

PROPAGATION OF ANALYTICITY FOR ESSENTIALLY FINITE C^∞ -SMOOTH CR MAPPINGS

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ABSTRACT. An analytico-geometric reflection principle is established by means of normal deformations of analytic discs.

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[With 5 figures]

§1. INTRODUCTION

The analyticity of local C^∞ -smooth CR diffeomorphisms between two essentially finite generic real analytic submanifolds of \mathbb{C}^n is established in [3], as a kind of reflection principle, provided that all the components of the CR diffeomorphism extend holomorphically to a fixed wedge (the so-called notion of *essential finiteness* appeared in the work [14] by K. Diederich and S.M. Webster; a further geometric reflection principle for locally finite C^∞ -smooth CR mappings appeared in the work [12] by K. Diederich and J.E. Fornæss). In [4], [5] and more recently in [8], [9], [10], [25] (*cf.* also the applications [7], [26]), separate assumptions on the map and on the target have been unified: instead, it is assumed that the so-called *characteristic variety* is zero-dimensional at the central point. However, in these references, it is always supposed that the source generic submanifold is minimal at the central point, whereas, in [3], no minimality assumption was needed. This article is devoted to fill the gap between these two trends of thought, applying the technique of normal deformations of analytic discs borrowed from [34], [23], [24].

§2. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREM

2.1. Initial data. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $\nu \in \mathbb{N}$. Let $x \in \mathbb{K}^\nu$. Set $|x| := \max_{1 \leq i \leq \nu} |x_i|$. For $\rho > 0$, denote $\Delta_n(\rho) := \{x \in \mathbb{K}^\nu : |x| < \rho\}$.

Consider a local C^∞ -smooth CR mapping between two local generic submanifolds M in \mathbb{C}^n and M' in $\mathbb{C}^{n'}$, defined *precisely* as follows (background material may be found in [2], [21]). The purpose is to avoid the (ambiguous) language of germs.

Definition 2.2. A local C^∞ -smooth CR mapping consists of the following data.

Date: 2004-7-7.

2000 Mathematics Subject Classification. Primary: 32H02. Secondary: 32H04, 32V20, 32V30, 32V35, 32V40.

- (I) A local generic submanifold M in \mathbb{C}^n of positive codimension $d \geq 1$ and of positive CR dimension $m := n - d \geq 1$ passing through a point $p_0 \in \mathbb{C}^n$ and defined in coordinates $t = (z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at p_0 as a graph:

$$(2.3) \quad M = \{(z, w) \in \Delta_n(\rho_1) : v_j = \varphi_j(x, y, u), j = 1, \dots, d\},$$

where the functions φ_j are real analytic on $\Delta_{2m+d}(2\rho_1)$, for some $\rho_1 > 0$. It is also required for all ρ with $0 \leq \rho \leq 2\rho_1$, that $|\varphi(x, y, u)| < \rho$ if $|(x, y, u)| < \rho$, namely M is a uniformly approximatively horizontal graph. Of course, after perhaps shrinking $\rho_1 > 0$, this condition is automatically satisfied if the coordinates are adjusted at the beginning in order that $T_0M = \{v = 0\}$. In fact, it is more convenient to work with a local representation of M by complex defining equations

$$(2.4) \quad M = \{(z, w) \in \Delta_n(\rho_1) : \bar{w}_j = \Theta_j(\bar{z}, z, w), j = 1, \dots, d\},$$

obtained by applying the implicit function theorem to (2.3), where of course, one may assume that the Θ_j converge normally for $|(\bar{z}, z, w)| < 2\rho_1$ and that for all ρ with $0 \leq \rho \leq 2\rho_1$, one has $|\Theta_j(\bar{z}, z, w)| < \rho$ if $|(\bar{z}, z, w)| < \rho$.

- (II) A local generic submanifold M' in $\mathbb{C}^{n'}$ with central point p'_0 of positive codimension $d' \geq 1$ and of positive CR dimension $m' := n' - d' \geq 1$ passing through a point $p'_0 \in \mathbb{C}^{n'}$ and defined in coordinates $t' = (z', w') \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}$ similarly as in (2.4).
 (III) A \mathcal{C}^∞ -smooth mapping $t' = h(t) = (f(t), g(t)) = (z', w')$ with $h(p_0) = p'_0$ which is defined in $M \cap \Delta_n(\rho_1) \equiv M$ and which satisfies for some two radii ρ_2, ρ'_2 with $0 < \rho_2 < \rho_1, 0 < \rho'_2 < \rho'_1$, the condition

$$(2.5) \quad h(M \cap \Delta_n(\rho_2)) \subset M' \cap \Delta_{n'}(\rho'_2).$$

By [6] (and also [2]), after shrinking (if necessary) $\rho_1 > 0$ and $\rho_2 > 0$ with $0 < \rho_2 < \rho_1$:

- (IV) every \mathcal{C}^∞ -smooth CR function defined on $M \cap \Delta_n(\rho_1)$ (and in particular the n' components $h_1, \dots, h_{n'}$ of h) is a uniform limit of polynomials on $M \cap \Delta_n(\rho_2)$.

Definition 2.6. A complete wedge in $\Delta_n(\rho_2)$ with edge $M \cap \Delta_n(\rho_2)$ is a subset of \mathbb{C}^n of the form $\mathcal{W} = \mathcal{W}(M, C, \Delta_n(\rho_2)) = \{(z, w) \in \Delta_n(\rho_2); v - \varphi(x, y, u) \in C\}$, where C is some open strictly convex infinite (i.e. not truncated) cone in \mathbb{R}^d .

As in [3], it will be assumed that:

- (V) there exists a complete wedge \mathcal{W}_2 in $\Delta_n(\rho_2)$ with edge $M \cap \Delta_n(\rho_2)$ such that the n components of h extend holomorphically to \mathcal{W}_2 .

2.7. CR differentiations. Put $r_j(t, \bar{t}) := \bar{w}_j - \Theta_j(\bar{z}, z, w)$ for $j = 1, \dots, d$ and $r'_{j'}(t', \bar{t}') := \bar{w}'_{j'} - \Theta'_{j'}(\bar{z}', z', w')$ for $j' = 1, \dots, d'$. Let $\bar{L}_1, \dots, \bar{L}_m$ be an arbitrary basis of $(0, 1)$ vector fields tangent to M having real analytic coefficients (the most convenient is written in (3.2) below). Consider the first characteristic variety of h at p_0 to be the complex analytic subset \mathbb{V}'_0 of $\Delta'_{n'}(\rho'_2)$ consisting of elements t' satisfying the equations

$$(2.8) \quad \left[\bar{L}^\beta r'_{j'}(t', \bar{h}(t')) \right] \Big|_{\bar{t}=0} = 0, \text{ for all } j' = 1, \dots, d' \text{ and all } \beta \in \mathbb{N}^m.$$

It is indeed a complex analytic subset defined as the zero set of an infinite collection of functions which are holomorphic in $\Delta'_{n'}(\rho'_2)$. By (2.5), $r'_{j'}(h(t), \bar{h}(t)) = 0$, for $j' = 1, \dots, d'$ and for all $t \in M$. It follows that the origin p'_0 belongs to the complex analytic subset \mathbb{V}'_0 . The focus is on the dimension at p'_0 of \mathbb{V}'_0 . The map h will be called essentially finite at p_0 if $\dim_{p'_0} \mathbb{V}'_0 = 0$. Denote by $\mathcal{O}_{CR}(M, p_0)$ the (not local) CR orbit of p_0 in M . The main result is as follows.

Theorem 2.9. *Let $h : M \rightarrow M'$ be a local C^∞ -smooth CR mapping between two real analytic local generic submanifolds of \mathbb{C}^n and of $\mathbb{C}^{n'}$. Assume that there exists a complete wedge \mathcal{W}_2 in $\Delta_n(\rho_2)$ with edge $M \cap \Delta_n(\rho_2)$ such that the n' components of h extend holomorphically to \mathcal{W}_2 , assume that there exist points*

$$(2.10) \quad q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_2)$$

arbitrarily close to p_0 at which h is real analytic and assume that h is essentially finite at p_0 . Then there exists a radius $\rho_3 > 0$ with $0 < \rho_3 < \rho_2 < \rho_1$ such that h extends holomorphically to $\Delta_n(\rho_3)$.

In the version of Theorem 2.9 published in [10], [25], it is assumed that M is minimal at p_0 , which entails the holomorphic extendability assumption **(V)**, thanks to [33]. In [10], [25], there is a crucial proposition about envelopes of meromorphy (same statement and same proof in the two references – albeit notations differ), relying on subtle geometric arguments which stem from the theory of deformations of analytic discs developed in [1], [33], [23]. In this article, the purpose is to clean up and to simplify these geometric arguments, by means of the propagation of wedge extendability theorem established in [32], [34] (cf. [16] for a preliminary version). The stronger Theorem 2.9 will be established thanks to this change of geometric point of view. An elementary lemma applies to recover from Theorem 2.9 the main result of [10], [25].

Lemma 2.11. *Let $h : M \rightarrow M'$ be a local C^∞ -smooth CR mapping between two real analytic local generic submanifolds of \mathbb{C}^n and of $\mathbb{C}^{n'}$. Assume that M is minimal at p_0 , so that $\mathcal{O}_{CR}(M, p_0)$ contains $M \cap \Delta_n(\rho_2)$ for some $\rho_2 > 0$ and so that (thanks to [33]) after perhaps shrinking $\rho_2 > 0$, the assumption **(V)** holds. If h is essentially finite at p_0 , then there exist points $q_0 \in M \cap \Delta_n(\rho_2)$ arbitrarily close to p_0 at which h is real analytic.*

Finally, in order to recover the main result of [3], remind that the essential finiteness of M' at p'_0 together with the CR diffeomorphism assumption entails the essential finiteness of h at p_0 ([10], Lemma 4.1; a more general version is Corollary 1.3 in [25]; the most general version appears as Theorem 4.3.1 **(3)** in [21], in which it is shown that CR-transversality of h at p_0 together with essential finiteness of M' at p'_0 implies that h is essentially finite at p_0). In [3], it is observed that essential finiteness of a hypersurface M at one of its points p_0 implies its minimality (finite type in the sense of Lie-Chow-Kohn-Bloom-Graham) at p_0 . Here are further observations.

Lemma 2.12. *If M' is a local generic submanifold of $\mathbb{C}^{n'}$ passing through a point p'_0 which is essentially finite at p'_0 , then $\dim_{\mathbb{R}} \mathcal{O}_{CR}(M', p'_0) \geq 2\text{CRdim } M' + 1$ and the CR orbit $\mathcal{O}_{CR}(M', p'_0)$ itself is essentially finite at p'_0 . Furthermore, in the case where M is a real analytic hypersurface, essential finiteness of h at p_0 implies that M is minimal at p_0 .*

Assume that $n = n'$ and that h is a CR diffeomorphism. Then there exists a Zariski-dense open subset of points $q'_0 \in \mathcal{O}_{CR}(M', p'_0)$ at which M' is finitely nondegenerate. It follows that h itself is finitely nondegenerate at $q_0 := h^{-1}(q'_0)$, and by a known result ([28], [15], [2], [19]), h is real analytic at q_0 . Applying then Theorem 2.9:

Corollary 2.13. ([3]) *Let $h : M \rightarrow M'$ be a local C^∞ -smooth CR mapping which is a CR diffeomorphism. If M is essentially finite at p_0 and if the components of h extend holomorphically to a wedge at p_0 , then h is real analytic at p_0 .*

Further applications (in the spirit of [7], [26]) that may be stated are left to the interested reader. The remainder of this article is devoted to the proofs of Theorem 2.9, of Lemma 2.11 and of Lemma 2.12.

§3. POLYNOMIAL IDENTITIES

3.1. Differentiations. Denote by $\bar{L}_1, \dots, \bar{L}_m$ the basis of $(0, 1)$ vector fields tangent to M defined explicitly by

$$(3.2) \quad \bar{L}_k := \frac{\partial}{\partial \bar{z}_k} + \sum_{j=1}^d \frac{\partial \Theta_j}{\partial \bar{z}_k}(\bar{z}, z, w) \frac{\partial}{\partial \bar{w}_j}.$$

Here, the coefficients of the vector fields \bar{L}_k are holomorphic with respect to w . Since it will be more convenient for later use to have antiholomorphic dependence with respect to w , replace w by $\bar{\Theta}(z, \bar{z}, \bar{w})$ (which is possible when (z, w) belongs to M), and write the vector fields under the form

$$(3.3) \quad \bar{L}_k := \frac{\partial}{\partial \bar{z}_k} + \sum_{j=1}^d \frac{\partial \Theta_j}{\partial \bar{z}_k}(z, \bar{z}, \bar{\Theta}(z, \bar{z}, \bar{w})) \frac{\partial}{\partial \bar{w}_j}.$$

Apply the derivations $(\bar{L})^\beta$, $\beta \in \mathbb{N}^m$, to $r'_{j'}(h(t), \overline{h(t)}) = 0$, which yields

$$(3.4) \quad (\bar{L})^\beta r'_{j'}(h(t), \overline{h(t)}) = 0,$$

for $t \in M \cap \Delta_n(\rho_2)$ and for $j' = 1, \dots, d'$.

As the coefficients of the vector fields \bar{L}_k are holomorphic with respect to (z, \bar{t}) , the differentiated equations (3.4) may be rewritten under the developed form:

$$(3.5) \quad R'_{j', \beta}(z, \bar{t}, J_{\bar{t}}^{|\beta|} \overline{h(t)} : h(t)) = 0.$$

Here, $J_{\bar{t}}^\ell \overline{h(t)} := \left(\frac{\partial^\alpha \overline{h_{i'}}(t)}{\partial \bar{t}^\alpha} \right)_{1 \leq i' \leq n', |\alpha| \leq \ell}$ denotes the ℓ -th jet of $\overline{h(t)}$. By construction, the functions $R'_{j', \beta} = R'_{j', \beta}(z, \bar{t}, J_{\bar{t}}^{|\beta|} : t')$ are holomorphic with respect to z, \bar{t} with $|z| < \rho_2, |\bar{t}| < \rho_2$, they are holomorphic with respect to the zero-th order jet $J_{\bar{t}}^0 \overline{h(t)} \equiv \overline{h(t)}$ with $|J_{\bar{t}}^0| < \rho_2'$, they are relatively polynomial with respect to the nonzero derivatives $\left(\frac{\partial^\alpha \overline{h_{i'}}(t)}{\partial \bar{t}^\alpha} \right)_{1 \leq i' \leq n', 1 \leq |\alpha| \leq |\beta|}$, and they are relatively holomorphic with respect to t' with $|t'| < \rho_2'$.

By the main assumption of essential finiteness, there exists an integer $\ell_0 \geq 1$ such that the complex analytic subset defined by the equations

$$(3.6) \quad R'_{j', \beta}(0, 0, J_{\bar{t}}^{|\beta|} \overline{h(0)} : t') = 0, \quad j' = 1, \dots, d', \quad |\beta| \leq \ell_0,$$

which passes through the origin in $\mathbb{C}^{n'}$, is of dimension zero at the origin.

By [2], chapter 5, it follows that there exists ρ_3 with $0 < \rho_3 < \rho_2$, there exists ρ_3' with $0 < \rho_3' < \rho_2'$, there exists ε with $\varepsilon > 0$ and for all $i' = 1, \dots, n'$, there exist monic Weierstrass polynomials $P_{i'}(z, \bar{t}, J_{\bar{t}}^{\ell_0} : t')$ of the form

$$(3.7) \quad P_{i'}(z, \bar{t}, J_{\bar{t}}^{\ell_0} : t') = (t_{i'})^{N_{i'}} + \sum_{1 \leq I' \leq N_{i'}} H_{i', I'}(z, \bar{t}, J_{\bar{t}}^{\ell_0}) (t_{i'})^{N_{i'} - I'},$$

with coefficients $H_{i', I'}$ being holomorphic with respect to z, \bar{t} with $|z| < \rho_3, |\bar{t}| < \rho_3$ and with respect to $J_{\bar{t}}^{\ell_0}$ with $|J_{\bar{t}}^{\ell_0} - J_{\bar{t}}^{\ell_0} \overline{h(0)}| < \varepsilon$, with moreover $|J_{\bar{t}}^{\ell_0} \overline{h(t)} - J_{\bar{t}}^{\ell_0} \overline{h(0)}| < \varepsilon$ for all $t \in M \cap \Delta_n(\rho_3)$, such that the complex analytic set

$$(3.8) \quad \left\{ (z, \bar{t}, J_{\bar{t}}^{\ell_0}, t') : R'_{j', \beta}(z, \bar{t}, J_{\bar{t}}^{|\beta|} : t') = 0, \quad j' = 1, \dots, d', \quad |\beta| \leq \ell_0 \right\}$$

is contained in the zero-set of all Weierstrass polynomials $P_{i'}$, namely the set:

$$(3.9) \quad \left\{ \left(z, \bar{t}, J_{\bar{t}}^{\ell_0}, t' \right) : P_{i'} \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} : t' \right) = 0, i' = 1, \dots, n' \right\}.$$

Thanks to Hilbert's Nullstellensatz, there exists an integer $\nu \geq 1$ such that for $i' = 1, \dots, n'$, the powers $(P_{i'})^\nu$ belong to the ideal generated by the $R'_{j', \beta}$. Consequently, each component $h_{i'}(t)$ satisfies the monic polynomial equation

$$(3.10) \quad (h_{i'}(t))^{N'_{i'}} + \sum_{1 \leq I' \leq N'_{i'}} H_{i', I'} \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} \right) (h_{i'}(t))^{N'_{i'} - I'} = 0,$$

for all $t \in M \cap \Delta_n(\rho_3)$. Here, notably, each component $h_{i'}(t)$ is separated from the others.

§4. PROOF OF LEMMA 2.11

To establish Lemma 2.11, let ρ_2 be as in its statement and let ρ_3 be as in §3 just above. Shrinking ρ_3 if necessary, assume $0 < \rho_3 < \rho_2$ to fix ideas.

Fix $i' := 1$ and consider the following trivial dichotomy: either

$$(4.1) \quad \frac{\partial P_1}{\partial t_1} \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} : h_1(t) \right) = 0,$$

for all $t \in M \cap \Delta_n(\rho_3)$ or

$$(4.2) \quad \frac{\partial P_1}{\partial t_1} \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} : h_1(t) \right) \neq 0,$$

over $M \cap \Delta_n(\rho_3)$. In the first case, replace the equation $P_1 \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} : h(t) \right) = 0$ by the equation (4.1), which is a monic polynomial of degree $N'_{i'} - 1$ in $h_{i'}(t)$. After a finite number of steps, the second case (4.2) holds, with a monic polynomial of lower degree, still denoted by P_1 . Pick a point $q_1 \in M \cap \Delta_n(\rho_3)$ together with an open neighborhood $\omega_1 \subset M \cap \Delta_n(\rho_3)$ of q_1 so that $\frac{\partial P_1}{\partial t_{i'}} \left(z_{q_1}, \bar{t}_{q_1}, J_{\bar{t}}^{\ell_0} \overline{h(t_{q_1})} : h(t_{q_1}) \right) \neq 0$ for all $t \in \omega_1$, whereas $P_1 \left(z_{q_1}, \bar{t}_{q_1}, J_{\bar{t}}^{\ell_0} \overline{h(t_{q_1})} : h(t_{q_1}) \right) \equiv 0$ over ω_1 . Fix now $i' := 2$. Apply the same dichotomy as (4.1), (4.2) to $\frac{\partial P_2}{\partial t_2}$, but for t running only over ω_1 . Process as in the case $i'_1 = 1$ to replace P_2 by a monic polynomial of minimal degree, chose q_2 and ω_2 , etc.

After n' steps, there exists a point $q_0 \in M \cap \Delta_n(\rho_3)$, there exists an open neighborhood $\omega_0 \subset M \cap \Delta_n(\rho_3)$ of q_0 , there exists monic polynomials, still denoted by $P_1, \dots, P_{n'}$ which are of the form (3.7), such that

$$(4.3) \quad P_{i'} \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} : h_{i'}(t) \right) = 0,$$

for all $t \in \omega_0$ but

$$(4.4) \quad \frac{\partial P_{i'}}{\partial t_{i'}} \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} : h_{i'}(t) \right) \neq 0,$$

for all $t \in \omega_0$. Apply then the implicit function theorem to the identities (4.3), to solve

$$(4.5) \quad h(t) = \Psi \left(z, \bar{t}, J_{\bar{t}}^{\ell_0} \overline{h(t)} \right),$$

for all t in a (possibly smaller) neighborhood of q_0 , where Ψ is a certain complex analytic $\mathbb{C}^{n'}$ -valued mapping defined in a neighborhood of $\left(z_{q_0}, \bar{t}_{q_0}, J_{\bar{t}}^{\ell_0} \overline{h(t_{q_0})} \right)$ in $\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^{N_{n', n, \ell_0}}$. It then follows from standard arguments which are easy modifications of the original phenomenon

discovered in [28] that h extends holomorphically to a neighborhood of q_0 in \mathbb{C}^n , or equivalently that h is real analytic in a neighborhood of q_0 in M .

The positive radius ρ_3 could have been shrunk from the beginning to be arbitrarily small, and the same reasoning provides a point q_0 , arbitrarily close to p_0 , at which h is real analytic.

The proof of Lemma 2.11 is complete. \square

§5. DESCRIPTION OF THE PROOF OF THEOREM 2.9

5.1. Summary and statement of Main Proposition 5.2. Let ρ_1 be as in Definition 2.2, let ρ_2 be as in the assumptions of Theorem 2.9 and let ρ_3 be as in §3 above. By hypothesis, there exists at least one point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ at which h is real analytic (it will not be necessary to deal with such points which are closer to p_0).

By assumption, the components $h_{i'}(t)$ extend holomorphically to the complete wedge \mathcal{W}_2 in $\Delta_n(\rho_2)$ with edge $M \cap \Delta_n(\rho_2)$. Define a complete wedge \mathcal{W}_3 in $\Delta_n(\rho_3)$ with edge $M \cap \Delta_n(\rho_3)$ as the intersection of \mathcal{W}_2 with $\Delta_n(\rho_3)$. Here is an illustration, where \mathcal{W}_3 is on the top and its symmetric $\widetilde{\mathcal{W}}_3$ (to be introduced later) is on the bottom:

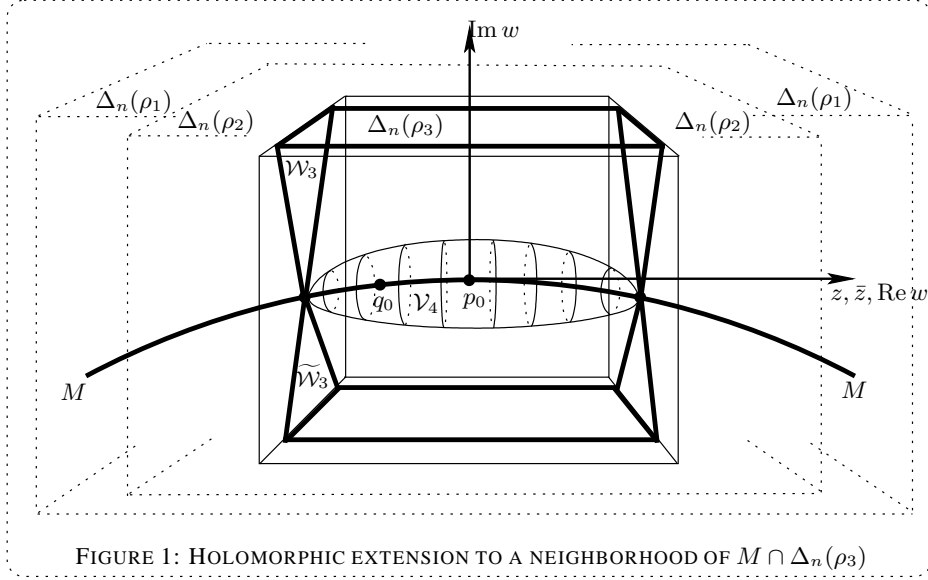


FIGURE 1: HOLOMORPHIC EXTENSION TO A NEIGHBORHOOD OF $M \cap \Delta_n(\rho_3)$

In these conditions, the components $h_{i'}(t)$ of h extend holomorphically to \mathcal{W}_3 . Also, the jet $J_{\bar{t}}^{\ell_0} \overline{h(t)}$ in the arguments of the functions $H_{i', I'}$ in (3.10) extends antiholomorphically to \mathcal{W}_3 .

It is now time to state an independent general proposition, where M' disappears, where each component $h_{i'}$ is replaced by a C^∞ -smooth CR function a , not necessarily coming from a CR mapping, where the jet $J_{\bar{t}}^{\ell_0} \overline{h(t)}$ is replaced by an independent vector valued mapping \bar{b} which extends antiholomorphically to \mathcal{W}_3 . For similar statements, see [30], [31], [8], [9], [10], [25], [7], [26].

Main Proposition 5.2. *As above, let M be a real analytic local generic submanifold defined as a graph in $\Delta_n(\rho_1)$ by (2.4), let ρ_3 with $0 < \rho_3 < \rho_1$ and let \mathcal{W}_3 be a complete wedge in $\Delta_n(\rho_3)$ with edge $M \cap \Delta_n(\rho_3)$. Let $a(t)$ be a C^∞ -smooth CR function defined over $M \cap \Delta_n(\rho_3)$ which extends holomorphically to the complete wedge \mathcal{W}_3 and which is real analytic in a neighborhood*

of at least one point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$. Let $\nu \in \mathbb{N}$, let $\varepsilon > 0$ and let $b(t)$ be a \mathbb{C}^ν -valued C^∞ -smooth CR mapping defined on $M \cap \Delta_n(\rho_3)$ which satisfies $|b(t) - b(0)| < \varepsilon$ for all $t \in M \cap \Delta_n(\rho_3)$, which extends holomorphically to \mathcal{W}_3 and which is real analytic in a neighborhood of the same point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$. Let $N \in \mathbb{N}$ with $N \geq 1$, and for $\ell = 0, 1, \dots, N$, let $H_\ell = H_\ell(z, \bar{t}, \bar{b})$ be some functions which are holomorphic for $|z| < \rho_3$, for $|\bar{t}| < \rho_3$ and for $|\bar{b} - \bar{b}(0)| < \varepsilon$. Assume that $a(t)$ satisfies the (not necessarily monic) polynomial equation

$$(5.3) \quad \sum_{0 \leq \ell \leq N} H_\ell \left(z, \bar{t}, \overline{b(t)} \right) a(t)^{N-\ell} = 0,$$

for all $t \in M \cap \Delta_n(\rho_3)$ and that the C^∞ -smooth functions $H_\ell \left(z, \bar{t}, \overline{b(t)} \right)$ are not all identically zero. Then there exists an open neighborhood \mathcal{V}_4 of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n to which $a(t)$ extends holomorphically.

This proposition, applied to each $h_{i'}$, completes the proof of Theorem 2.9. The remainder of this section is devoted to describe its proof, intuitively speaking.

5.4. Heuristic. Although the equations (3.10) (of a form similar to (5.3)) were obtained by applying the $(0, 1)$ vector fields tangent to M , it will be crucial to reapply the vector fields \bar{L}_k to (5.3).

But before applying the \bar{L}_k , it is also crucial to assume that N is the smallest possible integer with the property that there exists a relation of the form (5.3) on $M \cap \Delta_n(\rho_3)$. This assumption is of course free. Since $a(t)$ is CR, it may be considered as a constant by the derivations \bar{L}_k . Suppose for a while that dividing by $H_0 \left(z, \bar{t}, \overline{b(t)} \right)$ is allowed in some sense. Then rewrite (5.3) as follows:

$$(5.5) \quad a(t)^N + \sum_{1 \leq \ell \leq N} \frac{H_\ell \left(z, \bar{t}, \overline{b(t)} \right)}{H_0 \left(z, \bar{t}, \overline{b(t)} \right)} a(t)^{N-\ell} = 0.$$

Applying now the derivations \bar{L}_k , the term $\bar{L}_k [a(t)^N]$ vanishes (crucial fact), which yields the identities

$$(5.6) \quad \sum_{1 \leq \ell \leq N} \frac{H_0 \left(z, \bar{t}, \overline{b(t)} \right) \bar{L}_k [H_\ell \left(z, \bar{t}, \overline{b(t)} \right)] - H_\ell \left(z, \bar{t}, \overline{b(t)} \right) \bar{L}_k [H_0 \left(z, \bar{t}, \overline{b(t)} \right)]}{[H_0 \left(z, \bar{t}, \overline{b(t)} \right)]^2} a(t)^{N-\ell} = 0,$$

for $k = 1, \dots, m$. Now, after chasing the unnecessary denominator, observe that since the coefficients of the \bar{L}_k are holomorphic in (z, \bar{t}) , there exist new holomorphic functions $H_{1,\ell}$, $1 \leq \ell \leq N$, such that the identity (5.6) may be rewritten as

$$(5.7) \quad \sum_{1 \leq \ell \leq N} H_{1,\ell} \left(z, \bar{t}, \overline{b_1(t)} \right) a(t)^{N-\ell} = 0,$$

where of course

$$(5.8) \quad \overline{b_1(t)} := J_t^1 \overline{b(t)} = \left(\partial_{\bar{t}_i} \overline{b_j(t)} \right)_{1 \leq i \leq n, 1 \leq j \leq \nu}.$$

Setting $\nu_1 := \nu(n+1)$, the relation (5.7) is totally similar to (5.3), if some freedom is allowed about the number of functions $\overline{b_j}$. However, the degree $N-1$ of the relation (5.7) is strictly less than the degree N of the relation (5.3). *Because the degree N was chosen to be the smallest possible one, this relation (5.7) has to be trivial.*

Equivalently:

$$(5.9) \quad \bar{L}_k \left(\frac{H_\ell \left(z, \bar{t}, \overline{b(t)} \right)}{H_0 \left(z, \bar{t}, \overline{b(t)} \right)} \right) \equiv 0,$$

over $M \cap \Delta_n(\rho_3)$, for $k = 1, \dots, m$ and $\ell = 1, \dots, N$. In other words, the quotient H_ℓ/H_0 (which has to be defined carefully in some sense) is CR over $M \cap \Delta_n(\rho_3)$. Informally speaking, such an identity should be exceptional, because the term $\overline{b(t)}$ is anti-CR. In fact, one may expect intuitively that if the relation (5.9) is satisfied, then there are no terms $\overline{b(t)}$ at all, and hence the quotient H_ℓ/H_0 extends meromorphically (but not holomorphically, because of the presence of a quotient) to a neighborhood \mathcal{V}_3 of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n . This is true and will be proved below in Main Lemma 7.1 formulated below, where the assumption that $a(t)$ and $b(t)$ are already real analytic in a neighborhood of the point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ is used.

According to the works of K. Oka and E.E. Levi, a meromorphic function defined in a domain of \mathbb{C}^n is always the global quotient of two holomorphic functions (*see* [18] for instance). It follows that there exist functions $R_\ell = R_\ell(t)$ holomorphic in \mathcal{V}_3 such that

$$(5.10) \quad \frac{H_\ell \left(z, \bar{t}, \overline{b(t)} \right)}{H_0 \left(z, \bar{t}, \overline{b(t)} \right)} \equiv \frac{R_\ell(t)}{R_0(t)},$$

for all $t \in \mathcal{V}_3$. Replacing then (5.10) in (5.5) and chasing the denominator:

$$(5.11) \quad R_0(t) a(t)^N + \sum_{1 \leq \ell \leq N} R_\ell(t) a(t)^{N-\ell} = 0,$$

for all $t \in \mathcal{V}_3$. Here, the function $R_0(t)$ is not identically zero. Finally, by reproducing Section 3.3 of [10] or Proposition 6.4 of [25] (*see also* [27]), it follows that $a(t)$ is real analytic at every point of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$, hence (thanks to the Severi-Tomassini theorem) it extends holomorphically to an open neighborhood $\mathcal{V}_4 \subset \mathcal{V}_3$ of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n .

Sections 6, 7, 8, 9 and 10 are devoted to complete rigorously all the details of this strategy of proof.

§6. RINGS OF CR FUNCTIONS AND THEIR FIELDS OF FRACTIONS

Inspired by the preceding discussion, introduce the set $\mathcal{H}(M, \rho_3, \varepsilon)$ of functions of the form $H \left(z, \bar{t}, \overline{b(t)} \right)$, where $\nu \in \mathbb{N}$, where $b(t)$ is a \mathbb{C}^ν -valued \mathcal{C}^∞ -smooth CR mapping defined on $M \cap \Delta_n(\rho_3)$ which extends holomorphically to \mathcal{W}_3 and which is real analytic at the point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ and where $H = H(z, \bar{t}, \bar{b})$ is holomorphic for $|z| < \rho_3$, $|\bar{t}| < \rho_3$ and for $|\bar{b} - \overline{b(0)}| < \varepsilon$ and where $|\overline{b(t)} - \overline{b(0)}| < \varepsilon$ for all $t \in M \cap \Delta_n(\rho_3)$. The main feature of $\mathcal{H}(M, \rho_3, \varepsilon)$ is the following.

Lemma 6.1. *Every function $H \left(z, \bar{t}, \overline{b(t)} \right) \in \mathcal{H}(M, \rho_3, \varepsilon)$ admits a real analytic extension to \mathcal{W}_3 which is antiholomorphic with respect to the complex transversal coordinates $w = (w_1, \dots, w_d) \in \mathbb{C}^d$.*

Proof. Indeed, $\overline{b(t)}$ admits an antiholomorphic extension to \mathcal{W}_3 by assumption, the function $H \left(z, \bar{t}, \overline{b(t)} \right)$ is holomorphic with respect to its variables, but it also depends on the holomorphic variables $z = (z_1, \dots, z_m)$ in general. Consequently, the antiholomorphicity with respect to z (only) is lost. \square

Lemma 6.2. *The set $\mathcal{H}(M, \rho_3, \varepsilon)$ is an entire ring which is stable under differentiation by the $(0, 1)$ vector fields tangent to M :*

$$(6.3) \quad \bar{L}_k \mathcal{H}(M, \rho_3, \varepsilon) \subset \mathcal{H}(M, \rho_3, \varepsilon), \quad k = 1, \dots, m.$$

Proof. The set $\mathcal{H}(M, \rho_3, \varepsilon)$ is obviously stable under addition and multiplication. Suppose that there exist two elements H_1, H_2 such that

$$(6.4) \quad H_1 \left(z, \bar{t}, \overline{b_1(t)} \right) \cdot H_2 \left(z, \bar{t}, \overline{b_2(t)} \right) = 0,$$

for all $t \in M \cap \Delta_n(\rho_3)$. Then there exists a nonempty open subset V of $M \cap \Delta_n(\rho_3)$ on which $H_1(z, \bar{t}, \overline{b_1(t)})$ or $H_2(z, \bar{t}, \overline{b_2(t)})$ – say $H_1(z, \bar{t}, \overline{b_1(t)})$ to fix ideas – vanishes identically. One must show that that $H_1(z, \bar{t}, \overline{b_1(t)})$ vanishes identically on $M \cap \Delta_n(\rho_3)$.

According to Lemma 6.1, the function $H_1(z, \bar{t}, \overline{b_1(t)})$ extends real analytically and antiholomorphically with respect to w into the wedge \mathcal{W}_3 . By the principle of analytic continuation (for real analytic functions), it suffices to show that $H_1(z, \bar{t}, \overline{b_1(t)})$ vanishes identically on a nonempty open subset of \mathcal{W}_3 .

Let $p \in V$ and let \mathcal{V}_p be an open polydisc centered at p with $\mathcal{V}_p \cap M \subset V$. For $q = (z_q, w_q) \in \mathcal{V}_p$, the intersection $M \cap \{z = z_q\} \cap \mathcal{V}_p$ is maximally real in the slice $\{z = z_q\} \cap \mathcal{V}_p$. Also, the function $H_1(z, \bar{t}, \overline{b_1(t)})$ extends antiholomorphically with respect to w into the sliced wedge $\mathcal{W}_3 \cap \{z = z_q\} \cap \mathcal{V}_p$. By the generic uniqueness principle (for antiholomorphic functions), it follows that $H_1(z, \bar{t}, \overline{b_1(t)})$ vanishes identically in the sliced wedge $\mathcal{W}_3 \cap \{z = z_q\} \cap \mathcal{V}_p$. Since z_q was arbitrary, it follows that $H_1(z, \bar{t}, \overline{b_1(t)})$ vanishes identically in the nonempty open subset $\mathcal{W}_3 \cap \mathcal{V}_p$ of \mathcal{W}_3 , as desired.

Finally, since the coefficients of the vector fields \bar{L}_k are holomorphic for $|z| < \rho_1$ and $|\bar{t}| < \rho_1$, the second property follows from the chain rule. The proof of Lemma 6.2 is complete. \square

A slightly more precise result has in fact been established.

Corollary 6.5. *The zero-set of a nonzero function $H(z, \bar{t}, \overline{b(t)})$ in $\mathcal{H}(M, \rho_3, \varepsilon)$ is a closed subset of $M \cap \Delta_n(\rho_3)$ with empty interior.* \square

In the sequel, denote by \mathcal{Z}_H the zero set of $H(z, \bar{t}, \overline{b(t)})$ on $M \cap \Delta_n(\rho_3)$. Since the ring $\mathcal{H}(M, \rho_3, \varepsilon)$ is entire, it is allowed to consider its *field of fractions* $\mathcal{R}(M, \rho_3, \varepsilon)$ which consists formally of quotients of the form

$$(6.6) \quad \frac{H_1(z, \bar{t}, \overline{b_1(t)})}{H_2(z, \bar{t}, \overline{b_2(t)})},$$

and which may be viewed as a standard complex-valued function on the dense open subset $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_{H_2}$.

6.7. Two notions of algebraic dependence. Since $\mathcal{R}(M, \rho_3, \varepsilon)$ is a field, the notion of algebraic dependence makes sense. Precisely, a C^∞ -smooth CR function $a(t)$ which is defined on $M \cap \Delta_n(\rho_3)$ is called *algebraic over* $\mathcal{R}(M, \rho_3, \varepsilon)$ if there exists a nonzero polynomial

$$(6.8) \quad \sum_{0 \leq \ell \leq N} \frac{H_{1,\ell}(z, \bar{t}, \overline{b_{1,\ell}(t)})}{H_{2,\ell}(z, \bar{t}, \overline{b_{2,\ell}(t)})} X^{N-\ell},$$

with coefficients in $\mathcal{R}(M, \rho_3, \varepsilon)$ which annihilates $a(t)$, *i.e.* such that

$$(6.9) \quad \sum_{0 \leq \ell \leq N} \frac{H_{1,\ell} \left(z, \bar{t}, \overline{b_{1,\ell}(t)} \right)}{H_{2,\ell} \left(z, \bar{t}, \overline{b_{2,\ell}(t)} \right)} a(t)^{N-\ell} = 0,$$

for all $t \in (M \cap \Delta_n(\rho_3)) \setminus \bigcup_{0 \leq \ell \leq N} \mathcal{Z}_{H_{2,\ell}}$.

Equivalently, after chasing the denominators, the \mathcal{C}^∞ -smooth CR function $a(t)$ is algebraically dependent over the ring $\mathcal{H}(M, \rho_3, \varepsilon)$.

Another notion of algebraic dependence is the following. Let $\text{Mer}(\mathcal{O}_{CR}(M, p_0), \rho_3)$ denote the field of functions which are meromorphic in some connected open neighborhood of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n . In this field, the classical algebraic operations are defined up to some shrinking of the domains of definition. Then a \mathcal{C}^∞ -smooth CR function $a(t)$ defined on $M \cap \Delta_n(\rho_3)$ is called *algebraic over* $\text{Mer}(\mathcal{O}_{CR}(M, p_0), \rho_3)$ if there exists a nonzero polynomial

$$(6.10) \quad \sum_{0 \leq \ell \leq N} R_\ell(t) X^{N-\ell},$$

with coefficients $R_\ell(t) \in \text{Mer}(\mathcal{O}_{CR}(M, p_0), \rho_3)$ which annihilates $a(t)$, *i.e.* such that

$$(6.11) \quad \sum_{0 \leq \ell \leq N} R_\ell(t) a(t)^{N-\ell} = 0,$$

for all t in some neighborhood \mathcal{V}_3 of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n , outside the union of the polar sets of the $R_\ell(t)$.

It is now time to reformulate and to slightly generalize the heuristic discussion of §5.4. In the following lemma, a Main Lemma 7.1 (to be formulated and to be proved later), is hidden.

Lemma 6.12. *Let $a(t)$ be a \mathcal{C}^∞ -smooth CR function on $M \cap \Delta_n(\rho_3)$ which extends holomorphically to the complete wedge \mathcal{W}_3 in $\Delta_n(\rho_3)$ with edge $M \cap \Delta_n(\rho_3)$. Assume that $a(t)$ is real analytic at one point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$. If the function $a(t)$ is algebraic over the field $\mathcal{R}(M, \rho_3, \varepsilon)$ (or equivalently over the ring $\mathcal{H}(M, \rho_3, \varepsilon)$), then it is algebraic over the field $\text{Mer}(\mathcal{O}_{CR}(M, p_0), \rho_3)$.*

Proof. Without loss of generality, assume that $a(t)$ satisfies a polynomial relation

$$(6.13) \quad \sum_{0 \leq \ell \leq N} H_\ell \left(z, \bar{t}, \overline{b(t)} \right) a(t)^{N-\ell} = 0,$$

for all $t \in M \cap \Delta_n(\rho_3)$, whose degree $N \geq 1$ is the smallest possible. Hence $H_0 \left(z, \bar{t}, \overline{b(t)} \right)$ does not vanish identically. Proceeding exactly as in the heuristic discussion of §5.4, divide by $H_0 \left(z, \bar{t}, \overline{b(t)} \right)$, which yields

$$(6.14) \quad a(t)^N + \sum_{1 \leq \ell \leq N} \frac{H_\ell \left(z, \bar{t}, \overline{b(t)} \right)}{H_0 \left(z, \bar{t}, \overline{b(t)} \right)} a(t)^{N-\ell} = 0.$$

Apply now the derivations \overline{L}_k , which yields

$$(6.15) \quad \sum_{1 \leq \ell \leq N} \frac{H_0 \left(z, \bar{t}, \overline{b(t)} \right) \overline{L}_k \left[H_\ell \left(z, \bar{t}, \overline{b(t)} \right) \right] - H_\ell \left(z, \bar{t}, \overline{b(t)} \right) \overline{L}_k \left[H_0 \left(z, \bar{t}, \overline{b(t)} \right) \right]}{\left[H_0 \left(z, \bar{t}, \overline{b(t)} \right) \right]^2} a(t)^{N-\ell} = 0.$$

Now, after chasing the unnecessary denominator, and using (6.3), rewrite this identity under the form

$$(6.16) \quad \sum_{1 \leq \ell \leq N} H_{1,\ell} \left(z, \bar{t}, \overline{b_1(t)} \right) a(t)^{N-\ell} = 0,$$

where $H_{1,\ell} \left(z, \bar{t}, \overline{b_1(t)} \right)$ belongs to $\mathcal{H}(M, \rho_3, \varepsilon)$. Thus a relation which is totally similar to (10.1.47), but which is of degree strictly less than N , has been constructed. Because the degree N was chosen to be the smallest possible one, this relation has to be trivial. Consequently, the $(0, 1)$ vector fields \bar{L}_k annihilate the quotients H_ℓ/H_0 outside the zero-set \mathcal{Z}_{H_0} . By Main Lemma 7.1 (to be discussed later), this implies that for each $\ell = 1, \dots, N$, the quotient H_ℓ/H_0 coincides in some neighborhood \mathcal{V}_3 of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n with the restriction to $[\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)] \setminus \mathcal{Z}_{H_0}$ of a meromorphic function $R_\ell(t)/R_0(t)$. Finally, replacing H_ℓ/H_0 by $R_\ell/R_0(t)$ in the original relation (6.13), it follows that $a(t)$ is algebraic over $\text{Mer}(\mathcal{O}_{CR}(M, p_0), \rho_3)$. The proof of Lemma 6.12 is complete. \square

Thanks to this Lemma 6.12, the proof of Main Proposition 5.2 is complete, as explained after (5.11). \square

§7. STATEMENT OF MAIN LEMMA 7.1

In sum, everything relies upon a statement, interesting in itself, which is reformulated carefully, including all the assumptions.

Main Lemma 7.1. *As in Main Proposition 5.2, let M be a real analytic local generic submanifold defined in $\Delta_n(\rho_1)$, as in Definition 2.2 (I). Let ρ_3 with $0 < \rho_3 < \rho_2 < \rho_1$ and let \mathcal{W}_3 be a complete wedge in $\Delta_n(\rho_3)$ with edge $M \cap \Delta_n(\rho_3)$. Let $\nu_1, \nu_2 \in \mathbb{N}$, let $\varepsilon > 0$ and let $b_1(t), b_2(t)$ be two \mathbb{C}^{ν_1} - and \mathbb{C}^{ν_2} -valued C^∞ -smooth CR mappings defined on $M \cap \Delta_n(\rho_3)$ which satisfy $|b_1(t) - b_1(0)| < \varepsilon$, $|b_2(t) - b_2(0)| < \varepsilon$, which extend both holomorphically to \mathcal{W}_3 and which are real analytic in a neighborhood of the same point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$. Let $H_1(z, \bar{t}, \bar{b}_1)$ and $H_2(z, \bar{t}, \bar{b}_2)$ be two holomorphic functions defined for $|z| < \rho_3$, for $|\bar{t}| < \rho_3$ and for $|\bar{b}_1 - \bar{b}_1(0)| < \varepsilon$, $|\bar{b}_2 - \bar{b}_2(0)| < \varepsilon$. Assume that $H_2 \left(z, \bar{t}, \overline{b_2(t)} \right)$ does not vanish identically on $M \cap \Delta_n(\rho_3)$ and consider the quotient*

$$(7.2) \quad \frac{H_1 \left(z, \bar{t}, \overline{b_1(t)} \right)}{H_2 \left(z, \bar{t}, \overline{b_2(t)} \right)},$$

which belongs to $\mathcal{R}(M, \rho_3, \varepsilon)$ and remind that by Corollary 6.5, the zero-set \mathcal{Z}_{H_2} of $H_2 \left(z, \bar{t}, \overline{b_2(t)} \right)$ is a closed subset of $M \cap \Delta_n(\rho_3)$ with empty interior. Then the following three conditions are equivalent:

- (1) *there exists a meromorphic function in $\text{Mer}(\mathcal{O}_{CR}(M, p_0), \rho_3)$ of the form $P_1(t)/P_2(t)$, where $P_1(t)$ and $P_2(t)$ are holomorphic functions defined in some open connected neighborhood \mathcal{V}_3 of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n with $P_2(t) \not\equiv 0$ such that*

$$(7.3) \quad \frac{H_1 \left(z, \bar{t}, \overline{b_1(t)} \right)}{H_2 \left(z, \bar{t}, \overline{b_2(t)} \right)} = \frac{P_1(t)}{P_2(t)},$$

for all $t \in \mathcal{V}_3 \cap M$ outside the zero set of H_2 ;

- (2) *the quotient H_1/H_2 is CR over the dense open subset $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_{H_2}$;*

(3) *There exists a nonempty open subset V of the dense open subset $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_{H_2}$ on which the quotient H_1/H_2 is CR.*

Proof. Obviously, (2) \Rightarrow (3). Treat first the two implications (1) \Rightarrow (2) and (3) \Rightarrow (2), which are easy. Indeed, by Lemma 6.1, for $k = 1, \dots, m$, the CR derivated functions $\bar{L}_k(H_1/H_2)$ also belong to $\mathcal{R}(M, \rho_3, \varepsilon)$. If (1) or (3) holds, then H_1/H_2 is CR on a nonempty open subset V of $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_{H_2}$, so Corollary 6.5 implies that $\bar{L}_k(H_1/H_2)$ vanishes identically on $M \cap \Delta_n(\rho_3) \setminus \mathcal{Z}_{H_2}$, which yields (2).

Next, begin the proof of the implication (2) \Rightarrow (1), which is by far the main task. Even if it is already known that (2) and (3) are equivalent, property (1) will be established assuming only that H_1/H_2 is CR over a nonempty open subset V , as in (3).

Define the Schwarz reflection across M which stabilizes the slices $\{(z, w) \in \Delta_n(\rho_1) : z = ct.\}$ explicitly by

$$(7.4) \quad s : (z, w) \mapsto (z, \bar{\Theta}(z, \bar{z}, \bar{w})).$$

Shrinking ρ_3 a bit if necessary, one can construct a slightly smaller complete wedge $\widetilde{\mathcal{W}}_3$ which is contained in $s(\mathcal{W}_3)$, as depicted in FIGURE 1 above. What is important is that \mathcal{W}_3 and $\widetilde{\mathcal{W}}_3$ are directed by two opposite vectors at every point of $M \cap \Delta_n(\rho_3)$, because a version of the edge of the wedge theorem will be applied in the end of the proof.

Since s is antiholomorphic with respect to w and since $H_1(z, \bar{t}, \overline{b_1(t)})$ and $H_2(z, \bar{t}, \overline{b_2(t)})$ extend as real analytic functions in \mathcal{W}_3 which are antiholomorphic with respect to w , it follows by composition with the reflection $s(z, w)$ that H_1 and H_2 extend to be real analytic in $\widetilde{\mathcal{W}}_3$ and holomorphic with respect to w . Denote by $\widetilde{H}_1(t, \bar{t})$ and $\widetilde{H}_2(t, \bar{t})$ these two real analytic extensions. Since the antiholomorphic reflection coincides with the identity mapping on $M \cap \Delta_n(\rho_3)$, the C^∞ boundary values of the real analytic extensions $\widetilde{H}_1, \widetilde{H}_2$ are just H_1 and H_2 .

Since $\widetilde{H}_1/\widetilde{H}_2$ is CR on the nonempty open subset V of $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_2$, and since it is holomorphic with respect to the variable w in the vertical conelike slices $\{(z, w) \in \Delta_n(\rho_1) : z = ct.\} \cap \widetilde{\mathcal{W}}_3$, it follows from an elementary (and known) separate Cauchy-Riemann principle that there exists a neighborhood \mathcal{V} of V in \mathbb{C}^n such that $\widetilde{H}_1/\widetilde{H}_2$ extends holomorphically to $\mathcal{V} \cap \widetilde{\mathcal{W}}_3$. Apply now the following general lemma to deduce that $\widetilde{H}_1/\widetilde{H}_2$ can be represented as a quotient P_1/P_2 , where P_1 and $P_2 \neq 0$ are holomorphic in $\widetilde{\mathcal{W}}_3$.

Lemma 7.5. *Let $\Omega \subset \mathbb{C}^n$ be a domain, let A_1 and $A_2 \neq 0$ be two real analytic functions in Ω . Suppose that there exist a point $p \in \Omega \setminus \mathcal{Z}_{A_2}$ and a nonempty open neighborhood V_p of p contained in $\Omega \setminus \mathcal{Z}_{A_2}$ such that the restriction to V_p of A_1/A_2 is holomorphic in V_p . Then there exist two holomorphic functions P_1 and $P_2 \neq 0$ in Ω such that*

$$(7.6) \quad \frac{P_1}{P_2} \Big|_{\Omega \setminus \mathcal{Z}_{A_2}} = \frac{A_1}{A_2} \Big|_{\Omega \setminus \mathcal{Z}_{A_2}}.$$

Proof. In this proof (only), denote some complex coordinates on \mathbb{C}^n by $z = (z_1, \dots, z_n)$ and $z = x + iy$. By assumption, for $i = 1, \dots, n$ and for $z = x + iy \in V_p$, it holds that

$$(7.7) \quad \frac{\partial}{\partial \bar{z}_i} \left(\frac{A_1(x, y)}{A_2(x, y)} \right) \equiv 0,$$

which yields by analytic continuation:

$$(7.8) \quad A_2(x, y) \frac{\partial A_1}{\partial \bar{z}_i}(x, y) - A_1(x, y) \frac{\partial A_2}{\partial \bar{z}_i}(x, y) \equiv 0,$$

for all $z \in \Omega$. It follows that the quotient A_1/A_2 is holomorphic on $\Omega \setminus \mathcal{Z}_{A_2}$, even if $\Omega \setminus \mathcal{Z}_{A_2}$ is not (locally) connected. In fact, to achieve the proof, the local connectedness of $\Omega \setminus \mathcal{Z}_{A_2}$ is needed. To fix this point, it is sufficient to show that \mathcal{Z}_{A_2} is in fact of real dimension $\leq 2n - 2$, hence in particular $\Omega \setminus \mathcal{Z}_{A_2}$ is locally connected.

Indeed, proceeding by contradiction, in a neighborhood of a point $q \in \mathcal{Z}_{A_2}$ at which \mathcal{Z}_{A_2} is geometrically smooth and of dimension $2n - 1$, make a translation of coordinates so that q is the origin and \mathcal{Z}_{A_2} is locally represented by $\{x_1 = 0\}$. Then write

$$(7.9) \quad A_2(x, y) = (x_1)^\kappa [A_{2,0}(x_2, \dots, x_n, y_1, \dots, y_n) + x_1 A_{2,1}(x, y)],$$

for some integer $\kappa \geq 1$ and some real analytic function $A_{2,0}$ independent of x_1 which is not identically zero. Without loss of generality, assume that $A_1(x, y) \not\equiv 0$, hence

$$(7.10) \quad A_1(x, y) = (x_1)^\lambda [A_{1,0}(x_2, \dots, x_n, y_1, \dots, y_n) + x_1 A_{1,1}(x, y)],$$

for some integer $\lambda \geq 0$, where the function $A_{1,0}$ is also not identically zero. Since by assumption $q \in \mathcal{Z}_{A_2}$, it holds that $\kappa \geq \lambda + 1$. Compute then (7.8) for $i = 1$, which yields

$$(7.11) \quad 0 \equiv \left(\frac{\lambda - \kappa}{2} \right) x_1^{\kappa + \lambda - 1} [A_{1,0} A_{2,0} + O(x_1)],$$

a contradiction.

Thus A_1/A_2 extends holomorphically to the *locally connected* open set $\Omega \setminus \mathcal{Z}_{A_2}$.

To pursue the proof, choose an arbitrary point $q \in \Omega$, center the coordinates z at q and consider the polydisc $\Delta_n(\rho_q) \subset \Omega$, where $\rho_q = \inf_{r \in \partial\Omega} |r - q|$. Then $A_1(x, y)$ and $A_2(x, y)$ may be developed in power series with respect to x and y with radius of convergence at least equal to ρ_q . After perhaps making a unitary transformation, assume that the maximally real submanifold $[\mathbb{R}^n \times \{0\}] \cap \Delta_n(\rho_q)$ is not contained in \mathcal{Z}_{A_2} . Then complexify the variable $x \in \mathbb{R}^n$ to be the complex variable $z \in \mathbb{C}^n$ and introduce the element of meromorphic function

$$(7.12) \quad R_q(z) := \frac{A_1(z, 0)}{A_2(z, 0)},$$

which is defined in $\Delta_n(\rho_q)$. By construction, the restrictions to the maximally real subspace

$$(7.13) \quad [(\mathbb{R}^n \times \{0\}) \cap \Delta_n(\rho_q)] \setminus \mathcal{Z}_{A_2}$$

of $R_q(z)$ and of $A_1(x, y)/A_2(x, y)$ coincide. Because $\Delta_n(\rho_q) \setminus \mathcal{Z}_{A_2}$ is locally connected, it follows from the principle of analytic continuation that $R_q(z)$ and $A_1(x, y)/A_2(x, y)$ coincide all over $\Delta_n(\rho_q) \setminus \mathcal{Z}_{A_2}$.

In summary, at every point q of Ω , a meromorphic extension of the holomorphic function $A_1(x, y)/A_2(x, y)$ defined in $\Omega \setminus \mathcal{Z}_{A_2}$ has been constructed. It follows from [18] that this meromorphic function can be represented as a quotient of holomorphic functions in Ω , which completes the proof of the lemma. \square

7.14. Continuation of the proof. Thus $\widetilde{H}_1/\widetilde{H}_2$ extends meromorphically to $\widetilde{\mathcal{W}}_3 \cup \mathcal{V}$. Of course, neither \mathcal{V} nor a neighborhood (in \mathbb{C}^n) of the dense subset $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_{H_2}$ need contain a point of the (thin in the nonminimal case) subset $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$.

Here comes the assumption that the CR mappings $b_1(t)$ and $b_2(t)$ extend holomorphically to a neighborhood \mathcal{V}_{q_0} (in \mathbb{C}^n) of some point $q_0 \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$. It follows that $H_1(z, \bar{t}, \overline{b_1(t)})$ and $H_2(z, \bar{t}, \overline{b_2(t)})$ extend real analytically to \mathcal{V}_{q_0} . Thanks to the (already established) equivalence between (2) and (3), the quotient H_1/H_2 is CR over the dense open subset $(M \cap \Delta_n(\rho_3)) \setminus \mathcal{Z}_{H_2}$. In particular, there exists a point $r_0 \in \mathcal{V}_{q_0}$ in a neighborhood of which H_2 is nonzero. Then, thanks to the Severi-Tomassini extension theorem, H_1/H_2 extends holomorphically to a neighborhood \mathcal{V}_{r_0} (in \mathbb{C}^n) of r_0 . Applying Lemma 7.5 just above, it follows

that H_1/H_2 extends meromorphically to \mathcal{V}_{q_0} . Forget the open subset \mathcal{V} and summarize the obtained extension result.

Lemma 7.15. *The CR quotient H_1/H_2 extends meromorphically to $\widetilde{\mathcal{W}}_3 \cup \mathcal{V}_{q_0}$.*

The goal, to which the remainder of the paper is devoted, is to prove that H_1/H_2 extends meromorphically to some open neighborhood \mathcal{V}_3 of $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ in \mathbb{C}^n .

To this aim, define the set D of points $q \in \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ such that there exists a small nonempty open polydisc \mathcal{V}_q centered at q and a meromorphic extension $R_q(t)$ of $H_1/H_2|_{(\mathcal{V}_q \cap M) \setminus \mathcal{Z}_{H_2}}$ to \mathcal{V}_q . This set is nonempty, since q_0 belongs to D by assumption. It follows from the uniqueness principle on a generic edge that the various meromorphic functions $R_q(t)$ glue together to provide a well defined meromorphic function $R_D(t)$ defined in the open neighborhood $\mathcal{V}_D := \bigcup_{p \in D} \mathcal{V}_p$ of D in \mathbb{C}^n . State this property as a step lemma.

Lemma 7.16. *The CR quotient extends meromorphically to $\widetilde{\mathcal{W}}_3 \cup \mathcal{V}_D$.*

If $D = \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$, Main Lemma 7.1 would be proved, almost gratuitously. Suppose therefore that the complement E of D in $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ is nonempty. To conclude the proof of Main Lemma 7.1, it will suffice to derive a contradiction in the following form: establish that there exists in fact *at least one point* $p_1 \in E$ at which H_1/H_2 extends meromorphically.

§8. LOCALIZATION AT A NICE BOUNDARY POINT

So, assume that E (the bad set) and D (the good set) are nonempty, with $E \cap D = \emptyset$ and with $E \cup D = \mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$. For technical convenience, it is better to pick a special point $p_1 \in E$ so that E lies behind a real analytic generic “wall” M_1 passing through p_1 , as depicted in FIGURE 2 below (cf. [23], [24]).

Lemma 8.1. *There exists a point $p_1 \in E$ and a real analytic hypersurface $M_1 \subset M$ passing through p_1 which is generic in \mathbb{C}^n such that $E \setminus \{p_1\}$ lies, near p_1 , in one side of M_1 .*

Proof. Let $q \in E \neq \emptyset$ be an arbitrary point and let γ be a piecewise real analytic curve running in complex tangential directions to M (CR-curve) which links the point q with the point $q_0 \in D$. Such a curve exists because the CR orbit $\mathcal{O}_{CR}(M, p_0) \cap \Delta_n(\rho_3)$ is locally minimal at every point. After shortening γ and changing the points q and q_0 if necessary, one can assume that γ is a smoothly embedded segment, that q and q_0 belong to γ and are close to each other. Therefore γ can be described as a part of an integral curve of some nonvanishing real analytic section L of T^cM defined in a neighborhood of q .

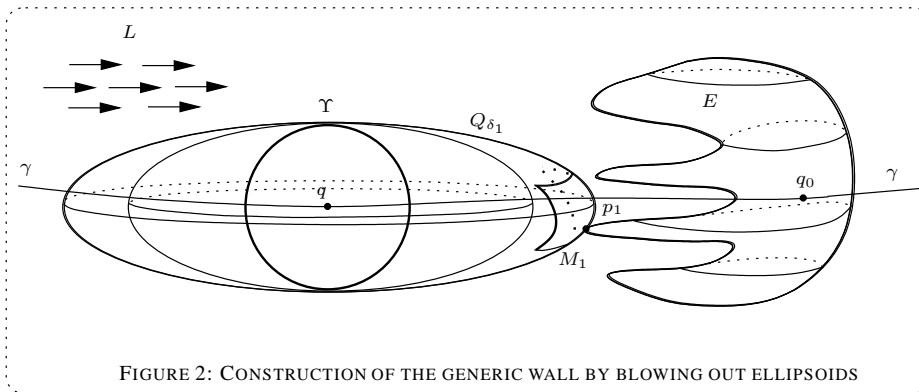


FIGURE 2: CONSTRUCTION OF THE GENERIC WALL BY BLOWING OUT ELLIPSOIDS

Let $H \subset M$ be a small $(2m + d - 1)$ -dimensional real analytic hypersurface passing through q such that $L(q)$ is not tangent to H at q . Integrating L with initial conditions in H , one obtains real analytic coordinates $(s_1, s_2) \in \mathbb{R} \times \mathbb{R}^{2m+d-1}$ so that for fixed $s_{2,0}$, the segments $(s_1, s_{2,0})$ are contained in the trajectories of L . After a translation, one may assume that the origin $(0, 0)$ corresponds to a point of γ close to q which is not contained in E , again denoted by q . Fix a small $\varepsilon > 0$ and for real $\delta \geq 1$, define the ellipsoids (see FIGURE 2 above)

$$(8.2) \quad Q_\delta := \{(s_1, s_2) : |s_1|^2/\delta + |s_2|^2 < \varepsilon\}.$$

There exists the smallest $\delta_1 > 1$ with $\overline{Q_{\delta_1}} \cap E \neq \emptyset$. Then $\overline{Q_{\delta_1}} \cap E = \partial Q_{\delta_1} \cap E$ and $Q_{\delta_1} \cap E = \emptyset$. Observe that every boundary ∂Q_δ is transverse to the trajectories of L out off the equatorial set $\Upsilon := \{(0, s_2) : |s_2|^2 = \varepsilon\}$ which is contained in D . Hence ∂Q_{δ_1} is transverse to L at every point of $\partial Q_{\delta_1} \cap E$. So $\partial Q_{\delta_1} \setminus \Upsilon$ is generic in \mathbb{C}^n , since L is a section of $T^c M$.

To conclude, it suffices to choose a point $p_1 \in \partial Q_{\delta_1} \cap E$ and to take for M_1 a small real analytic hypersurface passing through p_1 which is tangent to ∂Q_{δ_1} and satisfies $M_1 \setminus \{p_1\} \subset Q_{\delta_1}$. The proof of Lemma 8.1. is complete. \square

8.3. Localization. Choose now a point $p_1 \in E$ as in Lemma 10.1 and choose $\varepsilon_1 > 0$ such that the polydisc $\Delta_n(\varepsilon_1)$ (in the new coordinates centered at p_1) with center p_1 is contained in the polydisc $\Delta_n(\rho_3)$ (in the old coordinates centered at p_0) with center p_0 and localize everything in $\Delta_n(\varepsilon_1)$ (cf. FIGURE 3 below, where E has been redrawn on the left).

Denote by M_1^- the negative, left open one-sided neighborhood of M_1 in M such that $E \setminus \{p_1\}$ lies in M_1^- in a neighborhood of p_1 , and denote by M_1^+ the other side, which is contained in D by assumption. Choose affine coordinates vanishing at p_1 , still denoted by $t = (z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ in order that $T_0 M = \{v = 0\}$ and $T_0 M_1 = \{v = 0, x_1 = 0\}$. Denote by $z_\sharp \in \mathbb{C}^{m-1}$ the coordinates (z_2, \dots, z_m) , which are decomposed in real and imaginary part as $z_\sharp = x_\sharp + iy_\sharp$. Then M is defined by $v_j = \varphi_j(x, y, u)$, $j = 1, \dots, d$ with $\varphi_j(0) = 0$, $d\varphi_j(0) = 0$ and M_1 is defined by a supplementary equation $x_1 = \psi(y_1, x_\sharp, y_\sharp, u)$, with $\psi(0) = 0$, $d\psi(0) = 0$.

After possibly replacing M_1 by a new hypersurface which is contained in $M_1^+ \cup \{p_1\}$ (cf. the end of the proof of Lemma 8.1) and after possibly making a dilatation of coordinates, one may assume that the supplementary equation of M_1 is simply given by $x_1 = y_1^2 + |z_\sharp|^2 + |u|^2$, and that M_1^+ is given by

$$(8.4) \quad M_1^+ : \quad x_1 > y_1^2 + |z_\sharp|^2 + |u|^2.$$

(For the readability of FIGURE 3 below, the curvature of M_1 has been reversed, hence the picture of M_1 is slightly incorrect).

Intersect everything with the polydisc $\Delta_n(\varepsilon_1)$ centered at p_1 . In particular, define

$$(8.5) \quad \widetilde{W}_{\varepsilon_1} := \widetilde{W}_3 \cap \Delta_n(\varepsilon_1) \quad \text{and} \quad \mathcal{V}_{\varepsilon_1} := \mathcal{V}_D \cap \Delta_n(\varepsilon_1).$$

To reach the desired contradiction (cf. the last sentence of §7) which achieves the proof of Main Lemma 7.1, the principal goal is now to show that the quotient

$$(8.6) \quad \frac{H_1(z, \bar{t}, \overline{b_1(t)})}{H_2(z, \bar{t}, \overline{b_2(t)})},$$

which already extends holomorphically to $\widetilde{W}_{\varepsilon_1} \cup \mathcal{V}_{\varepsilon_1}$ extends holomorphically to a neighborhood of p_1 in \mathbb{C}^n . For this purpose, the technique of normal deformations of analytic discs enters the scene ([34], [23], [24]).

§9. CONSTRUCTION OF ANALYTIC DISCS

Let ρ_0 with $0 < \rho_0 \leq \varepsilon_1/4$ and for every ρ with $0 \leq \rho \leq \rho_0$, consider an analytic disc

$$(9.1) \quad A_\rho(\zeta) = (Z_\rho(\zeta), W_\rho(\zeta)) = (X_\rho(\zeta) + iY_\rho(\zeta), U_\rho(\zeta) + iV_\rho(\zeta)),$$

where $Z_\rho(\zeta)$ is given by

$$(9.2) \quad Z_\rho(\zeta) = (\rho(1 - \zeta), 0, \dots, 0).$$

The disc A_ρ should be attached to $M \cap \Delta_n(\varepsilon_1)$ and should satisfy $A_\rho(1) = p_1 = 0$ (remind that p_1 is the origin in the chosen coordinates). A necessary and sufficient condition is that U_ρ satisfies the so-called Bishop functional equation

$$(9.3) \quad U_\rho(\zeta) = -[T_1(\varphi(X_\rho(\cdot), Y_\rho(\cdot), U_\rho(\cdot)))](\zeta),$$

for all $\zeta \in \partial\Delta$. Here, T_1 denotes the harmonic conjugate operator (Hilbert transform on $\partial\Delta$) normalized at $\zeta = 1$, namely satisfying $T_1 u(1) = 0$ for every $u \in C^\infty(\partial\Delta, \mathbb{R}^d)$. By [2], the solution exists, is unique and yields a family of analytic discs $A_\rho(\zeta)$ which is smooth (and in fact real analytic) with respect to ρ and ζ . Of course, for $\rho = 0$, the disc A_0 is the constant disc which maps $\bar{\Delta}$ to p_1 .

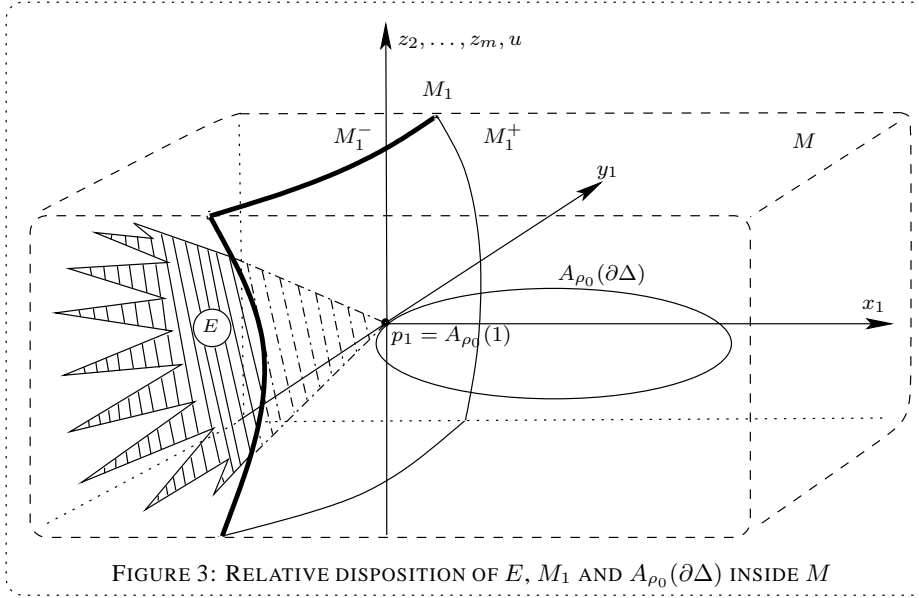


FIGURE 3: RELATIVE DISPOSITION OF E , M_1 AND $A_{\rho_0}(\partial\Delta)$ INSIDE M

The following elementary lemma shows that the boundary $A_{\rho_0}(\partial\Delta)$ of the disc A_{ρ_0} meets the wall M_1 only at p_1 , as shown in FIGURE 3 just above. The notion of analytic isotopy is appropriate to apply the continuity principle (cf. [23], [24]).

Lemma 9.4. *There exists ρ_0 with $0 < \rho_0 < \varepsilon_1/4$ such that the following two properties are satisfied:*

- (1) *for every ρ with $0 < \rho \leq \rho_0$, the mapping $A_\rho : \bar{\Delta} \rightarrow \Delta_n(\varepsilon_1)$ is an embedding; moreover, each A_ρ is analytically isotopic to the point p_1 ;*
- (2) *for every ρ with $0 < \rho \leq \rho_0$:*

$$(9.5) \quad A_\rho(\partial\Delta) \setminus \{1\} \subset M_1^+.$$

Proof. The fact that A_ρ is an embedding for $\rho > 0$ is obvious, since this is the case for Z_ρ . Then the analytic isotopy is obtained by letting ρ decrease to 0. This proves part **(1)**.

For part **(2)**, put $\zeta = re^{i\theta}$, where $|\theta| \leq \pi$ and compute

$$(9.6) \quad \begin{cases} X_{1,\rho}(\zeta) = \frac{\rho(1-\zeta) + \rho(1-\bar{\zeta})}{2} = \frac{\rho}{2} |1 - e^{i\theta}|^2, \\ Y_{1,\rho}(\zeta) = \frac{\rho(1-\zeta) - \rho(1-\bar{\zeta})}{2i} = -\rho \sin \theta. \end{cases}$$

Since the real analytic solution $U_\rho(\zeta)$ vanishes identically for $\rho = 0$ and vanishes at $\zeta = 1$, there exists a constant $C > 0$ such that

$$(9.7) \quad |U_\rho(\zeta)| \leq C \rho |1 - \zeta|,$$

for all $\rho \leq \varepsilon_1/4$. One then deduces from the elementary inequalities $\frac{2|\theta|}{\pi} \leq |1 - e^{i\theta}| \leq |\theta|$ and $|\sin \theta| \leq |\theta|$ that

$$(9.8) \quad X_{1,\rho}(e^{i\theta}) \geq \frac{\rho}{2\pi^2} \theta^2, \quad |Y_{1,\rho}(e^{i\theta})|^2 \leq \rho^2 \theta^2, \quad |U_\rho(e^{i\theta})|^2 \leq C^2 \rho^2 \theta^2.$$

Recall that $Z_{\sharp,\rho}(re^{i\theta}) \equiv 0$. Hence it suffices to choose

$$(9.9) \quad \rho_0 < \frac{1}{2\pi^2(1 + C^2)}$$

in order to insure that

$$(9.10) \quad X_{1,\rho}(e^{i\theta}) > (Y_{1,\rho}(e^{i\theta}))^2 + |U_\rho(e^{i\theta})|^2,$$

for all $\theta \neq 0$, which completes the proof. \square

§10. MEROMORPHIC EXTENSION

The goal is to construct a meromorphic extension of H_1/H_2 to a neighborhood of p_1 . Let $\Omega \subset \mathcal{V}_{\varepsilon_1}$ be an open neighborhood of the point $A_{\rho_0}(-1)$. Thanks to Lemma 2.7 in [23] (a slight modification of the geometric constructions in [34]), it is possible to include the disc $A_{\rho_0}(\zeta)$ in a *regular* (in the sense of Definition 1.8 in [23]; see also p. 493 of [24]) family of analytic discs $A_{\rho_0,s,v}(\zeta)$, where $s \in \mathbb{R}^{2m+d-1}$ satisfies $|s| < s_0$ for some $s_0 > 0$, where $v \in \mathbb{R}^{d-1}$ satisfies $|v| < v_0$ for some $v_0 > 0$ and where $A_{\rho_0,s,v}(\partial\Delta)$ is contained in

$$(10.1) \quad (M \cap \Delta_n(\varepsilon_1)) \cup (\mathcal{W}_{\varepsilon_1} \cap \Omega).$$

Essentially, the disc is deformed near $A_{\rho_0}(-1)$ in order that its direction at exit at $A_{\rho_0}(1) = p_1$ covers an open cone at p_1 , by means of the parameter v , as in [34], [23], [24]. Then the parameter s achieves translation along M . By an application of the continuity principle (where property **(1)** of Lemma 9.4 is needed), there exist θ_0 with $\theta_0 > 0$ and r_0 with $r_0 > 0$, $1 - r_0 < 1$ such that the following set covered by pieces of analytic discs

$$(10.2) \quad \mathcal{W}_4 := \{A_{\rho_0,s,v}(re^{i\theta}) : |s| < s_0, |v| < v_0, |\theta| < \theta_0, 1 - r_0 < r < 1\}$$

is a (curved) local wedge of edge M at p_1 (see FIGURE 4 just below) to which meromorphic functions in $\widetilde{\mathcal{W}}_{\varepsilon_1} \cup \mathcal{V}_{\varepsilon_1}$ extend meromorphically.

smaller. Importantly, there exists a unitary vector $v_4 \in T_{p_1} \mathbb{C}^n$ with coordinates of the form $(0, iv_4) \in \mathbb{C}^m \times \mathbb{C}^d$ such that $v_4 \in T_{p_1} \mathcal{W}_4$ and $-v_4 \in T_{p_1} \widetilde{\mathcal{W}}_4$ (see again FIGURE 4).

Finally, prove **(3)** of Lemma 10.3. In the domain $\widetilde{\mathcal{W}}_{\varepsilon_1} \cup \mathcal{V}_{\varepsilon_1}$, the meromorphic extension of the CR quotient H_1/H_2 , can be represented as a quotient P_1/P_2 . Since for every $\eta < 0$, the discs $A_{\rho_0, s, v}(\zeta) + \eta v_1$ have their boundaries contained in $\widetilde{\mathcal{W}}_{\varepsilon_1} \cup \mathcal{V}_{\varepsilon_1}$ and are analytically isotopic to a point in $\widetilde{\mathcal{W}}_{\varepsilon_1} \cup \mathcal{V}_{\varepsilon_1}$, it follows from the continuity principle that P_1 and P_2 extend holomorphically to $\mathcal{W}_4 + \eta v_1$. By letting η tend to zero, one deduces that P_1 and P_2 extend holomorphically to \mathcal{W}_4 . Hence the CR quotient H_1/H_2 extends meromorphically to \mathcal{W}_4 .

The proof of Lemma 10.3 is complete. \square

The proof of Main Lemma 7.1 is almost achieved. By Lemma 10.3, the CR quotient H_1/H_2 extends meromorphically to the union

$$(10.6) \quad \mathcal{W}_4 \cup \widetilde{\mathcal{W}}_4 \cup \mathcal{V}_{\varepsilon_1}.$$

This union is connected (cf. FIGURE 5 just below. Denote again by P_1/P_2 this meromorphic extension, where P_1 and P_2 are holomorphic in the domain $\mathcal{W}_4 \cup \widetilde{\mathcal{W}}_4 \cup \mathcal{V}_{\varepsilon_1}$.

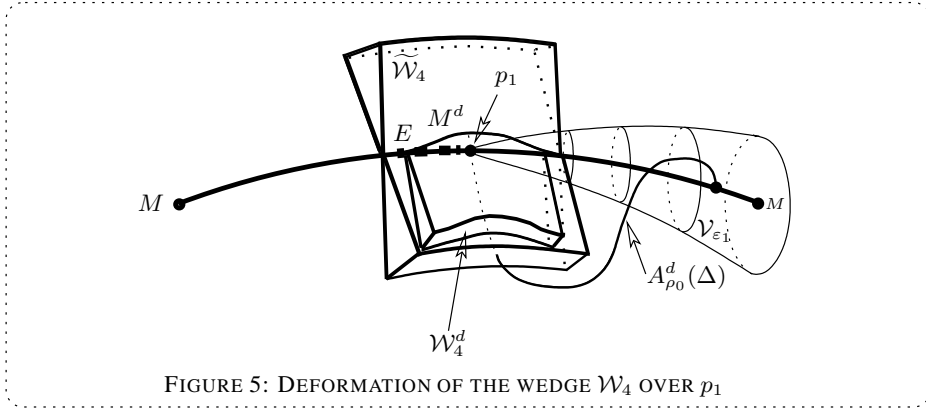


FIGURE 5: DEFORMATION OF THE WEDGE \mathcal{W}_4 OVER p_1

Introduce a one parameter family of smooth deformations M^d , $d \geq 0$, of M localized in a neighborhood of p_1 , with $M^0 = M$, by pushing M near p_1 inside $\widetilde{\mathcal{W}}_4$. For this, it suffices to use a cut-off function $\chi(x, y, u)$ with support in a neighborhood of the origin, and for $d \in \mathbb{R}$, $d \geq 0$, to define M^d by the vectorial equation

$$(10.7) \quad v = \varphi(x, y, u) + d \chi(x, y, u) v_4,$$

where $v_4 \in T_{p_1} \mathcal{W}_4$ and $-v_4 \in T_{p_1} \widetilde{\mathcal{W}}_4$.

Since the resolution of Bishop's equation is stable under perturbation, there exists a smoothly deformed family $A_{\rho_0, s, v}^d(\zeta)$ of analytic discs with $A_{\rho_0, s, v}^0(\zeta) = A_{\rho_0, s, v}(\zeta)$ and a deformed wedge

$$(10.8) \quad \mathcal{W}_4^d := \{A_{\rho_0, s, v}^d(re^{i\theta}) : |s| < s_0, |v| < v_0, |\theta| < \theta_0, 1 - r_0 < r < 1\}.$$

Since \mathcal{W}_4 and $\widetilde{\mathcal{W}}_4$ have opposite directions, it is clear that one can insure that \mathcal{W}_4^d contains the point p_1 .

For $d > 0$, let the parameter ρ vary in the interval $[0, \rho_0]$ to deduce that the discs $A_{\rho_0, s, v}^d$ are analytically isotopic to a point in $\mathcal{W}_4 \cup \widetilde{\mathcal{W}}_4 \cup \mathcal{V}_{\varepsilon_1}$. By a further application of the continuity

principle, it follows that P_1 and P_2 extend holomorphically to \mathcal{W}_4^d , hence to a neighborhood of p_1 . In conclusion, the CR quotient H_1/H_2 extends meromorphically to a neighborhood of p_1 .

The proof of Main Lemma 7.1 is complete. \square

In conclusion, the proof of Theorem 2.9 is complete.

§11 PROOF OF LEMMA 2.12

Establish first the second sentence. Assume that the source M is a hypersurface and that h is essentially finite at p_0 . Proceeding by contradiction, suppose that M is not minimal at p_0 . Equivalently, the CR orbit of p_0 is an $(n-1)$ -dimensional complex hypersurface passing through p_0 , which coincides in fact with the Segre variety $S_{\bar{p}_0}$. Choose holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p_0 in which $S_{\bar{p}_0} = \{w = 0\}$ and represent M by a single complex equation $\bar{w} = \Theta(\bar{z}, z, w)$, where $S_{\bar{p}_0} = \{(z, 0)\}$, whence $\Theta(\bar{z}, z, 0) \equiv 0$. It follows that the usual basis of $(0, 1)$ vector fields tangent to M satisfies $\bar{L}_k = \partial/\partial \bar{z}_k + O(w) \partial/\partial \bar{w}$.

In the fundamental equations

$$(11.1) \quad r'_{j'}(h(t), \bar{h}(\bar{t})) = 0,$$

for $j' = 1, \dots, d'$, specify $t = (z, 0) \in M$, which yields:

$$(11.2) \quad r'_{j'}(h(z, 0), \bar{h}(\bar{z}, 0)) \equiv 0, \quad j' = 1, \dots, d'.$$

Since h is CR and $\{(z, 0)\}$ is contained in M , the mapping $z \mapsto h(z, 0)$ is holomorphic, hence it is justified to write $\bar{h}(\bar{z}, 0)$ instead of $\bar{h}(z, 0)$.

By assumption, h is essentially finite at p_0 . Hence apply the derivations \bar{L}^β , for $\beta \in \mathbb{N}^{n-1}$, to (11.1), which amounts to applying the derivations $\partial_{\bar{z}}^\beta$ to (11.2), which yields expressions of the form

$$(11.3) \quad S'_{j', \beta} \left(J_{\bar{z}}^{|\beta|} \bar{h}(\bar{z}, 0) : h(z, 0) \right) \equiv 0.$$

Exactly as in §3, it follows from the essential finiteness assumption that there exist Weierstrass polynomials such that

$$(11.4) \quad (h_{i'}(z, 0))^{N'_{i'}} + \sum_{1 \leq I' \leq N'_{i'}} H_{i', I'} \left(J_{\bar{z}}^{\ell_0} \bar{h}(\bar{z}, 0) \right) (h_{i'}(z, 0))^{N'_{i'} - I'} = 0,$$

Putting $\bar{z} = 0$, it follows that $h_{i'}(z, 0)$ is a constant for $i' = 1, \dots, n'$, hence vanishes identically. However, if $h(z, 0)$ vanishes identically, it is clearly impossible for h to be essentially finite at p_0 , since differentiation with respect to \bar{z} of $r'_{j'}(t', \bar{h}(\bar{z}, 0))$ gives nothing else than the constant zero, so \mathbb{V}'_0 is defined by $\{w' = 0\}$, hence is positive-dimensional. This establishes the second sentence of Lemma 2.12.

To establish the first sentence, remind again that if $\dim \mathcal{O}_{CR}(M', p'_0) = 2m'$, then the CR orbit of p'_0 is an m' -dimensional complex manifold passing through p'_0 , which coincides in fact with the Segre variety $S'_{\bar{p}'_0}$. Assume that the coordinates $(z', w') \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}$ are such that $S'_{\bar{p}'_0} = \{w' = 0\}$. Then the complex defining equations of M' are of the form $\bar{w}'_{j'} = \Theta'_{j'}(\bar{z}', z', w')$, where $\Theta'_{j'}(\bar{z}', z', 0) \equiv 0$ for $j' = 1, \dots, d'$. However, this clearly contradicts the two classical characterizations of essential finiteness ([2]).

In general, there exist coordinates $(z', w'_1, w'_2) \in \mathbb{C}^{m'} \times \mathbb{C}^{d'_1} \times \mathbb{C}^{d'_2}$ centered at p'_0 , where d'_2 is the holomorphic codimension of the intrinsic complexification $(\mathcal{O}_{CR}(M', p'_0))^{ic}$ and where

$d'_1 = d' - d'_2$, such that M' is represented by

$$(11.5) \quad \begin{cases} \bar{w}_{1,j'_1} = \Theta'_{1,j'_1}(\bar{z}', z', w'_1, w'_2), & j'_1 = 1, \dots, d'_1, \\ \bar{w}_{2,j'_2} = \Theta'_{2,j'_2}(\bar{z}', z', w'_1, w'_2), & j'_2 = 1, \dots, d'_2, \end{cases}$$

where $\Theta'_{2',j'_2}(\bar{z}', z', w'_1, 0) \equiv 0$ for $j'_2 = 1, \dots, d'_2$. Without loss of generality, assume that the coordinates are normal. By the characterization of essential finiteness in normal coordinates, the ideal in $\mathbb{C}\{z'\}$ generated by the partial derivatives $\partial_{\bar{z}'}^{\beta'} \Theta'_{1,j'_1}(z', 0, 0, 0)$, $j_1 = 1, \dots, d'_1$, and the partial derivatives $\partial_{\bar{z}'}^{\beta'} \Theta'_{2',j'_2}(z', 0, 0, 0)$, $j'_2 = 1, \dots, d'_2$, should be of finite codimension, where β' runs in $\mathbb{N}^{m'}$. However, the second collection vanishes identically. Thus, the ideal in $\mathbb{C}\{z'\}$ generated by the partial derivatives $\partial_{\bar{z}'}^{\beta'} \Theta'_{1,j'_1}(z', 0, 0, 0)$ is of finite codimension. This shows that the CR orbit $M' \cap \{w'_2 = 0\}$ is essentially finite at the origin. The proof of Lemma 2.12 is complete.

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