ & - \sum_{k=0; k \neq j}^m \square_{y^{l_2} y^k}^k \square_{y^{l_1} y^{l_3}}^j + \sum_{k=0; k \neq j}^m \square_{y^{l_1} y^{l_3}}^k \square_{y^{l_2} y^k}^j + \\ & + \sum_{k=0}^m \square_{y^{l_1} y^{l_3}}^j \square_{y^{l_2} y^k}^k. \end{aligned} \right.$$

Finally, we observe that in the two pairs of sums having $k \neq j$ appearing in the lines 2 and 4 just above, we can include the term $k = j$ in each pair, because these two terms are immediately killed inside the corresponding pair. In conclusion, after a final obvious killing of four (among six) complete sums in this modification of (3.56), we obtain the desired formula (3.41), with two sums. This completes the proof of Lemma 3.40 and also at the same occasion, the proof of Lemma 2.31. \square

3.57. Compatibility conditions for the first auxiliary system. According to the (approximate) identities (3.3), taking account of the explicit definitions (3.30) of the square functions, we have

$$(3.58) \quad \left\{ \begin{array}{lll} \square_{xx}^0 \cong X_{xx}, & \square_{xy^{l_1}}^0 \cong X_{xy^{l_1}}, & \square_{y^{l_1} y^{l_2}}^0 \cong X_{y^{l_1} y^{l_2}}, \\ \square_{xx}^j \cong Y_{xx}^j, & \square_{xy^{l_1}}^j \cong Y_{xy^{l_1}}^j, & \square_{y^{l_1} y^{l_2}}^j \cong Y_{y^{l_1} y^{l_2}}^j. \end{array} \right.$$

Consequently, the first auxiliary system (3.38) looks approximately like a complete second order system of partial differential equations in the $(m+1)$ independent variables (x, y) and in the $(m+1)$ dependent variables (X, Y) . By means of elementary algebraic operations, one may transform this system in a true second order *complete* system, solved with respect to the top order derivatives, namely of the form

$$(3.59) \quad \left\{ \begin{array}{lll} X_{xx} = \Lambda_{0,0}^0, & X_{xy^{l_1}} = \Lambda_{0,l_1}^0, & X_{y^{l_1} y^{l_2}} = \Lambda_{l_1,l_2}^0, \\ Y_{xx}^j = \Lambda_{0,0}^j, & Y_{xy^{l_1}}^j = \Lambda_{0,l_1}^j, & Y_{y^{l_1} y^{l_2}}^j = \Lambda_{l_1,l_2}^j, \end{array} \right.$$

where the $\Lambda_{j_1, j_2}^{k_1}$ are local \mathbb{K} -analytic functions of $(x, y^{l_1}, X, Y^j, X_x, X_{y^{l_1}}, Y_x^j, Y_{y^{l_1}}^j)$. For such a system, the compatibility conditions [which are necessary and sufficient for the existence of a solution (X, Y)] follow by obvious cross differentiation. Coming back to the system 3.38, these compatibility conditions amount to the quadratic-like compatibility conditions expressed in Lemma 3.40. In conclusion, we have proved the following intermediate statement.

Proposition 3.60. *There exist functions X, Y^j solving the first auxiliary system (3.38) of nonlinear second order partial differential equations if and only if the right-hand*

side functions $\Pi_{l_1, l_2}^j(x, y)$ satisfy the quadratic compatibility conditions

$$(3.61) \quad \frac{\partial \Pi_{l_1, l_2}^j}{\partial y^{l_3}} - \frac{\partial \Pi_{l_1, l_3}^j}{\partial y^{l_2}} = - \sum_{k=0}^m \Pi_{l_1, l_2}^k \cdot \Pi_{l_3, k}^j + \sum_{k=0}^m \Pi_{l_1, l_3}^k \cdot \Pi_{l_2, k}^j,$$

for $j, l_1, l_2, l_3 = 0, 1, \dots, m$.

3.62. Principal unknowns. As there are $(m + 1)$ more square (or Pi) functions than the functions $G^j, H_{l_1}^j, L_{l_1, l_2}^j$ and M_{l_1, l_2} defined by (3.35), we cannot invert directly the linear system (3.35) (which is of maximal rank). Hence we must choose $(m + 1)$ specific square functions, calling them *principal unknowns*, and similarly as in §2.34, the best choice is to choose \square_{xx}^0 and \square_{xx}^j , for $j = 1, \dots, m$. For clarity, it will be useful to adopt the notational equivalences

$$(3.63) \quad \Theta^0 \equiv \Pi_{0,0}^0 \quad \text{and} \quad \Theta^j \equiv \Pi_{j,j}^j.$$

Then we may quasi-inverse the system (3.35), which yields :

$$(3.64) \quad \left\{ \begin{array}{l} \Pi_{0,0}^j = \square_{xx}^j = -G^j, \\ \Pi_{0,l_1}^j = \square_{xy^{l_1}}^j = -\frac{1}{2} H_{l_1}^j + \frac{1}{2} \delta_{l_1}^j \Theta^0, \\ \Pi_{l_1, l_2}^j = \square_{y^{l_1} y^{l_2}}^j = -L_{l_1, l_2}^j + \frac{1}{2} \delta_{l_1}^j L_{l_2, l_2}^{l_2} + \frac{1}{2} \delta_{l_2}^j L_{l_1, l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^j \Theta^{l_2} + \frac{1}{2} \delta_{l_2}^j \Theta^{l_1}, \\ \Pi_{0, l_1}^0 = \square_{xy^{l_1}}^0 = \frac{1}{2} L_{l_1, l_1}^{l_1} + \frac{1}{2} \Theta^{l_1}, \\ \Pi_{l_1, l_2}^0 = \square_{y^{l_1} y^{l_2}}^0 = M_{l_1, l_2}. \end{array} \right.$$

Before replacing these new expressions of the functions $\Pi_{0,0}^j, \Pi_{0,l_1}^j, \Pi_{l_1, l_2}^j, \Pi_{l_1, l_2}^0$ and $\Pi_{0, l_1}^{l_1}$ into the compatibility conditions (3.61), it is necessary to expound first (3.61), taking account of the original splitting of the indices in the two sets $\{0\}$ and $\{1, 2, \dots, m\}$. This yields six families of compatibility conditions, totally equivalent to the compact identities (3.61):

$$(3.65) \quad \left\{ \begin{array}{l} \left(\Pi_{0,0}^j \right)_{y^{l_1}} - \left(\Pi_{0, l_1}^j \right)_x = -\Pi_{0,0}^0 \Pi_{l_1,0}^j - \sum_{k=1}^m \Pi_{0,0}^k \Pi_{l_1, k}^j + \Pi_{0, l_1}^0 \Pi_{0,0}^j + \sum_{k=1}^m \Pi_{0, l_1}^k \Pi_{0, k}^j, \\ \left(\Pi_{l_1, l_2}^j \right)_x - \left(\Pi_{l_1, 0}^j \right)_{y^{l_2}} = -\Pi_{l_1, l_2}^0 \Pi_{0,0}^j - \sum_{k=1}^m \Pi_{l_1, l_2}^k \Pi_{0, k}^j + \Pi_{l_1, 0}^0 \Pi_{l_2, 0}^j + \sum_{k=1}^m \Pi_{l_1, 0}^k \Pi_{l_2, k}^j, \\ \left(\Pi_{l_1, l_2}^j \right)_{y^{l_3}} - \left(\Pi_{l_1, l_3}^j \right)_{y^{l_2}} = -\Pi_{l_1, l_2}^0 \Pi_{l_3, 0}^j - \sum_{k=1}^m \Pi_{l_1, l_2}^k \Pi_{l_3, k}^j + \Pi_{l_1, l_3}^0 \Pi_{l_2, 0}^j + \sum_{k=1}^m \Pi_{l_1, l_3}^k \Pi_{l_2, k}^j, \\ \left(\Pi_{0,0}^0 \right)_{y^{l_1}} - \left(\Pi_{0, l_1}^0 \right)_x = \underbrace{-\Pi_{0,0}^0 \Pi_{l_1, 0}^0}_{\textcircled{a}} - \sum_{k=1}^m \Pi_{0,0}^k \Pi_{l_1, k}^0 + \underbrace{\Pi_{0, l_1}^0 \Pi_{0,0}^0}_{\textcircled{a}} + \sum_{k=1}^m \Pi_{0, l_1}^k \Pi_{0, k}^0, \\ \left(\Pi_{l_1, l_2}^0 \right)_x - \left(\Pi_{l_1, 0}^0 \right)_{y^{l_2}} = -\Pi_{l_1, l_2}^0 \Pi_{0,0}^0 - \sum_{k=1}^m \Pi_{l_1, l_2}^k \Pi_{0, k}^0 + \Pi_{l_1, 0}^0 \Pi_{l_2, 0}^0 + \sum_{k=1}^m \Pi_{l_1, 0}^k \Pi_{l_2, k}^0, \\ \left(\Pi_{l_1, l_2}^0 \right)_{y^{l_3}} - \left(\Pi_{l_1, l_3}^0 \right)_{y^{l_2}} = -\Pi_{l_1, l_2}^0 \Pi_{l_3, 0}^0 - \sum_{k=1}^m \Pi_{l_1, l_2}^k \Pi_{l_3, k}^0 + \Pi_{l_1, l_3}^0 \Pi_{l_2, 0}^0 + \sum_{k=1}^m \Pi_{l_1, l_3}^k \Pi_{l_2, k}^0. \end{array} \right.$$

3.66. Convention about sums. Up to the end of Section 4, we shall abbreviate any sum $\sum_{k=1}^m$ or $\sum_{p=1}^m$ as \sum_k or \sum_p . Such sums will appear very frequently. For all other sums, we shall precisely write down the domain of variation of the summation index.

3.67. Continuation. Thus, we have to replace (3.64) in the six identities (3.65). Firstly, let us expose all the intermediate steps in dealing with the first identity (3.65)₁. Replacing plainly (3.64) in (3.65)₁, we get:

$$(3.68) \quad \left\{ \begin{aligned} & \left(\Pi_{0,0}^j \right)_{y^{l_1}} - \left(\Pi_{0,l_1}^j \right)_x = G_{y^{l_1}}^j + \frac{1}{2} H_{l_1,x}^j - \frac{1}{2} \delta_{l_1}^j \Theta_x^0 = \\ & = \frac{1}{2} \Theta^0 H_{l_1}^j - \frac{1}{2} \delta_{l_1}^j \Theta^0 \Theta^0 - \\ & \quad - \sum_k \left(-G^k \right) \left(-L_{l_1,k}^j + \frac{1}{2} \delta_{l_1}^j L_{k,k}^k + \frac{1}{2} \delta_k^j L_{l_1,l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^j \Theta^k + \frac{1}{2} \delta_k^j \Theta^k \right) + \\ & \quad + \left(\frac{1}{2} L_{l_1,l_1}^{l_1} + \frac{1}{2} \Theta^{l_1} \right) (-G^j) + \\ & \quad + \sum_k \left(-\frac{1}{2} H_{l_1}^k + \frac{1}{2} \delta_{l_1}^k \Theta^0 \right) \left(-\frac{1}{2} H_k^j + \frac{1}{2} \delta_k^j \Theta^0 \right) = \\ & = \frac{1}{2} \underbrace{H_{l_1}^j \Theta^0}_{\text{(a)}} - \frac{1}{2} \delta_{l_1}^j \Theta^0 \Theta^0 - \sum_k G^k L_{l_1,k}^j + \frac{1}{2} \delta_{l_1}^j \sum_k G^k L_{k,k}^k + \frac{1}{2} \underbrace{G^j L_{l_1,l_1}^{l_1}}_{\text{(b)}} + \\ & \quad + \frac{1}{2} \delta_{l_1}^j \sum_k G^k \Theta^k + \frac{1}{2} \underbrace{G^j \Theta^{l_1}}_{\text{(c)}} - \frac{1}{2} \underbrace{G^j L_{l_1,l_1}^{l_1}}_{\text{(b)}} - \frac{1}{2} \underbrace{G^j \Theta^{l_1}}_{\text{(c)}} + \frac{1}{4} \sum_k H_{l_1}^k H_k^j - \\ & \quad - \frac{1}{4} \underbrace{H_{l_1}^j \Theta^0}_{\text{(a)}} - \frac{1}{4} \underbrace{H_{l_1}^j \Theta^0}_{\text{(a)}} + \frac{1}{4} \delta_{l_1}^j \Theta^0 \Theta^0. \end{aligned} \right.$$

Eliminating the underlined vanishing terms with the letters a , b , c and d appended, multiplying by -2 and reorganizing the identity so as to put the term $\delta_{l_1}^j \Theta_x^0$ solely in the left-hand side, we obtain the relation

$$(3.69) \quad \left\{ \begin{aligned} & \delta_{l_1}^j \Theta_x^0 = -2 G_{y^{l_1}}^j + H_{l_1,x}^j + 2 \sum_k G^k L_{l_1,k}^j - \delta_{l_1}^j \sum_k G^k L_{k,k}^k - \\ & \quad - \frac{1}{2} \sum_k H_{l_1}^k H_k^j - \delta_{l_1}^j \sum_k G^k \Theta^k + \frac{1}{2} \delta_{l_1}^j \Theta^0 \Theta^0. \end{aligned} \right.$$

3.70. Conventions for simplifications of formal expressions. Before proceeding further, let us explain how we will organize the computations with the formal expressions we shall encounter until the end of Section 4. *Our main goal is to devise a methodology of writing formal computations which enables to check every computation visually, without being forced to rebuild any intermediate step.* In fact, it would be unsatisfactory to just claim that Theorem 1.7 (3) follows by hidden massive formal computations, so that we have to guide the rigorous and demanding reader until the very extremal branches of our coral tree of formal computations.

As an example, suppose that we have to simplify the equation $0 = A_x + B_y + A - B + 2C - \frac{1}{3}D - \frac{2}{3}A + \frac{1}{6}D + E + B - 2C$. In the beginning, the terms A_x and B_y are differentiated once and they do not simplify with other terms. To distinguish them,

we underline them plainly and we copy the nine remaining terms afterwards:

$$(3.71) \quad \left\{ \begin{array}{l} 0 = \underline{A_x + B_y} + \\ + \underline{A}_{\square 1} - \underline{B}_{\circ a} + \underline{2C}_{\circ b} - \frac{1}{3} \underline{D}_{\square 2} - \frac{2}{3} \underline{A}_{\square 1} + \frac{1}{6} \underline{D}_{\square 2} + \underline{E}_{\square 3} + \underline{B}_{\circ a} - \underline{2C}_{\circ b}. \end{array} \right.$$

Here, each remaining term is also underlined, with a number or with a letter appended. For reasons of typographical readability, we never underline the sign, + or – of each term; however, it should be understood that *every term always includes its* (not underlined) *sign*. Until the end of Section 4, we shall use the roman alphabetic letters a, b, c , etc. inside an octagon \circ to exhibit the vanishing terms. As readily checked by the eyes, we indeed have $-\underline{B}_{\circ a} + \underline{B}_{\circ a} = 0$ and $\underline{2C}_{\circ b} - \underline{2C}_{\circ b} = 0$. Also, until the end of Section 4, we shall use the numbers 1, 2, 3, etc. inside a square \square to exhibit the remaining terms, collected in a certain order. The numbers have the following signification: after the simplifications, the equation (3.71) may be written

$$(3.72) \quad \left\{ \begin{array}{l} 0 = \underline{A_x + B_y} + \\ + \frac{1}{3} \underline{A} - \frac{1}{6} \underline{D} + \underline{E}. \end{array} \right.$$

Here, the plainly underlined terms $\underline{A_x + B_y}$ do not count in the numbering (their number is zero, for instance) and *the first term of the second line* $\frac{1}{3} \underline{A}$ *correspond to the addition of all terms* $\square 1$ *in (3.69)*. Analogously, the second term $-\frac{1}{6} \underline{D}$ correspond to the addition of all terms $\square 2$ in (3.69). Again, this guiding facilitates the checking of the correctness of the computation, using simply the eyes. No hidden delicate computational step is “left to the reader” for the convenience of the writer.

This principle will be constantly used until the end of Section 4; it has been systematically used in [M2004] and it could be applied in various other contexts. Again, the advantage is that it enables to check the correctness of all the formal computations just by reading, without having to write anything more. This is also useful for the author.

3.73. Choice of an ordering. Until the end of Section 4, we shall have to deal with terms G, H, L, M, Θ together with indices and partial derivatives up to order two. In order to organize the formal expressions in a way which provides an easier deciphering, it is convenient to introduce an order between these differential monomials. In a symbolic index-free notation, we choose:

$$(3.74) \quad G < H < L < M < \Theta.$$

It follows for instance that $G < GH < GL < HHL < HLM\Theta$. Also, if a sum appears, we choose: $GM < \sum GM$.

Here, we have only considered terms of order zero, without partial differentiation. The first order partial differentiations are $(\cdot)_x$ and $(\cdot)_y$, again in symbolic notation, dropping the indices. We choose:

$$(3.75) \quad G_x < G_y < H_x < H_y < L_x < L_y < M_x < M_y < \Theta_x < \Theta_y < G < \dots .$$

For second order derivatives, we choose:

$$(3.76) \quad G_{xx} < G_{xy} < G_{yy} < H_{xx} < \dots < \Theta_{yy} < G_x < \dots .$$

As a final general example including indices we have the inequalities

$$(3.77) \quad H_{l_1, y^{l_2}}^j < L_{l_1, l_2, x}^j < G^j M_{l_1, l_2} < \sum_{k=1}^m G^k M_{l_2, k} < \sum_{k=1}^m H_k^{l_2} H_{l_2, l_2}^k,$$

extracted from (II) of Theorem 1.7 (3).

In the sequel, we shall call

- terms of order 0 monomials like G, H, LM, GHM ;
- terms of order 1 monomials like $G_x, G_x M, L_x \Theta$;
- terms of order 2 monomials like G_{xy}, L_{yy}, M_{xx} (our terms of order two will always be linear),

according to the top order partial derivatives.

3.78. A mean of checking intuitively the validity of partial differential relations.

Before replacing (3.64) in the five remaining identities (3.65)₂, (3.65)₃, (3.65)₄, (3.65)₅ and (3.65)₆, let us observe that if we assume that $j \neq l_1$ in (3.69), then all the terms involving Θ vanish, so that we obtain the *nontrivial partial differential equations*:

$$(3.79) \quad 0 = \underline{-2G_{y^{l_1}}^j + H_{l_1,x}^j} + 2 \sum_k G^k L_{l_1,k}^j - \frac{1}{2} \sum_k H_{l_1}^k H_k^j,$$

for $j \neq l_1$. Here, we have underlined the first order terms plainly, in order to distinguish them from the terms of order zero. These equations coincides with (I) of Theorem 1.7 (3), again specialized with $j \neq l_1$. Importantly, we notice that the choice of indices $j \neq l_1$ is possible only if $m \geq 2$. Thus, we have derived *a subpart of (I) as a necessary condition for the point equivalence to $Y_{XX}^j = 0, j = 1, \dots, m \geq 2$* . These first order equations show at once that there is a strong difference with the case $m = 1$.

How can we confirm (at least informally) that the functions $G^j, H_{l_1}^j$ and L_{l_1,l_2}^j given by (3.35) in terms of X and Y^j do indeed satisfy these equations for $j \neq l_1$? Dropping the zero order terms in (3.79) above, we obtain an approximated equation

$$(3.80) \quad 0 \equiv -2G_{y^{l_1}}^j + H_{l_1,x}^j.$$

Here, the sign \equiv precisely means: “*modulo zero order terms*”. We claim that this approximated equation is a consequence of the existence of X, Y^j .

Indeed, according to the approximation (3.58), together with the definition (3.35) of the functions G^j and $H_{l_1}^j$, we have

$$(3.81) \quad \begin{cases} G^j = -\square_{xx}^j \cong -Y_{xx}^j, \\ H_{l_1}^j = -2\square_{xy^{l_1}}^j \cong -2Y_{xy^{l_1}}^j. \end{cases}$$

Differentiation of the first line with respect to y^{l_1} and of the second line with respect to x yields:

$$(3.82) \quad G_{y^{l_1}}^j \cong -Y_{xxy^{l_1}}^j \quad \text{and} \quad H_{l_1,x}^j \cong -2Y_{xy^{l_1}x}^j,$$

so that we indeed have $0 \equiv -2G_{y^{l_1}}^j + H_{l_1,x}^j$, approximatively and modulo the derivatives of order 0, 1 and 2 of the functions X, Y^j .

Similar verifications have been effected constantly in our manuscript in order to control the truth of the formal computations that we shall expose until the end of Section 4.

3.83. Continuation. From now on and up to the end of Section 4, the hardest computational core of the proof may — at last — be developed. Further amazing computational obstacles will be encountered.

Replacing plainly (3.64) in (3.65)₂, we get:

$$\begin{aligned}
& \left(\Pi_{l_1, l_2}^j \right)_x - \left(\Pi_{l_1, 0}^j \right)_{y^{l_2}} = -L_{l_1, l_2, x}^j + \frac{1}{2} \delta_{l_1}^j L_{l_2, l_2, x}^{l_2} + \frac{1}{2} \delta_{l_2}^j L_{l_1, l_1, x}^{l_1} + \frac{1}{2} \delta_{l_1}^j \Theta_x^{l_2} + \\
& \quad + \frac{1}{2} \delta_{l_2}^j \Theta_x^{l_1} + \frac{1}{2} H_{l_1, y^{l_2}}^j - \frac{1}{2} \delta_{l_1}^j \Theta_{y^{l_2}}^0 = \\
& = -\Pi_{l_1, l_2}^0 \cdot \Pi_{0, 0}^j - \sum_k \Pi_{l_1, l_2}^k \cdot \Pi_{0, k}^j + \Pi_{l_1, 0}^0 \cdot \Pi_{l_2, 0}^j + \sum_k \Pi_{l_1, 0}^k \cdot \Pi_{l_2, k}^j = \\
& = M_{l_1, l_2} G^j - \sum_k \left(-L_{l_1, l_2}^k + \frac{1}{2} \delta_{l_1}^k L_{l_2, l_2}^{l_2} + \frac{1}{2} \delta_{l_2}^k L_{l_1, l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^k \Theta^{l_2} + \frac{1}{2} \delta_{l_2}^k \Theta^{l_1} \right) \cdot \\
(3.84) \quad & \quad \cdot \left(-\frac{1}{2} H_k^j + \frac{1}{2} \delta_k^j \Theta^0 \right) + \\
& + \left(\frac{1}{2} L_{l_1, l_1}^{l_1} + \frac{1}{2} \Theta^{l_1} \right) \cdot \left(-\frac{1}{2} H_{l_2}^j + \frac{1}{2} \delta_{l_2}^j \Theta^0 \right) + \\
& + \sum_k \left(-\frac{1}{2} H_{l_1}^k + \frac{1}{2} \delta_{l_1}^k \Theta^0 \right) \cdot \\
& \quad \cdot \left(-L_{l_2, k}^j + \frac{1}{2} \delta_{l_2}^j L_{k, k}^k + \frac{1}{2} \delta_k^j L_{l_2, l_2}^{l_2} + \frac{1}{2} \delta_{l_2}^j \Theta^k + \frac{1}{2} \delta_k^j \Theta^{l_2} \right).
\end{aligned}$$

Developing the products and ordering each monomial, we get:

$$\begin{aligned}
& = \frac{G^j M_{l_1, l_2}}{\boxed{1}} - \frac{1}{2} \sum_k \frac{H_k^j L_{l_1, l_2}^k}{\boxed{2}} + \frac{1}{4} \frac{H_{l_1}^j L_{l_2, l_2}^{l_2}}{\textcircled{a}} + \frac{1}{4} \frac{H_{l_2}^j L_{l_1, l_1}^{l_1}}{\textcircled{b}} + \\
& + \frac{1}{4} \frac{H_{l_1}^j \Theta^{l_2}}{\textcircled{c}} + \frac{1}{4} \frac{H_{l_2}^j \Theta^{l_1}}{\textcircled{d}} + \frac{1}{2} \frac{L_{l_1, l_2}^j \Theta^0}{\textcircled{e}} - \frac{1}{4} \frac{\delta_{l_1}^j L_{l_2, l_2}^{l_2} \Theta^0}{\textcircled{f}} - \\
& - \frac{1}{4} \frac{\delta_{l_2}^j L_{l_1, l_1}^{l_1} \Theta^0}{\textcircled{g}} - \frac{1}{4} \frac{\delta_{l_1}^j \Theta^0 \Theta^{l_2}}{\textcircled{h}} - \frac{1}{4} \frac{\delta_{l_2}^j \Theta^0 \Theta^{l_1}}{\textcircled{i}} - \frac{1}{4} \frac{H_{l_2}^j L_{l_1, l_1}^{l_1}}{\textcircled{b}} + \\
(3.85) \quad & + \frac{1}{4} \frac{\delta_{l_2}^j L_{l_1, l_1}^{l_1} \Theta^0}{\boxed{6}} - \frac{1}{4} \frac{H_{l_2}^j \Theta^{l_1}}{\textcircled{d}} + \frac{1}{4} \frac{\delta_{l_2}^j \Theta^0 \Theta^{l_1}}{\textcircled{i}} + \frac{1}{2} \sum_k \frac{H_{l_1}^k L_{l_2, k}^j}{\boxed{3}} - \\
& - \frac{1}{4} \sum_k \frac{H_{l_1}^k L_{k, k}^k}{\boxed{4}} - \frac{1}{4} \frac{H_{l_1}^j L_{l_2, l_2}^{l_2}}{\textcircled{a}} - \frac{1}{4} \frac{\delta_{l_2}^j \sum_k H_{l_1}^k \Theta^k}{\boxed{5}} - \frac{1}{4} \frac{H_{l_1}^j \Theta^{l_2}}{\textcircled{c}} - \\
& - \frac{1}{2} \frac{L_{l_2, l_1}^j \Theta^0}{\textcircled{e}} + \frac{1}{4} \frac{\delta_{l_2}^j L_{l_1, l_1}^{l_1} \Theta^0}{\textcircled{g}} + \frac{1}{4} \frac{\delta_{l_1}^j L_{l_2, l_2}^{l_2} \Theta^0}{\textcircled{f}} + \frac{1}{4} \frac{\delta_{l_2}^j \Theta^0 \Theta^{l_1}}{\boxed{7}} + \\
& + \frac{1}{4} \frac{\delta_{l_1}^j \Theta^0 \Theta^{l_2}}{\textcircled{h}}.
\end{aligned}$$

We simplify according to our general principles and we reorganize the equality between the first two lines of (3.84) and (3.85) so as to put all terms Θ_x in the left-hand side of the equality and to put all remaining terms in the right-hand side, respecting the

order of §3.73. We get:

$$\begin{aligned}
& \frac{1}{2} \delta_{l_1}^j \Theta_x^{l_2} + \frac{1}{2} \delta_{l_2}^j \Theta_x^{l_1} - \frac{1}{2} \delta_{l_1}^j \Theta_{y^{l_2}}^0 = \\
& = \underline{-\frac{1}{2} H_{l_1, y^{l_2}}^j + L_{l_1, l_2, x}^j - \frac{1}{2} \delta_{l_1}^j L_{l_2, l_2, x}^{l_2} - \frac{1}{2} \delta_{l_2}^j L_{l_1, l_1, x}^{l_1}} + \\
(3.86) \quad & + G^j M_{l_1, l_2} - \frac{1}{2} \sum_k H_k^j L_{l_1, l_2}^k + \frac{1}{2} \sum_k H_{l_1}^k L_{l_2, k}^j - \\
& - \frac{1}{4} \delta_{l_2}^j \sum_k H_{l_1}^k L_{k, k}^k - \frac{1}{4} \delta_{l_2}^j \sum_k H_{l_1}^k \Theta^k + \frac{1}{4} \delta_{l_2}^j L_{l_1, l_1}^{l_1} \Theta^0 + \\
& + \frac{1}{4} \delta_{l_2}^j \Theta^0 \Theta^{l_1}.
\end{aligned}$$

Here, we have underlined plainly the four first order terms appearing in the second line.

Next, replacing plainly (3.64) in (3.65)₃, we get:

$$\begin{aligned}
& \left(\Pi_{l_1, l_2}^j \right)_{y^{l_3}} - \left(\Pi_{l_1, l_3}^j \right)_{y^{l_2}} = \\
& = -L_{l_1, l_2, y^{l_3}}^j + \frac{1}{2} \delta_{l_1}^j L_{l_2, l_2, y^{l_3}}^{l_2} + \frac{1}{2} \delta_{l_2}^j L_{l_1, l_1, y^{l_3}}^{l_1} + \frac{1}{2} \delta_{l_1}^j \Theta_{y^{l_3}}^{l_2} + \frac{1}{2} \delta_{l_2}^j \Theta_{y^{l_3}}^{l_1} + \\
& + L_{l_1, l_3, y^{l_2}}^j - \frac{1}{2} \delta_{l_1}^j L_{l_3, l_3, y^{l_2}}^{l_3} - \frac{1}{2} \delta_{l_3}^j L_{l_1, l_1, y^{l_2}}^{l_1} - \frac{1}{2} \delta_{l_1}^j \Theta_{y^{l_2}}^{l_3} - \frac{1}{2} \delta_{l_3}^j \Theta_{y^{l_2}}^{l_1} = \\
& = -\Pi_{l_1, l_2}^0 \cdot \Pi_{l_3, 0}^j - \sum_k \Pi_{l_1, l_2}^k \cdot \Pi_{l_3, k}^j + \Pi_{l_1, l_3}^0 \cdot \Pi_{l_2, 0}^j + \sum_k \Pi_{l_1, l_3}^k \cdot \Pi_{l_2, k}^j = \\
(3.87) \quad & = -M_{l_1, l_2} \cdot \left(-\frac{1}{2} H_{l_3}^j + \frac{1}{2} \delta_{l_3}^j \Theta^0 \right) - \sum_k \left(-L_{l_1, l_2}^k + \frac{1}{2} \delta_{l_1}^k L_{l_2, l_2}^{l_2} + \right. \\
& + \frac{1}{2} \delta_{l_2}^k L_{l_1, l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^k \Theta^{l_2} + \frac{1}{2} \delta_{l_2}^k \Theta^{l_1} \left. \right) \cdot \left(-L_{l_3, k}^j + \frac{1}{2} \delta_{l_3}^j L_{k, k}^k + \right. \\
& + \frac{1}{2} \delta_k^j L_{l_3, l_3}^{l_3} + \frac{1}{2} \delta_{l_3}^j \Theta^k + \frac{1}{2} \delta_k^j \Theta^{l_3} \left. \right) + \\
& + M_{l_1, l_3} \cdot \left(-\frac{1}{2} H_{l_2}^j + \frac{1}{2} \delta_{l_2}^j \Theta^0 \right) + \sum_k \left(-L_{l_1, l_3}^k + \frac{1}{2} \delta_{l_1}^k L_{l_3, l_3}^{l_3} + \right. \\
& + \frac{1}{2} \delta_{l_3}^k L_{l_1, l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^k \Theta^{l_3} + \frac{1}{2} \delta_{l_3}^k \Theta^{l_1} \left. \right) \cdot \left(-L_{l_2, k}^j + \frac{1}{2} \delta_{l_2}^j L_{k, k}^k + \right. \\
& + \frac{1}{2} \delta_k^j L_{l_2, l_2}^{l_2} + \frac{1}{2} \delta_{l_2}^j \Theta^k + \frac{1}{2} \delta_k^j \Theta^{l_2} \left. \right).
\end{aligned}$$

Developing the products and ordering each monomial, we get:

$$\begin{aligned}
(3.88) \quad &= \frac{1}{2} \underline{H_{l_3}^j M_{l_1, l_2}}_{\boxed{1}} - \frac{1}{2} \delta_{l_3}^j \underline{M_{l_1, l_2} \Theta^0}_{\boxed{16}} - \sum_k \underline{L_{l_1, l_2}^k L_{l_3, k}^j}_{\boxed{6}} + \frac{1}{2} \delta_{l_3}^j \sum_k \underline{L_{l_1, l_2}^k L_{k, k}^k}_{\boxed{7}} + \\
&+ \frac{1}{2} \underline{L_{l_1, l_2}^j L_{l_3, l_3}^{l_3}}_{\textcircled{a}} + \frac{1}{2} \delta_{l_3}^j \sum_k \underline{L_{l_1, l_2}^k \Theta^k}_{\boxed{13}} + \frac{1}{2} \underline{L_{l_1, l_2}^j \Theta^{l_3}}_{\textcircled{b}} + \frac{1}{2} \underline{L_{l_2, l_2}^{l_2} L_{l_3, l_1}^j}_{\textcircled{c}} - \\
&- \frac{1}{4} \delta_{l_3}^j \underline{L_{l_2, l_2}^{l_2} L_{l_1, l_1}^{l_1}}_{\boxed{4}} - \frac{1}{4} \delta_{l_1}^j \underline{L_{l_2, l_2}^{l_2} L_{l_3, l_3}^{l_3}}_{\textcircled{d}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_2, l_2}^{l_2} \Theta^{l_1}}_{\textcircled{e}} - \frac{1}{4} \delta_{l_1}^j \underline{L_{l_2, l_2}^{l_2} \Theta^{l_3}}_{\textcircled{f}} + \\
&+ \frac{1}{2} \underline{L_{l_1, l_1}^{l_1} L_{l_3, l_2}^j}_{\textcircled{g}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_1, l_1}^{l_1} L_{l_2, l_2}^{l_2}}_{\textcircled{h}} - \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} L_{l_3, l_3}^{l_3}}_{\textcircled{i}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_2}}_{\textcircled{j}} - \\
&- \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_3}}_{\textcircled{k}} + \frac{1}{2} \underline{L_{l_3, l_1}^j \Theta^{l_2}}_{\textcircled{l}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_2}}_{\boxed{10}} - \frac{1}{4} \delta_{l_1}^j \underline{L_{l_3, l_3}^{l_3} \Theta^{l_2}}_{\textcircled{m}} - \\
&- \frac{1}{4} \delta_{l_3}^j \underline{\Theta^{l_2} \Theta^{l_1}}_{\textcircled{n}} - \frac{1}{4} \delta_{l_1}^j \underline{\Theta^{l_2} \Theta^{l_3}}_{\textcircled{o}} + \frac{1}{2} \underline{L_{l_3, l_2}^j \Theta^{l_1}}_{\textcircled{p}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_2, l_2}^{l_2} \Theta^{l_1}}_{\boxed{12}} - \\
&- \frac{1}{4} \delta_{l_2}^j \underline{L_{l_3, l_3}^{l_3} \Theta^{l_1}}_{\textcircled{q}} - \frac{1}{4} \delta_{l_3}^j \underline{\Theta^{l_1} \Theta^{l_2}}_{\boxed{18}} - \frac{1}{4} \delta_{l_2}^j \underline{\Theta^{l_1} \Theta^{l_3}}_{\textcircled{r}} - \\
&- \frac{1}{2} \underline{H_{l_2}^j M_{l_1, l_3}}_{\boxed{2}} + \frac{1}{2} \delta_{l_2}^j \underline{M_{l_1, l_3} \Theta^0}_{\boxed{15}} + \sum_k \underline{L_{l_1, l_3}^k L_{l_2, k}^j}_{\boxed{5}} - \frac{1}{2} \delta_{l_2}^j \sum_k \underline{L_{l_1, l_3}^k L_{k, k}^k}_{\boxed{8}} - \\
&- \frac{1}{2} \underline{L_{l_1, l_3}^j L_{l_2, l_2}^{l_2}}_{\textcircled{c}} - \frac{1}{2} \delta_{l_2}^j \sum_k \underline{L_{l_1, l_3}^k \Theta^k}_{\boxed{14}} - \frac{1}{2} \underline{L_{l_1, l_3}^j \Theta^{l_2}}_{\textcircled{l}} - \frac{1}{2} \underline{L_{l_3, l_3}^{l_3} L_{l_2, l_1}^j}_{\textcircled{a}} + \\
&+ \frac{1}{4} \delta_{l_2}^j \underline{L_{l_3, l_3}^{l_3} L_{l_1, l_1}^{l_1}}_{\boxed{3}} + \frac{1}{4} \delta_{l_1}^j \underline{L_{l_3, l_3}^{l_3} L_{l_2, l_2}^{l_2}}_{\textcircled{d}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_3, l_3}^{l_3} \Theta^{l_1}}_{\textcircled{q}} + \frac{1}{4} \delta_{l_1}^j \underline{L_{l_3, l_3}^{l_3} \Theta^{l_2}}_{\textcircled{m}} - \\
&- \frac{1}{2} \underline{L_{l_1, l_1}^{l_1} L_{l_2, l_3}^j}_{\textcircled{g}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} L_{l_3, l_3}^{l_3}}_{\textcircled{i}} + \frac{1}{4} \delta_{l_3}^j \underline{L_{l_1, l_1}^{l_1} L_{l_2, l_2}^{l_2}}_{\textcircled{h}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_3}}_{\boxed{9}} + \\
&+ \frac{1}{4} \delta_{l_3}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_2}}_{\textcircled{j}} - \frac{1}{2} \underline{L_{l_2, l_1}^j \Theta^{l_3}}_{\textcircled{b}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_3}}_{\textcircled{k}} + \frac{1}{4} \delta_{l_1}^j \underline{L_{l_2, l_2}^{l_2} \Theta^{l_3}}_{\textcircled{f}} + \\
&+ \frac{1}{4} \delta_{l_2}^j \underline{\Theta^{l_3} \Theta^{l_1}}_{\boxed{17}} + \frac{1}{4} \delta_{l_1}^j \underline{\Theta^{l_3} \Theta^{l_2}}_{\textcircled{o}} - \frac{1}{2} \underline{L_{l_2, l_3}^j \Theta^{l_1}}_{\textcircled{p}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_3, l_3}^{l_3} \Theta^{l_1}}_{\boxed{11}} + \\
&+ \frac{1}{4} \delta_{l_3}^j \underline{L_{l_2, l_2}^{l_2} \Theta^{l_1}}_{\textcircled{e}} + \frac{1}{4} \delta_{l_2}^j \underline{\Theta^{l_1} \Theta^{l_3}}_{\textcircled{r}} + \frac{1}{4} \delta_{l_3}^j \underline{\Theta^{l_1} \Theta^{l_2}}_{\textcircled{n}} .
\end{aligned}$$

We simplify and we reorganize the equality between the second and third lines of (3.88) and (3.85) so as to put all terms Θ_y in the left-hand side of the equality and to put all

remaining terms in the right-hand side, respecting the order of §3.73. We get:

$$\begin{aligned}
& \frac{1}{2} \delta_{l_1}^j \Theta_{y^{l_3}}^{l_2} - \frac{1}{2} \delta_{l_1}^j \Theta_{y^{l_2}}^{l_3} + \frac{1}{2} \delta_{l_2}^j \Theta_{y^{l_3}}^{l_1} - \frac{1}{2} \delta_{l_3}^j \Theta_{y^{l_2}}^{l_1} = \\
& = \frac{L_{l_1, l_2, y^{l_3}}^j - L_{l_1, l_3, y^{l_2}}^j + \frac{1}{2} \delta_{l_1}^j L_{l_3, l_3, y^{l_2}}^{l_3} - \frac{1}{2} \delta_{l_1}^j L_{l_2, l_2, y^{l_3}}^{l_2} +}{\hline} \\
& \quad + \frac{1}{2} \delta_{l_3}^j L_{l_1, l_1, y^{l_2}}^{l_1} - \frac{1}{2} \delta_{l_2}^j L_{l_1, l_1, y^{l_3}}^{l_1} + \\
& \quad + \frac{1}{2} H_{l_3}^j M_{l_1, l_2} - \frac{1}{2} H_{l_2}^j M_{l_1, l_3} + \frac{1}{4} \delta_{l_2}^j L_{l_3, l_3}^{l_3} L_{l_1, l_1}^{l_1} - \frac{1}{4} \delta_{l_3}^j L_{l_2, l_2}^{l_2} L_{l_1, l_1}^{l_1} + \\
(3.89) \quad & \quad + \sum_k L_{l_1, l_3}^k L_{l_2, k}^j - \sum_k L_{l_1, l_2}^k L_{l_3, k}^j + \\
& \quad + \frac{1}{2} \delta_{l_3}^j \sum_k L_{l_1, l_2}^k L_{k, k}^k - \frac{1}{2} \delta_{l_2}^j \sum_k L_{l_1, l_3}^k L_{k, k}^k + \\
& \quad + \frac{1}{4} \delta_{l_2}^j L_{l_1, l_1}^{l_1} \Theta^{l_3} - \frac{1}{4} \delta_{l_3}^j L_{l_1, l_1}^{l_1} \Theta^{l_2} + \frac{1}{4} \delta_{l_2}^j L_{l_3, l_3}^{l_3} \Theta^{l_1} - \frac{1}{4} \delta_{l_3}^j L_{l_2, l_2}^{l_2} \Theta^{l_1} + \\
& \quad + \frac{1}{2} \delta_{l_3}^j \sum_k L_{l_1, l_2}^k \Theta^k - \frac{1}{2} \delta_{l_2}^j \sum_k L_{l_1, l_3}^k \Theta^k + \\
& \quad + \frac{1}{2} \delta_{l_2}^j M_{l_1, l_3} \Theta^0 - \frac{1}{2} \delta_{l_3}^j M_{l_1, l_2} \Theta^0 + \frac{1}{4} \delta_{l_2}^j \Theta^{l_3} \Theta^{l_1} - \frac{1}{4} \delta_{l_3}^j \Theta^{l_2} \Theta^{l_1}.
\end{aligned}$$

Next, replacing plainly (3.64) in (3.65)₄, we get:

$$\begin{aligned}
& (\Pi_{0,0}^0)_{y^{l_1}} - (\Pi_{0,l_1}^0)_x = \\
& = \Theta_{y^{l_1}}^0 - \frac{1}{2} L_{l_1, l_1, x}^{l_1} - \frac{1}{2} \Theta_x^{l_1} = \\
& = -\frac{\Pi_{0,0}^0 \cdot \Pi_{l_1,0}^0}{\textcircled{a}} - \sum_k \Pi_{0,0}^k \cdot \Pi_{l_1,k}^0 + \frac{\Pi_{0,l_1}^0 \cdot \Pi_{0,0}^0}{\textcircled{a}} + \sum_k \Pi_{0,l_1}^k \cdot \Pi_{0,k}^0 = \\
(3.90) \quad & = -\sum_k \left(-G^k \right) \cdot (M_{l_1,k}) + \sum_k \left(-\frac{1}{2} H_{l_1}^k + \frac{1}{2} \delta_{l_1}^k \Theta^0 \right) \cdot \\
& \quad \cdot \left(\frac{1}{2} L_{k,k}^k + \frac{1}{2} \Theta^k \right) = \\
& = \sum_k G^k M_{l_1,k} - \frac{1}{4} \sum_k H_{l_1}^k L_{k,k}^k - \frac{1}{4} \sum_k H_{l_1}^k \Theta^k + \frac{1}{4} L_{l_1, l_1}^{l_1} \Theta^0 + \frac{1}{4} \Theta^0 \Theta^{l_1}.
\end{aligned}$$

Reorganizing the equality so as to put the terms Θ_x and Θ_y alone in the left-hand side, we get:

$$\begin{aligned}
(3.91) \quad & -\frac{1}{2} \Theta_x^{l_1} + \Theta_{y^{l_1}}^0 = \frac{1}{2} L_{l_1, l_1, x}^{l_1} + \\
& \quad + \sum_k G^k M_{l_1,k} - \frac{1}{4} \sum_k H_{l_1}^k L_{k,k}^k - \frac{1}{4} \sum_k H_{l_1}^k \Theta^k + \\
& \quad + \frac{1}{4} L_{l_1, l_1}^{l_1} \Theta^0 + \frac{1}{4} \Theta^0 \Theta^{l_1}.
\end{aligned}$$

Next, replacing plainly (3.64) in (3.65)₅, we get:

$$\begin{aligned}
 & (\Pi_{l_1, l_2}^0)_x - (\Pi_{l_1, 0}^0)_{y^{l_2}} = \\
 & = M_{l_1, l_2, x} - \frac{1}{2} L_{l_1, l_1, y^{l_2}}^{l_1} - \frac{1}{2} \Theta_{y^{l_2}}^{l_1} = \\
 & = -\Pi_{l_1, l_2} \cdot \Pi_{0, 0}^0 - \sum_k \Pi_{l_1, l_2}^k \cdot \Pi_{0, k}^0 + \Pi_{l_1, 0}^0 \cdot \Pi_{l_2, 0}^0 + \sum_k \Pi_{l_1, 0}^k \cdot \Pi_{l_2, k}^0 = \\
 (3.92) \quad & = -M_{l_1, l_2} \Theta^0 - \sum_k \left(-L_{l_1, l_2}^k + \frac{1}{2} \delta_{l_1}^k L_{l_2, l_2}^{l_2} + \frac{1}{2} \delta_{l_2}^k L_{l_1, l_1}^{l_1} + \right. \\
 & \quad \left. + \frac{1}{2} \delta_{l_1}^k \Theta^{l_2} + \frac{1}{2} \delta_{l_2}^k \Theta^{l_1} \right) \cdot \left(\frac{1}{2} L_{k, k}^k + \frac{1}{2} \Theta^k \right) + \\
 & \quad + \left(\frac{1}{2} L_{l_1, l_1}^{l_1} + \frac{1}{2} \Theta^{l_1} \right) \cdot \left(\frac{1}{2} L_{l_2, l_2}^{l_2} + \frac{1}{2} \Theta^{l_2} \right) + \\
 & \quad + \sum_k \left(-\frac{1}{2} H_{l_1}^k + \frac{1}{2} \delta_{l_1}^k \Theta^0 \right) \cdot M_{l_2, k} = \\
 & = -\frac{M_{l_1, l_2} \Theta^0}{\boxed{7}} + \frac{1}{2} \sum_k \frac{L_{l_1, l_2}^k L_{k, k}^k}{\boxed{3}} + \frac{1}{2} \sum_k \frac{L_{l_1, l_2}^k \Theta^k}{\boxed{6}} - \frac{1}{4} \frac{L_{l_2, l_2}^{l_2} L_{l_1, l_1}^{l_1}}{\boxed{2}} - \\
 & \quad - \frac{1}{4} \frac{L_{l_2, l_2}^{l_2} \Theta^{l_1}}{\boxed{a}} - \frac{1}{4} \frac{L_{l_1, l_1}^{l_1} L_{l_2, l_2}^{l_2}}{\boxed{b}} - \frac{1}{4} \frac{L_{l_1, l_1}^{l_1} \Theta^{l_2}}{\boxed{c}} - \frac{1}{4} \frac{L_{l_1, l_1}^{l_1} \Theta^{l_2}}{\boxed{4}} - \\
 & \quad - \frac{1}{4} \frac{\Theta^{l_1} \Theta^{l_2}}{\boxed{d}} - \frac{1}{4} \frac{L_{l_2, l_2}^{l_2} \Theta^{l_1}}{\boxed{5}} - \frac{1}{4} \frac{\Theta^{l_1} \Theta^{l_2}}{\boxed{8}} + \frac{1}{4} \frac{L_{l_1, l_1}^{l_1} L_{l_2, l_2}^{l_2}}{\boxed{b}} + \\
 & \quad + \frac{1}{4} \frac{L_{l_1, l_1}^{l_1} \Theta^{l_2}}{\boxed{c}} + \frac{1}{4} \frac{L_{l_2, l_2}^{l_2} \Theta^{l_1}}{\boxed{a}} + \frac{1}{4} \frac{\Theta^{l_1} \Theta^{l_2}}{\boxed{d}} - \frac{1}{2} \sum_k \frac{H_{l_1}^k M_{l_2, k}}{\boxed{1}} + \\
 & \quad + \frac{1}{2} \frac{M_{l_2, l_1} \Theta^0}{\boxed{7}}.
 \end{aligned}$$

Multiplying by -2 and reorganizing the equality, we get:

$$\begin{aligned}
 (3.93) \quad & \Theta_{y^{l_2}}^{l_1} = -\frac{L_{l_1, l_1, y^{l_2}}^{l_1} + 2 M_{l_1, l_2, x}}{\boxed{7}} \\
 & + \sum_k H_{l_1}^k M_{l_2, k} + \frac{1}{2} L_{l_1, l_1}^{l_1} L_{l_2, l_2}^{l_2} - \sum_k L_{l_1, l_2}^k L_{k, k}^k + \\
 & + \frac{1}{2} L_{l_1, l_1}^{l_1} \Theta^{l_2} + \frac{1}{2} L_{l_2, l_2}^{l_2} \Theta^{l_1} - \sum_k L_{l_1, l_2}^k \Theta^k + \\
 & + M_{l_1, l_2} \Theta^0 + \frac{1}{2} \Theta^{l_1} \Theta^{l_2}.
 \end{aligned}$$

Next, replacing plainly (3.64) in (3.65)₄, we get:

$$\begin{aligned}
& (\Pi_{l_1, l_2}^0)_{y^{l_3}} - (\Pi_{l_1, l_3}^0)_{y^{l_2}} = \\
& = M_{l_1, l_2, y^{l_3}} - M_{l_1, l_3, y^{l_2}} = \\
& = -\Pi_{l_1, l_2}^0 \cdot \Pi_{l_3, 0}^0 - \sum_k \Pi_{l_1, l_2}^k \cdot \Pi_{l_3, k}^0 + \Pi_{l_1, l_3}^0 \cdot \Pi_{l_2, 0}^0 + \sum_k \Pi_{l_1, l_3}^k \cdot \Pi_{l_2, k}^0 = \\
(3.94) \quad & = -M_{l_1, l_2} \left(\frac{1}{2} L_{l_3, l_3}^{l_3} + \frac{1}{2} \Theta^{l_3} \right) - \sum_k M_{l_3, k} \left(-L_{l_1, l_2}^k + \frac{1}{2} \delta_{l_1}^k L_{l_2, l_2}^{l_2} + \right. \\
& \quad \left. + \frac{1}{2} \delta_{l_2}^k L_{l_1, l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^k \Theta^{l_2} + \frac{1}{2} \delta_{l_2}^k \Theta^{l_1} \right) + \\
& \quad + M_{l_1, l_3} \left(\frac{1}{2} L_{l_2, l_2}^{l_2} + \frac{1}{2} \Theta^{l_2} \right) + \sum_k M_{l_2, k} \left(-L_{l_1, l_3}^k + \frac{1}{2} \delta_{l_1}^k L_{l_3, l_3}^{l_3} + \right. \\
& \quad \left. + \frac{1}{2} \delta_{l_3}^k L_{l_1, l_1}^{l_1} + \frac{1}{2} \delta_{l_1}^k \Theta^{l_3} + \frac{1}{2} \delta_{l_3}^k \Theta^{l_1} \right).
\end{aligned}$$

Developing the products and ordering each monomial, we get:

$$\begin{aligned}
(3.95) \quad & = -\frac{1}{2} L_{l_3, l_3}^{l_3} M_{l_1, l_2} \text{ (a)} - \frac{1}{2} M_{l_1, l_2} \Theta^{l_3} \text{ (b)} + \underbrace{\sum_k L_{l_1, l_2}^k M_{l_3, k}}_{\boxed{1}} - \frac{1}{2} L_{l_2, l_2}^{l_2} M_{l_3, l_1} \text{ (c)} - \\
& - \frac{1}{2} L_{l_1, l_1}^{l_1} M_{l_3, l_2} \text{ (d)} - \frac{1}{2} M_{l_3, l_1} \Theta^{l_2} \text{ (e)} - \frac{1}{2} M_{l_3, l_2} \Theta^{l_1} \text{ (f)} + \frac{1}{2} L_{l_2, l_2}^{l_2} M_{l_3, l_1} \text{ (c)} + \\
& + \frac{1}{2} M_{l_1, l_3} \Theta^{l_2} \text{ (e)} - \underbrace{\sum_k L_{l_1, l_3}^k M_{l_2, k}}_{\boxed{2}} + \frac{1}{2} L_{l_3, l_3}^{l_3} M_{l_2, l_1} \text{ (a)} + \frac{1}{2} L_{l_1, l_1}^{l_1} M_{l_2, l_3} \text{ (d)} + \\
& + \frac{1}{2} M_{l_2, l_1} \Theta^{l_3} \text{ (b)} + \frac{1}{2} M_{l_2, l_3} \Theta^{l_1} \text{ (f)}.
\end{aligned}$$

Simplifying, we obtain the family (IV) in the statement of Theorem 1.7 (3):

$$(3.96) \quad 0 = \underbrace{M_{l_1, l_2, y^{l_3}} - M_{l_1, l_3, y^{l_2}}}_{\boxed{2}} - \sum_k L_{l_1, l_2}^k M_{l_3, k} + \sum_k L_{l_1, l_3}^k M_{l_2, k}.$$

3.97. Solving Θ_x^0 , $\Theta_{y^{l_1}}^0$, $\Theta_x^{l_1}$ and $\Theta_{y^{l_2}}^{l_1}$. It is now easy to solve all first order partial derivatives of the functions Θ^0 and Θ^l . Equation (3.93) already provides the solution for $\Theta_{y^{l_2}}^{l_1}$. We state the result as an independent proposition.

Proposition 3.98. *As a consequence of the six families of equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96) the first order derivatives Θ_x^0 , $\Theta_{y^{l_1}}^0$, $\Theta_x^{l_1}$ and $\Theta_{y^{l_2}}^{l_1}$ of the principal unknowns are given by:*

$$(3.99) \quad \left\{ \begin{aligned} \Theta_x^0 &= \frac{-2 G_{y^{l_1}}^{l_1} + H_{l_1, x}^{l_1}}{2} + \\ & + 2 \sum_k G^k L_{l_1, k}^{l_1} - \sum_k G^k L_{k, k}^k - \frac{1}{2} \sum_k H_{l_1}^k H_k^{l_1} - \\ & - \sum_k G^k \Theta^k + \frac{1}{2} \Theta^0 \Theta^0. \end{aligned} \right.$$

$$(3.100) \quad \left\{ \begin{array}{l} \Theta_{y^{l_1}}^0 = \frac{2}{3} L_{l_1, l_1, x}^{l_1} - \frac{1}{3} H_{l_1, y^{l_1}}^{l_1} + \\ \quad + \frac{2}{3} G^{l_1} M_{l_1, l_1} + \frac{4}{3} \sum_k G^k M_{l_1, k} - \frac{1}{3} \sum_k H_k^{l_1} L_{l_1, l_1}^k + \\ \quad + \frac{1}{3} \sum_k H_{l_1}^k L_{l_1, k}^{l_1} - \frac{1}{2} \sum_k H_{l_1}^k L_{k, k}^k - \frac{1}{2} \sum_k H_{l_1}^k \Theta^k + \\ \quad + \frac{1}{2} L_{l_1, l_1}^{l_1} \Theta^0 + \frac{1}{2} \Theta^0 \Theta^{l_1}. \end{array} \right.$$

$$(3.101) \quad \left\{ \begin{array}{l} \Theta_x^{l_1} = -\frac{2}{3} H_{l_1, y^{l_1}}^{l_1} + \frac{1}{3} L_{l_1, l_1, x}^{l_1} + \\ \quad + \frac{4}{3} G^{l_1} M_{l_1, l_1} + \frac{2}{3} \sum_k G^k M_{l_1, k} - \frac{2}{3} \sum_k H_k^{l_1} L_{l_1, l_1}^k + \\ \quad + \frac{2}{3} \sum_k H_{l_1}^k L_{l_1, k}^{l_1} - \frac{1}{2} \sum_k H_{l_1}^k L_{k, k}^k - \frac{1}{2} \sum_k H_{l_1}^k \Theta^k + \\ \quad + \frac{1}{2} L_{l_1, l_1}^{l_1} \Theta^0 + \frac{1}{2} \Theta^0 \Theta^{l_1}. \end{array} \right.$$

$$(3.102) \quad \left\{ \begin{array}{l} \Theta_{y^{l_2}}^{l_1} = -L_{l_1, l_1, y^{l_2}}^{l_1} + 2 M_{l_1, l_2, x} + \\ \quad + \sum_k H_{l_1}^k M_{l_2, k} + \frac{1}{2} L_{l_1, l_1}^{l_1} L_{l_2, l_2}^{l_2} - \sum_k L_{l_1, l_2}^k L_{k, k}^k + \\ \quad + \frac{1}{2} L_{l_1, l_1}^{l_1} \Theta^{l_2} + \frac{1}{2} L_{l_2, l_2}^{l_2} \Theta^{l_1} - \sum_k L_{l_1, l_2}^k \Theta^k + \\ \quad + M_{l_1, l_2} \Theta^0 + \frac{1}{2} \Theta^{l_1} \Theta^{l_2}. \end{array} \right.$$

We notice that the right-hand side of (3.99) should be independent of l_1 ; this phenomenon will be explained in a while.

Proof. For Θ_x^0 in (3.99), it suffices to put $j := l_1$ in (3.69).

To obtain $\Theta_{y^{l_1}}^0$, we put $j := l_2$ and $l_2 := l_1$ in (3.86), which yields:

$$(3.103) \quad \begin{aligned} \Theta_x^{l_1} - \frac{1}{2} \Theta_{y^{l_1}}^0 &= -\frac{1}{2} H_{l_1, y^{l_1}}^{l_1} + \\ &\quad + G^{l_1} M_{l_1, l_1} - \frac{1}{2} \sum_k H_k^{l_1} L_{l_1, l_1}^k + \\ &\quad + \frac{1}{2} \sum_k H_{l_1}^k L_{l_1, k}^{l_1} - \frac{1}{4} \sum_k H_{l_1}^k L_{k, k}^k - \\ &\quad - \frac{1}{4} \sum_k H_{l_1}^k \Theta^k + \frac{1}{4} L_{l_1, l_1}^{l_1} \Theta^0 + \frac{1}{4} \Theta^0 \Theta^{l_1}. \end{aligned}$$

We may easily solve $\Theta_{y^{l_1}}^0$ and $\Theta_x^{l_1}$ thanks to this equation (3.103) and thanks to (3.91): indeed, to obtain (3.100), it suffices to compute $\frac{4}{3} \cdot (3.91) + \frac{2}{3} \cdot (3.103)$; to obtain (3.101), it suffices to compute $\frac{2}{3} \cdot (3.91) + \frac{4}{3} \cdot (3.103)$. Finally, (3.102) is a copy of (3.93). This completes the proof. \square

3.104. Appearance of the crucial four families of first order partial differential relations (I), (II), (III) and (IV) of Theorem 1.7 (3). However, in solving Θ_x^0 , $\Theta_{y^{l_1}}^0$, $\Theta_x^{l_1}$ and $\Theta_{y^{l_2}}^{l_1}$ from our six families of equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96), only a subpart of these equations has been used. We notice that the two families of equations (3.91) and (3.93) have been used completely and that the family of equations (3.96), which does not involve Θ , coincides precisely with the system (IV) of Theorem 1.7 (3). To insure that Θ_x^0 , $\Theta_{y^{l_1}}^0$, $\Theta_x^{l_1}$ and $\Theta_{y^{l_2}}^{l_1}$ as written in Proposition 3.98 are true solutions, it is necessary and sufficient that they satisfy the remaining equations. Thus, we have to replace these solutions (3.99), (3.100), (3.101) and (3.102) in the three remaining families (3.69), (3.86) and (3.89).

Firstly, let us insert inside (3.69) the value of Θ_x^0 given by the equation (3.99), in which the index l_1 is replaced in advance by an arbitrary index l_2 . We get:

$$\begin{aligned}
(3.105) \quad 0 = & \frac{-2G_{y^{l_1}}^j + 2\delta_{l_1}^j G_{y^{l_2}}^{l_2} + H_{l_1,x}^j - \delta_{l_1}^j H_{l_2,x}^{l_2}}{+ 2 \sum_k G^k L_{l_1,k}^j - 2\delta_{l_1}^j \sum_k G^k L_{l_2,k}^{l_2} - \delta_{l_1}^j \sum_k G^k L_{k,k}^k} + \\
& \frac{\delta_{l_1}^j \sum_k G^k L_{k,k}^k}{- \frac{1}{2} \sum_k H_{l_1}^k H_k^j + \frac{1}{2} \delta_{l_1}^j \sum_k H_{l_2}^k H_k^{l_2}} - \\
& \frac{\delta_{l_1}^j \sum_k G^k \Theta^k}{+ \delta_{l_1}^j \sum_k G^k \Theta^k} + \frac{\frac{1}{2} \delta_{l_1}^j \Theta^0 \Theta^0}{- \frac{1}{2} \delta_{l_1}^j \Theta^0 \Theta^0}.
\end{aligned}$$

We simplify, which yields the family (I) of partial differential relations of Theorem 1.7 (3):

$$\begin{aligned}
(3.106) \quad 0 = & \frac{-2G_{y^{l_1}}^j + 2\delta_{l_1}^j G_{y^{l_2}}^{l_2} + H_{l_1,x}^j - \delta_{l_1}^j H_{l_2,x}^{l_2}}{+ 2 \sum_k G^k L_{l_1,k}^j - 2\delta_{l_1}^j \sum_k G^k L_{l_2,k}^{l_2}} + \\
& - \frac{1}{2} \sum_k H_{l_1}^k H_k^j + \frac{1}{2} \delta_{l_1}^j \sum_k H_{l_2}^k H_k^{l_2}.
\end{aligned}$$

Secondly, let us insert inside (3.86) the values of $\Theta_x^{l_1}$, $\Theta_x^{l_2}$ given by (3.101) and the value of $\Theta_{y^{l_1}}^0$ given by (3.100). We place all the terms in the right-hand side of the equality and we place the first order terms in the beginning (first three lines just

below). We obtain:

(3.107)

$$\begin{aligned}
 0 = & -\frac{1}{2} \underline{H_{l_1, y^{l_2}}^j}_{[1]} + \underline{L_{l_1, l_2, x}^j}_{[4]} - \frac{1}{2} \delta_{l_1}^j \underline{L_{l_2, l_2, x}^{l_2}}_{[5]} - \frac{1}{2} \delta_{l_1}^j \underline{L_{l_1, l_1, x}^{l_1}}_{[6]} + \\
 & + \frac{1}{3} \delta_{l_1}^j \underline{H_{l_2, y^{l_2}}^{l_2}}_{[2]} - \frac{1}{6} \delta_{l_1}^j \underline{L_{l_2, l_2, x}^{l_2}}_{[5]} + \frac{1}{3} \delta_{l_2}^j \underline{H_{l_1, y^{l_1}}^{l_1}}_{[3]} - \frac{1}{6} \delta_{l_2}^j \underline{L_{l_1, l_1, x}^{l_1}}_{[6]} + \\
 & + \frac{1}{3} \delta_{l_1}^j \underline{L_{l_2, l_2, x}^{l_2}}_{[5]} - \frac{1}{6} \delta_{l_1}^j \underline{H_{l_2, y^{l_2}}^{l_2}}_{[2]} + \\
 & + \underline{G^j M_{l_1, l_2}}_{[7]} - \frac{1}{2} \sum_k \underline{H_k^j L_{l_1, l_2}^k}_{[12]} + \frac{1}{2} \sum_k \underline{H_{l_1}^k L_{l_2, k}^j}_{[13]} - \frac{1}{4} \delta_{l_2}^j \sum_k \underline{H_{l_1}^k L_{k, k}^k}_{[a]} - \\
 & - \frac{1}{4} \delta_{l_2}^j \sum_k \underline{H_{l_1}^k \Theta^k}_{[b]} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} \Theta^0}_{[c]} + \frac{1}{4} \delta_{l_2}^j \underline{\Theta^0 \Theta^{l_1}}_{[d]} - \\
 & - \frac{2}{3} \delta_{l_1}^j \underline{G^{l_2} M_{l_2, l_2}}_{[8]} - \frac{1}{3} \delta_{l_1}^j \sum_k \underline{G^k M_{l_2, k}}_{[10]} + \frac{1}{3} \delta_{l_1}^j \sum_k \underline{H_k^{l_2} L_{l_2, l_2}^k}_{[14]} - \frac{1}{3} \delta_{l_1}^j \sum_k \underline{H_{l_2}^k L_{l_2, k}^{l_2}}_{[15]} +
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4} \delta_{l_1}^j \sum_k \underline{H_{l_2}^k L_{k, k}^k}_{[e]} + \frac{1}{4} \delta_{l_1}^j \sum_k \underline{H_{l_2}^k \Theta^k}_{[f]} - \frac{1}{4} \delta_{l_1}^j \underline{L_{l_2, l_2}^{l_2} \Theta^0}_{[g]} - \frac{1}{4} \delta_{l_1}^j \underline{\Theta^0 \Theta^{l_2}}_{[h]} - \\
 & - \frac{2}{3} \delta_{l_2}^j \underline{G^{l_1} M_{l_1, l_1}}_{[9]} - \frac{1}{3} \delta_{l_2}^j \sum_k \underline{G^k M_{l_1, k}}_{[11]} + \frac{1}{3} \delta_{l_2}^j \sum_k \underline{H_k^{l_1} L_{l_1, l_1}^k}_{[16]} - \frac{1}{3} \delta_{l_2}^j \sum_k \underline{H_{l_1}^k L_{l_1, k}^{l_1}}_{[17]} + \\
 & + \frac{1}{4} \delta_{l_2}^j \sum_k \underline{H_{l_1}^k L_{k, k}^k}_{[a]} + \frac{1}{4} \delta_{l_2}^j \sum_k \underline{H_{l_1}^k \Theta^k}_{[b]} - \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} \Theta^0}_{[c]} - \frac{1}{4} \delta_{l_2}^j \underline{\Theta^0 \Theta^0}_{[d]} + \\
 & + \frac{1}{3} \delta_{l_1}^j \underline{G^{l_2} M_{l_2, l_2}}_{[8]} + \frac{2}{3} \delta_{l_1}^j \sum_k \underline{G^k M_{l_2, k}}_{[10]} - \frac{1}{6} \delta_{l_1}^j \sum_k \underline{H_k^{l_2} L_{l_2, l_2}^k}_{[14]} + \frac{1}{6} \delta_{l_1}^j \sum_k \underline{H_{l_2}^k L_{l_2, k}^{l_2}}_{[15]} - \\
 & - \frac{1}{4} \delta_{l_1}^j \sum_k \underline{H_{l_2}^k L_{k, k}^k}_{[e]} - \frac{1}{4} \delta_{l_1}^j \sum_k \underline{H_{l_2}^k \Theta^k}_{[f]} + \frac{1}{4} \delta_{l_1}^j \underline{L_{l_2, l_2}^{l_2} \Theta^0}_{[g]} + \frac{1}{4} \delta_{l_1}^j \underline{\Theta^0 \Theta^{l_2}}_{[h]}.
 \end{aligned}$$

Simplifying and ordering, we obtain the family (II) of partial differential relations of Theorem 1.7 (3):

(3.108)

$$\begin{aligned}
0 = & -\frac{1}{2} H_{l_1, y^{l_2}}^j + \frac{1}{6} \delta_{l_1}^j H_{l_2, y^{l_2}}^{l_2} + \frac{1}{3} \delta_{l_2}^j H_{l_1, y^{l_1}}^{l_1} + \\
& + \underline{L_{l_1, l_2, x}^j - \frac{1}{3} \delta_{l_1}^j L_{l_2, l_2, x}^{l_2} - \frac{2}{3} \delta_{l_2}^j L_{l_1, l_1, x}^{l_1}} + \\
& + G^j M_{l_1, l_2} - \frac{1}{3} \delta_{l_1}^j G^{l_2} M_{l_2, l_2} - \frac{2}{3} \delta_{l_2}^j G^{l_1} M_{l_1, l_1} + \frac{1}{3} \delta_{l_1}^j \sum_k G^k M_{l_2, k} - \\
& - \frac{1}{3} \delta_{l_2}^j \sum_k G^k M_{l_1, k} - \frac{1}{2} \sum_k H_k^j L_{l_1, l_2}^k + \frac{1}{2} \sum_k H_{l_1}^k L_{l_2, k}^j + \\
& + \frac{1}{6} \delta_{l_1}^j \sum_k H_k^{l_2} L_{l_2, l_2}^k - \frac{1}{6} \delta_{l_1}^j \sum_k H_{l_2}^k L_{l_2, k}^{l_2} + \\
& + \frac{1}{3} \delta_{l_2}^j \sum_k H_k^{l_1} L_{l_1, l_1}^k - \frac{1}{3} \delta_{l_2}^j \sum_k H_{l_1}^k L_{l_1, k}^{l_1}.
\end{aligned}$$

Thirdly, let us insert inside (3.89) the values of $\Theta_{y^{l_3}}^{l_2}$, of $\Theta_{y^{l_2}}^{l_3}$, of $\Theta_{y^{l_3}}^{l_1}$ and of $\Theta_{y^{l_2}}^{l_1}$ given by (3.102). We place all the terms in the right-hand side of the equality and we place the first order terms in the beginning (first four lines just below). We obtain:

$$\begin{aligned}
(3.109) \quad 0 = & \frac{1}{2} \delta_{l_1}^j \underline{L_{l_2, l_2, y^{l_3}}^{l_2}}_{\text{(a)}} - \delta_{l_1}^j \underline{M_{l_2, l_3, x}}_{\text{(b)}} + \frac{1}{2} \delta_{l_2}^j \underline{L_{l_1, l_1, y^{l_3}}^{l_1}}_{\text{(c)}} - \delta_{l_2}^j \underline{M_{l_1, l_3, x}}_{\text{(4)}} - \\
& - \frac{1}{2} \delta_{l_1}^j \underline{L_{l_3, l_3, y^{l_2}}^{l_3}}_{\text{(d)}} + \delta_{l_1}^j \underline{M_{l_3, l_2, x}}_{\text{(b)}} - \frac{1}{3} \delta_{l_3}^j \underline{L_{l_1, l_1, y^{l_2}}^{l_1}}_{\text{(e)}} + \delta_{l_3}^j \underline{M_{l_1, l_2, x}}_{\text{(3)}} + \\
& + \underline{L_{l_1, l_2, y^{l_3}}^j}_{\text{(1)}} - \underline{L_{l_1, l_3, y^{l_2}}^j}_{\text{(2)}} + \frac{1}{2} \delta_{l_1}^j \underline{L_{l_3, l_3, y^{l_2}}^{l_3}}_{\text{(d)}} - \frac{1}{2} \delta_{l_1}^j \underline{L_{l_2, l_2, y^{l_3}}^{l_2}}_{\text{(a)}} + \\
& + \frac{1}{3} \delta_{l_3}^j \underline{L_{l_1, l_1, y^{l_2}}^{l_1}}_{\text{(e)}} - \frac{1}{2} \delta_{l_2}^j \underline{L_{l_1, l_1, y^{l_3}}^{l_1}}_{\text{(c)}} + \\
& + \frac{1}{2} \underline{H_{l_3}^j M_{l_1, l_2}}_{\text{(5)}} - \frac{1}{2} \underline{H_{l_2}^j M_{l_1, l_3}}_{\text{(6)}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_3, l_3}^{l_3} L_{l_1, l_1}^{l_1}}_{\text{(f)}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_2, l_2}^{l_2} L_{l_1, l_1}^{l_1}}_{\text{(g)}} + \\
& + \underline{\sum_k L_{l_1, l_3}^k L_{l_2, k}^j}_{\text{(11)}} - \underline{\sum_k L_{l_1, l_2}^k L_{l_3, k}^j}_{\text{(12)}} + \\
& + \frac{1}{2} \delta_{l_3}^j \underline{\sum_k L_{l_1, l_2}^k L_{k, k}^k}_{\text{(h)}} - \frac{1}{2} \delta_{l_2}^j \underline{\sum_k L_{l_1, l_3}^k L_{k, k}^k}_{\text{(i)}} + \\
& + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_3}}_{\text{(j)}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_1, l_1}^{l_1} \Theta^{l_2}}_{\text{(k)}} + \frac{1}{4} \delta_{l_2}^j \underline{L_{l_3, l_3}^{l_3} \Theta^{l_1}}_{\text{(l)}} - \frac{1}{4} \delta_{l_3}^j \underline{L_{l_2, l_2}^{l_2} \Theta^{l_1}}_{\text{(m)}} + \\
& + \frac{1}{2} \delta_{l_3}^j \underline{\sum_k L_{l_1, l_2}^k \Theta^k}_{\text{(n)}} - \frac{1}{2} \delta_{l_2}^j \underline{\sum_k L_{l_1, l_3}^k \Theta^k}_{\text{(o)}} + \\
& + \frac{1}{2} \delta_{l_2}^j \underline{M_{l_1, l_3} \Theta^0}_{\text{(p)}} - \frac{1}{2} \delta_{l_3}^j \underline{M_{l_1, l_2} \Theta^0}_{\text{(q)}} + \frac{1}{4} \delta_{l_2}^j \underline{\Theta^{l_3} \Theta^{l_1}}_{\text{(r)}} - \frac{1}{4} \delta_{l_3}^j \underline{\Theta^{l_2} \Theta^{l_1}}_{\text{(s)}} -
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2} \delta_{l_1}^j \sum_k H_{l_2}^k M_{l_3,k} \quad \text{[8]} \quad - \frac{1}{4} \delta_{l_1}^j L_{l_2,l_2}^{l_2} L_{l_3,l_3}^{l_3} \quad \text{[t]} \quad + \frac{1}{2} \delta_{l_1}^j \sum_k L_{l_2,l_3} L_{k,k}^k \quad \text{[u]} \\
& - \frac{1}{4} \delta_{l_1}^j L_{l_2,l_2}^{l_2} \Theta^{l_3} \quad \text{[v]} \quad - \frac{1}{4} \delta_{l_1}^j L_{l_3,l_3}^{l_3} \Theta^{l_2} \quad \text{[w]} \quad + \frac{1}{2} \delta_{l_1}^j \sum_k L_{l_2,l_3}^k \Theta^k \quad \text{[x]} \\
& - \frac{1}{2} \delta_{l_1}^j M_{l_2,l_3} \Theta^0 \quad \text{[y]} \quad - \frac{1}{4} \delta_{l_1}^j \Theta^{l_2} \Theta^{l_3} \quad \text{[z]} \\
& - \frac{1}{2} \delta_{l_2}^j \sum_k H_{l_1}^k M_{l_3,k} \quad \text{[10]} \quad - \frac{1}{4} \delta_{l_2}^j L_{l_1,l_1}^{l_1} L_{l_3,l_3}^{l_3} \quad \text{[f]} \quad + \frac{1}{2} \delta_{l_2}^j \sum_k L_{l_1,l_3}^k L_{k,k}^k \quad \text{[i]} \\
& - \frac{1}{4} \delta_{l_2}^j L_{l_1,l_1}^{l_1} \Theta^{l_3} \quad \text{[j]} \quad - \frac{1}{4} \delta_{l_2}^j L_{l_3,l_3}^{l_3} \Theta^{l_1} \quad \text{[l]} \quad + \frac{1}{2} \delta_{l_2}^j \sum_k L_{l_1,l_3}^k \Theta^k \quad \text{[o]} \\
& - \frac{1}{2} \delta_{l_2}^j M_{l_1,l_3} \Theta^0 \quad \text{[p]} \quad - \frac{1}{4} \delta_{l_2}^j \Theta^{l_3} \Theta^{l_1} \quad \text{[r]} \quad + \\
& + \frac{1}{2} \delta_{l_1}^j \sum_k H_{l_3}^k M_{l_2,k} \quad \text{[7]} \quad + \frac{1}{4} \delta_{l_1}^j L_{l_3,l_3}^{l_3} L_{l_2,l_2}^{l_2} \quad \text{[t]} \quad - \frac{1}{2} \delta_{l_1}^j \sum_k L_{l_3,l_2} L_{k,k}^k \quad \text{[u]} \\
& + \frac{1}{4} \delta_{l_1}^j L_{l_3,l_3}^{l_3} \Theta^{l_2} \quad \text{[w]} \quad + \frac{1}{4} \delta_{l_1}^j L_{l_2,l_2}^{l_2} \Theta^{l_3} \quad \text{[v]} \quad - \frac{1}{2} \delta_{l_1}^j \sum_k L_{l_3,l_2}^k \Theta^k \quad \text{[x]} \\
& + \frac{1}{2} \delta_{l_1}^j M_{l_3,l_2} \Theta^0 \quad \text{[y]} \quad + \frac{1}{4} \delta_{l_1}^j \Theta^{l_3} \Theta^{l_2} \quad \text{[z]} \\
& + \frac{1}{2} \delta_{l_3}^j \sum_k H_{l_1}^k M_{l_2,k} \quad \text{[9]} \quad + \frac{1}{4} \delta_{l_3}^j L_{l_1,l_1}^{l_1} L_{l_2,l_2}^{l_2} \quad \text{[g]} \quad - \frac{1}{2} \delta_{l_3}^j \sum_k L_{l_1,l_2}^k L_{k,k}^k \quad \text{[h]} \\
& + \frac{1}{4} \delta_{l_3}^j L_{l_1,l_1}^{l_1} \Theta^{l_2} \quad \text{[k]} \quad + \frac{1}{4} \delta_{l_3}^j L_{l_2,l_2}^{l_2} \Theta^{l_1} \quad \text{[m]} \quad - \frac{1}{2} \delta_{l_3}^j \sum_k L_{l_1,l_2}^k \Theta^k \quad \text{[n]} \\
& + \frac{1}{2} \delta_{l_3}^j M_{l_1,l_2} \Theta^0 \quad \text{[q]} \quad + \frac{1}{4} \delta_{l_3}^j \Theta^{l_1} \Theta^{l_2} \quad \text{[s]} .
\end{aligned}$$

Simplifying and ordering, we obtain the family (III) of partial differential relations of Theorem 1.7 (3):

$$\begin{aligned}
(3.110) \quad 0 = & \frac{L_{l_1,l_2,y}^j L_{l_3}^j - L_{l_1,l_3,y}^j L_{l_2}^j + \delta_{l_3}^j M_{l_1,l_2,x} - \delta_{l_2}^j M_{l_1,l_3,x}}{} \\
& + \frac{1}{2} H_{l_3}^j M_{l_1,l_2} - \frac{1}{2} H_{l_2}^j M_{l_1,l_3} + \\
& + \frac{1}{2} \delta_{l_1}^j \sum_k H_{l_3}^k M_{l_2,k} - \frac{1}{2} \delta_{l_1}^j \sum_k H_{l_2}^k M_{l_3,k} + \\
& + \frac{1}{2} \delta_{l_3}^j \sum_k H_{l_1}^k M_{l_2,k} - \frac{1}{2} \delta_{l_2}^j \sum_k H_{l_1}^k M_{l_3,k} + \\
& + \sum_k L_{l_1,l_3}^k L_{l_2,k}^j - \sum_k L_{l_1,l_2}^k L_{l_3,k}^j .
\end{aligned}$$

3.111. Arguments for the proof of Theorem 1.7 (3): necessity and sufficiency of (I), (II), (III), (IV). Let us summarize the implications that have been established so far, from the beginning of Section 3. Recall that $m \geq 2$.

- There exist functions X, Y^j of (x, y) transforming the system $y_{xx}^j = F^j(x, y, y_x), j = 1, \dots, m$, to the free particle system $Y_{XX}^j = 0, j = 1, \dots, m$.
- ↓
- There exist functions Π_{l_1, l_2}^j of $(x, y), 0 \leq j, l_1, l_2 \leq m$, satisfying the first auxiliary system (3.38) of partial differential equations.
- ↓
- There exist (principal unknowns) functions Θ^0, Θ^j satisfying the six families of partial differential equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96).
- ↓
- The functions $G^j, H_{l_1}^j, L_{l_1, l_2}^j$ and M_{l_1, l_2} satisfy the four families of partial differential equations (I), (II), (III) and (IV) of Theorem 1.7 (3).

The four families of first order partial differential equations (3.99), (3.100), (3.101) and (3.102) satisfied by the principal unknowns will be called the *second auxiliary system*. It is a complete system.

To achieve the proof of Theorem 1.7 (3), we have to establish the reverse implications. More precisely:

- Some given functions $G^j, H_{l_1}^j, L_{l_1, l_2}^j = L_{l_2, l_1}^j$ and $M_{l_1, l_2} = M_{l_2, l_1}$ of (x, y) satisfy the four families of partial differential equations (I), (II), (III) and (IV) of Theorem 1.7 (3), or equivalently, the partial differential equations (3.106), (3.108), (3.110) and (3.96).
- ↓
- There exist functions Θ^0, Θ^j satisfying the second auxiliary system (3.99), (3.100), (3.101) and (3.102).
- ↓
- These solution functions Θ^0, Θ^j satisfy the six families of partial differential equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96).
- ↓
- There exist functions Π_{l_1, l_2}^j of $(x, y), 0 \leq j, l_1, l_2 \leq m$, satisfying the first auxiliary system (3.38) of partial differential equations.
- ↓
- There exist functions X, Y^j of (x, y) transforming the system $y_{xx}^j = F^j(x, y, y_x), j = 1, \dots, m$, to the free particle system $Y_{XX}^j = 0, j = 1, \dots, m$.

The above last three implications have been already implicitly established in the preceding paragraphs, as may be checked by inspecting Lemma 3.40 and the formal computations after §3.62.

Thus, *it remains only to establish the first implication in the above reverse list*. Since the second auxiliary system (3.99), (3.100), (3.101) and (3.102) is complete and of first order, a necessary and sufficient condition for the existence of solutions follows

by writing out the following four families of cross-differentiations:

$$(3.112) \quad \begin{cases} 0 = (\Theta_x^0)_{y^{l_1}} - (\Theta_{y^{l_1}}^0)_x, \\ 0 = (\Theta_{y^{l_1}}^0)_{y^{l_2}} - (\Theta_{y^{l_2}}^0)_{y^{l_1}}, \\ 0 = (\Theta_x^{l_1})_{y^{l_2}} - (\Theta_{y^{l_2}}^{l_1})_x, \\ 0 = (\Theta_{y^{l_2}}^{l_1})_{y^{l_3}} - (\Theta_{y^{l_3}}^{l_1})_{y^{l_2}}. \end{cases}$$

In the hardest technical part of this paper (Section 4 below), we verify that these four families of compatibility conditions are a consequence of (I), (II), (III) and (IV). For reasons of space, we shall in fact only study the first family of compatibility conditions, *i.e.* the first line of (3.112). In our manuscript, we have treated the remaining three families of compatibility conditions similarly and completely, up to the very end of every branch of the coral tree of computations. However, we would like to mention that typesetting the remaining three cases would add at least fifty pages of Latex to Section 4. Thus, we prefer to expose thoroughly the treatment of the first family of compatibility conditions, explaining implicitly how to guess the treatment of the remaining three.

§4. COMPATIBILITY CONDITIONS FOR THE SECOND AUXILIARY SYSTEM

So, we have to develop the first line of (3.112): we replace Θ_x^0 by its expression (3.99), we differentiate it with respect to y^{l_1} , we replace $\Theta_{y^{l_1}}^0$ by its expression (3.100), we differentiate it with respect to x and we subtract. We get:

(4.1)

$$\begin{aligned} 0 &= (\Theta_x^0)_{y^{l_1}} - (\Theta_{y^{l_1}}^0)_x \\ &= \underline{-2G_{y^{l_1}y^{l_1}}^{l_1} + H_{l_1,xy^{l_1}}^{l_1}} + \\ &\quad + 2 \sum_k G_{y^{l_1}}^k L_{l_1,k}^{l_1} + 2 \sum_k G^k L_{l_1,k,y^{l_1}}^{l_1} - \sum_k G_{y^{l_1}}^k L_{k,k}^k - \sum_k G^k L_{k,k,y^{l_1}}^k - \\ &\quad - \frac{1}{2} \sum_k H_{l_1,y^{l_1}}^k H_k^{l_1} - \frac{1}{2} \sum_k H_{l_1}^k H_{k,y^{l_1}}^{l_1} - \sum_k G_{y^{l_1}}^k \Theta^k - \sum_k G^k \underline{\Theta_{y^{l_1}}^k} + \\ &\quad + \Theta^0 \underline{\Theta_{y^{l_1}}^0} - \\ &\quad - \underline{\frac{2}{3} L_{l_1,l_1,xx}^{l_1} + \frac{1}{3} H_{l_1,y^{l_1}x}^{l_1}} - \\ &\quad - \frac{2}{3} G_x^{l_1} M_{l_1,l_1} - \frac{2}{3} G^{l_1} M_{l_1,l_1,x} - \frac{4}{3} \sum_k G_x^k M_{l_1,k} - \frac{4}{3} \sum_k G^k M_{l_1,k,x} + \\ &\quad + \frac{1}{3} \sum_k H_{k,x}^{l_1} L_{l_1,l_1}^k + \frac{1}{3} \sum_k H_k^{l_1} L_{l_1,l_1,x}^k - \frac{1}{3} \sum_k H_{l_1,x}^k L_{l_1,k}^{l_1} - \frac{1}{3} \sum_k H_{l_1}^k L_{l_1,k,x}^{l_1} + \\ &\quad + \frac{1}{2} \sum_k H_{l_1,x}^k L_{k,k}^k + \frac{1}{2} \sum_k H_{l_1}^k L_{k,k,x}^k + \frac{1}{2} \sum_k H_{l_1,x}^k \Theta^k + \frac{1}{2} \sum_k H_{l_1}^k \underline{\Theta_x^k} - \\ &\quad - \frac{1}{2} L_{l_1,l_1,x} \Theta^0 - \frac{1}{2} L_{l_1,l_1} \underline{\Theta_x^0} - \frac{1}{2} \underline{\Theta_x^0} \Theta^{l_1} - \frac{1}{2} \Theta^0 \underline{\Theta_x^{l_1}}. \end{aligned}$$

Here, we underline twice the second order terms. Also, we have underlined once the six terms: $\underline{\Theta_{y^{l_1}}^k}$, $\underline{\Theta_{y^{l_1}}^0}$, $\underline{\Theta_x^k}$, $\underline{\Theta_x^0}$, $\underline{\Theta_x^0}$ and $\underline{\Theta_x^{l_1}}$. They must be replaced by their values given in (3.99), (3.100), (3.101) and (3.102). In this replacement, some double sums appear. As before, we use the first index $k = 1, \dots, m$ for single summation and then the second index $p = 1, \dots, m$ for double summation. Finally, we put all the second order terms in the first line, not disturbing the order of appearance of the 73 remaining terms. We get:

(4.2)

$$\begin{aligned}
0 = & \underline{\underline{-2 G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3} H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1} +}} \\
& \underline{\underline{+ 2 \sum_k G_{y^{l_1}}^k L_{l_1, k}^{l_1}}} \quad \underline{\underline{+ 2 \sum_k G^k L_{l_1, k, y^{l_1}}^{l_1}}} \quad - \underline{\underline{\sum_k G_{y^{l_1}}^k L_{k, k}^k}} \quad - \underline{\underline{\sum_k G^k L_{k, k, y^{l_1}}^k}} \quad - \\
& \underline{\underline{- \frac{1}{2} \sum_k H_{l_1, y^{l_1}}^k H_k^{l_1}}} \quad - \underline{\underline{- \frac{1}{2} \sum_k H_{l_1}^k H_{k, y^{l_1}}^{l_1}}} \quad - \underline{\underline{\sum_k G_{y^{l_1}}^k \Theta^k}} \quad + \\
& \underline{\underline{+ \sum_k G^k L_{k, k, y^{l_1}}^k}} \quad - \underline{\underline{2 \sum_k G^k M_{k, l_1, x}}} \quad - \underline{\underline{\sum_k \sum_p G^k H_k^p M_{l_1, p}}} \quad - \\
& \underline{\underline{- \frac{1}{2} \sum_k G^k L_{k, k}^k L_{l_1, l_1}^{l_1}}} \quad + \underline{\underline{\sum_k \sum_p G^k L_{k, l_1}^p L_{p, p}^p}} \quad - \underline{\underline{\frac{1}{2} \sum_k G^k L_{k, k}^k \Theta^{l_1}}} \quad - \\
& \underline{\underline{- \frac{1}{2} \sum_k G^k L_{l_1, l_1}^{l_1} \Theta^k}} \quad + \underline{\underline{\sum_k \sum_p G^k L_{k, l_1}^p \Theta^p}} \quad - \underline{\underline{\sum_k G^k M_{k, l_1} \Theta^0}} \quad - \underline{\underline{\frac{1}{2} \sum_k G^k \Theta^k \Theta^{l_1}}} \quad + \\
& \underline{\underline{+ \frac{2}{3} L_{l_1, l_1, x}^{l_1} \Theta^0}} \quad - \underline{\underline{\frac{1}{3} H_{l_1, y^{l_1}}^{l_1} \Theta^0}} \quad + \\
& \underline{\underline{+ \frac{2}{3} G^{l_1} M_{l_1, l_1} \Theta^0}} \quad + \underline{\underline{\frac{4}{3} \sum_k G^k M_{l_1, k} \Theta^0}} \quad - \underline{\underline{\frac{1}{3} \sum_k H_k^{l_1} L_{l_1, l_1}^{l_1} \Theta^0}} \quad + \underline{\underline{\frac{1}{3} \sum_k H_{l_1}^k L_{l_1, k}^{l_1} \Theta^0}} \quad - \\
& \underline{\underline{- \frac{1}{2} \sum_k H_{l_1}^k L_{k, k}^k \Theta^0}} \quad - \underline{\underline{\frac{1}{2} \sum_k H_{l_1}^k \Theta^k \Theta^0}} \quad + \underline{\underline{\frac{1}{2} L_{l_1, l_1}^{l_1} \Theta^0 \Theta^0}} \quad + \underline{\underline{\frac{1}{2} \Theta^0 \Theta^0 \Theta^{l_1}}} \quad - \\
& \underline{\underline{- \frac{2}{3} G_x^{l_1} M_{l_1, l_1}}} \quad - \underline{\underline{\frac{2}{3} G^{l_1} M_{l_1, l_1, x}}} \quad - \underline{\underline{\frac{4}{3} \sum_k G_x^k M_{l_1, k}}} \quad - \underline{\underline{\frac{4}{3} \sum_k G^k M_{l_1, k, x}}} \quad + \\
& \underline{\underline{+ \frac{1}{3} \sum_k H_{k, x}^{l_1} L_{l_1, l_1}^k}} \quad + \underline{\underline{\frac{1}{3} \sum_k H_k^{l_1} L_{l_1, l_1, x}^k}} \quad - \underline{\underline{\frac{1}{3} \sum_k H_{l_1, x}^{l_1} L_{l_1, k}^{l_1}}} \quad - \underline{\underline{\frac{1}{3} \sum_k H_{l_1}^k L_{l_1, k, x}^{l_1}}} \quad + \\
& \underline{\underline{+ \frac{1}{2} \sum_k H_{l_1, x}^k L_{k, k}^k}} \quad + \underline{\underline{\frac{1}{2} \sum_k H_{l_1}^k L_{k, k, x}^k}} \quad + \underline{\underline{\frac{1}{2} \sum_k H_{l_1, x}^k \Theta^k}} \quad -
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{3} \sum_k \underline{H_{l_1}^k H_{k,y^k}^k} \quad \underline{12} \quad + \frac{1}{6} \sum_k \underline{H_{l_1}^k L_{k,k,x}^k} \quad \underline{15} \quad + \frac{2}{3} \sum_k \underline{H_{l_1}^k G^k M_{k,k}} \quad \underline{20} \quad + \\
& + \frac{1}{3} \sum_k \sum_p \underline{H_{l_1}^k G^p M_{k,p}} \quad \underline{21} \quad - \frac{1}{3} \sum_k \sum_p \underline{H_{l_1}^k H_p^p L_{k,k}^p} \quad \underline{24} \quad + \frac{1}{3} \sum_k \sum_p \underline{H_{l_1}^k H_k^p L_{k,p}^k} \quad \underline{25} \quad - \\
& - \frac{1}{4} \sum_k \sum_p \underline{H_{l_1}^k H_k^p L_{p,p}^p} \quad \underline{26} \quad - \frac{1}{4} \sum_k \sum_p \underline{H_{l_1}^k H_k^p \Theta^p} \quad \underline{\eta} \quad + \frac{1}{4} \sum_k \underline{H_{l_1}^k L_{k,k}^k \Theta^0} \quad \underline{\textcircled{1}} \quad + \\
& + \frac{1}{4} \sum_k \underline{H_{l_1}^k \Theta^0 \Theta^k} \quad \underline{\textcircled{m}} \quad - \frac{1}{2} \underline{L_{l_1,l_1,x}^{l_1} \Theta^0} \quad \underline{\textcircled{g}} \quad + \\
& + \frac{G_{y^{l_1}}^{l_1} L_{l_1,l_1}^{l_1}}{\underline{3}} \quad \underline{3} \quad - \frac{1}{2} \underline{H_{l_1,x}^{l_1} L_{l_1,l_1}^{l_1}} \quad \underline{6} \quad - \\
& - \sum_k \underline{G^k L_{l_1,l_1}^{l_1} L_{l_1,k}^{l_1}} \quad \underline{22} \quad + \frac{1}{2} \sum_k \underline{G^k L_{l_1,l_1}^{l_1} L_{k,k}^k} \quad \underline{\textcircled{b}} \quad + \frac{1}{4} \sum_k \underline{H_{l_1}^k H_k^{l_1} L_{l_1,l_1}^{l_1}} \quad \underline{27} \quad + \\
& + \frac{1}{2} \sum_k \underline{G^k L_{l_1,l_1}^{l_1} \Theta^k} \quad \underline{\textcircled{d}} \quad - \frac{1}{4} \underline{L_{l_1,l_1}^{l_1} \Theta^0 \Theta^0} \quad \underline{\textcircled{n}} \quad + \\
& + \frac{G_{y^{l_1}}^{l_1} \Theta^{l_1}}{\underline{\beta}} \quad \underline{\beta} \quad - \frac{1}{2} \underline{H_{l_1,x}^{l_1} \Theta^{l_1}} \quad \underline{\delta} \quad - \\
& - \sum_k \underline{G^k L_{l_1,k}^{l_1} \Theta^{l_1}} \quad \underline{\zeta} \quad + \frac{1}{2} \sum_k \underline{G^k L_{k,k}^k \Theta^{l_1}} \quad \underline{\textcircled{c}} \quad + \frac{1}{4} \sum_k \underline{H_{l_1}^k H_k^{l_1} \Theta^{l_1}} \quad \underline{\theta} \quad + \frac{1}{2} \sum_k \underline{G^k \Theta^k \Theta^{l_1}} \quad \underline{\textcircled{f}} \quad - \\
& - \frac{1}{4} \underline{\Theta^0 \Theta^0 \Theta^{l_1}} \quad \underline{\textcircled{o}} \quad + \\
& + \frac{1}{3} \underline{H_{l_1,y^{l_1}}^{l_1} \Theta^0} \quad \underline{\textcircled{h}} \quad - \frac{1}{6} \underline{L_{l_1,l_1,x}^{l_1} \Theta^0} \quad \underline{\textcircled{g}} \quad - \\
& - \frac{2}{3} \underline{G^{l_1} M_{l_1,l_1} \Theta^0} \quad \underline{\textcircled{i}} \quad - \frac{1}{3} \sum_k \underline{G^k M_{l_1,k} \Theta^0} \quad \underline{\textcircled{e}} \quad + \frac{1}{3} \sum_k \underline{H_k^{l_1} L_{l_1,l_1}^k \Theta^0} \quad \underline{\textcircled{j}} \quad - \frac{1}{3} \sum_k \underline{H_{l_1}^k L_{l_1,k}^{l_1} \Theta^0} \quad \underline{\textcircled{k}} \quad + \\
& + \frac{1}{4} \sum_k \underline{H_{l_1}^k L_{k,k}^k \Theta^0} \quad \underline{\textcircled{1}} \quad + \frac{1}{4} \sum_k \underline{H_{l_1}^k \Theta^0 \Theta^k} \quad \underline{\textcircled{m}} \quad - \frac{1}{4} \underline{L_{l_1,l_1}^{l_1} \Theta^0 \Theta^0} \quad \underline{\textcircled{n}} \quad - \frac{1}{4} \underline{\Theta^0 \Theta^0 \Theta^{l_1}} \quad \underline{\textcircled{o}} \quad .
\end{aligned}$$

As usual, all the terms underlined with the 15 roman alphabetic letters a, b, . . . , n, o appended vanish evidently. Furthermore, we claim that the eight terms underlined with the 8 Greek alphabetic letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ and θ also vanish:

(4.3)

$$\begin{aligned}
0 = ? = & - \sum_k G_{y^{l_1}}^k \Theta^k + G_{y^{l_1}}^{l_1} \Theta^{l_1} + \frac{1}{2} \sum_k H_{l_1,x}^k \Theta^k - \frac{1}{2} H_{l_1,x}^{l_1} \Theta^{l_1} + \\
& + \sum_k \sum_p G^k L_{k,l_1}^p \Theta^p - \sum_k G^k L_{l_1,k}^{l_1} \Theta^{l_1} - \frac{1}{4} \sum_k \sum_p H_{l_1}^k H_k^p \Theta^p + \frac{1}{4} \sum_k H_{l_1}^k H_k^{l_1} \Theta^{l_1} .
\end{aligned}$$

Indeed, it suffices to observe that this identity coincides with

$$(4.4) \quad 0 = \frac{1}{2} \sum_k \Theta^k \left((3.106)|_{j:=k; l_1:=l_1; l_2:=l_1} \right) .$$

Simplifying then (4.2), we get the explicit formulation of the first family of compatibility conditions for the second auxiliary system:

$$\begin{aligned}
(4.5) \quad 0 =? &= \frac{2G_{y^{l_1}y^{l_1}}^{l_1} + \frac{4}{3}H_{l_1,xy^{l_1}}^{l_1} - \frac{2}{3}L_{l_1,l_1,xx}^{l_1}}{} \\
&- \frac{2}{3}G_x^{l_1}M_{l_1,l_1} - \frac{4}{3}\sum_k G_x^k M_{l_1,k} + G_{y^{l_1}}^{l_1}L_{l_1,l_1}^k + \\
&+ 2\sum_k G_{y^{l_1}}^k L_{l_1,k}^{l_1} - \sum_k G_{y^{l_1}}^k L_{k,k}^k - \\
&- \frac{1}{2}H_{l_1,x}^{l_1}L_{l_1,l_1}^{l_1} - \frac{1}{3}\sum_k H_{l_1,x}^k L_{l_1,k}^{l_1} + \frac{1}{2}\sum_k H_{l_1,x}^k L_{k,k}^k + \\
&+ \frac{1}{3}\sum_k H_{k,x}^{l_1}L_{l_1,l_1}^k - \frac{1}{2}\sum_k H_{l_1,y^{l_1}}^k H_k^{l_1} - \frac{1}{2}\sum_k H_{k,y^{l_1}}^{l_1} H_{l_1}^k - \\
&- \frac{1}{3}\sum_k H_{k,y^k}^k H_{l_1}^k - \\
&- \frac{1}{3}\sum_k L_{l_1,k,x}^{l_1} H_{l_1}^k + \frac{1}{3}\sum_k L_{l_1,l_1,x}^k H_k^{l_1} + \frac{2}{3}\sum_k L_{k,k,x}^k H_{l_1}^k + \\
&+ 2\sum_k L_{l_1,k,y^{l_1}}^{l_1} G^k - \\
&- \frac{2}{3}M_{l_1,l_1,x} G^{l_1} - \frac{10}{3}\sum_k M_{k,l_1,x} G^k - \\
&- \sum_k \sum_p G^k H_k^p M_{l_1,p} + \frac{2}{3}\sum_k G^k H_{l_1}^k M_{k,k} + \frac{1}{3}\sum_k \sum_p G^p H_{l_1}^k M_{k,p} - \\
&- \sum_k G^k L_{l_1,k}^{l_1} L_{l_1,l_1}^{l_1} + \sum_k \sum_p G^k L_{k,l_1}^p L_{p,p}^p - \\
&- \frac{1}{3}\sum_k \sum_p H_{l_1}^k H_p^k L_{k,k}^p + \frac{1}{3}\sum_k \sum_p H_{l_1}^k H_k^p L_{k,p}^k - \\
&- \frac{1}{4}\sum_k \sum_p H_{l_1}^k H_k^p L_{p,p}^p + \frac{1}{4}\sum_k H_{l_1}^k H_k^{l_1} L_{l_1,l_1}^{l_1}.
\end{aligned}$$

We can now state the main technical lemma of this section and of this paper.

Lemma 4.6. *The second order partial differential relations (4.5) hold true for $l = 1, \dots, m$, and they are a consequence, by differentiations and by linear combinations, of the fundamental first order partial differential equations (3.106), (3.108), (3.110) and (3.96).*

4.7. Reconstitution of the appropriate linear combinations. The remaining of Section 4 is entirely devoted to the proof of this statement. From the manual computational point of view, the difficulty of the task is due to the fact that one has to manipulate formal expressions having from 10 to 50 terms. So the real question is: *how can we reconstitute the linear combinations and the differentiations which lead to the goal (4.5) from the data (3.106), (3.108), (3.110) and (3.96)?*

The main trick is to first neglect the first order and the zero order terms in the goal (4.5). Using the symbol “ \equiv ” to denote “*modulo first order and the zero order*”

terms”, we formulate the following sub-goal:

$$(4.8) \quad 0 \stackrel{?}{\equiv} \underline{\underline{-2 G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3} H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1}}}}$$

for $l_1 = 1, \dots, m$. Before establishing that these partial differential relations are a consequence of the data (3.106), (3.108), (3.110) and (3.96) (written with a similar sign \equiv), let us check that they are a consequence of the existence of the change of coordinates $(x, y) \mapsto (X, Y)$ (however, recall that, in establishing the reverse implications of §3.111, we still do not know that such a change of coordinates really exists); this will confirm the coherence and the validity of our computations. Importantly, we have been able to achieve systematic corrections of our computations by always checking them alongside with the existence of the change of coordinates $(x, y) \mapsto (X, Y)$.

Coming back to the definition (3.35) and to the approximation (3.58), we have:

$$(4.9) \quad \begin{aligned} G^{l_1} &= -\square_{xx}^{l_1} \cong -Y_{xx}^{l_1}, \\ H_{l_1}^{l_1} &= -2\square_{xy^{l_1}}^{l_1} + \square_{xx}^0 \cong -2Y_{xy^{l_1}}^{l_1} + X_{xx}, \\ L_{l_1, l_1}^{l_1} &= -\square_{y^{l_1} y^{l_1}}^{l_1} + 2\square_{xy^{l_1}}^0 \cong -Y_{y^{l_1} y^{l_1}}^{l_1} + 2X_{xy^{l_1}}. \end{aligned}$$

Differentiating the first two lines with respect to y^{l_1} and the third line with respect to x , and replacing the sign \cong by the sign \equiv (in a non-rigorous way, this corresponds essentially to neglecting the derivatives of order 0, 1, 2 and 3 of X, Y^j and to neglecting the difference between the Jacobian matrix of the transformation and the identity matrix), we get:

$$(4.10) \quad \begin{aligned} \underline{\underline{G_{y^{l_1} y^{l_1}}^{l_1}}} &\equiv -Y_{x y^{l_1} y^{l_1}}^{l_1}, \\ \underline{\underline{H_{l_1, x y^{l_1}}^{l_1}}} &\equiv -2Y_{x y^{l_1} x y^{l_1}}^{l_1} + X_{x x x y^{l_1}}, \\ \underline{\underline{L_{l_1, l_1, x x}^{l_1}}} &\equiv -Y_{y^{l_1} y^{l_1} x x}^{l_1} + 2X_{x y^{l_1} x x}. \end{aligned}$$

Hence the linear combination $-2 \cdot (4.10)_1 + \frac{4}{3} \cdot (4.10)_2 - \frac{2}{3} (4.10)_3$ yields the desired result:

$$(4.11) \quad \begin{aligned} 0 \stackrel{?}{\equiv} &\underline{\underline{-2 G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3} H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1}}} \\ &\equiv \underline{\underline{2 Y_{x y^{l_1} y^{l_1}}^{l_1}}} \textcircled{a} - \underline{\underline{\frac{8}{3} Y_{x y^{l_1} x y^{l_1}}^{l_1}}} \textcircled{a} + \underline{\underline{\frac{4}{3} X_{x x x y^{l_1}}}} \textcircled{b} + \underline{\underline{\frac{2}{3} Y_{y^{l_1} y^{l_1} x x}^{l_1}}} \textcircled{a} - \underline{\underline{\frac{4}{3} X_{x y^{l_1} x x}}} \textcircled{b} \\ &\equiv 0, \quad \text{indeed!} \end{aligned}$$

Thanks to this straightforward computation, we guess that the approximate partial differential relations (4.8) are a consequence of the approximate relations (3.106), (3.108), (3.110) and (3.96), namely:

$$(4.12) \quad \begin{aligned} (3.106)^{\text{mod}} : \quad &0 \equiv -2 G_{y^{l_1}}^j + 2 \delta_{l_1}^j G_{y^{l_2}}^{l_2} + H_{l_1, x}^j - \delta_{l_1}^j H_{l_2, x}^{l_2}, \\ (3.108)^{\text{mod}} : \quad &0 \equiv -\frac{1}{2} H_{l_1, y^{l_2}}^j + \frac{1}{6} \delta_{l_1}^j H_{l_2, y^{l_2}}^{l_2} + \frac{1}{3} \delta_{l_2}^j H_{l_1, y^{l_1}}^{l_1} + \\ &+ L_{l_1, l_2, x}^j - \frac{1}{3} \delta_{l_1}^j L_{l_2, l_2, x}^{l_2} - \frac{2}{3} \delta_{l_2}^j L_{l_1, l_1, x}^{l_1}, \\ (3.110)^{\text{mod}} : \quad &0 \equiv L_{l_1, l_2, y^{l_3}}^j - L_{l_1, l_3, y^{l_2}}^j + \delta_{l_3}^j M_{l_1, l_2, x} - \delta_{l_2}^j M_{l_1, l_3, x}, \\ (3.96)^{\text{mod}} : \quad &0 \equiv M_{l_1, l_2, y^{l_3}} - M_{l_1, l_3, y^{l_2}}. \end{aligned}$$

Here, the sign \equiv means “*modulo zero order terms*”. Before proceeding further, recall the correspondence between partial differential relations:

$$(4.13) \quad \begin{aligned} \text{(I)} &= (3.106), \\ \text{(II)} &= (3.108), \\ \text{(III)} &= (3.110), \\ \text{(IV)} &= (3.96). \end{aligned}$$

However, these couples of equivalent identities are written slightly differently, as may be read by comparison. To fix ideas and to facilitate the eye-checking of our subsequent computations, *we shall only use and refer to the exact writing of (3.106), of (3.108), of (3.110) and of (3.96).*

4.14. Construction of a guide. So we want to show that the approximate relation (4.8) is a consequence, by differentiations and by linear combinations, of the approximate identities (4.12). The interest of working with approximate identities is that formal computations are lightened substantially. After having discovered which linear combinations and which differentiations are appropriate, *i.e.* after having constructed a “guide”, in §4.22 below, we shall write down the complete computations, including all zero order terms, following our guide.

We shall use two indices l_1 and l_2 with $1 \leq l_1, l_2 \leq m$ and, crucially, $l_2 \neq l_1$. Again, the assumption $m \geq 2$ is used strongly.

Firstly, put $j := l_1$ in (3.106)^{mod} with $l_2 \neq l_1$ and differentiate with respect to y^{l_1} :

$$(4.15) \quad 0 \equiv -2G_{y^{l_1}y^{l_1}}^{l_1} + 2G_{y^{l_2}y^{l_1}}^{l_2} + H_{l_1,xy^{l_1}}^{l_1} - H_{l_2,xy^{l_1}}^{l_2}.$$

Secondly, put $j := l_2$ in (3.106)^{mod} with $l_1 \neq l_2$ and differentiate with respect to y^{l_2} :

$$(4.16) \quad 0 \equiv -2G_{y^{l_1}y^{l_2}}^{l_2} + H_{l_1,xy^{l_2}}^{l_2}.$$

Thirdly, put $j := l_2$ in (3.108)^{mod} with $l_1 \neq l_2$ and differentiate with respect to x :

$$(4.17) \quad 0 \equiv -\frac{1}{2}H_{l_1,y^{l_2}x}^{l_2} + \frac{1}{3}H_{l_1,y^{l_1}x}^{l_1} + L_{l_1,l_2,xx}^{l_2} - \frac{2}{3}L_{l_1,l_1,xx}^{l_1}.$$

Fourthly, put $j := l_1$ in (3.108)^{mod} with $l_2 \neq l_1$ and differentiate with respect to x :

$$(4.18) \quad 0 \equiv -\frac{1}{2}H_{l_1,y^{l_2}x}^{l_1} + \frac{1}{6}H_{l_2,y^{l_2}x}^{l_2} + L_{l_1,l_2,xx}^{l_1} - \frac{1}{3}L_{l_2,l_2,xx}^{l_2}.$$

Fifthly, permute the indices $(l_1, l_2) \mapsto (l_2, l_1)$:

$$(4.19) \quad 0 \equiv -\frac{1}{2}H_{l_2,y^{l_1}x}^{l_2} + \frac{1}{6}H_{l_1,y^{l_1}x}^{l_1} + L_{l_2,l_1,xx}^{l_2} - \frac{1}{3}L_{l_1,l_1,xx}^{l_1}.$$

Finally, compute the linear combination (4.15) + (4.16) + 2 · (4.17) – 2 · (4.19):

$$(4.20) \quad \begin{aligned} 0 \equiv & \underbrace{-2G_{y^{l_1}y^{l_1}}^{l_1}}_{\boxed{1}} + \underbrace{2G_{y^{l_1}y^{l_2}}^{l_2}}_{\textcircled{a}} + \underbrace{H_{l_1,xy^{l_1}}^{l_1}}_{\boxed{2}} - \underbrace{H_{l_2,xy^{l_1}}^{l_2}}_{\textcircled{b}} - \\ & \underbrace{-2G_{y^{l_1}y^{l_2}}^{l_2}}_{\textcircled{a}} + \underbrace{H_{l_1,xy^{l_2}}^{l_2}}_{\textcircled{c}} - \\ & \underbrace{-H_{l_1,xy^{l_2}}^{l_2}}_{\textcircled{c}} + \underbrace{\frac{2}{3}H_{l_1,l_1,xy^{l_1}}^{l_1}}_{\boxed{2}} + \underbrace{2L_{l_1,l_2,xx}^{l_2}}_{\textcircled{d}} - \underbrace{\frac{4}{3}L_{l_1,l_1,xx}^{l_1}}_{\boxed{3}} + \\ & \underbrace{+H_{l_2,xy^{l_1}}^{l_2}}_{\textcircled{b}} - \underbrace{\frac{1}{3}H_{l_1,xy^{l_1}}^{l_1}}_{\boxed{2}} - \underbrace{2L_{l_2,l_1,xx}^{l_2}}_{\textcircled{d}} + \underbrace{\frac{2}{3}L_{l_1,l_1,xx}^{l_1}}_{\boxed{3}}. \end{aligned}$$

We indeed get the desired approximate identity:

$$(4.21) \quad 0 \equiv -2 G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3} H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1}.$$

4.22. Complete computation. Now that the guide is constructed, we can achieve the complete computations.

Firstly, put $j := l_1$ in (3.106) with $l_2 \neq l_1$ and differentiate with respect to y^{l_1} :

$$(4.23) \quad \begin{aligned} 0 \equiv & \underline{\underline{-2 G_{y^{l_1} y^{l_1}}^{l_1} + 2 G_{y^{l_2} y^{l_1}}^{l_2} + H_{l_1, x y^{l_1}}^{l_1} - H_{l_2, x y^{l_1}}^{l_2} +}} \\ & + 2 \sum_k G_{y^{l_1}}^k L_{l_1, k}^{l_1} + 2 \sum_k G^k L_{l_1, k, y^{l_1}}^{l_1} - 2 \sum_k G_{y^{l_1}}^k L_{l_2, k}^{l_2} - 2 \sum_k G^k L_{l_2, k, y^{l_1}}^{l_2} \\ & - \frac{1}{2} \sum_k H_{l_1, y^{l_1}}^k H_k^{l_1} - \frac{1}{2} \sum_k H_{l_1}^k H_{k, y^{l_1}}^{l_1} + \frac{1}{2} \sum_k H_{l_2, y^{l_1}}^k H_k^{l_2} + \frac{1}{2} \sum_k H_{l_2}^k H_{k, y^{l_1}}^{l_2}. \end{aligned}$$

Secondly, put $j := l_2$ in (3.106) with $l_1 \neq l_2$ and differentiate with respect to y^{l_2} :

$$(4.24) \quad \begin{aligned} 0 \equiv & \underline{\underline{-2 G_{y^{l_1} y^{l_2}}^{l_2} + H_{l_1, x y^{l_2}}^{l_2} +}} \\ & 2 \sum_k G_{y^{l_2}}^k L_{l_1, k}^{l_2} + 2 \sum_k G^k L_{l_1, k, y^{l_2}}^{l_2} - \frac{1}{2} \sum_k H_{l_1, y^{l_2}}^k H_k^{l_2} - \frac{1}{2} \sum_k H_{l_1}^k H_{k, y^{l_2}}^{l_2}. \end{aligned}$$

Thirdly, put $j := l_2$ in (3.108) with $l_1 \neq l_2$ and differentiate with respect to x :

$$(4.25) \quad \begin{aligned} 0 \equiv & \underline{\underline{-\frac{1}{2} H_{l_1, y^{l_2} x}^{l_2} + \frac{1}{3} H_{l_1, y^{l_1} x}^{l_1} + L_{l_1, l_2, x x}^{l_2} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1} +}} \\ & + G_x^{l_2} M_{l_1, l_2} + G_x^{l_2} M_{l_1, l_2, x} - \frac{2}{3} G_x^{l_1} M_{l_1, l_1} - \frac{2}{3} G^{l_1} M_{l_1, l_1, x} - \\ & - \frac{1}{3} \sum_k G_x^k M_{l_1, k} - \frac{1}{3} \sum_k G^k M_{l_1, k, x} - \frac{1}{2} \sum_k H_{k, x}^{l_2} L_{l_1, l_2}^k - \frac{1}{2} \sum_k H_k^{l_2} L_{l_1, l_2, x}^k + \\ & + \frac{1}{2} \sum_k H_{l_1, x}^k L_{l_2, k}^{l_2} + \frac{1}{2} \sum_k H_{l_1}^k L_{l_2, k, x}^{l_2} + \frac{1}{3} \sum_k H_{k, x}^{l_1} L_{l_1, l_1}^k + \frac{1}{3} \sum_k H_k^{l_1} L_{l_1, l_1, x}^k - \\ & - \frac{1}{3} \sum_k H_{l_1, x}^k L_{l_1, k}^{l_1} - \frac{1}{3} \sum_k H_{l_1}^k L_{l_1, k, x}^{l_1}. \end{aligned}$$

Fourthly, put $j := l_1$ in (3.108) with $l_2 \neq l_1$, differentiate with respect to x and permute the indices $(l_1, l_2) \mapsto (l_2, l_1)$:

$$(4.26) \quad \begin{aligned} 0 \equiv & \underline{\underline{-\frac{1}{2} H_{l_2, y^{l_1} x}^{l_2} + \frac{1}{6} H_{l_1, y^{l_1} x}^{l_1} + L_{l_2, l_1, x x}^{l_2} - \frac{1}{3} L_{l_1, l_1, x x}^{l_1} +}} \\ & + G_x^{l_2} M_{l_2, l_1} + G^{l_2} M_{l_2, l_1, x} - \frac{1}{3} G_x^{l_1} M_{l_1, l_1} - \frac{1}{3} G^{l_1} M_{l_1, l_1, x} + \\ & + \frac{1}{3} \sum_k G_x^k M_{l_1, k} + \frac{1}{3} \sum_k G^k M_{l_1, k, x} - \frac{1}{2} \sum_k H_{k, x}^{l_2} L_{l_2, l_1}^k - \frac{1}{2} \sum_k H_k^{l_2} L_{l_2, l_1, x}^k + \\ & + \frac{1}{2} \sum_k H_{l_2, x}^k L_{l_1, k}^{l_2} + \frac{1}{2} \sum_k H_{l_2}^k L_{l_1, k, x}^{l_2} + \frac{1}{6} \sum_k H_{k, x}^{l_1} L_{l_1, l_1}^k + \frac{1}{6} \sum_k H_k^{l_1} L_{l_1, l_1, x}^k - \\ & - \frac{1}{6} \sum_k H_{l_1, x}^k L_{l_1, k}^{l_1} - \frac{1}{6} \sum_k H_{l_1}^k L_{l_1, k, x}^{l_1}. \end{aligned}$$

Finally, compute the linear combination (4.23) + (4.24) + 2 · (4.25) – 2 · (4.26):
(4.27)

$$\begin{aligned}
0 = & \underline{\underline{-2 G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3} H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1} -}} \\
& - \frac{2}{3} G_x^{l_1} M_{l_1, l_1} - \frac{4}{3} \sum_k G_x^k M_{l_1, k} + \\
& + 2 \sum_k G_{y^{l_1}}^k L_{l_1, k}^{l_1} - 2 \sum_k G_{y^{l_1}}^k L_{l_2, k}^{l_2} + 2 \sum_k G_{y^{l_2}}^k L_{l_1, k}^{l_2} + \\
& + \sum_k H_{l_1, x}^k L_{l_2, k}^{l_2} + \frac{1}{3} \sum_k H_{k, x}^{l_1} L_{l_1, l_1}^k - \frac{1}{3} \sum_k H_{l_1, x}^k L_{l_1, k}^{l_1} - \sum_k H_{l_2, x}^k L_{l_1, k}^{l_2} + \\
& + \frac{1}{2} \sum_k H_{l_2, y^{l_1}}^k H_k^{l_2} + \frac{1}{2} \sum_k H_{k, y^{l_1}}^{l_2} H_{l_2}^k - \frac{1}{2} \sum_k H_{l_1, y^{l_1}}^k H_k^{l_1} - \\
& - \frac{1}{2} \sum_k H_{l_1}^k H_{k, y^{l_1}}^{l_1} - \frac{1}{2} \sum_k H_{l_1, y^{l_2}}^k H_k^{l_1} - \frac{1}{2} \sum_k H_{l_1}^k H_{k, y^{l_2}}^{l_2} + \\
& + \sum_k L_{l_2, k, x}^{l_2} H_{l_1}^k + \frac{1}{3} \sum_k L_{l_1, l_1, x}^k H_k^{l_1} - \frac{1}{3} \sum_k L_{l_1, k, x}^{l_1} H_{l_1}^k - \sum_k L_{l_1, k, x}^{l_2} H_{l_2}^k + \\
& + 2 \sum_k L_{l_1, k, y^{l_1}}^{l_1} G^k + 2 \sum_k L_{l_1, k, y^{l_2}}^{l_2} G^k - 2 \sum_k L_{l_2, k, y^{l_1}}^{l_2} G^k - \\
& - \frac{2}{3} M_{l_1, l_1, x} G^{l_1} - \frac{4}{3} \sum_k M_{l_1, k, x} G^k.
\end{aligned}$$

In this partial differential relation, importantly, the second order terms are exactly the same as in our goal (4.5). Unfortunately, the first order and the zero order terms are not the same.

4.28. Formulation of a new goal. Thus, in order to get rid of the second order expression $2 G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3} H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3} L_{l_1, l_1, x x}^{l_1}$, we subtract: (4.5) – (4.27). In the result, we write the first order terms in a certain way, adapted in advance to our subsequent computations. For this subtraction yielding (4.29) just below, we have not underlined the terms in (4.5) and in (4.27). However, they may be underlined with a pencil to

check that the result (4.29) is correct. We get:

$$\begin{aligned}
(4.29) \quad 0 & \stackrel{?}{=} \\
& = - \underbrace{\sum_k G_{y^{l_1}}^k L_{k,k}^k + G_{y^{l_1}}^{l_1} L_{l_1,l_1}^{l_1} + \frac{1}{2} \sum_k H_{l_1,x}^k L_{k,k}^k - \frac{1}{2} H_{l_1,x}^{l_1} L_{l_1,l_1}^{l_1}}_+ \\
& \quad + 2 \underbrace{\sum_k G_{y^{l_1}}^k L_{l_2,k}^{l_2} - \sum_k H_{l_1,x}^k L_{l_2,k}^{l_2}}_- \\
& \quad - 2 \underbrace{\sum_k G_{y^{l_2}}^k L_{l_1,k}^{l_2} + \sum_k H_{l_2,x}^k L_{l_1,k}^{l_2}}_+ \\
& \quad + \frac{1}{2} \sum_k H_{k,y^{l_2}}^{l_2} H_{l_1}^k - \frac{1}{3} \sum_k H_{k,y^k}^k H_{l_1}^k - \sum_k L_{l_2,k,x}^{l_2} H_{l_1}^k + \frac{2}{3} \sum_k L_{k,k,x}^k H_{l_1}^k - \\
& \quad - \frac{1}{2} \sum_k H_{k,y^{l_1}}^{l_2} H_{l_2}^k + \sum_k L_{l_1,k,x}^{l_2} H_{l_2}^k + \\
& \quad + \frac{1}{2} \sum_k H_{l_1,y^{l_2}}^k H_k^{l_2} - \frac{1}{2} \sum_k H_{l_2,y^{l_1}}^k H_k^{l_2} + \\
& \quad + 2 \underbrace{\sum_k L_{l_2,k,y^{l_1}}^{l_2} G^k - 2 \sum_k L_{l_1,k,y^{l_2}}^{l_2} G^k - 2 \sum_k M_{k,l_1,x} G^k}_- \\
& \quad - \sum_k \sum_p G^k H_k^p M_{l_1,p} + \frac{2}{3} \sum_k G^k H_{l_1}^k M_{k,k} + \frac{1}{3} \sum_k \sum_p G^p H_{l_1}^k M_{k,p} - \\
& \quad - \sum_k G^k L_{l_1,l_1}^{l_1} L_{l_1,k}^{l_1} + \sum_k \sum_p G^k L_{k,l_1}^p L_{p,p}^p - \frac{1}{3} \sum_k \sum_p H_{l_1}^k H_p^k L_{k,k}^p + \\
& \quad + \frac{1}{3} \sum_k \sum_p H_{l_1}^k H_k^p L_{k,p}^k - \frac{1}{4} \sum_k \sum_p H_{l_1}^k H_k^p L_{p,p}^p + \frac{1}{4} \sum_k H_{l_1}^k H_k^{l_1} L_{l_1,l_1}^{l_1}.
\end{aligned}$$

We have underlined plainly the first order terms appearing in lines 1, 2, 3, 4, 5, 6 and 7.

4.30. Reconstitution of the subgoal (4.29) from (3.106), from (3.108) and from (3.110). Now, it suffices to establish that the first order partial differential relations (4.29) for $1 \leq l_1, l_2 \leq m$ and $l_2 \neq l_1$ (crucial assumption) are a consequence of (3.106), of (3.108) and of (3.110) by linear combinations. The auxiliary index l_2 , which is absent in the goal (4.5), will disappear at the end. Differentiations will not be applied anymore. Also, the partial differential relations (3.96), which were not used above, will neither be used in the sequel. However, they are strongly used in the treatment of the remaining three compatibility conditions (3.112)₂, (3.112)₃ and (3.112)₄, the detail of which we do not copy in the typesetted paper. Finally, the construction of a guide for the subgoal (4.29) may be guessed similarly as in §4.14 above. We shall provide the final computations directly, without any guide: they consists of the *seven* partial differential relations (4.32), (4.33), (4.35), (4.37), (4.39), (4.43) and (4.45) below. At the end, we shall make the addition (4.47) below, producing the desired subgoal (4.29) := (4.32) + (4.33) + (4.35) + (4.37) + (4.39) + (4.43) + (4.45), with the numerotation of terms corresponding to the order of appearance of the terms of (4.29), as usual.

Firstly, put $j := k$, $l_1 := l_1$ and $l_2 := l_1$ in (3.106):

$$(4.31) \quad 0 = \frac{-2G_{y^{l_1}}^k + 2\delta_{l_1}^k G_{y^{l_1}}^{l_1} + H_{l_1,x}^k - \delta_{l_1}^k H_{l_1,x}^{l_1}}{1} + 2 \sum_p G^p L_{l_1,p}^k - 2\delta_{l_1}^k \sum_p G^p L_{l_1,p}^{l_1} - \frac{1}{2} \sum_p H_{l_1}^p H_p^k + \frac{1}{2} \delta_{l_1}^k \sum_p H_{l_1}^p H_p^{l_1}.$$

Apply the operator $\frac{1}{2} \sum_k L_{k,k}^k(\cdot)$ to the preceding equality, namely compute $\frac{1}{2} \sum_k L_{k,k}^k \cdot (4.31)$. This yields:

$$(4.32) \quad 0 = \frac{-\sum_k G_{y^{l_1}}^k L_{k,k}^k + G_{y^{l_1}}^{l_1}}{1} + \frac{1}{2} \sum_k H_{l_1,x}^k L_{k,k}^k - \frac{1}{2} H_{l_1,x}^{l_1} L_{l_1,l_1}^{l_1} + \sum_k \sum_p G^p L_{k,k}^k L_{l_1,p}^k - \sum_p G^p L_{l_1,l_1}^{l_1} L_{l_1,p}^k - \frac{1}{4} \sum_k \sum_p H_{l_1}^p H_p^k L_{k,k}^k + \frac{1}{4} \sum_p H_{l_1}^p H_p^{l_1} L_{l_1,l_1}^{l_1}.$$

Secondly, apply the operator $-\sum_k L_{l_2,k}^{l_2}(\cdot)$ to (4.32), namely compute $-\sum_k L_{l_2,k}^{l_2} \cdot (4.32)$. This yields:

$$(4.33) \quad 0 = \frac{2 \sum_k G_{y^{l_1}}^k L_{l_2,k}^{l_2} - 2G_{y^{l_1}}^{l_1} L_{l_2,l_1}^{l_2} - \sum_k H_{l_1,x}^k L_{l_2,k}^{l_2} + H_{l_1,x}^{l_1} L_{l_2,l_1}^{l_2}}{1} - 2 \sum_k \sum_p G^p L_{l_2,k}^{l_2} L_{l_1,p}^k + 2 \sum_p G^p L_{l_2,l_1}^{l_2} L_{l_1,p}^{l_1} + \frac{1}{2} \sum_k \sum_p H_{l_1}^p H_p^k L_{l_2,k}^{l_2} - \frac{1}{2} \sum_p H_{l_1}^p H_p^{l_1} L_{l_2,l_1}^{l_2}.$$

Thirdly, put $j := k$, $l_1 := l_2$ and $l_2 := l_1$ with $l_2 \neq l_1$ in (3.106):

$$(4.34) \quad 0 = \frac{-2G_{y^{l_2}}^k + 2\delta_{l_2}^k G_{y^{l_1}}^{l_1} + H_{l_2,x}^k - \delta_{l_2}^k H_{l_1,x}^{l_1}}{1} + 2 \sum_p G^p L_{l_2,p}^k - 2\delta_{l_2}^k \sum_p G^p L_{l_1,p}^{l_1} - \frac{1}{2} \sum_p H_{l_2}^p H_p^k + \frac{1}{2} \delta_{l_2}^k \sum_p H_{l_1}^p H_p^{l_1}.$$

Next, apply the operator $\sum_k L_{l_1,k}^{l_2}(\cdot)$ to (4.34), namely compute $\sum_k L_{l_1,k}^{l_2} \cdot (4.34)$. This yields:

$$(4.35) \quad 0 = \frac{-2 \sum_k G_{y^{l_2}}^k L_{l_1,k}^{l_2} + 2G_{y^{l_1}}^{l_1} L_{l_1,l_2}^{l_2} + \sum_k H_{l_2,x}^k L_{l_1,k}^{l_2} - H_{l_1,x}^{l_1} L_{l_1,l_2}^{l_2}}{1} + 2 \sum_k \sum_p G^p L_{l_2,p}^k L_{l_1,k}^{l_2} - 2 \sum_p G^p L_{l_1,p}^{l_1} L_{l_1,l_2}^{l_2} - \frac{1}{2} \sum_k \sum_p H_{l_2}^p H_p^k L_{l_1,k}^{l_2} + \frac{1}{2} \sum_p H_{l_1}^p H_p^{l_1} L_{l_1,l_2}^{l_2}.$$

Fourthly, put $j := l_2$, $l_1 := k$ and $l_2 := l_2$ in (3.108):

(4.36)

$$\begin{aligned}
0 = & \underbrace{-\frac{1}{2} H_{k,y^{l_2}}^{l_2} + \frac{1}{6} \delta_k^{l_2} H_{l_2,y^{l_2}}^{l_2} + \frac{1}{3} H_{k,y^k}^k + L_{k,l_2,x}^{l_2}}_{-} \\
& - \frac{1}{3} \delta_{l_2}^k L_{l_2,l_2,x}^{l_2} - \frac{2}{3} L_{k,k,x}^k + \\
& + G^{l_2} M_{k,l_2} - \frac{1}{3} \delta_{l_2}^k G^{l_2} M_{l_2,l_2} - \frac{2}{3} G^k M_{k,k} + \frac{1}{3} \delta_k^{l_2} \sum_p G^p M_{l_2,p} - \\
& - \frac{1}{3} \sum_p G^p M_{k,p} - \frac{1}{2} \sum_p H_p^{l_2} L_{k,l_2}^p + \frac{1}{2} \sum_p H_k^p L_{l_2,p}^{l_2} + \frac{1}{6} \delta_k^{l_2} \sum_p H_p^{l_2} L_{l_2,l_2}^p - \\
& - \frac{1}{6} \delta_k^{l_2} \sum_p H_{l_2}^p L_{l_2,p}^{l_2} + \frac{1}{3} \sum_p H_p^k L_{k,k}^p - \frac{1}{3} \sum_p H_k^p L_{k,p}^k.
\end{aligned}$$

Next, apply the operator $-\sum_k H_{l_1}^k(\cdot)$ to (4.), namely compute $-\sum_k H_{l_1}^k \cdot (4.36)$. This yields:

(4.37)

$$\begin{aligned}
0 = & \frac{1}{2} \sum_k H_{k,y^{l_2}}^{l_2} H_{l_1}^k - \frac{1}{6} H_{l_2,y^{l_2}}^{l_2} H_{l_1}^{l_2} - \frac{1}{3} \sum_k H_{k,y^k}^k H_{l_1}^k - \\
& \underbrace{- \sum_k L_{k,l_2,x}^{l_2} H_{l_1}^k + \frac{1}{3} L_{l_2,l_2,x}^{l_2} H_{l_1}^{l_2} + \frac{2}{3} \sum_k L_{k,k,x}^k H_{l_1}^k}_{-} - \\
& - \sum_k G^{l_2} H_{l_1}^k M_{k,l_2} + \frac{1}{3} G^{l_2} H_{l_1}^{l_2} M_{l_2,l_2} + \frac{2}{3} \sum_k G^k H_{l_1}^k M_{k,k} - \\
& - \frac{1}{3} \sum_p G^p H_{l_1}^{l_2} M_{l_2,p} + \frac{1}{3} \sum_k \sum_p G^p H_{l_1}^k M_{k,p} + \frac{1}{2} \sum_k \sum_p H_{l_1}^k H_p^{l_2} L_{k,l_2}^p - \\
& - \frac{1}{2} \sum_k \sum_p H_{l_1}^k H_k^p L_{l_2,p}^{l_2} - \frac{1}{6} \sum_p H_{l_1}^{l_2} H_p^{l_2} L_{l_2,l_2}^p + \frac{1}{6} \sum_p H_{l_1}^{l_2} H_{l_2}^p L_{l_2,p}^{l_2} - \\
& - \frac{1}{3} \sum_k \sum_p H_{l_1}^k H_p^k L_{k,k}^p + \frac{1}{3} \sum_k \sum_p H_{l_1}^k H_k^p L_{k,p}^k.
\end{aligned}$$

Fifthly, put $j := l_2$, $l_1 := k$ and $l_2 := l_1$ in (3.108):

$$\begin{aligned}
(4.38) \quad 0 = & \underbrace{-\frac{1}{2} H_{k,y^{l_1}}^{l_2} + \frac{1}{6} \delta_k^{l_2} H_{l_1,y^{l_1}}^{l_1} + L_{k,l_1,x}^{l_2} - \frac{1}{3} \delta_k^{l_2} L_{l_1,l_1,x}^{l_1}}_{-} + \\
& + G^{l_2} M_{k,l_1} - \frac{1}{3} \delta_k^{l_2} G^{l_1} M_{l_1,l_1} + \frac{1}{3} \delta_k^{l_2} \sum_p G^p M_{l_1,p} - \frac{1}{2} \sum_p H_p^{l_2} L_{k,l_1}^p + \\
& + \frac{1}{2} \sum_p H_k^p L_{l_1,p}^{l_2} + \frac{1}{6} \delta_k^{l_2} \sum_p H_p^{l_1} L_{l_1,l_1}^p - \frac{1}{6} \delta_k^{l_2} \sum_p H_{l_1}^p L_{l_1,p}^{l_1}.
\end{aligned}$$

Next, apply the operator $\sum_k H_{l_2}^k(\cdot)$ to (4.38), namely compute $\sum_k H_{l_2}^k \cdot (4.38)$. This yields:

(4.39)

$$\begin{aligned}
0 = & -\frac{1}{2} \sum_k H_{k,y^{l_1}}^{l_2} H_{l_2}^k + \frac{1}{6} H_{l_1,y^{l_1}}^{l_1} H_{l_2}^{l_2} + \sum_k L_{k,l_1,x}^{l_2} H_{l_2}^k - \\
& - \frac{1}{3} L_{l_1,l_1,x}^{l_1} H_{l_2}^{l_2} + \\
& + \sum_k G^{l_2} H_{l_2}^k M_{k,l_1} - \frac{1}{3} G^{l_1} H_{l_2}^{l_2} M_{l_1,l_1} + \frac{1}{3} \sum_p G^p H_{l_2}^{l_2} M_{l_1,p} - \\
& - \frac{1}{2} \sum_k \sum_p H_{l_2}^k H_p^{l_2} L_{k,l_1}^p + \frac{1}{2} \sum_k \sum_p H_{l_2}^k H_k^p L_{l_1,p}^{l_2} + \frac{1}{6} \sum_p H_{l_2}^{l_2} H_p^{l_1} L_{l_1,l_1}^p - \\
& - \frac{1}{6} \sum_p H_{l_2}^{l_2} H_{l_1}^p L_{l_1,p}^{l_1}.
\end{aligned}$$

Sixthly, we form the expression:

$$(4.40) \quad (3.108)|_{j:=k; l_1:=l_1; l_2:=l_2} - (3.108)|_{j:=k; l_1:=l_2; l_2:=l_1}.$$

Writing term by term the subtractions, we get:

(4.41)

$$\begin{aligned}
0 = & -\frac{1}{2} H_{l_1,y^{l_2}}^k \boxed{1} + \frac{1}{2} H_{l_2,y^{l_1}}^k \boxed{2} + \frac{1}{6} \delta_{l_1}^k H_{l_2,y^{l_2}}^{l_2} \boxed{3} - \frac{1}{6} \delta_{l_2}^k H_{l_1,y^{l_1}}^{l_1} \boxed{4} + \\
& + \frac{1}{3} \delta_{l_2}^k H_{l_1,y^{l_1}}^{l_1} \boxed{4} - \frac{1}{3} \delta_{l_1}^k H_{l_2,y^{l_2}}^{l_2} \boxed{3} + \frac{L_{l_1,l_2,x}^k}{\textcircled{a}} - \frac{L_{l_2,l_1,x}^k}{\textcircled{a}} - \\
& - \frac{1}{3} \delta_{l_1}^k L_{l_2,l_2,x}^{l_2} \boxed{5} + \frac{1}{3} \delta_{l_2}^k L_{l_1,l_1,x}^{l_1} \boxed{6} - \frac{2}{3} \delta_{l_2}^k L_{l_1,l_1,x}^{l_1} \boxed{6} + \frac{2}{3} \delta_{l_1}^k L_{l_2,l_2,x}^{l_2} \boxed{5} + \\
& + \frac{G^k M_{l_1,l_2}}{\textcircled{b}} - \frac{G^k M_{l_2,l_1}}{\textcircled{b}} - \frac{1}{3} \delta_{l_1}^k G^{l_2} M_{l_2,l_2} \boxed{7} + \frac{1}{3} \delta_{l_2}^k G^{l_1} M_{l_1,l_1} \boxed{8} - \\
& - \frac{2}{3} \delta_{l_2}^k G^{l_1} M_{l_1,l_1} \boxed{8} + \frac{2}{3} \delta_{l_1}^k G^{l_2} M_{l_2,l_2} \boxed{7} + \frac{1}{3} \delta_{l_1}^k \sum_p G^p M_{l_2,p} \boxed{9} - \frac{1}{3} \delta_{l_2}^k \sum_p G^p M_{l_1,p} \boxed{10} - \\
& - \frac{1}{3} \delta_{l_2}^k \sum_p G^p M_{l_1,p} \boxed{10} + \frac{1}{3} \delta_{l_1}^k \sum_p G^p M_{l_2,p} \boxed{9} - \frac{1}{2} \sum_p H_p^k L_{l_1,l_2}^p \textcircled{c} + \frac{1}{2} \sum_p H_p^k L_{l_2,l_1}^p \textcircled{c} + \\
& + \frac{1}{2} \sum_p H_{l_1}^p L_{l_2,p}^k \boxed{11} - \frac{1}{2} \sum_p H_{l_2}^p L_{l_1,p}^k \boxed{12} + \frac{1}{6} \delta_{l_1}^k \sum_p H_p^{l_2} L_{l_2,l_2}^p \boxed{13} - \frac{1}{6} \delta_{l_2}^k \sum_p H_p^{l_1} L_{l_1,l_1}^p \boxed{14} - \\
& - \frac{1}{6} \delta_{l_1}^k \sum_p H_{l_2}^p L_{l_2,p}^{l_2} \boxed{15} + \frac{1}{6} \delta_{l_2}^k \sum_p H_{l_1}^p L_{l_1,p}^{l_1} \boxed{16} + \frac{1}{3} \delta_{l_2}^k \sum_p H_p^{l_1} L_{l_1,l_1}^p \boxed{14} - \\
& - \frac{1}{3} \delta_{l_1}^k \sum_p H_p^{l_2} L_{l_2,p}^p \boxed{13} - \frac{1}{3} \delta_{l_2}^k \sum_p H_{l_1}^p L_{l_1,p}^{l_1} \boxed{16} + \frac{1}{3} \delta_{l_1}^k \sum_p H_{l_2}^p L_{l_2,p}^{l_2} \boxed{15}.
\end{aligned}$$

Simplifying, we get:

(4.42)

$$\begin{aligned}
0 = & \underbrace{-\frac{1}{2} H_{l_1, y^{l_2}}^k + \frac{1}{2} H_{l_2, y^{l_1}}^k - \frac{1}{6} \delta_{l_1}^k H_{l_2, y^{l_2}}^{l_2} + \frac{1}{6} \delta_{l_2}^k H_{l_1, y^{l_1}}^{l_1}}_{\text{}} \\
& + \frac{1}{3} \delta_{l_1}^k L_{l_2, l_2, x}^{l_2} - \frac{1}{3} \delta_{l_2}^k L_{l_1, l_1, x}^{l_1} \\
& + \frac{1}{3} \delta_{l_1}^k G^{l_2} M_{l_2, l_2} - \frac{1}{3} \delta_{l_2}^k G^{l_1} M_{l_1, l_1} + \frac{2}{3} \delta_{l_1}^k \sum_p G^p M_{l_2, p} - \frac{2}{3} \delta_{l_2}^k \sum_p G^p M_{l_1, p} + \\
& + \frac{1}{2} \sum_p H_{l_1}^p L_{l_2, p}^k - \frac{1}{2} \sum_p H_{l_2}^p L_{l_1, p}^k - \frac{1}{6} \delta_{l_1}^k \sum_p H_p^{l_2} L_{l_2, l_2}^p + \frac{1}{6} \delta_{l_2}^k \sum_p H_p^{l_1} L_{l_1, l_1}^p + \\
& + \frac{1}{6} \delta_{l_1}^k \sum_p H_{l_2}^p L_{l_2, p}^{l_2} - \frac{1}{6} \delta_{l_2}^k \sum_p H_{l_1}^p L_{l_1, p}^{l_1}.
\end{aligned}$$

Next, apply the operator $-\sum_k H_k^{l_2}(\cdot)$ to (4.42), namely compute $-\sum_k H_k^{l_2} \cdot (4.42)$.

This yields:

(4.43)

$$\begin{aligned}
0 = & \underbrace{\frac{1}{2} \sum_k H_{l_1, y^{l_2}}^k H_k^{l_2} - \frac{1}{2} \sum_k H_{l_2, y^{l_1}}^k H_k^{l_2} + \frac{1}{6} H_{l_2, y^{l_2}}^{l_2} H_{l_1}^{l_2}}_{\text{}} - \\
& \underbrace{\frac{1}{6} H_{l_1, y^{l_1}}^{l_1} H_{l_2}^{l_2} - \frac{1}{3} L_{l_2, l_2, x}^{l_2} H_{l_1}^{l_2} + \frac{1}{3} L_{l_1, l_1, x}^{l_1} H_{l_2}^{l_2}}_{\text{}} - \\
& - \frac{1}{3} G^{l_2} H_{l_1}^{l_2} M_{l_2, l_2} + \frac{1}{3} G^{l_1} H_{l_2}^{l_2} M_{l_1, l_1} - \frac{2}{3} \sum_p G^p H_{l_1}^{l_2} M_{l_2, p} + \\
& + \frac{2}{3} \sum_p G^p H_{l_2}^{l_2} M_{l_1, p} - \frac{1}{2} \sum_k \sum_p H_k^{l_2} H_{l_1}^p L_{l_2, p}^k + \frac{1}{2} \sum_k \sum_p H_k^{l_2} H_{l_2}^p L_{l_1, p}^k + \\
& + \frac{1}{6} \sum_p H_{l_1}^{l_2} H_p^{l_2} L_{l_2, l_2}^p - \frac{1}{6} \sum_p H_{l_2}^{l_2} H_p^{l_1} L_{l_1, l_1}^p - \frac{1}{6} \sum_p H_{l_1}^{l_2} H_{l_2}^p L_{l_2, p}^{l_2} + \\
& + \frac{1}{6} \sum_p H_{l_2}^{l_2} H_{l_1}^p L_{l_1, p}^{l_1}.
\end{aligned}$$

Seventhly, put $j := l_2, l_1 := k, l_2 := l_2$ and $l_3 := l_1$ in (3.108):

$$\begin{aligned}
(4.44) \quad 0 = & \underbrace{L_{k, l_2, y^{l_1}}^{l_2} - L_{k, l_1, y^{l_2}}^{l_2} - M_{k, l_1, x}}_{\text{}} + \\
& + \frac{1}{2} H_{l_1}^{l_2} M_{k, l_2} - \frac{1}{2} H_{l_2}^{l_2} M_{k, l_1} + \frac{1}{2} \delta_k^{l_2} \sum_p H_{l_1}^p M_{l_2, p} - \frac{1}{2} \delta_k^{l_2} \sum_p H_{l_2}^p M_{l_1, p} - \\
& - \frac{1}{2} \sum_p H_k^p M_{l_1, p} + \sum_p L_{k, l_1}^p L_{l_2, p}^{l_2} - \sum_p L_{k, l_2}^p L_{l_1, p}^{l_1},
\end{aligned}$$

and then apply the operator $2 \sum_k G^k(\cdot)$:

$$\begin{aligned}
(4.45) \quad 0 &= 2 \sum_k L_{k,l_2,y^1}^{l_2} G^k - 2 \sum_k L_{k,l_1,y^2}^{l_2} G^k - 2 \sum_k M_{k,l_1,x} G^k + \\
&+ \sum_k G^k H_{l_1}^{l_2} M_{l_1,l_2} - \sum_k G^k H_{l_2}^{l_2} M_{k,l_1} + \sum_p G^{l_2} H_{l_1}^p M_{l_2,p} - \\
&- \sum_p G^{l_2} H_{l_2}^p M_{l_1,p} - \sum_k \sum_p G^k H_k^p M_{l_1,p} + 2 \sum_k \sum_p G^k L_{k,l_1}^p L_{l_2,p}^{l_2} - \\
&- 2 \sum_k \sum_p G^k L_{k,l_2}^p L_{l_1,p}^{l_2}.
\end{aligned}$$

Finally, achieve the addition

$$(4.46) \quad (4.32) + (4.33) + (4.35) + (4.37) + (4.39) + (4.43) + (4.45).$$

We copy these seven formal expression, we underline the vanishing terms and we number the remaining terms so as to respect the order of appearance of the terms of the subgoal (4.29):

$$\begin{aligned}
(4.47) \quad 0 &= \underbrace{- \sum_k G_{y^1}^k L_{k,k}^k}_{\boxed{1}} + \underbrace{G_{y^1}^{l_1}}_{\boxed{2}} + \underbrace{\frac{1}{2} \sum_k H_{l_1,x}^k L_{k,k}^k}_{\boxed{3}} - \underbrace{\frac{1}{2} H_{l_1,x}^{l_1} L_{l_1,l_1}^{l_1}}_{\boxed{4}} + \\
&+ \underbrace{\sum_k \sum_p G^p L_{k,k}^k L_{l_1,p}^k}_{\boxed{24}} - \underbrace{\sum_p G^p L_{l_1,l_1}^{l_1} L_{l_1,p}^k}_{\boxed{23}} - \underbrace{\frac{1}{4} \sum_k \sum_p H_{l_1}^p H_p^k L_{k,k}^k}_{\boxed{27}} + \\
&+ \underbrace{\frac{1}{4} \sum_p H_{l_1}^p H_p^{l_1} L_{l_1,l_1}^{l_1}}_{\boxed{28}} + \\
&+ 2 \underbrace{\sum_k G_{y^1}^k L_{l_2,k}^{l_2}}_{\boxed{5}} - 2 \underbrace{G_{y^1}^{l_1} L_{l_2,l_1}^{l_2}}_{\textcircled{a}} - \underbrace{\sum_k H_{l_1,x}^k L_{l_2,k}^{l_2}}_{\boxed{6}} + \underbrace{H_{l_1,x}^{l_1} L_{l_2,l_1}^{l_2}}_{\textcircled{b}} - \\
&- 2 \underbrace{\sum_k \sum_p G^p L_{l_2,k}^{l_2} L_{l_1,p}^k}_{\textcircled{g}} + 2 \underbrace{\sum_p G^p L_{l_2,l_1}^{l_2} L_{l_1,p}^{l_1}}_{\textcircled{h}} + \underbrace{\frac{1}{2} \sum_k \sum_p H_{l_1}^p H_p^k L_{l_2,k}^{l_2}}_{\textcircled{i}} - \\
&- \underbrace{\frac{1}{2} \sum_p H_{l_1}^p H_p^{l_1} L_{l_2,l_1}^{l_2}}_{\textcircled{j}} - \\
&- 2 \underbrace{\sum_k G_{y^2}^k L_{l_1,k}^{l_2}}_{\boxed{7}} + 2 \underbrace{G_{y^1}^{l_1} L_{l_1,l_2}^{l_2}}_{\textcircled{a}} + \underbrace{\sum_k H_{l_2,x}^k L_{l_1,k}^{l_2}}_{\boxed{8}} - \underbrace{H_{l_1,x}^{l_1} L_{l_1,l_2}^{l_2}}_{\textcircled{b}} + \\
&+ 2 \underbrace{\sum_k \sum_p G^p L_{l_2,p}^k L_{l_1,k}^{l_2}}_{\textcircled{k}} - 2 \underbrace{\sum_p G^p L_{l_1,p}^{l_1} L_{l_1,l_2}^{l_2}}_{\textcircled{h}} - \underbrace{\frac{1}{2} \sum_k \sum_p H_{l_2}^p H_p^k L_{l_1,k}^{l_2}}_{\textcircled{l}} + \\
&+ \underbrace{\frac{1}{2} \sum_p H_{l_1}^p H_p^{l_1} L_{l_1,l_2}^{l_2}}_{\textcircled{j}} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_k H_{k,y^{l_2}}^{l_2} H_{l_1}^k - \frac{1}{6} H_{l_2,y^{l_2}}^{l_2} H_{l_1}^{l_2} - \frac{1}{3} \sum_k H_{k,y^k}^k H_{l_1}^k - \\
& \quad \frac{1}{2} \sum_k L_{k,l_2,x}^{l_2} H_{l_1}^k + \frac{1}{3} L_{l_2,l_2,x}^{l_2} H_{l_1}^{l_2} + \frac{2}{3} \sum_k L_{k,k,x}^k H_{l_1}^k - \\
& \quad - \sum_k G^{l_2} H_{l_1}^k M_{k,l_2} + \frac{1}{3} G^{l_2} H_{l_1}^{l_2} M_{l_2,l_2} + \frac{2}{3} \sum_k G^k H_{l_1}^k M_{k,k} - \\
& \quad - \frac{1}{3} \sum_p G^p H_{l_1}^{l_2} M_{l_2,p} + \frac{1}{3} \sum_k \sum_p G^p H_{l_1}^k M_{k,p} + \frac{1}{2} \sum_k \sum_p H_{l_1}^k H_p^{l_2} L_{k,l_2}^p - \\
& \quad - \frac{1}{2} \sum_k \sum_p H_{l_1}^k H_p^p L_{l_2,p}^{l_2} - \frac{1}{6} \sum_p H_{l_1}^{l_2} H_p^{l_2} L_{l_2,l_2}^p + \frac{1}{6} \sum_p H_{l_1}^{l_2} H_p^p L_{l_2,p}^{l_2} - \\
& \quad - \frac{1}{3} \sum_k \sum_p H_{l_1}^k H_p^p L_{k,k}^{l_2} + \frac{1}{3} \sum_k \sum_p H_{l_1}^k H_p^p L_{k,p}^k - \\
& \quad - \frac{1}{2} \sum_k H_{k,y^{l_1}}^{l_2} H_{l_2}^k + \frac{1}{6} H_{l_1,y^{l_1}}^{l_1} H_{l_2}^{l_2} + \sum_k L_{k,l_1,x}^{l_2} H_{l_2}^k - \\
& \quad - \frac{1}{3} L_{l_1,l_1,x}^{l_1} H_{l_2}^{l_2} + \\
& \quad + \sum_k G^{l_2} H_{l_2}^k M_{k,l_1} - \frac{1}{3} G^{l_1} H_{l_2}^{l_2} M_{l_1,l_1} + \frac{1}{3} \sum_p G^p H_{l_2}^{l_2} M_{l_1,p} - \\
& \quad - \frac{1}{2} \sum_k \sum_p H_{l_2}^k H_p^{l_2} L_{k,l_1}^p + \frac{1}{2} \sum_k \sum_p H_{l_2}^k H_p^p L_{l_1,p}^{l_2} + \frac{1}{6} \sum_p H_{l_2}^{l_2} H_p^{l_1} L_{l_1,l_1}^p - \\
& \quad - \frac{1}{6} \sum_p H_{l_2}^{l_2} H_{l_1}^p L_{l_1,p}^{l_1} + \\
& \quad + \frac{1}{2} \sum_k H_{l_1,y^{l_2}}^k H_k^{l_2} - \frac{1}{2} \sum_k H_{l_2,y^{l_1}}^k H_k^{l_2} + \frac{1}{6} H_{l_2,y^{l_2}}^{l_2} H_{l_1}^{l_2} - \\
& \quad - \frac{1}{6} H_{l_1,y^{l_1}}^{l_1} H_{l_2}^{l_2} - \frac{1}{3} L_{l_2,l_2,x}^{l_2} H_{l_1}^{l_2} + \frac{1}{3} L_{l_1,l_1,x}^{l_1} H_{l_2}^{l_2} - \\
& \quad - \frac{1}{3} G^{l_2} H_{l_1}^{l_2} M_{l_2,l_2} + \frac{1}{3} G^{l_1} H_{l_2}^{l_2} M_{l_1,l_1} - \frac{2}{3} \sum_p G^p H_{l_1}^{l_2} M_{l_2,p} + \\
& \quad + \frac{2}{3} \sum_p G^p H_{l_2}^{l_2} M_{l_1,p} - \frac{1}{2} \sum_k \sum_p H_{l_2}^{l_2} H_{l_1}^p L_{l_2,p}^k + \frac{1}{2} \sum_k \sum_p H_{l_2}^{l_2} H_{l_2}^p L_{l_1,p}^k + \\
& \quad + \frac{1}{6} \sum_p H_{l_1}^{l_2} H_p^{l_2} L_{l_2,l_2}^p - \frac{1}{6} \sum_p H_{l_2}^{l_2} H_p^{l_1} L_{l_1,l_1}^p - \frac{1}{6} \sum_p H_{l_1}^{l_2} H_{l_2}^p L_{l_2,p}^{l_2} + \\
& \quad + \frac{1}{6} \sum_p H_{l_2}^{l_2} H_{l_1}^p L_{l_1,p}^{l_1} +
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_k L_{k,l_2,y^{l_1}}^{l_2} G^k \quad \underline{\text{[17]}} \quad - 2 \sum_k L_{k,l_1,y^{l_2}}^{l_2} G^k \quad \underline{\text{[18]}} \quad - 2 \sum_k M_{k,l_1,x} G^k \quad \underline{\text{[19]}} \quad + \\
& + \sum_k G^k H_{l_1}^{l_2} M_{l_1,l_2} \quad \underline{\text{[o]}} \quad - \sum_k G^k H_{l_2}^{l_2} M_{k,l_1} \quad \underline{\text{[u]}} \quad + \sum_p G^{l_2} H_{l_1}^p M_{l_2,p} \quad \underline{\text{[m]}} \quad - \\
& - \sum_p G^{l_2} H_{l_2}^p M_{l_1,p} \quad \underline{\text{[s]}} \quad - \sum_k \sum_p G^k H_k^p M_{l_1,p} \quad \underline{\text{[20]}} \quad + 2 \sum_k \sum_p G^k L_{k,l_1}^p L_{l_2,p}^{l_2} \quad \underline{\text{[g]}} \quad - \\
& - 2 \sum_k \sum_p G^k L_{k,l_2}^p L_{l_1,p}^{l_2} \quad \underline{\text{[k]}} \quad .
\end{aligned}$$

In conclusion, there is exact coincidence with the subgoal (4.29). The proof that the first family $(3.112)_1$ of compatibility conditions of the second auxiliary system (3.99), (3.100), (3.101) and (3.102) are a consequence of (I), (II), (III) and (IV) of Theorem 1.7 **(3)** is complete. Granted that the treatment of the other three families of compatibility conditions $(3.112)_2$, $(3.112)_3$ and $(3.112)_4$ is similar (and as well painful), we consider that the proof of the equivalence between **(1)** and **(3)** in Theorem 1.7 is complete, now. \square

§5. GENERAL FORM OF THE POINT TRANSFORMATION OF THE FREE PARTICLE SYSTEM

This section is devoted to the exposition of a complete proof of Lemma 3.32. To start with, we must develop the fundamental equations (3.10), for $j = 1, \dots, m$. Recalling that the total differentiation operator is given by $D = \frac{\partial}{\partial x} + \sum_{l_1=1}^m y_x^{l_1} \cdot \frac{\partial}{\partial y^{l_1}} + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot \frac{\partial}{\partial y_x^{l_1}}$, we compute first

$$(5.1) \quad \left\{ \begin{aligned} DDX &= D \left[X_x + \sum_{l_1=1}^m y_x^{l_1} \cdot X_{y^{l_1}} \right] \\ &= X_{xx} + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot X_{xy^{l_1}} + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot X_{y^{l_1}y^{l_2}} + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot X_{y^{l_1}}, \end{aligned} \right.$$

and

$$(5.2) \quad \left\{ \begin{aligned} DDY^j &= D \left[Y_x^j + \sum_{l_1=1}^m y_x^{l_1} \cdot Y_{y^{l_1}}^j \right] \\ &= Y_{xx}^j + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot Y_{xy^{l_1}}^j + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot Y_{y^{l_1}y^{l_2}}^j + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot Y_{y^{l_1}}^j. \end{aligned} \right.$$

Now, we can develop the equation $0 = -DY^j \cdot DDX + DX \cdot DDY^j$, which yields

$$(5.3) \quad 0 = - \left[Y_x^j + \sum_{l_1=1}^m y_x^{l_1} \cdot Y_{y^{l_1}}^j \right] \cdot \left[X_{xx} + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot X_{xy^{l_1}} + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot X_{y^{l_1} y^{l_2}} + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot X_{y^{l_1}} \right] + \left[X_x + \sum_{l_1=1}^m y_x^{l_1} \cdot X_{y^{l_1}} \right] \cdot \left[Y_{xx}^j + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot Y_{xy^{l_1}}^j + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot Y_{y^{l_1} y^{l_2}}^j + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot Y_{y^{l_1}}^j \right] =$$

$$\begin{aligned} &= - X_{xx} Y_x^j + Y_{xx}^j X_x + \\ &\quad + \sum_{l_1=1}^m y_x^{l_1} \cdot \left[-2 X_{xy^{l_1}} Y_x^j + 2 Y_{xy^{l_1}}^j X_x - \right. \\ &\quad \quad \quad \left. - X_{xx} Y_{y^{l_1}}^j + Y_{xx}^j X_{y^{l_1}} \right] + \\ &\quad + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[-X_{y^{l_1} y^{l_2}} Y_x^j + Y_{y^{l_1} y^{l_2}}^j X_x - \right. \\ &\quad \quad \quad \left. - 2 X_{xy^{l_2}} Y_{y^{l_1}}^j + 2 Y_{xy^{l_2}}^j X_{y^{l_1}} \right] + \\ &\quad + \sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l_3=1}^m y_x^{l_1} y_x^{l_2} y_x^{l_3} \cdot \left[-X_{y^{l_2} y^{l_3}} Y_{y^{l_1}}^j + Y_{y^{l_2} y^{l_3}}^j X_{y^{l_1}} \right] + \\ &\quad + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot \left[-X_{y^{l_1}} Y_x^j + Y_{y^{l_1}}^j X_x \right] + \\ &\quad + \sum_{l_1=1}^m \sum_{l_2=1}^m y_{xx}^{l_1} y_x^{l_2} \cdot \left[-X_{y^{l_1}} Y_{y^{l_2}}^j + Y_{y^{l_1}}^j X_{y^{l_2}} \right]. \end{aligned}$$

The goal is to show that after solving these m equations for $j = 1, \dots, m$ with respect to the y_{xx}^l , $l = 1, \dots, m$, one obtains the expression (3.33) of Lemma 3.32, or

equivalently, using the Δ notation instead of the square notation, one obtains

$$(5.4) \quad \left\{ \begin{aligned} 0 &= y_{xx}^j \cdot \Delta(x|y^1| \cdots |y^m) + \Delta(x|y^1| \cdots |^j xx| \cdots |y^m) + \\ &+ \sum_{l_1=1}^m y_x^{l_1} \cdot [2 \Delta(x|y^1| \cdots |^j xy^{l_1}| \cdots |y^m) - \\ &\quad - \delta_{l_1}^j \Delta(xx|y^1| \cdots |y^m)] + \\ &+ \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot [\Delta(x|y^1| \cdots |^j y^{l_1} y^{l_2}| \cdots |y^m) - \\ &\quad - 2 \delta_{l_1}^j \Delta(xy^{l_2}|y^1| \cdots |y^m)] + \\ &+ \sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l_3=1}^m y_x^{l_1} y_x^{l_2} y_x^{l_3} [-\delta_{l_1}^j \Delta(y^{l_2} y^{l_3}|y^1| \cdots |y^m)]. \end{aligned} \right.$$

Unfortunately, the equations (5.3) are not solved with respect to the y_{xx}^j , because in its last line, we notice that the $y_{xx}^{l_2}$ are mixed with the $y_x^{l_1}$. Consequently, we have to solve a linear system of m equations with the unknowns y_{xx}^j of the form

$$(5.5) \quad \left\{ 0 = A^j + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot \left[-X_{y^{l_1}} Y_x^j + Y_{y^{l_1}}^j X_x + \sum_{l_2=1}^m y_x^{l_2} \cdot \left[-X_{y^{l_1}} Y_{y^{l_2}}^j + Y_{y^{l_1}}^j X_{y^{l_2}} \right] \right], \right.$$

for $j = 1, \dots, m$, where A^j is an abbreviation for the terms appearing in the lines 5, 6, 7, 8, 9 and 10 of (5.3), or even more compactly, changing the index j to the index k

$$(5.6) \quad \left\{ 0 = A^k + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot B_{l_1}^k, \right.$$

for $k = 1, \dots, m$, where $B_{l_1}^k$ is an abbreviation for the terms in the brackets in (5.5).

Thanks to the assumption that the determinant (3.2) is the identity determinant at $(x, y) = (0, 0)$, we deduce that the determinant of the $m \times m$ matrix $(B_{l_1}^k)_{1 \leq l_1 \leq m}^{1 \leq k \leq m}$ is also the identity determinant at $(x, y, y_x) = (0, 0, 0)$. It follows that the determinant of the $m \times m$ matrix $(B_{l_1}^k)_{1 \leq l_1 \leq m}^{1 \leq k \leq m}$ is nonvanishing in a neighborhood of the origin in the first order jet space. Consequently, we can apply the rule of Cramer to solve the y_{xx}^j explicitly in terms of the A^k and of the $B_{l_1}^k$ as follows

$$(5.7) \quad \left\{ y_{xx}^j = - \frac{\begin{vmatrix} B_1^1 & \cdots & A^1 & \cdots & B_m^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_1^m & \cdots & A^m & \cdots & B_m^m \end{vmatrix}}{\begin{vmatrix} B_1^1 & \cdots & B^1 & \cdots & B_m^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_1^m & \cdots & B^m & \cdots & B_m^m \end{vmatrix}}, \right.$$

where on the numerator, the only modification of the determinant of the matrix $(B_{l_1}^k)_{1 \leq l_1 \leq m}^{1 \leq k \leq m}$ is the replacement of its j -th column by the column vector A . We have to show that after replacing the A^k and the $B_{l_1}^k$ by their complete expressions, one indeed obtains the desired equation (5.4). As in (3.43), we shall introduce a notation for the

two $m \times m$ determinants appearing in (5.6): we write this quotient under the form

$$(5.8) \quad \left\{ \begin{array}{l} y_{xx}^j = - \frac{\|B_1^k \cdots |^j A^k \cdots B_m^k\|}{\|B_1^k \cdots |^j B_j^k \cdots B_m^k\|}, \end{array} \right.$$

where it is understood that $B_1^k, \dots, B_j^k, \dots, B_m^k$ and A^k are column vectors whose index k (for their lines) varies from 1 to m . This representation of determinants emphasizing only its columns will be appropriate for later manipulations.

Our first task is to compute the determinant in the denominator of (5.8). Recalling that we make the notational identification $y^0 \equiv x$, it will be convenient to reexpress the $B_{l_1}^k$ in a slightly compacter form, using the total differentiation operator D :

$$(5.9) \quad \left\{ \begin{array}{l} B_{l_1}^k = -X_{y^{l_1}} Y_x^k + Y_{y^{l_1}}^k X_x + \sum_{l_2=1}^m y_x^{l_2} \cdot [-X_{y^{l_1}} Y_{y^{l_2}}^k + Y_{y^{l_1}}^k X_{y^{l_2}}] = \\ = Y_{y^{l_1}}^k \cdot DX - X_{y^{l_1}} \cdot DY^k. \end{array} \right.$$

Lemma 5.10. *We have the following expression for the determinant of the matrix $(B_{l_1}^k)_{\substack{1 \leq k \leq m \\ 1 \leq l_1 \leq m}}$:*

$$(5.11) \quad \left\{ \begin{array}{l} \|Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k\| \cdots \|Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k\| = \\ = [DX]^{m-1} \cdot \Delta(x|y^1| \cdots |y^m). \end{array} \right.$$

Proof. By multilinearity, we may develop the determinant written in the first line of (5.11). Since it contains two terms in each columns, we should obtain a sum of 2^m determinants. However, since the obtained determinants vanish as soon as the column DY^k (multiplied by various factors $X_{y^{l_1}}$) appears at least two different places, it remains only $(m+1)$ nonvanishing determinants, those for which the column DY^k appears at most once:

$$(5.12) \quad \left\{ \begin{array}{l} \|Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k\| \cdots \|Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k\| = \\ = [DX]^m \cdot \|Y_{y^1}^k\| \cdots \|Y_{y^m}^k\| - [DX]^{m-1} X_{y^1} \cdot \|DY^k\| \|Y_{y^2}^k\| \cdots \|Y_{y^m}^k\| - \cdots - \\ - [DX]^{m-1} X_{y^m} \cdot \|Y_{y^1}^k\| \cdots \|Y_{y^{m-1}}^k\| \|DY^k\|. \end{array} \right.$$

To establish the desired expression appearing in the second line of (5.11), we factor out by $[DX]^{m-1}$ and we develop all the remaining total differentiation operators D . Since $y^0 \equiv x$, we have $y_x^0 = 1$, and this enables us to contract $X_x + \sum_{l_1=1}^m y_x^{l_1} X_{y^{l_1}}$ as $\sum_{l_1=0}^m y_x^{l_1} X_{y^{l_1}}$. So, we achieve the following computation (further explanations and

comments just afterwards):

(5.13)

$$\left\{ \begin{aligned}
 &= [DX]^{m-1} \cdot \left\{ \sum_{l_1=0}^m y_x^{l_1} X_{y^{l_1}} \cdot \|Y_{y^1}^k | \cdots | Y_{y^m}^k\| - \right. \\
 &\quad \left. - \sum_{l_1=0}^m y_x^{l_1} X_{y^1} \cdot \|Y_{y^{l_1}}^k | Y_{y^2}^k | \cdots | Y_{y^m}^k\| - \right. \\
 &\quad \left. - \cdots - \sum_{l_1=0}^m y_x^{l_1} X_{y^m} \cdot \|Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_{y^{l_1}}^k\| \right\} \\
 &= [DX]^{m-1} \cdot \left\{ \sum_{l_1=0}^m y_x^{l_1} X_{y^{l_1}} \cdot \|Y_{y^1}^k | \cdots | Y_{y^m}^k\| - X_{y^1} \cdot \|Y_x^k | Y_{y^2}^k | \cdots | Y_{y^m}^k\| - \right. \\
 &\quad \left. - y_x^1 X_{y^1} \cdot \|Y_{y^1}^k | Y_{y^2}^k | \cdots | Y_{y^m}^k\| - \cdots - \right. \\
 &\quad \left. - X_{y^m} \cdot \|Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_x^k\| - y_x^m X_{y^m} \cdot \|Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_{y^m}^k\| \right\} \\
 &= [DX]^{m-1} \cdot \left\{ X_x \cdot \|Y_{y^1}^k | \cdots | Y_{y^m}^k\| - X_{y^1} \cdot \|Y_x^k | Y_{y^2}^k | \cdots | Y_{y^m}^k\| - \cdots - \right. \\
 &\quad \left. - X_{y^m} \cdot \|Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_x^k\| \right\} \\
 &= [DX]^{m-1} \cdot \left\{ \Delta(x|y^1 | \cdots | y^m) \right\}.
 \end{aligned} \right.$$

For the passage to the equality of line 4, using the fact that a determinant having two identical columns vanishes, we observe that in each of the m sums $\sum_{l_1=0}^m$ appearing in lines 2 and 3 (including the cdots), there remains only two non-vanishing determinants. For the passage to the equality of line 7, we just sum up all the linear combinations of determinants appearing in lines 4, 5 and 6. Finally, for the passage to the equality of line 9, we recognize the development of the fundamental Jacobian determinant (3.2) along its first line $(X_x, X_{y^1}, \dots, X_{y^m})$, modulo some permutations of columns in the $m \times m$ minors. The proof is complete. \square

Our second task, similar but computationnally more heavy, is to compute the determinant in the numerator of (5.8). First of all, we have to re-express the A^k defined implicitly between (5.3) and (5.5) using the total differentiation operator to contract them as follows

(5.14)

$$\left\{ \begin{aligned}
 A^k &= DX \cdot Y_{xx}^k - DY^k \cdot X_{xx} + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot \left[DX \cdot Y_{xy^{l_1}}^k - DY^k \cdot X_{xy^{l_1}} \right] + \\
 &\quad + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \left[DX \cdot Y_{y^{l_1}y^{l_2}}^k - DY^k \cdot X_{y^{l_1}y^{l_2}} \right].
 \end{aligned} \right.$$

Replacing this expression of A^k in (5.8), taking account of the expression of the denominator already obtained in the second line of (5.11) and abbreviating

$\Delta(x|y^1|\cdots|y^m)$ as Δ , we may write (5.8) in length and then develop it by linearity as follows

(5.15)

$$\left. \begin{aligned}
 y_{xx}^j &= \frac{-1}{[DX]^{m-1} \cdot \Delta} \cdot \left[\begin{aligned}
 &\|Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k| \cdots \\
 &\cdots |^j DX \cdot Y_{xx}^k - DY^k \cdot Y_{xx}^k + \\
 &\quad + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot [DX \cdot Y_{xy^{l_1}}^k - DY^k \cdot X_{xy^{l_1}}] + \\
 &\quad + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot [DX \cdot Y_{y^{l_1}y^{l_2}}^k - DY^k \cdot X_{y^{l_1}y^{l_2}}] | \cdots \\
 &\cdots \|Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k\|
 \end{aligned} \right] \\
 &= \frac{-1}{[DX]^{m-1} \cdot \Delta} \cdot \left[\begin{aligned}
 &\|Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k| \cdots \\
 &\cdots |^j DX \cdot Y_{xx}^k - DY^k \cdot X_{xx}| \cdots \\
 &\quad \cdots \|Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k\| + \\
 &\quad + 2 \sum_{l_1=1}^m y_x^{l_1} \cdot \|Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k| \cdots \\
 &\quad \cdots |^j DX \cdot Y_{xy^{l_1}}^k - DY^k \cdot X_{xy^{l_1}}| \cdots \\
 &\quad \cdots \|Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k\| + \\
 &\quad + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \|Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k| \cdots \\
 &\quad \cdots |^j DX \cdot Y_{y^{l_1}y^{l_2}}^k - DY^k \cdot X_{y^{l_1}y^{l_2}}| \cdots \\
 &\quad \cdots \|Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k\|
 \end{aligned} \right] .
 \end{aligned}$$

As it is delicate to read, let us say that lines 2, 3 and 4 just express the j -th column $|^j A^k|$ of the determinant $\|B_1^k| \cdots |^j A^k| \cdots |B_m^k\|$, after replacement of A^k by its complete expression (5.14).

In lines 6, 7, 8; in lines 9, 10, 11; and in lines 12, 13, 14, there are three families of $m \times m$ determinants containing a linear combination (soustraction) having exactly two terms in each column. As in the proof of Lemma 5.10, by multilinearity, we have to develop each such determinant. In principle, for each development, we should get 2^m terms, but since the obtained determinants vanish as soon as the column DY^k (modulo a multiplication by some factor) appears at least twice, it remains only $(m+1)$ nonvanishing determinants, those for which the column DY^k appears at most once. In addition, for each of the obtained determinant, the factor $[DX]^{m-1}$ appears (sometimes even the factor $[DX]^m$), so that this factor compensates the factor $[DX]^{m-1}$ in the

numerator. In sum, the continuation of the huge computation yields:

(5.16)

$$y_{xx}^j = -\frac{1}{\Delta} \cdot \left[\begin{array}{l} DX \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\| - \\ - X_{y^1} \cdot \|DY^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\| - \cdots - \\ - X_{xx} \cdot \|Y_{y^1}^k\| \cdots |^j DY^k| \cdots |Y_{y^m}^k\| - \cdots - \\ - X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |DY^k\| + \\ + 2 \sum_{l_1=1}^m y_x^{l_1} DX \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\| - \\ - 2 \sum_{l_1=1}^m y_x^{l_1} X_{y^1} \cdot \|DY^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\| - \cdots - \\ - 2 \sum_{l_1=1}^m y_x^{l_1} X_{xy^{l_1}} \cdot \|Y_{y^1}^k\| \cdots |^j DY^k| \cdots |Y_{y^m}^k\| - \cdots - \\ - 2 \sum_{l_1=1}^m y_x^{l_1} X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |DY^k\| + \\ + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} DX \cdot \|Y_{y^1}^k\| \cdots |^j Y_{y^{l_1}y^{l_2}}^k| \cdots |Y_{y^m}^k\| - \\ - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^1} \cdot \|DY^k\| \cdots |^j Y_{y^{l_1}y^{l_2}}^k| \cdots |Y_{y^m}^k\| - \cdots - \\ - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^{l_1}y^{l_2}} \cdot \|Y_{y^1}^k\| \cdots |^j DY^k| \cdots |Y_{y^m}^k\| - \cdots - \\ - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{y^{l_1}y^{l_2}}^k| \cdots |DY^k\| \end{array} \right].$$

To establish the desired expression (5.4), we must develop all the total differentiation operators D of the terms DX placed as factor and of the terms DY^k placed in various columns of determinants. We notice that in developing DY^k , we obtain columns $Y_{y^l}^k$ (multiplied by the factor y_x^l) and for *only* three (or two) values of $l = 0, 1, \dots, m$, this column does not already appear in the corresponding determinant, so that $(m - 1)$ determinants vanish and only 3 (or 2) remain nonzero. Taking account of these simplifications, we have the continuation

$$(5.17) \quad -y_{xx}^j \cdot \Delta = \text{I} + \text{II} + \text{III},$$

where the term I is the development of lines 1, 2, 3, 4 of (5.16); the term II is the development of lines 5, 6, 7, 8 of (5.16); and the term III is the development of lines 9,

10, 11, 12 of (5.16). So we get firstly (further explanations follows):

$$\begin{aligned}
& \frac{\sum_{l=0}^m y_x^l X_{y^l} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\|}{\boxed{1}} - \\
& - X_{y^1} \cdot \|Y_x^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\| - \\
& - \frac{y_x^1 X_{y^1} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\|}{\boxed{1}} - \\
(5.18) \quad \text{I} := & - y_x^j X_{y^j} \cdot \|Y_{y^j}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\| - \cdots - \\
& - X_{xx} \cdot \|Y_{y^1}^k\| \cdots |^j Y_x^k| \cdots |Y_{y^m}^k\| - \\
& - y_x^j X_{xx} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{y^j}^k| \cdots |Y_{y^m}^k\| - \cdots - \\
& - X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_x^k\| - \\
& - \frac{y_x^m X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^m}^k\|}{\boxed{1}} - \\
& - y_x^j X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xx}^k| \cdots |Y_{y^j}^k\|,
\end{aligned}$$

and secondly (we discuss afterwards the annihilation of the underlined terms):

$$\begin{aligned}
& \frac{2 \sum_{l_1=1}^m \sum_{l=0}^m y_x^{l_1} y_x^l X_{y^l} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\|}{\boxed{2}} - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} X_{y^1} \cdot \|Y_x^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\| - \\
& - \frac{2 \sum_{l_1=1}^m y_x^{l_1} y_x^1 X_{y^1} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\|}{\boxed{2}} - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{y^1} \cdot \|Y_{y^j}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\| - \cdots - \\
(5.19) \quad \text{II} := & - 2 \sum_{l_1=1}^m y_x^{l_1} X_{xy^{l_1}} \cdot \|Y_{y^1}^k\| \cdots |^j Y_x^k| \cdots |Y_{y^m}^k\| - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{xy^{l_1}} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{y^j}^k| \cdots |Y_{y^m}^k\| - \cdots - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_x^k\| - \\
& - \frac{2 \sum_{l_1=1}^m y_x^{l_1} y_x^m X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^m}^k\|}{\boxed{2}} - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{y^m} \cdot \|Y_{y^1}^k\| \cdots |^j Y_{xy^{l_1}}^k| \cdots |Y_{y^j}^k\|,
\end{aligned}$$

and where thirdly (we are nearly the end of the proof):

$$\begin{aligned}
& \underbrace{\sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l=0}^m y_x^{l_1} y_x^{l_2} y_x^l X_{y^l} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{3}} - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^1} \cdot \left\| Y_x^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^1 X_{y^1} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^1} \cdot \left\| Y_{y^j}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
(5.20) \quad \text{III} := & - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^{l_1} y^{l_2}} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_x^k | \cdots | Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^{l_1} y^{l_2}} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^j}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_x^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^m X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^j}^k \right\|.
\end{aligned}$$

Now, we explain the annihilation of the underlined terms. Consider I: in the first sum $\sum_{l=0}^m$, all the terms except only the two corresponding to $l = 0$ and to $l = j$ are annihilated by the other terms with $\boxed{1}$ appended: indeed, one must take account of the fact that in the expression of I, we have two sums represented by some \cdots s, the nature of which was defined without ambiguity in the passage from (5.15) to (5.16).

Similar simplifications occur for II and for III. Consequently, we obtain firstly:

$$\begin{aligned}
& X_x \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xx}^k | \cdots | Y_{y^m}^k \right\| + \\
& + y_x^j X_{y^j} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xx}^k | \cdots | Y_{y^m}^k \right\| - \\
& - X_{y^1} \cdot \left\| Y_x^k | \cdots |^j Y_{xx}^k | \cdots | Y_{y^m}^k \right\| - \\
(5.21) \quad \text{I} := & - y_x^j X_{y^j} \cdot \left\| Y_{y^j}^k | \cdots |^j Y_{xx}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
& - X_{xx} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_x^k | \cdots | Y_{y^m}^k \right\| - \\
& - y_x^j X_{xx} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^j}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
& - X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xx}^k | \cdots | Y_x^k \right\| - \\
& - y_x^j X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xx}^k | \cdots | Y_{y^j}^k \right\| ;
\end{aligned}$$

just above, the first two lines consist of the two terms in the sum underlined at the first line of (5.18) which are not annihilated; secondly we obtain:

$$\begin{aligned}
& 2 \sum_{l_1=1}^m y_x^{l_1} X_x \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\| + \\
& + 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{y^j} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\| - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} X_{y^1} \cdot \left\| Y_x^k | \cdots |^j Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\| - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{y^1} \cdot \left\| Y_{y^j}^k | \cdots |^j Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
(5.22) \quad \text{II} := & - 2 \sum_{l_1=1}^m y_x^{l_1} X_{xy^{l_1}} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_x^k | \cdots | Y_{y^m}^k \right\| - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{xy^{l_1}} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_x^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xy^{l_1}}^k | \cdots | Y_x^k \right\| - \\
& - 2 \sum_{l_1=1}^m y_x^{l_1} y_x^j X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{xy^{l_1}}^k | \cdots | Y_{y^j}^k \right\| ;
\end{aligned}$$

similarly, the first two lines above consist of the two terms in the sum underlined at the first line of (5.19) which are not annihilated; and thirdly we obtain:

$$\begin{aligned}
& \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_x \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1}y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| + \\
& + \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^j} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1}y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \\
(5.23) \quad \text{III} := & - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^1} \cdot \left\| Y_x^k | \cdots |^j Y_{y^{l_1}y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^1} \cdot \left\| Y_{y^j}^k | \cdots |^j Y_{y^{l_1}y^{l_2}}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^{l_1}y^{l_2}} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_x^k | \cdots | Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^{l_1}y^{l_2}} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^j}^k | \cdots | Y_{y^m}^k \right\| - \cdots -
\end{aligned}$$

$$\begin{aligned}
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_x^k \right\| - \\
& - \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} y_x^j X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots |^j Y_{y^{l_1} y^{l_2}}^k | \cdots | Y_{y^j}^k \right\|.
\end{aligned}$$

Collecting the odd lines of (5.21), we obtain exactly $(m+1)$ terms which correspond to the development of the determinant $\Delta(x | \cdots |^j xx | \cdots | y^m)$ along its first line, modulo permutations of columns of the associated $m \times m$ minors; collecting the even lines of (5.21), we obtain exactly $(m+1)$ terms which correspond to the development of the determinant $-y_x^j \cdot \Delta(xx | y^1 | \cdots | y^m)$ along its first lines, modulo permutations of columns of the associated $m \times m$ minors. Similar observations hold about II and III.

In , we may rewrite the final expressions of these three terms: firstly

$$(5.24) \quad \left\{ \begin{array}{l}
\text{I} = \Delta(x | \cdots |^j xx | \cdots | y^m) - y_x^j \cdot \Delta(xx | y^1 | \cdots | y^m), \\
\text{II} = 2 \sum_{l_1=1}^m y_x^{l_1} \cdot \Delta(x | \cdots |^j x y^{l_1} | \cdots | y^m) - \\
\quad - 2 y_x^j \sum_{l_1=1}^m y_x^{l_1} \cdot \Delta(x y^{l_1} | y^1 | \cdots | y^m), \\
\text{III} = \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \Delta(x | \cdots |^j y^{l_1} y^{l_2} | \cdots | y^m) - \\
\quad - y_x^j \sum_{l_1=1}^m \sum_{l_2=1}^m y_x^{l_1} y_x^{l_2} \cdot \Delta(y^{l_1} y^{l_2} | y^1 | \cdots | y^m).
\end{array} \right.$$

Coming back to (5.17), we obtain the desired expression (5.4).

The proof of the — technical, though involving only linear algebra — Lemma 3.32 is complete. \square

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CNRS, UNIVERSITÉ DE PROVENCE, LATP, UMR 6632, CMI, 39 RUE JOLIOT-CURIE, F-13453 MARSEILLE CEDEX 13, FRANCE. , Internet: <http://www.cmi.univ-mrs.fr/~merker/index.html>

E-mail address: merker@cmi.univ-mrs.fr