CHARACTERISTIC FOLIATIONS ON MAXIMALLY REAL SUBMANIFOLDS OF $\mathbb{C}^n$ AND REMOVABLE SINGULARITIES FOR CR FUNCTIONS

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[25 colored illustrations]

INTRODUCTION

The study of removable singularities for solutions of (partial) differential equations, deeply rooted in the classical theory of holomorphic functions of one complex variable, plays a major rôle in contemporary Analysis. Historically, the subject was initiated by Riemann's basic removability theorem (1854) — stating that $\mathcal{O}(\mathbb{C}\backslash\{\rm pt\}) \cap L^\infty_{loc}(\mathbb{C}\backslash\{\rm pt\}) = \mathcal{O}(\mathbb{C})$ — and in the last few years, the research field has enjoyed quite spectacular advances. For instance, Painlevé's long outstanding problem (*see* the Bourbaki survey [Pa2005]) about characterizing *geometrically* the compact sets $K \subset \mathbb{C}$ for which $\mathcal{O}(\mathbb{C}\backslash K) \cap L^\infty_{loc}(\mathbb{C}\backslash K) = \mathcal{O}(\mathbb{C})$, is nowadays considered to be essentially solved ([To2003]) in terms of the average Menger curvature of Radon measures supported on K.

For functions of several complex variables, the subject is even richer, because in higher dimensions, existing geometrical concepts and refined cohomological tools broaden considerably the research perspectives. Also, an adequate approach to removable singularities for operators of multi-dimensional complex analysis must certainly take account of the compulsory Hartogs-type extension phenomena that are widely known and still deeply studied in contemporary Cauchy-Riemann Geometry.

It is worth mentioning that since the 1990's, singularities of CR functions on boundaries of domains in complex manifolds attracted much attention. An intensively studied question was to provide criterions insuring that the Hartogs-Kneser extension theorem still holds true, when considering CR functions that are defined only

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in the complement $\partial\Omega\backslash K$ of a compact subset K of a connected boundary $\partial\Omega \Subset \mathbb{C}^n$ ($n \geqslant 2$). For $\mathcal{C}^2$ strongly pseudoconvex boundaries contained in two-dimensional Stein manifolds, a satisfactory function-theoretical characterization was obtained by Stout ([Stu1993]), namely K is removable if and only if it is $\mathcal{O}(\overline{\Omega})$-convex. Slightly after, for arbitrary complex dimension $n \geqslant 3$, a complete cohomological characterization of different nature was obtained by Lupacciolu ([Lu1994]). A survey of Chirka-Stout ([CSt1994]), a monograph of Kytmanov ([Ky1995]) and lecture notes by Laurent-Thiébaut ([Lt1997]) already restitute these aspects excellently and provide a valuable introduction to $\overline{\partial}$ techniques in removable singularities.

In 1988, opening a broad new geometric trend with totally different techniques, Jöricke established an outstanding removability theorem: *closed maximally real $\mathcal{C}^2$ discs in strongly pseudoconvex boundaries $\partial\Omega \Subset \mathbb{C}^2$ are removable*. This was the first CR version of Denjoy's approach to Painlevé's problem, where the singularity is assumed to be one-codimensional. Compared to other results, it was particularly satisfactory to devise the geometrical structure of removable sets, often invisible in functional-theoretic and in cohomological characterizations.

In the late 1990's, within the general framework of CR extension theory that reached a considerable degree of achievement thanks to the works of Trépreau and of Tumanov, it became mathematically accessible to endeavour the general study of (geometrically) removable singularities, for CR functions defined on embedded CR manifolds $M \subset \mathbb{C}^n$ that have arbitrary CR dimension and arbitrary codimension. In recent years, rather (almost) complete removability results have been published by Jöricke for hypersurfaces and by the two authors for general generic submanifolds. Usually, the given generic $M \subset \mathbb{C}^n$ is assumed to be *globally minimal, i.e.* to consist of a single CR orbit (a very weak assumption which allows M to possess quite large Levi-flat regions); in fact, such orbits are the intrinsic objects adequately linked to CR extension; also, they appear to be bricks that are essentially independent; and in the technical details, proofs do in fact proceed orbitwise, so that known corollaries valuable for not globally minimal M's follow from elementary arguments. Since it is wiser to refrain from formulating superficial corollaries, one usually assumes global minimality everywhere.

Towards a general unified theory of removable singularities for CR functions, our finest joint result ([MP2002]) states that closed sets whose $(\dim M - 2)$-dimensional Hausdorff measure vanishes are always removable on such globally minimal M's. Also, we obtained previously several positive results ([MP1999]) in the case where the illusory singularity is assumed to be *a priori* contained in a given submanifold N of M. Thanks to the guiding ideas of Jöricke, complete results were obtained for M of CR dimension $\geqslant 2$ (and of codimension $\geqslant 1$), with $\operatorname{codim}_M N = 1, 2$ or $\geqslant 3$, but in the much more delicate case where M has only CR dimension $=$ 1, in the existing literature, the codimension of the singularity is assumed to be $\geqslant$ 2 ([Me1997, Po1997, Jö1999a, Jö1999b, MP1999]), except notably when M is a hypersurface ([Jö1988, FS1991, Du1993]).

Thus, in the subject, there remained essentially one single principal (daring and difficult) open question raised by Jöricke in [Jö1999b], namely to study the (possibly very massive) singularities that are contained in a one-codimensional maximally real submanifold M^1 of a given generic submanifold $M \subset \mathbb{C}^n$ having CR dimension equal to 1 and codimension $\geqslant 2$ (whence $n \geqslant 3$). As already mentioned briefly, the original motivation was to elaborate CR versions of a celebrated characterization asserted by Denjoy in 1909, who obtained a partial solution to the Painlevé problem that was correct only in the case of a singularity contained in a real analytic curve; nowadays, the best generalization (solution of *Denjoy's conjecture*) says that a compact set $K \subset \mathbb{C}$ contained *a priori* in some Lipschitz curve is removable for bounded holomorphic functions *if and only if* it has zero length, viz. zero one-dimensional Hausdorff measure (*see* [Pa2005] for a precise historical account; recent results go far beyond Denjoy's original approach). In the expected CR generalization, M plays the rôle of a domain in $\mathbb{C}$ and M^1 plays the rôle of the curve.

As discovered by Jöricke in [Jö1988], unlike in the complex plane and thanks to the freedom offered by the various Hartogs-type extension phenomena, removability of illusory singularities may hold true even if they have nonempty interior (in M^1), and without requiring neither the vanishing of some metrical (Lebesgue, Minkowski, Hausdorff) content, nor of some auxiliary capacity. Jöricke also cleverly emphasized that the classical removability theorems enjoyed by general linear partial differential operators that were unified by Harvey and Polking in [HP1970] do only provide restricted insight into the nature of CR singularities. In fact, because these results are based on elementary metrical estimates showing that the singularity becomes innocuous through integration by parts, the formulation of these theorems does depend on the class (*e.g.* L^{p}_{loc} or $\mathcal{C}^{\kappa,\alpha}$) and also, it seems impossible to get L^1-removability without a strong assumption of growth near the singularity. On the contrary, Jöricke ([Jö1999a, Jö1999b]) and the two authors ([MP1999, MP2002]) obtained results formulated geometrically that are uniform with respect to the class — including L^1_{loc} — and that require no growth tameness.

Following this trend of thought, our main objective in the present research paper is to answer completely the first Problem 2.1 raised in [Jö1999b] (and mentioned above), with $M \subset \mathbb{C}^n$ ($n \geqslant 2$) of CR dimension 1 and $M^1 \subset M$ maximally real, both of class $\mathcal{C}^{2,\alpha}$ ($0 < \alpha < 1$). Since $M^1 \subset M$ has null CR dimension, the standard processus of sweeping out by wedges becomes void, because small Bishop discs attached to M^1 are not available. Accordingly, the proof of the main Proposition 1.13 below relies upon a new localization device, based on families of analytic discs that are only half-attached to M^1 (following Bishop and Pinchuk), the very gist of the argument being the selection of a special point to be removed. In the hypersurface case (only), previously known approaches relied upon a global *Kontinuitätssatz*, or upon a global filling of 2-spheres by Levi-flat 3-balls (following Bedford-Klingenberg and Kruzhilin), but both tools have no known controllable counterpart in higher codimension. So, as a final comment, we point out that it is satisfactory to bring in this paper a purely local framework for the treatment of one-codimensional singularities, even when M is a hypersuface of $\mathbb{C}^2$.

The results presented here are entirely new in codimension $\geqslant 2$.

§1. CHARACTERISTIC FOLIATION AND REMOVABILITY: MAIN RESULTS

1.1. Removability of totally real discs having hyperbolic complex tangencies. By means of a global *Kontinuitätssatz*, Jöricke ([Jö1988]) showed removability of closed maximally real smooth discs contained in strongly pseudoconvex boundaries $\partial\Omega \Subset \mathbb{C}^2$. Applying the filling of 2-spheres by Levi-flat 3-balls ([BK1991, Kr1991]), Forstnerič -Stout ([FS1991]) allowed finitely many complex tangencies of hyperbolic type (in the sense of Bishop) in the disc and established both its removability and its $\mathcal{O}(\overline{\Omega})$-convexity. Reasoning with Rossi's local maximum modulus principle and with Oka's criterion for holomorphic convexity, Duval ([Du1993]) re-obtained these result differently and generalized them to arbitrary surfaces $S \subset \partial\Omega$.

More recently, suppressing convexity hypotheses, the second author ([Po2004]) showed removability of closed maximally real discs contained in globally minimal hypersurfaces of $\mathbb{C}^2$. We point out that obtaining theorems without any assumption of (pseudo)convexity on CR manifolds leads to substantial difficulties, because one loses almost all of the strong interweavings between function-theoretic tools and geometric arguments which are valid in the pseudoconvex realm, for instance: Hopf lemma, plurisubharmonic exhaustions, envelopes of function spaces, local maximum modulus principle, Stein neighborhood basis and semi-global control of hulls.

Our first statement unifies the mentioned results, still without pseudoconvexity; importantly, we also establish removability of certain compact subsets of *arbitrary surfaces*, instead of plain discs, *see* Corollary 1.5 below. Throughout this article, *all* (sub)manifolds are assumed to be *embedded*.

Theorem 1.2. *Let M be a globally minimal $\mathcal{C}^{2,\alpha}$ $(0 < \alpha < 1)$ hypersurface in $\mathbb{C}^2$ and let $D \subset M$ be a $\mathcal{C}^{2,\alpha}$ surface which is*:

- *diffeomorphic to the open unit 2-disc of $\mathbb{R}^2$ and*:
- *totally real outside a discrete subset of isolated complex tangencies which are hyperbolic in the sense of Bishop.*

Then every compact subset K of D is CR-, $\mathcal{W}$- and L^{p}-removable[1].

The theorem holds with exactly the same proof if $\mathbb{C}^2$ is replaced by any two-dimensional complex manifold, not necessarily Stein. As a direct corollary, with $M = \partial\Omega \Subset \mathbb{C}^2$ being a $\mathcal{C}^{2,\alpha}$ compact boundary, hence automatically globally minimal ([Jö1999a, MP2006]), we obtain a Hartogs-Kneser extension theorem from $\partial\Omega \backslash K$. Also, the characterization of removable sets due to Stout yields that if $\partial\Omega \Subset \mathbb{C}^2$ is a $\mathcal{C}^{2,\alpha}$ boundary such that $\overline{\Omega}$ has a Stein neighborhood basis, then every $K \subset D \subset \partial\Omega$ as in the theorem is $\mathcal{O}(\overline{\Omega})$-convex.

As a more substantial application, reminding that satisfactory geometric criteria for polynomial convexity of general surfaces in $\mathbb{C}^2$ are far to be known, we derive new examples of polynomially convex sets contained in weakly pseudoconvex boundaries. The arguments of proof are postponed to Section 12.

Corollary 1.3. *Let $\Omega \Subset \mathbb{C}^2$ be a domain with $\mathcal{C}^{2,\alpha}$ boundary. Suppose that $\overline{\Omega}$ is polynomially convex* (*whence Ω is weakly pseudoconvex*). *Let $D \subset \partial\Omega$ be an embedded 2-disc of class $\mathcal{C}^{2,\alpha}$ which is totally real outside a discrete subset of hyperbolic complex tangencies. Then each compact set $K \subset D$ is polynomially convex.*

To describe briefly some aspects of the geometrical machinery underlying Theorem 1.2, we remind ([Jö1988, FS1991, Du1993]) that the totally real part of the 2-disc

[1]The classical notion of L^{p}-removability ([HP1970, Jö1988, Jö1999b, MP1999]) means that $L^{\mathsf{p}}_{loc}(M) \cap L^{\mathsf{p}}_{loc,CR}(M\backslash K) = L^{\mathsf{p}}_{loc,CR}(M)$; CR-removability means that $\mathcal{C}^0_{CR}(M\backslash K)$ extends holomorphically to some global one-sided neighborhood ω_M of M; $\mathcal{W}$-removability means essentially that the same extension property holds for $\mathcal{O}(\omega_{M\backslash K})$; the reader is referred to [MP2002] for rigorous and precise definitions, valuable in arbitrary codimension, and to [CSt1994, Jö1999a] for similar concepts, presented from the standard hypersurface perspective.

D is equipped with a so-called *characteristic foliation* F^c_D, obtained by integrating the line distribution $D \ni p \longmapsto T^c_p M \cap T_p D$, hence canonical. Then F^c_D has singularities exactly at the complex hyperbolic tangencies of D. If D is totally real at every point, the Poincaré-Bendixson theorem assures the inexistence of limit cycles as well as of foci, of centers and of saddle points, so that all characteristic curves must go from a point of the boundary of D to another boundary point; in the left diagram below, they are simply drawn as horizontal lines.

In [Du1993], Duval delineated a crucial, immediately seen geometric property: for every compact set $K \subset D$ (hence also trivially for every subcompact $K' \subset K \subset D$), there exists at least one characteristic curve γ touching K (resp. K') such that K (resp. K') is located in one closed side of γ in some thin, elongated neighborhood of γ. Also, the tips of γ being close to the boundary of D, they must lay at a positive distance from K. Moving then such a curve γ slightly up and down, one sees an intuitive processus of "erasing" the (bottom-left part in the picture) part of K which is very similar to the classical *Kontinuitätssatz*, alias *Continuity Principle*, in which one moves an analytic disc, keeping its boundary inside some (usually pseudoconcave) (Riemann) domain, in order to describe a part of an envelope of holomorphy.

Strikingly, this informal analogy underlies a true removability fact, which we formulate as an independent, main technical proposition, directly useful to the proof of Theorem 1.2. *All subsets C of a submanifold S of a manifold M that are called* closed *are assumed to be closed both in M and in S.* We point out that now D is replaced by a 2-surface S which may have arbitrary topology and that the removed set C is not necessarily compact, which will be needed.

Proposition 1.4. *Let M be a $\mathcal{C}^{2,\alpha}$ globally minimal hypersurface in $\mathbb{C}^2$, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$ surface, open or closed, with or without boundary, which is totally real at every point. Let C be a proper closed subset of S and assume that the following topological condition holds, meaning that C is nontransversal to F^c_S:*

- *for every closed subset $C' \subset C$, there exists a simple $\mathcal{C}^{2,\alpha}$ curve $\gamma : [-1,1] \to S$, whose range is contained in a single leaf of the characteristic foliation F^c_S (obtained by integrating the characteristic line field $T^cM|_S \cap TS$), with $\gamma(-1) \notin C'$, $\gamma(0) \in C'$ and $\gamma(1) \notin C'$, such that C' lies completely in one closed side of $\gamma[-1,1]$ with respect to the topology of S in a neighborhood of $\gamma[-1,1]$.*

Then C is CR-, $\mathcal{W}$- and L^{p}-removable.

In case $S = D$ is a 2-disc and $C = K$ is compact, the left diagram provides an illustration.

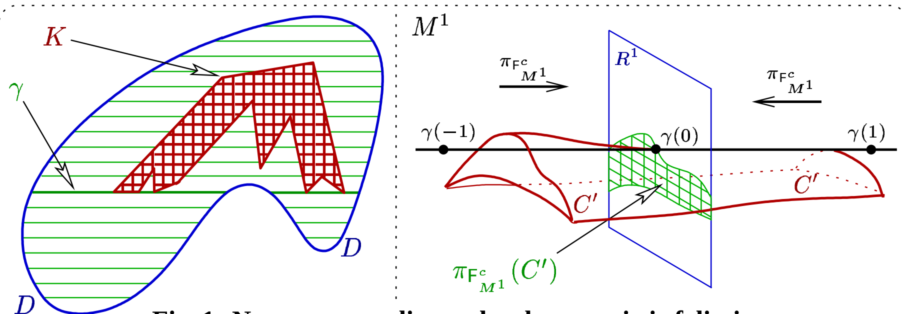

Fig. 1: Nontransversality to the characteristic foliation

In the absence of a Poincaré-Bendixson theorem taming the topology of the characteristic foliation (S is not a disc), this nontransversality condition appears to be the most adequate cause of removability. In fact, it is well known that boundaries of analytic discs or of Riemann surfaces attached to a 2-surface $S \subset \partial\Omega$ contained in some compact strongly pseudoconvex boundary $\partial\Omega \Subset \mathbb{C}^2$, are embedded circles everywhere transversal to F_S^c (because of Hopf's lemma), and it is clear that such holomorphic curves $\Lambda = \{g = 0\}$, with $g \in \mathcal{O}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, are never removable: it suffices to set $K := \{g = 0\} \cap \partial\Omega \subset S$ and to consider $\frac{1}{g}\big|_{\partial\Omega\backslash K}$. It is thus remarkable that the nontransversality of C to F_S^c appears again on globally minimal CR structures, where the distribution $p \mapsto T_p^c M$ is allowed to be very far from contact.

Let us now briefly explain why the main Proposition 1.4 is necessary to Theorem 1.2 (we recommend that the reader simultaneously watches Figure 22 in §11.3 below). At each hyperbolic point p, the phase diagram of F_D^c is saddle-like and contains two local separatrices intersecting at p which are smooth and transversal (cross-like). Hence we can decompose the 2-disc D as a union $D = T_D \cup D_o$, where T_D consists of the union of the hyperbolic points of D together with the separatrices issuing from them, and where $D_o := D\backslash T_D$ is the remaining open submanifold of D, obviously contained in the totally real part of D. By the theory of Poincaré-Bendixson ([HS1974, FS1991, Du1993]), since D is a disc, T_D must be a tree of $\mathcal{C}^{1,\alpha}$ curves which contains no subset homeomorphic to the unit circle. Accordingly, we set $K_{T_D} := K \cap T_D$ and $C_o := K \cap D_o$, so that $K = K_{T_D} \cup C_o$ decomposes in two parts. Then C_o is a relatively closed subset of D_o, and importantly, it is also closed in $M_o := M\backslash T_D$. Again thanks to Poincaré-Bendixson, C_o is nontransversal to the characteristic foliation of D_o. So Proposition 1.4 applies: we may remove C_o with respect to M_o, namely we get holomorphic extension to a global one-sided neighborhood ω_{M_o} of M_o in $\mathbb{C}^2$.

Deforming M slightly inside ω_{M_o}, we are left with the much thinner singularity K_{T_D}, of codimension $\leqslant 2$ in M. Since K_{T_D} contains no circle, its removal will follow from known theorems ([CSt1994, Jö1999a, MP1999]; however, a technical investigation of the behavior of the CR orbits near T_D will be required).

Section 2 describes and summarizes the proof of the main Proposition 1.4 in geometric and in conceptional terms.

The nontransversality assumption is a *common condition* on C and on the characteristic foliation F_S^c, namely on the relative disposition of C with respect to F_S^c; Figure 3 below provides a second illustration of it. As already mentioned, if S is diffeomorphic to a real 2-disc or if $S = D_o$ as above, then nontransversality holds true. Generally, it also holds when the characteristic foliation is given by the level sets of some $\mathcal{C}^{1,\alpha}$ real-valued function defined on S. More interestingly, to conclude with removal in $\mathbb{C}^2$, we formulate a consequence of Proposition 1.4 that is more general than Theorem 1.2, since it holds without the restricted assumption that S be diffeomorphic to a real 2-disc. We believe that this application shows well the strength of our new localization procedure.

Corollary 1.5. *Let M be a $\mathcal{C}^{2,\alpha}$ globally minimal hypersurface in $\mathbb{C}^2$, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$ real 2-surface, open or closed, with or without boundary which is totally real outside a discrete subset of isolated hyperbolic complex tangencies and let $K \subset S$ be a compact set. If T_S denotes the union of hyperbolic points of S together with all separatrices, assume that*:

- $K \cap (S \backslash T_S)$ *is nontransversal to* $\mathsf{F}^c_{S\backslash T_S}$;
- $K \cap T_S$ *does not contain any subset homeomorphic to the unit circle.*

Then K is CR-, $\mathcal{W}$- and L^{p}-removable.

1.6. Passage to arbitrary codimension. Our principal motivation for the present work was to devise a purely local strategy of proof for Theorem 1.2 in order to obtain higher codimensional removability results in the most delicate case of CR dimension 1. Accordingly, let M be a $\mathcal{C}^{2,\alpha}$ globally minimal generic submanifold of codimension $(n-1)$ in $\mathbb{C}^n$, with $n \geqslant 2$ arbitrary. Let M^1 be a $\mathcal{C}^{2,\alpha}$ one-codimensional submanifold of M which is generic in $\mathbb{C}^n$, hence maximally real. As in the $\mathbb{C}^2$ case, M^1 carries a *characteristic foliation* $\mathsf{F}^c_{M^1}$, whose leaves are the integral curves of the canonical line distribution $M^1 \ni p \longmapsto T_pM^1 \cap T_p^cM$.

Next, let $K \subset M^1$ be a compact set. Of course, the assumption that K locally lies in one closed side of some characteristic curve is meaningless inside M^1, when its dimension n is $\geqslant 3$. Taking inspiration from (pseudo)convexity theory, the appropriate condition requires that every compact $K' \subset K$ has at least one boundary point at which $M^1 \backslash K'$ becomes concave with respect to characteristic segments.

Definition 1.7. The complement $M^1 \backslash K$ is called *characteristically pseudoconcave* if for every subcompact $K' \subset K$, there is a $\mathcal{C}^1$ embedding $\Phi : [-1,1] \times [0,c_1] \to M^1$, $c_1 > 0$, such that:

- each horizontal leaf $\Phi\big([-1,1] \times \{\mathrm{cst}\}\big)$ is contained in a single characteristic curve;
- for $0 \leqslant \mathrm{cst} < c_1$, the intersection $\Phi\big([-1,1] \times \{\mathrm{cst}\}\big) \cap K' = \emptyset$ is void;
- $\Phi\big(\{-1\} \times [0,c_1]\big) \cap K' = \emptyset$ and $\Phi\big(\{1\} \times [0,c_1]\big) \cap K' = \emptyset$;
- $\Phi\big((-1,1) \times \{c_1\}\big) \cap K' \neq \emptyset$ is nonempty (but the two endpoints $\Phi(-1,c_1)$ and $\Phi(1,c_1)$ must, by the above item, lie at a positive distance from K').

In the left Figure 1 above, Φ amounts to translating γ downward. The reader will have noticed the close similarity with the classical continuity principle. In our setting, embedded families of holomorphic discs are replaced by families of characteristic segments, the endpoints of the segments corresponding to the boundary circles of the discs. We remind that to verify that a domain $\Omega \subset \mathbb{C}^n$ is pseudoconvex in the sense of Hartogs, one has to establish the *Kontinuitätssatz* for *all* appropriately embedded families of holomorphic discs. In our case the geometry is much more rigid because the directions of the embedded segments are already prescribed by the characteristic foliation.

Unexpectedly[2], our principal result in this paper establishes a deep link between the characteristic pseudoconcavity of $M^1 \backslash K$ in the real sense and the fact that the (partial) envelope of holomorphy of $M \backslash K$ is pseudoconcave enough to cover M.

Theorem 1.8. *Let $M \subset \mathbb{C}^n$ be generic, $\mathcal{C}^{2,\alpha}$, of codimension $(n-1)$ and globally minimal, let $M^1 \subset M$ be one-codimensional, $\mathcal{C}^{2,\alpha}$ and maximally real in $\mathbb{C}^n$, and let $K \subset M^1$ be a compact set. If $M^1 \backslash K$ is characteristically pseudoconcave, then it is is CR-, $\mathcal{W}$- and L^{p}-removable.*

[2]Needless to say, there is no direct direct magical translation from characteristic segments to holomorphic curves.

In fact, we recall from [Me1997, MP1999, MP2002] that the removability of K means (essentially) that the (partial) envelope of holomorphy of any wedgelike domain attached to $M \backslash K$ contains a complete wedge attached to M. Technically speaking, the proof is of high level and before launching the attack, we will formulate a more general main proposition, analogous to Proposition 1.4 and valid for certain closed sets that are nontransversal to $\mathsf{F}_{M^1}^c$ in a certain sense.

Meanwhile, we would like to mention that in the last Section 13 below (which may be read independently), we will exhibit a crucial example of a compact set $K \subset M^1 \subset M \subset \mathbb{C}^3$ diffeomorphic to a two-dimensional torus and everywhere transversal to $\mathsf{F}_{M^1}^c$, namely $T_pK \oplus \mathsf{F}_{M^1}^c(p) = T_pM^1$ for every $p \in K$, which is truly nonremovable. Since the embedded 2-torus K has no boundary, the complement $M^1 \backslash K$ cannot be characteristically pseudoconcave. This shows that the main geometrical assumption of the theorem above is adequate. In addition, similarly as in [JS2000], we may require (almost for free) that M and M^1 have the simplest possible topology. Recall that, according to a classical definition, type 4 at a point $p \in M$ means that the Lie brackets of the complex tangent bundle T^cM up to length 4 generate T_pM.

Theorem 1.9. *There exists a triple* (M, M^1, K)*, where*

- **(i)** M *is a* $\mathcal{C}^\infty$ *generic submanifold in* $\mathbb{C}^3$ *of CR dimension* 1*, diffeomorphic to a real* 4*-ball;*
- **(ii)** M^1 *is a* $\mathcal{C}^\infty$ *one-codimensional submanifold of* M *which is maximally real in* $\mathbb{C}^n$ *and diffeomorphic to a real* 3*-ball;*
- **(iii)** K *is a compact subset of* M^1 *diffeomorphic to a real* 2*-torus which is everywhere transversal to the characteristic foliation* $\mathsf{F}_{M^1}^c$*, hence* $M^1 \backslash K$ *cannot be characteristically pseudoconcave;*
- **(iv)** M *of finite type* 4 *at every point, hence globally minimal,*

such that K *is neither CR- nor* $\mathcal{W}$*- nor* L^{p}*-removable with respect to* M*.*

1.10. Characteristic nontransversality and main proposition. Let M, M^1, $\mathsf{F}_{M^1}^c$ be as before and let C be a proper subset of M^1, closed in M^1 and closed in M. Here is the higher dimensional notion of characteristic nontransversality, already illustrated by the right diagram above.

Definition 1.11. The closed set $C \subset M^1 \subset M$ is called *nontransversal to the characteristic foliation* if:

- for every closed subset $C' \subset C$, there exists a simple $\mathcal{C}^{2,\alpha}$ curve $\gamma : [-1,1] \to M^1$ whose range $\gamma[-1,1]$ is contained in a single leaf of the characteristic foliation $\mathsf{F}_{M^1}^c$ with $\gamma(-1) \notin C'$, $\gamma(0) \in C'$ and $\gamma(1) \notin C'$, there exists a local $(n-1)$-dimensional transversal $R^1 \subset M^1$ to γ passing through $\gamma(0)$ and there exists a thin elongated open neighborhood V_1 of $\gamma[-1,1]$ in M^1 such that if $\pi_{\mathsf{F}_{M^1}^c} : V_1 \to R^1$ denotes the semi-local projection parallel to the leaves of the characteristic foliation $\mathsf{F}_{M^1}^c$, then $\gamma(0)$ lies on the boundary, relatively to the topology of R^1, of $\pi_{\mathsf{F}_{M^1}^c}(C' \cap V_1)$.

Clearly, in the case $n = 2$, this amounts to say that $C' \cap V_1$ lies completely in one side of $\gamma[-1,1]$, as written in Proposition 1.4.

Lemma 1.12. *The two conditions introduced so far are in fact equivalent*:

- $M^1 \backslash C$ *is characteristically pseudoconcave* if and only if
- C *is nontransversal to* $\mathsf{F}^c_{M^1}$.

Furthermore, for every nonempty closed $C' \subset C$*, there exists* $p_1 \in C'$*, there exists a characteristic embedded* $\mathcal{C}^{2,\alpha}$ *curve* $\gamma := [-1,1] \to M^1$ *with* $\gamma(-1) \notin C'$, $\gamma(0) = p_1$ *and* $\gamma(1) \notin C'$ *and there exists a thin* $\mathcal{C}^{1,\alpha}$ *support hypersurface* $H^1 \subset M^1$ *containing* γ*, foliated by characteristic curves and elongated along* γ *such that* C' *is contained in one closed side of* H^1 *inside* M^1*, locally in a neighborhood of* γ*.*

The second assertion, proved in Proposition 5.2, entails immediately the equivalence. We may now formulate our main technical proposition (generalizing Proposition 1.4) upon which Theorem 1.8 relies.

Proposition 1.13. *If the closed set* $C \subset M^1 \subset M$ *is nontransversal to the characteristic foliation, it is CR-,* $\mathcal{W}$*- and* L^{p}*-removable.*

We emphasize that this concise statement constitutes the essential core of the present article. Sections 3, 4, 5, 6, 7, 8 and 9 are integrally devoted to its proof.

1.14. Comparison with a third hypothesis sufficient for removability. With $C = K$ compact, in §2.17 below, we compare our main nontransversality assumption to the following condition, suggested by a referee.

$\mathsf{H}\{K\}$: *there is an open neighborhood* U *of* K *in* M^1 *and a* $\mathcal{C}^1$ *submersion* $\rho : U \to V$ *with values in a* $\mathcal{C}^1$ *not necessarily connected* $(n-1)$*-dimensional manifold* V *without boundary and without compact components such that every level set* $\rho^{-1}(q)$, $q \in V$*, is a union of leaves of* $\mathsf{F}^c_U = \mathsf{F}^c_{M^1}\big|_U$.

We first verify that $\mathsf{H}\{K\}$ implies that K is nontransversal to $\mathsf{F}^c_{M^1}$. However, the reverse implication does not hold, so that for Theorem 1.8, $\mathsf{H}\{K\}$ is a strictly less general assumption than the characteristic pseudoconcavity of $M^1 \backslash K$. This was forseeable, since global foliations are rarely induced by a submersion. In dimension $n = 3$, we thus construct an example of (M, M^1, K) with $M^1 \backslash K$ characteristically pseudoconcave for which $\mathsf{H}\{K\}$ fails (*see* §2.17).

1.16. Application. For completeness, we formulate a higher codimensional version of Corollary 1.5.

Corollary 1.17. *Let* $M \subset \mathbb{C}^n$ *be generic,* $\mathcal{C}^{2,\alpha}$*, of codimension* $(n-1)$ *and minimal at every point, let* $\Lambda^1 \subset M$ *be a* $\mathcal{C}^{2,\alpha}$ *one-codimensional submanifold, totally real outside* $\Sigma \cup \Lambda^2$*, where* $\Sigma \subset M$ *is closed with vanishing* $(\dim M - 2)$*-dimensional Hausdorff measure and where*

$$\Lambda^2 = \bigcup_{j \in J} \Lambda^2_j \tag{1.18}$$

is a countable, locally finite union of disjoint connected 2*-codimensional* $\mathcal{C}^{2,\alpha}$ *submanifolds* $\Lambda^2_j \subset \Lambda^1$*, and let* $K \subset \Lambda^1$ *be a compact set. Assume that*

- $K \cap \left(\Lambda^1 \backslash \Lambda^2\right)$ *is nontransversal to* $\mathsf{F}^c_{\Lambda^1 \backslash \Lambda^2}$;
- $K \cap \Lambda^2_j$ *is a proper subset of* Λ^2_j *for every* $j \in J$.

Then K *is CR-,* $\mathcal{W}$*- and* L^{p}*-removable.*

Since M is (locally) minimal at every point, $M\backslash E$ is globally minimal, for every $E \subset M$. The removal of $K \cap (\Lambda^1 \backslash \Lambda^2)$ follows from the main Proposition 1.13, the removal of each $K \cap \Lambda_j^2$ is proved in [MP1999], and the removal of Σ is established in [MP2002].

1.19. Acknowledgments. We would like to express our sincere gratitude to Burglind Jöricke for her kind interest and for her clever suggestions that incited us to improve substantially the presentation of our results.

§2. Description of the proof of Proposition 1.4 and organization

In this preliminary section, we summarize the hypersurface version Proposition 1.4. Our goal is to provide a conceptional description of the basic geometric constructions, which should be helpful to read the proof of the general Proposition 1.13. Because precise, complete and rigorous formulations will be developed in the next sections, we allow here the use of a slightly informal language.

2.1. Strategy per absurdum. Let M, S, and C be as in Proposition 1.4. It is known that both the CR- and the L^{p}-removability of C are a (relatively mild) consequence of the $\mathcal{W}$-removability of C (*see* §3.14 and Section 10 below). Thus, we shall describe in this section only the $\mathcal{W}$-removability of C.

First of all, as M is globally minimal, it may be proved that for every closed subset $C' \subset C$, the complement $M\backslash C'$ is also globally minimal (*see* Lemma 3.5 below). As M is of codimension one in $\mathbb{C}^2$, a wedge attached to $M\backslash C$ is simply a connected one-sided neighborhood of $M\backslash C$ in $\mathbb{C}^2$. Let us denote such a one-sided neighborhood by ω_1. The goal is to prove that there exists a (bigger) one-sided neighborhood ω attached to M to which holomorphic functions in ω_1 extend holomorphically. By the definition of $\mathcal{W}$-removability, this will show that C is $\mathcal{W}$-removable.

Reasoning by contradiction, we shall denote by $C_{\rm nr}$ the smallest *nonremovable* subpart of C. By this we mean that holomorphic functions in ω_1 extend holomorphically to a one-sided neighborhood ω_2 of $M\backslash C_{\rm nr}$ in $\mathbb{C}^2$ and that $C_{\rm nr}$ is the smallest subset of C such that this extension property holds. If $C_{\rm nr}$ is empty, the conclusion of Proposition 1.4 holds, gratuitously: nothing has to be proved. If $C_{\rm nr}$ is nonempty, to come to an absurd, it suffices to show that at least one point of $C_{\rm nr}$ is *locally removable*. By this, we mean that there exists a local one-sided neighborhood ω_3 of at least one point of $C_{\rm nr}$ such that holomorphic functions in ω_2 extend holomorphically to ω_3. In fact, the choice of such a point will be the most delicate and the most tricky part of the proof.

In order to apply the continuity principle, as stated in [Me1997, MP2006], we now deform slightly M inside the one-sided neighborhood ω_2, keeping $C_{\rm nr}$ fixed, getting a hypersurface M^d (with d like "*d*eformed") satisfying $M^d\backslash C_{\rm nr} \subset \omega_2$. We point out that a local one-sided neighborhood of M^d at one point p of $C_{\rm nr}$ always contains a local one-sided neighborhood of M at p (the reader may draw a figure), so we may well work on M^d instead of working on M (however, the analogous property about wedges over deformed generic submanifolds is untrue in codimension $\geqslant 2$, *see* §3.16 below, where supplementary arguments are required).

Replacing the notation $C_{\rm nr}$ by the notation C, the notation M^d by the notation M and the notation ω_2 by the notation Ω, we see that Proposition 1.4 is reduced to the following main proposition, whose formulation is essentially analogous to that of Proposition 1.4, except that it suffices to remove at least one special point.

Proposition 2.2. *Let M be a $\mathcal{C}^{2,\alpha}$ globally minimal hypersurface in $\mathbb{C}^2$, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$ surface which is totally real at every point. Let C be a* nonempty *proper closed subset of S and assume that it is nontransversal to F_S^c. Let Ω be an arbitrary neighborhood of $M \backslash C$ in $\mathbb{C}^n$. Then there exists a special point $p_{\rm sp} \in C$ and there exists a local one-sided neighborhood $\omega_{p_{\rm sp}}$ of M in $\mathbb{C}^2$ at $p_{\rm sp}$ such that holomorphic functions in Ω extend holomorphically to $\omega_{p_{\rm sp}}$.*

2.3. Holomorphic extension to a half-one-sided neighborhood of M. The choice of the special point $p_{\rm sp}$ will be achieved in two main steps. According to the nontransversality assumption, there exists a characteristic segment $\gamma : [-1,1] \to S$ with $\gamma(-1) \notin C$, with $\gamma(0) \in C$ and with $\gamma(1) \notin C$ such that C lies in one (closed, semi-local) side of γ in S. As γ is a Jordan arc, we may orient S in M along γ, hence we may choose a semi-local open side $(S_\gamma)^+$ of S in M along γ. In the first main step (to be conducted in Section 4 in the context of the general Proposition 1.13), we shall construct what we call a *semi-local half-wedge $\mathcal{HW}_\gamma^+$ attached to $(S_\gamma)^+$ along* γ. By this, we mean the "half part" of a wedge attached to a neighborhood of the characteristic segment γ in M, which yields a wedge attached to the semi-local one-sided neighborhood $(S_\gamma)^+$. For an illustration, *see* Figure 8 below, in which one should replace the notation M^1 by the notation S. Such a half-wedge may also be interpreted as a wedge attached to a neighborhood of γ in S, but it should not be arbitrary, it should satisfy a further property: locally in a neighborhood of every point of γ, either the half-wedge contains $(S_\gamma)^+$ or one of its two ribs contains $(S_\gamma)^+$, as illustrated in Figure 8 below. Most importantly, *the cones of this attached half-wedge should vary continuously as we move along γ, cf.* again Figure 8.

The way how we will construct this half-wedge $\mathcal{HW}_\gamma^+$ is as follows. As illustrated in Figure 2 just below, we shall first construct a string of analytic discs $Z_{r:s}(\zeta)$, where r is the approximate radius of $Z_{r:s}(\partial\Delta)$, whose boundaries are contained in $(S_\gamma)^+ \subset M$ and which touch the curve γ only at the point $\gamma(s)$, for every $s \in [-1,1]$, namely $Z_{r:s}(1) = \gamma(s)$ and $Z_{r:s}(\partial\Delta\backslash\{1\}) \subset (S_\gamma)^+$.

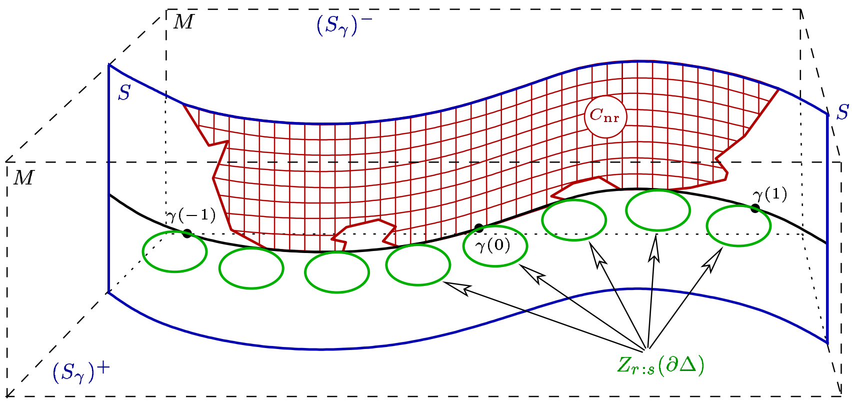

Fig. 2: String of analytic discs attached to $(M_\gamma^1)^+$

Next, we fix a small radius r_0. By deforming the discs $Z_{r_0:s}(\zeta)$ in Ω near their opposite points $Z_{r_0:s}(-1)$, which lie at a positive distance from the singularity C, we construct in Section 4 an extended family of analytic discs $Z_{r_0,t:s}(\zeta)$, where $t \in \mathbb{R}$ is a small parameter, so that the disc boundaries $Z_{r_0,t:s}(\partial\Delta)$ are pivoting tangentially to S at the point $\gamma(s) \equiv Z_{r_0,t:s}(1)$, which is assumed to remain fixed as t varies. Precisely, we mean that $\frac{\partial Z_{r_0,t:s}}{\partial\theta}(1) \in T_{\gamma(s)}S$ and that the mapping $t \longmapsto \frac{\partial Z_{r_0,t:s}}{\partial\theta}(1)$ is of rank 1 at $t = 0$. This construction and the next ones will be achieved thanks to perturbations of the Bishop equation, as in [Tu1994, MP1999]. Furthermore, we add a small parameter $\chi \in \mathbb{R}$ corresponding to vertical translations of the circles along S near γ, getting a family $Z_{r_0,t,\chi:s}(\zeta)$ with the property that the mapping $(\chi, s) \longmapsto Z_{r_0,t,\chi:s}(1) \in S$ is a diffeomorphism onto a neighborhood of $\gamma([-1,1])$ in S, still with the property that the point $Z_{r_0,t,\chi:s}(1)$ is fixed equal to the point $Z_{r_0,0,\chi:s}(1)$ as t varies. Finally, we add a small parameter $\nu \in \mathbb{R}$ with $\nu > 0$ corresponding to horizontal translations of the circles inside $(S_\gamma)^+$, getting a family $Z_{r_0,t,\chi,\nu:s}(\zeta)$ with $Z_{r_0,t,\chi,0:s}(\zeta) \equiv Z_{r_0,t,\chi:s}(\zeta)$, such that the mapping $(\chi,\nu,s) \longmapsto Z_{r_0,t,\chi,\nu:s}(1)$ is a diffeomorphism onto the semi-local one-sided neighborhood $(S_\gamma)^+$ of S along γ in M, provided $\nu > 0$. Then the semi-local attached half-wedge may be defined as

$$\begin{aligned}\mathcal{HW}_\gamma^+ := \Big\{ Z_{r_0,t,\chi,\nu:s}(\rho) :\ & |t| < \varepsilon,\ |\chi| < \varepsilon,\ 0 < \nu < \varepsilon, \\ & 1 - \varepsilon < \rho < 1,\ -1 \leqslant s \leqslant 1 \Big\},\end{aligned} \tag{2.4}$$

for some small $\varepsilon > 0$. In the first main technical step (Section 4), we shall show that every holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically to $\mathcal{HW}_\gamma^+$. Then to prove Proposition 2.2, we must find[3] a special point $p_{\rm sp} \in C$ such that there exists a local one-sided neighborhood $\omega_{p_{\rm sp}}$ at $p_{\rm sp}$ such that holomorphic functions in $\Omega \cup \mathcal{HW}_\gamma^+$ extend holomorphically to $\omega_{p_{\rm sp}}$.

2.5. Field of cones on S. We have to keep memory of the geometric disposal, of the orientation and of the size of $\mathcal{HW}_\gamma^+$. The way how $\mathcal{HW}_\gamma^+$ passes continuously above and under the half hypersurface $(S_\gamma)^+$ $\big($denoted $(M_\gamma^1)^+$ in Figure 8$\big)$ can be read off the full family of analytic discs $Z_{r_0,t,\chi,\nu:s}(\zeta)$.

Thanks to a technical application of the implicit function theorem, we can arrange from the beginning that the vectors $\frac{\partial Z_{r_0,t,\chi,0:s}}{\partial\theta}(1)$ are tangent to S at the point $Z_{r_0,0,\chi,0:s}(1) \in S$ when t varies, for all fixed s. Then by construction, when t varies, the disc boundaries $Z_{r_0,t,\chi,0:s}(\partial\Delta)$ are pivoting tangentially to S at the point $Z_{r_0,t,\chi,0:s}(1) \equiv Z_{r_0,0,\chi,0:s}(1)$. It follows that when t varies, the oriented half-lines $\mathbb{R}^+ \cdot \frac{\partial Z_{r_0,t,\chi,0:s}}{\partial\theta}(1)$ describe an open infinite oriented cone in the tangent space to S at the point $Z_{r_0,0,\chi,0:s}(1)$. Consequently, we may define a *field of cones* $p \mapsto \mathsf{C}_p$ as

$$\mathsf{C}_p := \left\{ \mathbb{R}^+ \cdot \frac{\partial Z_{r_0,t,\chi,0:s}}{\partial\theta}(1) :\ |t| < \varepsilon \right\}, \tag{2.6}$$

at every point $p = Z_{r_0,0,\chi,0:s}(1) \in S$ of a neighborhood of γ in S. The following figure provides an illustration. One should intuitively think that the small cones C_p are generated when the small discs boundaries of Figure 2 pivote tangentially to S.

[3]However, most points $p \in C$ are in fact *not* locally removable. Indeed, the simplest example of a local CR singularity being the intersection of M with some local holomorphic curve Σ, which yields a local real curve $\mu := \Sigma \cap M$, it may well happen that such a curve μ is fully contained in the closed set C, since C which might have nonempty interior in S (as in the figures).

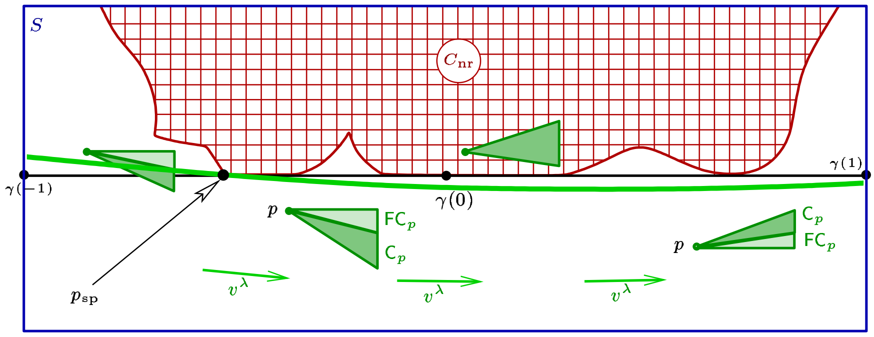

Fig. 3: Field of cones on TS and choice of a special point p_{sp}

After having defined this field of cones, we shall *fill* all the cones as follows. The motivation is to describe precisely what kinds of small Bishop discs half-attached to S (in the sense of Pinchuk) will surely have the other half of their boundaries contained in $\mathcal{HW}_\gamma^+$, so that a version of the continuity principle will be applicable to get Proposition 2.2.

Remind that a neighborhood of γ in S is foliated by characteristic segments, which are approximatively parallel to γ. In Figure 3 above, one should think that the characteristic leaves are all horizontal. So there exists a nowhere vanishing vector field $p \mapsto X_p$ defined in a neighborhood of γ whose integral curves are characteristic segments. We then define the *filled cone* FC_p by

$$\mathsf{FC}_p := \left\{\lambda \cdot X_p + (1-\lambda) \cdot v_p : \ 0 \leqslant \lambda < 1, \ v_p \in \mathsf{C}_p\right\}. \tag{2.7}$$

Geometrically, we rotate every half-line $\mathbb{R}^+ \cdot v_p$ towards the characteristic half-line $\mathbb{R}^+ \cdot X_p$ and we call the result the *filling* of C_p. In Figure 3 above, the cone drawn near $\gamma(0)$ coincides with its filling. Thus we have constructed a field of filled cones $p \longmapsto \mathsf{FC}_p$ over a neighborhood of γ in S.

2.8. Small analytic discs half-attached to S. The next main observation is that small analytic discs which are half-attached to S are essentially contained in the half-wedge $\mathcal{HW}_\gamma^+$, provided that they are approximatively directed by the cone C_p at the corresponding point $p \in S$. More is true: a similar property holds with the filled cone FC_p instead, and this fact will be used in an essential way, since we will need discs close to the characteristic direction.

Let us be more precise. Let $\partial^+\Delta := \{\zeta \in \partial\Delta : \ \mathrm{Re}\,\zeta \geqslant 0\}$ denote the *positive half part* of the unit circle $\partial\Delta$. We say that an analytic disc $A : \overline{\Delta} \to \mathbb{C}^2$ is *half-attached* to S if $A(\partial^+\Delta)$ is contained in S. Here, A is at least of class $\mathcal{C}^1$ over $\overline{\Delta}$ and holomorphic in Δ. In addition, we shall always assume that our discs A are embeddings of $\overline{\Delta}$ into $\mathbb{C}^2$. We shall say that A is *approximatively straight* (in an informal sense) if $A(\Delta)$ is close in $\mathcal{C}^1$-norm to an open subset of the complex line generated by the complex vector $\frac{\partial A}{\partial \zeta}(1)$. Finally, we say that A is *approximatively directed by the filled cone* FC_p *at* $p = A(1)$, if the vector $\frac{\partial A}{\partial \theta}(1) \in T_pS$ belongs to FC_p. Although this terminology will not be re-employed in the next sections, we may formulate a crucial geometric observation as follows.

Lemma 2.9. *A sufficiently small approximatively straight analytic disc $A : \overline{\Delta} \to \mathbb{C}^2$ of class at least $\mathcal{C}^1$ which is half-attached to S and which is approximatively directed* *by the filled cone* FC_p *at* $p = A(1) \in S$*, necessarily satisfies*

$$A\left(\overline{\Delta}\backslash\partial^+\Delta\right) \subset \mathcal{HW}_\gamma^+. \tag{2.10}$$

In the context of the general Proposition 1.13, this property (with more precisions) will be established in Section 8 below. Intuitively, the supplementary freedom offered by the filling FC_p comes from the fact that the half-wedge $\mathcal{HW}_\gamma^+$ is constructed by translating the discs horizontally (toward us in the two above figures) in (S_γ^+), the distribution of horizontal planes being approximatively equal to $T_p^c M$ in the illustrations. In Figure 8, one should think that a vector which varies in a vertical cone drawn there, when it is multiplied by i, will cover the whole aperture of the filled cone FC_p (not only of C_p).

2.11. Choice of a special point. In the second main step of the proof (to be conducted in Section 5 for the general Proposition 1.13), we shall choose the desired special point $p_{\rm sp}$ of Proposition 2.2 to be removed locally as follows. Since we shall use half-attached analytic discs (applying the continuity principle), we want to find a special point $p_{\rm sp} \in C$ so that the following two conditions hold true:

- **(i)** there exists a small approximatively straight analytic disc $A : \overline{\Delta} \to \mathbb{C}^2$ with $A(1) = p_{\rm sp}$ which is half-attached to S such that A is approximatively directed by the filled cone $\mathsf{FC}_{p_{\rm sp}}$ (so that the conclusion of Lemma 2.9 above holds true);
- **(ii)** the same disc satisfies $A\left(\partial^+\Delta\backslash\{1\}\right) \subset S\backslash C$.

In particular, since $M\backslash C$ is contained in Ω, it follows from these two conditions that the (excised) disc boundary $A\left(\partial\Delta\backslash\{1\}\right)$ is contained in the open subset $\Omega \cup \mathcal{HW}_\gamma^+$, a property that will be appropriate for the application of the continuity principle, as we shall explain in Section 9 below.

To fulfill conditions **(i)** and **(ii)** above, we first construct a supporting real segment at a special point of the nonempty closed subset $C \subset S$.

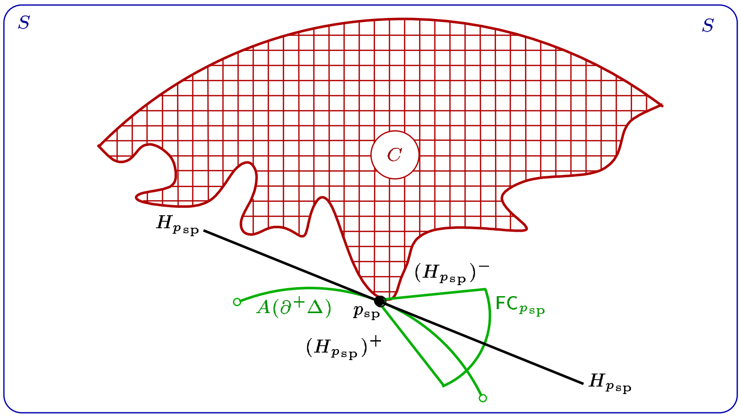

Fig. 4: Half-boundary of a disc directed by a cone and touching $C_{\rm nr}$

Lemma 2.12. *There exists at least one special point* $p_{\rm sp} \in C$ *arbitrarily close to* γ *in a neighborhood of which the following two properties hold true*:

- **(i')** *there exists a small* $\mathcal{C}^{2,\alpha}$ *open segment* $H_{p_{\rm sp}} \subset S$ *passing through* $p_{\rm sp}$ *such that an oriented tangent half-line to* $H_{p_{\rm sp}}$ *at* $p_{\rm sp}$ *is contained in the filled cone* $\mathsf{FC}_{p_{\rm sp}}$*, as illustrated in Figure 4 below;*

(ii') *the same segment is a supporting segment in the following sense*: *locally in a neighborhood of* $p_{\rm sp}$*, the set* $C\backslash\{p_{\rm sp}\}$ *is contained in one open side* $(H_{p_{\rm sp}})^-$ *if* $H_{p_{\rm sp}}$ *in* S.

The way how we prove Lemma 2.12 is illustrated intuitively in Figure 3 above. For $\lambda \in \mathbb{R}$ with $0 \leqslant \lambda < 1$ very close to 1, the vector field $p \longmapsto v_p^\lambda := \lambda \cdot X_p + (1-\lambda)\cdot v_p$ is very close to the characteristic vector field $p \mapsto X_p$. By construction, this vector field runs into the filled field of cones $p \mapsto \mathsf{FC}_p$. In Figure 3, the integral curves of $p \mapsto v_p^\lambda$ are almost horizontal if λ is very close to 1. If we choose the first integral curve (the bold one) from the lower part of Figure 3 which touches C at one special point $p_{\rm sp} \in C$ and if we choose for $H_{p_{\rm sp}}$ a small segment of this first integral curve, we may check that properties **(i')** and **(ii')** are satisfied, modulo some mild technicalities. A rigorous complete proof of Lemma 2.12 will be provided in Section 5 below.

2.13. Construction of analytic discs half-attached to S. Small analytic discs which are half-attached to a $\mathcal{C}^{2,\alpha}$ maximally real submanifold M^1 of $\mathbb{C}^n$ and which are approximatively straight will be constructed in Section 7 below. In fact, it is known ([Pi1974]) that one can prescribe arbitrarily the first order jet of a half-attached disc. However, prescribing $p_{\rm sp} = A(1)$ and $T_{p_{\rm sp}}H_{p_{\rm sp}} = \mathbb{R}\cdot \frac{\partial A}{\partial \theta}(1)$ does not suffices: it may well occur that $A(\partial^+\Delta)$ intersects the singularity C at several other points than $p_{\rm sp}$. Hopefully, $H_{p_{\rm sp}}$ being totally real, we may curve it much in advance in some good holomorphic system of coordinates so that the singularity C lies in the (closure of the) *convex* side of $H_{p_{\rm sp}}$, denoted by $(H_{p_{\rm sp}})^-$ in Figure 4. Thanks to this trick, provided only that the half boundary $A(\partial^+\Delta) \subset S$ is small and tangential to the convex side $(H_{p_{\rm sp}})^-$ at $p_{\rm sp} = A(1)$, it will follow just by a Taylor series argument that $A\big(\partial^+\Delta\backslash\{1\}\big)$ is contained in the good open side $(H_{p_{\rm sp}})^+$ not meeting C. In Figure 4 above, $H_{p_{\rm sp}}$ is straight and $A(\partial^+\Delta)$ is curved, which is equivalent. Thanks to this trick, we avoid having to construct discs with prescribed second order jet. Thus, the two geometric properties **(i')** and **(ii')** satisfied by the real segment $H_{p_{\rm sp}}$ may be realized by the half-boundary of a half-attached analytic disc.

2.14. Translation of half-attached discs and continuity principle. By means of the results of Section 7, we shall see that we may include the disc $A(\zeta)$ in a parametrized family $A_{x,v}(\zeta)$ of analytic discs half-attached to S, where $x \in \mathbb{R}^2$ and $v \in \mathbb{R}$ are small, so that the mapping $x \mapsto A_{x,0}(1) \in S$ is a local diffeomorphism onto a neighborhood of $p_{\rm sp}$ in S and so that the mapping $v \mapsto \frac{\partial A_{0,v}}{\partial \theta}(1)$ is of rank 1 at $v = 0$. Furthermore, we introduce a new parameter $u \in \mathbb{R}$ in order to "translate" the totally real surface S in M by means of a family $S_u \subset M$ with $S_0 = S$ and $S_u \subset (S_\gamma)^+$ for $u > 0$. Thanks to the flexibility of Bishop's equation, we deduce that there exists a deformed family of analytic discs $A_{x,v,u}(\zeta)$ which are half-attached to S_u and which satisfy $A_{x,v,0}(\zeta) \equiv A_{x,v}(\zeta)$. In particular, this family covers a local one-sided neighborhood $\omega_{p_{\rm sp}}$ of M at $p_{\rm sp}$ defined by

$$\omega_{p_{\rm sp}} := \{A_{x,v,u}(\rho) : \ |x| < \varepsilon, \ |v| < \varepsilon, \ |u| < \varepsilon, \ 1-\varepsilon < \rho < 1\}, \tag{2.15}$$

for some $\varepsilon > 0$.

In the third and last main step of the proof (to be conducted in Section 9 below), we shall prove that every disc $A_{x,v,u}(\zeta)$ with $u \neq 0$ is analytically isotopic to a point with the boundary of every disc of the isotopy being contained in $\Omega \cup \mathcal{HW}_\gamma^+$. In fact, for $u \neq 0$, the half-boundary $A_{x,v,u}(\partial^+\Delta)$ is contained in $S_u \subset \Omega$; the other half $A_{x,v,u}(\partial^-\Delta)$ remains stably inside $\mathcal{HW}_\gamma^+$, as was arranged in advance thanks to (2.10) (also for $u = 0$); and when $u > 0$, the whole disc $A_{x,v,u}(\overline{\Delta})$ is contained in $\Omega \cup \mathcal{HW}_\gamma^+$, hence analytically isotopic to a point there (just shrink its radius).

Thanks to the continuity principle, we will deduce that every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_\gamma^+\right)$ extends holomorphically to $\omega_{p_{\rm sp}}$ minus a certain thin closed subset $\mathcal{C}_{p_{\rm sp}}$ of $\omega_{p_{\rm sp}}$. Finally, we shall conclude both the proof of Proposition 2.2 and the proof of Proposition 1.4 by checking that the thin closed set $\mathcal{C}_{p_{\rm sp}}$ is in fact removable for holomorphic functions defined in $\omega_{p_{\rm sp}} \backslash \mathcal{C}_{p_{\rm sp}}$.

2.16. Organization. In Sections 3, 4, 5, 6, 7, 8 and 9, the proof of Proposition 1.13 will be endeavoured directly in arbitrary codimension, without any further reference to the hypersurface version. We point out that the crucial geometric argument which enables us to choose the desired special point will be conducted in the central Section 5 below.

In Section 10, we check that both the CR- and the $L^{\sf p}$-removability of C are a consequence of the $\mathcal{W}$-removability of C. In Section 11, we provide the proofs of Theorem 1.2 and of Corollary 1.5. Section 12 treats the criterion of polynomial convexity stated as Corollary 1.3. Finally, Section 13 proves Theorem 1.9.

2.17. Comparison of the nontransversality assumption with $\mathsf{H}\{K\}$. Firstly, we claim that $\mathsf{H}\{K\}$ implies that K is nontransversal to $\mathsf{F}_{M^1}^c$. Indeed, given a subcompact $K' \subset K$, we look at the compact $\rho(K') \subset V$. Considering a family of spheres of increasing radius centered at some point $r_1 \in V \backslash \rho(K')$ close to $\rho(K')$, we may find a first touched point $q_1 \in \rho(K')$ at which a small spherical cap of the limit sphere constitutes a *local support hypersurface* $N^1 \subset V$ with $q_1 \in N^1$; indeed, the interior of the limit ball being contained in $V \backslash \rho(K')$ by construction, it follows that $\rho(K')$ is situated only in the closed side exterior to the cap N^1. Then $H^1 := \rho^{-1}(N^1)$ constitutes a $\mathcal{C}^{1,\alpha}$ support hypersurface as in Lemma 1.12 which is foliated by characteristic curves whose endpoints lie in ∂U, at a positive distance from K, *q.e.d.*

Example 2.18. We produce an example contradicting the reverse implication.

a) Let $T := \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-dimensional real torus and let $\pi_T : \mathbb{R}^2 \to T$ denote the quotient map. For any slope α, the straight lines $\{y = \alpha x + b\}$ descend to a foliation F_α of T. We fix $\alpha \in (4, 8)$ and we set

$$K_T := \pi_T\big(\big\{(x+t, \alpha\, t) \in \mathbb{R}^2 : \, 1/2 \leqslant x \leqslant 3/4, \, 0 \leqslant t \leqslant 1/\alpha\big\}\big).$$

Geometrically, K_T is a closed parallelogram wrapped in y-direction once around T. Its long sides $\big($of length $\sqrt{1+\alpha^{-2}}\,\big)$ are contained in two leaves and its short sides $\big($of length $1/4\big)$ do meet along a segment of (small) length $1/4 - 1/\alpha > 0$.
Since $4 < \alpha < 8$, the intersections of K_T with the leaves are closed segments of length equal to $\sqrt{1+\alpha^{-2}}$ for $1/2 + 1/4 - 1/\alpha < x \leqslant 3/4$, or equal to $2\sqrt{1+\alpha^{-2}}$ for $1/2 \leqslant x \leqslant 1/2 + 1/4 - 1/\alpha$, as *e.g.* the green bold leaf in the left diagram. Note in particular that K_T does not contain any whole characteristic leaf.

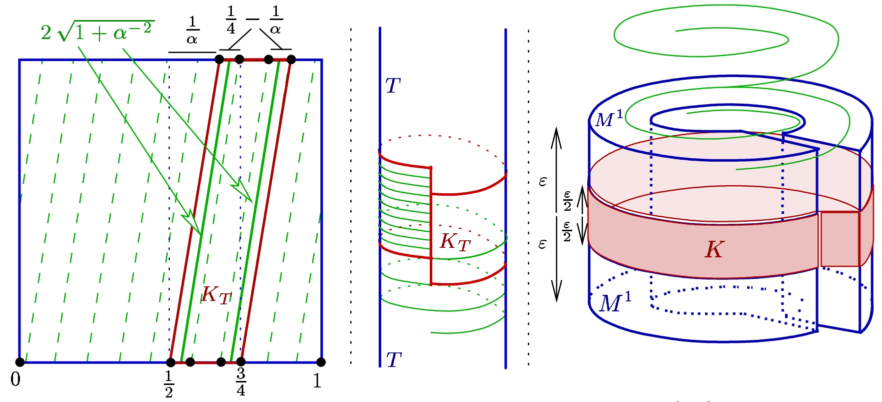

Fig. 5: A nontransversal compact K for which $H\{K\}$ fails

b) As was intended, we point out that for every (connected) neighborhood U_T of K_T in T, the restricted foliation $\mathsf{F}_\alpha\big|_{U_T}$ cannot be parametrized by some submersion $\rho_T : U_T \to V_T$. Indeed, for dimensional reasons, V_T should necessarily be a real interval and then, the restriction of ρ_T to the transversal $\pi_T\big([1/2, 3/4] \times \{0\}\big)$ should be strongly monotonous. But this is impossible, because the green bold leaf intersects twice this transversal at two different points.

c) Unfortunately, the parallelogram K_T is transversal to F_α, since for instance, the circle (of length 1) $\pi_T\big(\{3/4\} \times [0,1]\big) \subset K^T$ occurs to be everywhere transversal. Hopefully, we may increase the dimension by a unity. So we set $\widetilde{M}^1 := T \times (-\varepsilon, \varepsilon)$, and we embed it as a maximally real manifold $M^1 \subset \mathbb{C}^3$ by means of the obvious quotient of the map $\phi : \mathbb{R}^2 \times (-\varepsilon, \varepsilon) \to \mathbb{C}^3$ defined by $\phi(t_1, t_2, t_3) := \big(\exp(2\pi i\, t_1), \exp(2\pi i\, t_2), t_3\big)$. Equipping the fibers $T \times \{t_3\}$ with parallel copies of F_α yields on $\widetilde{M}^1$ a foliation by (quotiented) lines. We define $\mathsf{F}^c_{M^1}$ as its pushforward and we let K be the image of $\widetilde{K} := K_T \times [-\varepsilon/2, \varepsilon/2]$.

d) Let $T\mathsf{F}^c_{M^1} \subset TM^1 \subset T\mathbb{C}^3$ be the bundle of real lines tangent to $\mathsf{F}^c_{M^1}$. Since M^1 is totally real, the bundle $JT\mathsf{F}^c_{M^1} = \bigcup_{p \in M^1} J_p T_p \mathsf{F}^c_{M^1}$ obtained by complex multiplication is nowhere tangent to M^1. We choose a $\mathcal{C}^\infty$ manifold M containing M^1 such that at every $p \in M^1$, T_pM is spanned by T_pM^1 and $J_pT_p\mathsf{F}^c_{M^1}$. By construction, M is generic (provided it is defined to be a sufficiently thin strip along M^1) and the characteristic foliation of M^1 coincides with $\mathsf{F}^c_{M^1}$. Proceeding as in the the proof of Theorem 1.9**(iv)**, we can even arrange that M is of type 4 at every point, hence globally minimal.

e) We claim that $M^1 \backslash K$ is characteristically pseudoconcave. Indeed, let $K' \subset K$ be compact and let $h \in [-\varepsilon/2, \varepsilon/2]$ be maximal such that $\phi(T \times \{h\}) \cap K' \neq \emptyset$. Through any point $p \in \phi(T \times \{h\}) \cap K'$ there passes a compact characteristic segment I whose endpoints are not contained in K'. Translating I upwards (with respect to t_3), we derive the characteristic pseudoconcavity of $M^1 \backslash K$.

f) Finally, we claim that K does not satisfy $\mathsf{H}\{K\}$. Assume on the contrary that there exists a submersion $\rho : U \to V$ as required in $\mathsf{H}\{K\}$. For $\delta > 0$ very small, the δ-neighborhood U_δ of K in M^1 is contained in U. Restricting ρ to $U_\delta \cap \phi(T \cap \{0\})$ and pulling back via ϕ, we obtain a parametrization ρ_T of F_α in a very thin connected neighborhood U_T of K_T with values in a 1-dimensional manifold V_T without boundary. But **b)** already contradicted this.

§3. Strategy per absurdum for the proof of Proposition 1.13

3.1. Preliminary. As in [CSt1994, Me1997, MP1999, MP2002, Po2000], we shall proceed by contradiction. This strategy possesses a considerable advantage: it will not be necessary to control the size of the local subsets of C that are progressively removed, which will simplify substantially the presentation and the understandability of the reasonings. We shall explain how to reduce CR- and $L^{\sf p}$-removability of C to its $\mathcal{W}$-removability. Also, will show that the $\mathcal{W}$-removability of C can be reduced to the simpler case where the functions which we have to extend are even holomorphic in a neighborhood of $M\backslash C$ in $\mathbb{C}^n$. Although such a strategy is essentially carried out in detail in previous references (with some variations), we shall for completeness recall the complete reasonings briefly here, in §3.2 and in §3.16 below.

3.2. Global minimality of $M\backslash C$. Background about CR orbits may be found in [Jö1999a, MP2006]. Using the characteristic nontransversality, we shall apply the following two Lemmas 3.3 and 3.5 about the CR structure of the complement $M\backslash C'$, where $C' \subset C \subset M^1$ is an arbitrary proper closed subset of C.

Lemma 3.3. *Let M be a $\mathcal{C}^{2,\alpha}$ generic submanifold of $\mathbb{C}^n$ $(n \geqslant 2)$ of codimension $(n-1)$ and of CR dimension 1, let M^1 be a $\mathcal{C}^{2,\alpha}$ one-codimensional submanifold of M which is maximally real in $\mathbb{C}^n$ and let C' be an arbitrary proper closed subset of M^1. If C' is nontransversal to the characteristic foliation, then for every point $p' \in C'$, there exists a $\mathcal{C}^{2,\alpha}$ curve $\gamma : [0,1] \to M^1$ satisfying $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M\backslash\{0\}$ at every $s \in [0,1]$, such that $\gamma(0) = p'$ and $\gamma(1)$ does not belong to C'.*

Proof. We proceed by contradiction and we suppose that there exists a point $p' \in C'$ such that all $\mathcal{C}^{2,\alpha}$ curves $\gamma : [0,1] \to M^1$ with $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M\backslash\{0\}$ which have origin p' are entirely contained in C'. It follows immediately that all such γ are contained in a single characteristic leaf, and that the whole leaf is contained in C', contradicting the nontransversality assumption. □

Lemma 3.5. *With M, M^1, C and C' as in the preceding lemma, assume that for every point $q' \in C'$, there exists a $\mathcal{C}^{2,\alpha}$ curve $\gamma : [0,1] \to M^1$ with $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M\backslash\{0\}$ at every $s \in [0,1]$, such that $\gamma(0) = q'$ and $\gamma(1)$ does not belong to C'. Then the CR orbit in $M\backslash C'$ of every point $p \in M\backslash C'$ coincides with its CR orbit in M minus C', namely*

$$\mathcal{O}_{CR}\big(M\backslash C', p\big) = \mathcal{O}_{CR}(M,p)\backslash C'. \tag{3.6}$$

In particular, if M is globally minimal, then $M\backslash C'$ is also globally minimal.

Proof. We formulate a preliminary lemma.

Lemma 3.7. *Under the assumptions of Lemma 3.5, for every point $q' \in C' \subset M^1$, there exists a $\mathcal{C}^{1,\alpha}$ locally embedded submanifold $\Omega_{q'}$ of M passing through q' satisfying $T_{q'}\Omega_{q'} + T_{q'}M^1 = T_{q'}M$, such that*

- **(1)** *$\Omega_{q'}$ is a T^cM-integral submanifold, namely $T^c_pM \subset T_p\Omega_{q'}$, for every point $p \in \Omega_{q'}$;*
- **(2)** *$\Omega_{q'}\backslash C'$ is contained in a single CR orbit of M;*
- **(3)** *$\Omega_{q'}\backslash C'$ is also contained in a single CR orbit of $M\backslash C'$.*

Proof. So, let $q' \in C' \subset M^1$. Since M^1 is generic in $\mathbb{C}^n$, there exists a $\mathcal{C}^{1,\alpha}$ vector field Y defined in a neighborhood of q' which is complex tangential to M and locally transversal to M^1, *see* Figure 6 just below (for easier readability, we have erased the hatching of C' in a neighborhood of q').

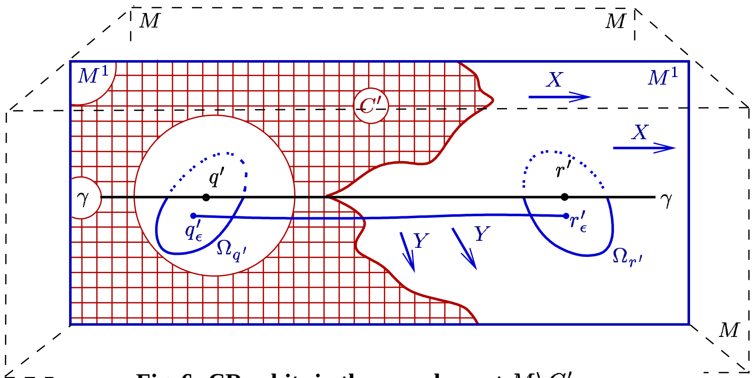

Fig. 6: CR orbits in the complement $M\backslash C'$

Following the integral curve of Y issued from q', we can define a point q'_ϵ in an ϵ-neighborhood of q' which does not belong to M^1. By assumption, there exists a $\mathcal{C}^{2,\alpha}$ curve $\gamma : [0,1] \to M^1$ with $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M\backslash\{0\}$ such that $\gamma(0) = q'$ and $\gamma(1)$ does not belong to C'. Furthermore, there exists a vector field X defined in a neighborhood of $\gamma([0,1])$ in M which is complex tangential to M, whose restriction to M^1 is a semi-local section of $TM^1 \cap T^cM|_{M^1}$, such that γ is an integral curve of X and such that $\gamma(1) = \exp(X)(q') \in M^1\backslash C'$. We can assume that the vector field Y is defined in the same neighborhood of $\gamma([0,1])$ in M and everywhere transversal to M^1. If ϵ is sufficiently small, *i.e.* if q'_ϵ is sufficiently close to q', the point $r'_\epsilon := \exp(X)(q'_\epsilon)$ is still very close to M^1. Thus, we can define a new point $r' \in M^1$ to be the unique intersection with M^1 of the integral curve of Y issued from r'_ϵ. By choosing ϵ small enough, the point r'_ϵ will be arbitrarily close to $\gamma(1) \notin C'$, and consequently, we can assume that r' also does not belong to C', as drawn in Figure 6. Notice that the integral curve of X from q'_ϵ to r'_ϵ is contained in $M\backslash M^1$, since the flow of X stabilizes M^1. We deduce that the two points r'_ϵ and r' belong to the CR orbit $\mathcal{O}_{CR}(M\backslash C', q'_\epsilon)$.

Let $\Omega_{r'}$ denote a small piece of the orbit (an immersed submanifold) $\mathcal{O}_{CR}(M\backslash C', r')$ passing through r'. By standard properties of CR orbits, $\Omega_{r'}$ is an embedded $\mathcal{C}^{1,\alpha}$ submanifold of $M\backslash C'$ of the same CR dimension as $M\backslash C'$. Of course, r'_ϵ belongs to $\Omega_{r'}$. Since Y is complex tangential to M, the submanifold $\Omega_{r'}$ is necessarily stretched along the flow lines of Y, hence it is transversal to M^1.

We then define the submanifold $\exp(-X)(\Omega_{r'})$, close to the point q' (we shall argue in a while that it passes in fact through q'). Since the flow of X stabilizes M^1, it follows that $\exp(-X)(\Omega_{r'})$ is transversal to M^1 and that $\exp(-X)(\Omega_{r'})$ is divided in two parts by its one-codimensional $\mathcal{C}^{1,\alpha}$ submanifold $M^1 \cap \exp(-X)(\Omega_{r'})$. Furthermore, we observe that the flow of X stabilizes the two sides of M^1 in M, semi-locally in a neighborhood of $\gamma([0,1])$, since it stabilizes M^1. Consequently, every integral curve of X issued from every point in $\Omega_{r'}\backslash M^1$ stays in $M\backslash M^1$, hence in $M\backslash C'$ and it follows that the submanifold

$$\exp(-X)(\Omega_{r'})\backslash M^1, \tag{3.8}$$

consisting of two connected pieces, is contained in the single CR orbit $\mathcal{O}_{CR}(M\backslash C', r')$. By the characteristic property of a CR orbit, this means that the two connected pieces of $\exp(-X)(\Omega_{r'})\backslash M^1$ are CR submanifolds of $M\backslash C'$ of the same CR dimension as $M\backslash C'$. Furthermore, since the intersection $M^1 \cap \exp(-X)(\Omega_{r'})$ is one-codimensional, it follows by continuity that *the $\mathcal{C}^{1,\alpha}$ submanifold $\exp(-X)(\Omega_{r'})$ is in fact a CR submanifold of M of the same CR dimension as M*.

Since q'_ϵ belongs to $\exp(-X)(\Omega_{r'})$ and since the flow of the complex tangent vector field Y necessarily stabilizes the T^cM-integral submanifold $\exp(-X)(\Omega_{r'})$, the point q' which belongs to an integral curve of Y issued from q'_ϵ, must belong to the submanifold $\exp(-X)(\Omega_{r'})$, which we can now denote by $\Omega_{q'} := \exp(-X)(\Omega_{r'})$, as in Figure 6. This finishes to prove property **(1)**.

Observe that locally in a neighborhood of q', the integral curves of Y are transversal to M^1 and meet M^1 only at one point. Shrinking if necessary $\Omega_{q'}$ a little bit and using integral curves of Y from both sides of M^1, we attain points in $(M^1\backslash C') \cap \Omega_{q'}$. Hence $\Omega_{q'}\backslash C'$ is contained in the single CR orbit $\mathcal{O}_{CR}(M\backslash C', r')$, which proves property **(3)**. Using again Y to attain points of $C' \cap \Omega_{q'}$, we deduce also that $\Omega_{q'}$ is contained in the single CR orbit $\mathcal{O}_{CR}(M, r')$, which proves property **(2)**.

The proof of Lemma 3.7 is complete. □

We can now prove Lemma 3.5. It suffices to establish that for every two points $p \in M\backslash C'$ and $q \in \mathcal{O}_{CR}(M,p)\backslash C'$, the point q belongs in fact to $\mathcal{O}_{CR}(M\backslash C', p)$.

Since $q \in \mathcal{O}_{CR}(M,p)$, there exists a piecewise $\mathcal{C}^{2,\alpha}$ curve $\lambda : [0,1] \to M$ with $\lambda(0) = p$, $\lambda(1) = q$ and $d\lambda(s)/ds \in T^c_{\lambda(s)}M\backslash\{0\}$ at every $s \in [0,1]$ at which λ is differentiable. For every s with $0 \leqslant s \leqslant 1$, we define a local $\mathcal{C}^{1,\alpha}$ submanifold $\Omega_{\lambda(s)}$ of M passing through $\lambda(s)$ as follows:

- if $\lambda(s)$ does not belong to C', choose for $\Omega_{\lambda(s)}$ a piece of the CR orbit of $\lambda(s)$ in $M\backslash C'$;
- if $\lambda(s)$ belongs to C', choose for $\Omega_{\lambda(s)}$ the submanifold constructed in Lemma 3.7 above.

Then for each s, the complement $\Omega_{\lambda(s)}\backslash C'$ is contained in a single CR orbit of $M\backslash C'$. Since each $\Omega_{\lambda(s)}$ is a T^cM-integral submanifold, a neighborhood of $\lambda(s)$ in the arc $\lambda([0,1])$ is necessarily contained $\Omega_{\gamma(s)}$. By compactness of $[0,1]$, we can therefore find an integer $k \geqslant 1$ and real numbers

(3.9)
$$0 = s_1 < r_1 < t_1 < s_2 < r_2 < t_2 < \cdots\cdots < s_{k-1} < r_{k-1} < t_{k-1} < s_k = 1,$$

such that $\lambda([0,1])$ is covered by $\Omega_{\lambda(0)} \cup \Omega_{\lambda(s_2)} \cup \cdots \cup \Omega_{\lambda(s_{k-1})} \cup \Omega_{\lambda(1)}$ and such that in addition, $\lambda([r_j, t_j]) \subset \Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}$ for $j = 1, \dots, k-1$.

Lemma 3.10. *The following union minus C'*

$$\Big(\Omega_{\lambda(0)} \cup \Omega_{\lambda(s_2)} \cup \cdots\cdots \cup \Omega_{\lambda(s_{k-1})} \cup \Omega_{\lambda(1)}\Big)\backslash C' \tag{3.11}$$

is contained in a single CR orbit of $M\backslash C'$.

Proof. It suffices to prove that for $j = 1, \dots, k-1$, the union $\big(\Omega_{\lambda(s_j)} \cup \Omega_{\lambda(s_{j+1})}\big)\backslash C'$ minus C' is contained in a single CR orbit of $M\backslash C'$.

Two cases are to be considered. Firstly, assume that $\lambda([r_j, t_j])$ is not contained in C', namely there exists u_j with $r_j \leqslant u_j \leqslant t_j$ such that

$$\gamma(u_j) \in \Big(\Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}\Big) \backslash C'. \tag{3.12}$$

Because $\Omega_{\lambda(s_j)} \backslash C'$ and $\Omega_{\lambda(s_{j+1})} \backslash C'$ are both contained in a single CR orbit of $M \backslash C'$, it follows from (3.12) that they are contained in the same CR orbit of $M \backslash C'$, as desired.

Secondly, assume that $\lambda([r_j, t_j])$ is contained in C'. Choose u_j arbitrary with $r_j \leqslant u_j \leqslant t_j$. By construction, $\lambda(u_j)$ belongs to $\Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}$ and both $\Omega_{\lambda(s_j)}$ and $\Omega_{\lambda(s_{j+1})}$ are T^cM-integral submanifolds of M passing through the point $\lambda(u_j)$. Let Y be a local section of T^cM defined in a neighborhood of $\lambda(u_j)$ which is not tangent to M^1 at $\lambda(u_j)$. On the integral curve of Y issued from $\lambda(u_j)$, we can choose a point $\lambda(u_j)_\epsilon$ arbitrarily close to $\lambda(u_j)$ which does not belong to C'. Since Y is a section of T^cM, it is tangent to both $\Omega_{\lambda(s_j)}$ and $\Omega_{\lambda(s_{j+1})}$, hence we deduce that

$$\gamma(u_j)_\epsilon \in \Big(\Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}\Big) \backslash C'. \tag{3.13}$$

Consequently, as in the first case, it follows that $\Omega_{\lambda(s_j)} \backslash C'$ and $\Omega_{\lambda(s_{j+1})} \backslash C'$ are both contained in the same CR orbit of $M \backslash C'$, as desired. □

Since p and q belong to the set (3.11), we deduce that $p = \lambda(0) \in M \backslash C'$ and $q = \lambda(1) \in \mathcal{O}_{CR}(M, p) \backslash C'$ belong to the same CR orbit of $M \backslash C'$, which completes the proof of Lemma 3.5. □

3.14. Reduction of CR- and of L^{p}-removability to $\mathcal{W}$-removability. Thus, in Proposition 1.13, $M \backslash C$ is globally minimal. It follows ([Me1994, Jö1996]) that there exists a wedgelike domain attached to $M \backslash C$ to which $\mathcal{C}^0_{CR}(M)$ extends holomorphically. Consequently, the CR-removability of $C \subset M^1$ claimed in Proposition 1.13 is an immediate consequence of its $\mathcal{W}$-removability. Based on the construction of analytic discs half-attached to M^1 which will be achieved in Section 7, we shall also be able to settle the reduction of L^{p}-removability in Section 10.

Lemma 3.15. *Under the assumptions of Proposition 1.13, if the closed subset $C \subset M^1$ is $\mathcal{W}$-removable, then it is L^{p}-removable, for all p with $1 \leqslant \mathsf{p} \leqslant \infty$.*

3.16. Strategy per absurdum: removal of a single point of the residual nonremovable subset. Thus, it suffices to establish that C is $\mathcal{W}$-removable. Let us fix a (nonempty) wedgelike domain $\mathcal{W}_1$ attached to $M \backslash C$. Our precise goal is to establish that there exists a wedgelike domain $\mathcal{W}_2$ attached to M (including C) and a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_2$ attached to $M \backslash C$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a holomorphic function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$. At first, we need some more definitions.

Let C' be an arbitrary closed subset of C. We shall say that $M \backslash C'$ *enjoys the wedge extension property* if there exist a wedgelike domain $\mathcal{W}_2'$ attached to $M \backslash C'$ and a wedgelike subdomain $\mathcal{W}_3' \subset \mathcal{W}_1 \cap \mathcal{W}_2'$ attached to $M \backslash C$ such that, for every function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a function $F' \in \mathcal{O}(\mathcal{W}_2')$ which coincides with f in $\mathcal{W}_3'$.

The notion of wedge removability can be localized as follows. Let again $C' \subset C$ be arbitrary. We shall say that a point $p' \in C'$ is *locally $\mathcal{W}$-removable with respect to C'* if for every wedgelike domain $\mathcal{W}_1'$ attached to $M \backslash C'$, there exists a neighborhood

U' of p' in M, there exists a wedgelike domain $\mathcal{W}_2'$ attached to $(M\backslash C')\cup U'$ and there exists a wedgelike subdomain $\mathcal{W}_3' \subset \mathcal{W}_1' \cap \mathcal{W}_2'$ attached to $M\backslash C'$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1')$, there exists a holomorphic function $F' \in \mathcal{O}(\mathcal{W}_2')$ which coincides with f in $\mathcal{W}_3'$.

Supppose now that $M\backslash C_1'$ and $M\backslash C_2'$ enjoy the wedge extension property, for some two closed subsets $C_1', C_2' \subset C$. Using the CR edge-of-the-wedge theorem ([Tu1994]), the two wedgelike domains attached to $M\backslash C_1'$ and to $M\backslash C_2'$ can be glued together (after appropriate shrinking) to produce a wedgelike domain $\mathcal{W}_1$ attached to $M\backslash(C_1' \cap C_2')$ in such a way that $M\backslash(C_1' \cap C_2')$ enjoys the $\mathcal{W}$-extension property. Also, if $M\backslash C'$ enjoys the wedge extension property and if $p' \in C'$ is locally $\mathcal{W}$-removable with respect to C', then again by means of the CR edge of the wedge theorem, it follows that there exists a neighborhood U' of p' in M such that $(M\backslash C')\cup U'$ enjoys the wedge extension property.

Based on these preliminary remarks, we define the following set of closed subsets of C:

$$\mathcal{C} := \Big\{ C' \subset C \text{ closed }; \ M\backslash C' \text{ enjoys the } \mathcal{W}\text{-extension property} \Big\}. \tag{3.17}$$

Then the residual set

$$C_{\rm nr} := \bigcap_{C' \in \mathcal{C}} C' \tag{3.18}$$

is a closed subset of M^1 contained in C. It follows from the above (abstract nonsense) considerations that $M\backslash C_{\rm nr}$ enjoys the wedge extension property and that no point of $C_{\rm nr}$ is locally $\mathcal{W}$-removable with respect to $C_{\rm nr}$. Here, we may think that the letters "nr" abbreviate "*n*on-*r*emovable", because by the very definition of $C_{\rm nr}$, none of its points should be locally $\mathcal{W}$-removable. Notice also that $M\backslash C_{\rm nr}$ is globally minimal, thanks to Lemma 3.5.

Clearly, to establish Proposition 1.4, *it is enough to show that* $C_{\rm nr} = \emptyset$.

We shall argue indirectly (by contradiction) and assume that $C_{\rm nr} \neq \emptyset$. In order to derive a contradiction, *it clearly suffices to show that there exists at least one point* $p \in C_{\rm nr}$ *which is in fact locally* $\mathcal{W}$*-removable with respect to* $C_{\rm nr}$.

At this point, we notice that the main assumption that C is nontransversal to $\mathsf{F}_{M^1}^c$ in Proposition 1.13 implies trivially that every closed subset C' of C is also nontransversal to $\mathsf{F}_{M^1}^c$. In particular $C_{\rm nr}$ is nontransversal to $\mathsf{F}_{M^1}^c$. Consequently, by following a *per absurdum* strategy, we are led to prove a statement wich is totally similar to Proposition 1.13 except that we now have only to establish that *a single point of* $C_{\rm nr}$ is locally $\mathcal{W}$-removable with respect to $C_{\rm nr}$. This preliminary logical consideration will simplify substantially the whole architecture of the proof. Another important advantage of this strategy is that we are even allowed to select a special point $p_{\rm sp}$ of $C_{\rm nr}$ by requiring some nice geometric disposition of $C_{\rm nr}$ in a neighborhood of $p_{\rm sp}$ before removing it. Sections 4 and 5 below are devoted to such a selection.

So we are led to show that for every wedgelike domain $\mathcal{W}_1$ attached to $M\backslash C_{\rm nr}$, there exists a special point $p_{\rm sp} \in C_{\rm nr}$, there exists a neighborhood $U_{p_{\rm sp}}$ of $p_{\rm sp}$ in M, there exists a wedgelike domain $\mathcal{W}_2$ attached to $(M\backslash C_{\rm nr}) \cup U_{p_{\rm sp}}$ and there exists a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_2$ attached to $M\backslash C_{\rm nr}$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$.

A further convenient simplification of the task may be achieved by deforming slightly M inside the wedge $\mathcal{W}_1$ attached to $M \backslash C_{\rm nr}$. Indeed, by means of a partition of unity, we may perform arbitrarily small $\mathcal{C}^{2,\alpha}$ deformations M^d of M leaving $C_{\rm nr}$ fixed and moving $M \backslash C_{\rm nr}$ inside the wedgelike domain $\mathcal{W}_1$. Furthermore, we can make M^d to depend on a single small real parameter $d \geqslant 0$ with $M^0 = M$ and $M^d \backslash C_{\rm nr} \subset \mathcal{W}_1$ for all $d > 0$. Now, *the wedgelike domain $\mathcal{W}_1$ becomes a neighborhood of M^d in $\mathbb{C}^n$*. Let us denote by Ω this neighborhood. After some substantial technical work has been performed, at the very end of the proof of Proposition 1.13 (Section 9), we shall construct a local wedge $\mathcal{W}_{p_{\rm sp}}^d$ of edge M^d at $p_{\rm sp}$ by means of small Bishop analytic discs glued to M^d, to Ω and to another subset (which we will call a *half-wedge*, *see* Section 4 below) such that every holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically to $\mathcal{W}_{p_{\rm sp}}^d$. Using the stability of Bishop's equation under perturbations, we shall argue in §9.23 below that all our constructions are stable under such small deformations[4], whence in the limit $d \to 0$, the wedges $\mathcal{W}_{p_{\rm sp}}^d$ tend smoothly to a local wedge $\mathcal{W}_{p_{\rm sp}} := \mathcal{W}_{p_{\rm sp}}^0$ of edge a neighborhood $U_{p_{\rm sp}}$ of $p_{\rm sp}$ in $M^0 \equiv M$. In addition, we shall derive univalent holomorphic extension to $\mathcal{W}_{p_{\rm sp}}$. Finally, using again the edge of the wedge theorem to fill in the space between $\mathcal{W}_1$ and $\mathcal{W}_{p_{\rm sp}}$, possibly after appropriate contractions of these two wedgelike domains, we may construct a wedgelike domain $\mathcal{W}_2$ attached to $(M \backslash C) \cup U_{p_{\rm sp}}$ and a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_{p_{\rm sp}}$ attached to $M \backslash C$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$. In conclusion, we will thus reach the desired contradiction to the definition of $C_{\rm nr}$.

To summarize, we have essentially shown that it suffices to prove Proposition 1.13 with two extra simplifying assumptions.

- Instead of functions which are holomorphic in a wedgelike domain attached to $M \backslash C_{\rm nr}$, we consider functions which are holomorphic in a neighborhood Ω of $M \backslash C_{\rm nr}$ in $\mathbb{C}^n$.
- Proceeding by contradiction, it suffices to remove at least one point of $C_{\rm nr}$.

After replacing $C_{\rm nr}$ by C and M^d by M, we are led to establish the following main assertion, to which Proposition 1.13 is reduced.

Theorem 3.19. *Let M be a $\mathcal{C}^{2,\alpha}$ globally minimal generic submanifold of $\mathbb{C}^n$ of codimension $(n-1)$ hence of CR dimension 1, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$ one-codimensional submanifold which is maximally real in $\mathbb{C}^n$, and let C be a* nonempty *proper closed subset of M^1. Assume that C is nontransversal to the characteristic foliation $\mathsf{F}_{M^1}^c$. Let Ω be an arbitrary neighborhood of $M \backslash C$ in $\mathbb{C}^n$. Then there exist a special point $p_{\rm sp} \in C$, there exists a local wedge $\mathcal{W}_{p_{\rm sp}}$ of edge M at $p_{\rm sp}$ and there exists a subneighborhood $\Omega' \subset \Omega$ of $M \backslash C$ in $\mathbb{C}^n$ with $\mathcal{W}_{p_{\rm sp}} \cap \Omega'$ connected such that for every holomorphic function $f \in \mathcal{O}(\Omega)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{W}_{p_{\rm sp}} \cup \Omega'\right)$ which coincides with f in Ω'.*

[4]We remind from [Me1997, MP1999, MP2002] that in codimension $\geqslant 2$, a wedge of edge a deformation M^d of M does not in general contain a wedge of edge M. This is why stability arguments are needed.

§4. CONSTRUCTION OF A SEMI-LOCAL HALF WEDGE

4.1. Preliminary. Later, in Section 5 below, we will analyze the assumption of characteristic nontransversality, but in the present Section 4, we shall not at all take account of it. With M and M^1 as above, let $\gamma : [-1,1] \to M^1$ be a $\mathcal{C}^{2,\alpha}$ curve, embedding the segment $[-1,1]$ into M, but not necessarily characteristic. In the present section, our goal is to construct a semi-local half-wedge attached to a one-sided neighborhood of M^1 along γ with the property that holomorphic functions in the neighborhood Ω of $M \backslash C$ in $\mathbb{C}^n$ do extend holomorphically to this half-wedge. First of all, we need to define what we understand by the term "half-wedge".

4.2. Three equivalent definitions of attached half-wedges. We shall denote by $\Delta_n(p,\delta)$ the open polydisc centered at $p \in \mathbb{C}^n$ of radius $\delta > 0$. Let $p_1 \in M^1$, and let C_1 be an open infinite cone in the normal space $T_{p_1}\mathbb{C}^n / T_{p_1}M$. Classically, a *local wedge of edge M at p_1* is a set of the form: $\mathcal{W}_{p_1} := \{p + \mathsf{c}_1 : p \in M, \mathsf{c}_1 \in \mathsf{C}_1\} \cap \Delta_n(p_1, \delta_1)$, for some $\delta_1 > 0$. Sometimes, we shall use the following terminology ([Tu1994, Me1994]): if v_1 is a nonzero vector in $T_{p_1}\mathbb{C}^n / T_{p_1}M$, we shall say that $\mathcal{W}_{p_1}$ is a *local wedge at (p_1, v_1)*. Thus, the positive half-line $\mathbb{R}^+ \cdot v_1$ generated by the vector v_1 is locally contained in the wedge $\mathcal{W}_{p_1}$.

For us, a *local half-wedge of edge M at p_1* will be a set of the form

$$\mathcal{HW}^+_{p_1} := \big\{p + \mathsf{c}_1 : p \in U_1 \cap (M^1)^+, \, \mathsf{c}_1 \in \mathsf{C}_1\big\} \cap \Delta_n(p_1, \delta_1). \tag{4.3}$$

This yields a first definition and we shall formulate two further equivalent definitions.

Let Δ denote the unit disc in $\mathbb{C}$, let $\partial\Delta$ denote its boundary, the unit circle and let $\overline{\Delta} = \Delta \cup \partial\Delta$ denote its closure. Throughout this article, we shall denote by $\zeta = \rho\, e^{i\theta}$ the variable of $\overline{\Delta}$ with $0 \leqslant \rho \leqslant 1$ and with $|\theta| \leqslant \pi$.

Concretely, our real local half-wedges (as to be constructed in this section) will be defined by means of a $\mathbb{C}^n$-valued map $(t, \chi, \nu, \rho) \longmapsto \mathcal{Z}_{t,\chi,\nu}(\rho)$ of class $\mathcal{C}^{2,\alpha-0} = \bigcap_{\beta<\alpha} \mathcal{C}^{2,\beta}$ which comes from a parametrized family of analytic discs of the form $\zeta \mapsto \mathcal{Z}_{t,\chi,\nu}(\zeta)$, where the parameters $t \in \mathbb{R}^{n-1}$, $\chi \in \mathbb{R}^n$, $\nu \in \mathbb{R}$ satisfy $|t| < \varepsilon$, $|\chi| < \varepsilon$, $|\nu| < \varepsilon$ for some small $\varepsilon > 0$, and where $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ is holomorphic with respect to ζ in Δ and $\mathcal{C}^{2,\alpha-0}$ in $\overline{\Delta}$. This mapping will satisfy the following three properties:

- **(i)** the map $\chi \mapsto \mathcal{Z}_{0,\chi,0}(1)$ is a diffeomorphism onto a neighborhood of p_1 in M^1, the map $(\chi, \nu) \mapsto \mathcal{Z}_{0,\chi,\nu}(1)$ is a diffeomorphism onto a neighborhood of p_1 in M, and $(M^1)^+$ corresponds to $\nu > 0$;
- **(ii)** $\mathcal{Z}_{t,0,0}(1) = p_1$ and the half-boundary $\mathcal{Z}_{t,\chi,\nu}\left(\left\{e^{i\theta} : |\theta| \leqslant \frac{\pi}{2}\right\}\right)$ is contained in M for all t, all χ and all ν;
- **(iii)** the vector $v_1 := \frac{\partial \mathcal{Z}_{0,0,0}}{\partial\theta}(1) \in T_{p_1}\mathbb{C}^n$ is nonzero and belongs to $T_{p_1}M^1$. Furthermore, the rank of the $\mathbb{R}^{n-1}$-valued $\mathcal{C}^{1,\alpha-0}$ mapping

$$\mathbb{R}^{n-1} \ni t \longmapsto \frac{\partial \mathcal{Z}_{t,0,0}}{\partial\theta}(1) \in T_{p_1}M^1 \text{ mod } \big(T_{p_1}M^1 \cap T^c_{p_1}M\big) \cong \mathbb{R}^{n-1} \tag{4.4}$$

  is maximal equal to $(n-1)$ at $t = 0$.

By holomorphicity of the map $\zeta \mapsto \mathcal{Z}_{t,\chi,\nu}(\zeta)$, we have $\frac{\partial \mathcal{Z}_{t,\chi,\nu}}{\partial\theta}(1) = J \cdot \frac{\partial \mathcal{Z}_{t,\chi,\nu}}{\partial\rho}(1)$, where J denotes the complex structure of $T\mathbb{C}^n$. Since J induces an isomorphism $T_{p_1}M / T^c_{p_1}M \xrightarrow{\sim} T_{p_1}\mathbb{C}^n / T_{p_1}M$, it follows from property **(iii)** above that the vectors

$\frac{\partial \mathcal{Z}_{t,0,0}}{\partial \rho}(1)$ cover an open cone containing Jv_1 in the quotient space $T_{p_1}M/T^c_{p_1}M$, as v varies. Then a *local half-wedge of edge* $(M^1)^+$ *at* p_1 will be a set of the form

$$\mathcal{HW}^+_{p_1} := \big\{\mathcal{Z}_{t,\chi,\nu}(\rho) \in \mathbb{C}^n : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ 0 < \nu < \varepsilon, \ 1-\varepsilon < \rho < 1\big\}. \tag{4.5}$$

We mention that a complete local wedge of edge M at p_1 can also be produced by such a family $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ and may be defined as $\mathcal{W}_{p_1} := \big\{\mathcal{Z}_{t,\chi,\nu}(\rho) : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ |\nu| < \varepsilon, \ 1-\varepsilon < \rho < 1\big\}$, the parameter ν being allowed to be negative (the points $\mathcal{Z}_{0,\chi,\nu}(1)$ then lie behind the "wall" M^1, namely in $(M^1)^-$).

As may be checked, this second definition of a half-wedge is *essentially equivalent* to the first one, in the sense that a half-wedge in the first sense always contains a half-wedge in the second sense, and vice versa, after appropriate shrinkings.

Furthermore, we may distinguish two cases: either the vector $v_1 = \frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ is not complex-tangential to M at p_1 (generically true) or it is. In the first case, after possibly shrinking $\varepsilon > 0$, it may be checked that a local half-wedge of edge $(M^1)^+$ coincides with the intersection of a (full) local wedge $\mathcal{W}_{p_1}$ of edge M at p_1 with a one-sided neighborhood $(N^1)^+$ of a local hypersurface N^1 which intersects M locally transversally along M^1 at p_1, as drawn in the left hand side of the following figure, where M is of codimension two.

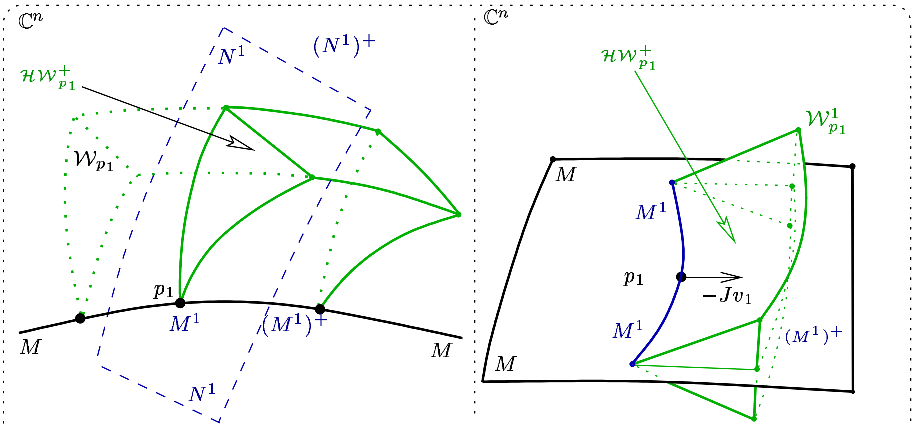

Figure 7: Geometric structure of a local half-wedge

In the second case, $v_1 = \frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ belongs to the characteristic direction $T_{p_1}M^1 \cap T^c_{p_1}M$, so the vector $-Jv_1$ which is interiorly tangent to the disc $\mathcal{Z}_{0,0,0}(\Delta)$, is tangent to M at p_1, is not tangent to M^1 at p_1, but points towards $(M^1)^+$ at p_1. It may then be checked that a local half-wedge of edge $(M^1)^+$ coincides with a local wedge $\mathcal{W}^1_{p_1}$ of edge M^1 at $(p_1, -Jv_1)$ containing the side $(M^1)^+$ in its interior, as drawn in the right hand side of Figure 7 above, in which M is of codimension one. This provides the third and the most intuitive definition of the notion of local half-wedge.

Finally, we may define the desired notion of a semi-local *attached* half-wedge. Let $\gamma : [-1,1] \to M^1$ be an embedded $\mathcal{C}^{2,\alpha}$ segment in M^1. We fix a coherent family of one-sided neighborhoods $(M^1_\gamma)^+$ of M^1 in M along γ. A *half-wedge attached to a one-sided neighborhood* $(M^1_\gamma)^+$ *of* M^1 *along* γ is a domain $\mathcal{HW}^+_\gamma$ which contains a local half-wedge of edge $(M^1)^+$ at $\gamma(s)$ for every $s \in [-1,1]$. Another essentially equivalent definition is to require that we have a family $\mathcal{Z}_{t,\chi,\nu:s}(\rho)$ of maps smoothly varying with the parameter s such that at each point $\gamma(s) = \mathcal{Z}_{t,\chi,\nu:s}(1)$, the three conditions **(i)**, **(ii)** and **(iii)** introduced above to define a local half-wedge are satisfied.

Intuitively speaking, the direction of the cone defining the local half wedge at the point $\gamma(s)$ varies smoothly with respect to s.

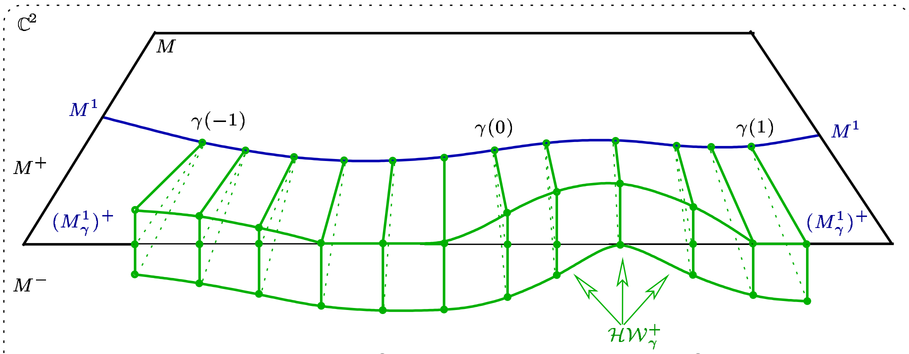

Fig. 8: Semi-local half-wedge attached to a hypersurface

Proposition 4.6. *Let M, M^1, C, Ω be as in Theorem 3.19 and let $\gamma : [-1, 1] \to M^1$ be an embedded $\mathcal{C}^{2,\alpha}$ curve. Then there exist a neighborhood V_γ of $\gamma[-1, 1]$ in M, there exists a semi-local one-sided neighborhood $(M^1_\gamma)^+$ of M^1 in M along γ and there exists a semi-local half-wedge $\mathcal{HW}^+_\gamma$ attached to $(M^1_\gamma)^+ \cap V_\gamma$ with $\Omega \cap \mathcal{HW}^+_\gamma$ connected (shrinking Ω if necessary) such that for every holomorphic function $f \in \mathcal{O}(\Omega)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{HW}^+_\gamma \cup \Omega\right)$ with $F|_\Omega = f$.*

To build $\mathcal{HW}^+_\gamma$, we shall construct families of analytic discs with boundaries in $(M^1_\gamma)^+$. First of all, we need to formulate a special, adapted version of the so-called *approximation theorem* ([BT1981]).

4.7. Local approximation theorem. As observed in [Me1997, MP1999, MP2002], when dealing with natural geometric assumptions on the singularity to be removed — for instance, a two-codimensional singularity $N \subset M$ with $T_pN \supset T^c_pM$ at some points $p \in N$ or metrically thin singularities $E \subset M$ with $\mathsf{H}^{\dim M - 2}(E) = 0$ — it is impossible to show *a priori* that continuous CR functions on M minus the singularity are approximable by polynomials, which justifies the introduction of deformations of M and the use of the continuity principle. But in the present situation, the genericity of M^1 helps much.

Lemma 4.8. *Let $p_1 \in M^1$ and denote by $(M^1)^\pm$ the two sides in which M is divided by M^1 near p_1. Then there exist two neighborhoods U_1 and V_1 of p_1 in M with $V_1 \subset\subset U_1$ such that for every continuous CR function $f \in \mathcal{C}^0_{CR}\left((M^1)^+ \cap U_1\right)$, there exists a sequence of holomorphic polynomials $(P_\nu)_{\nu\in\mathbb{N}}$ wich converges uniformly to f on $(M^1)^+ \cap V_1$.*

Proof. We adapt [BT1981]. In coordinates $z = (z_1, \dots, z_n) = x + iy \in \mathbb{C}^n$ vanishing at p_1, we can assume that the tangent plane to M^1 at p_1 is $\mathbb{R}^n = \{y = 0\}$. We include M^1 in a one-parameter family of maximally real submanifolds $M^1_u \subset M$, where $u \in \mathbb{R}^d$ is small, with $M^1_0 = M^1$, such that $M^1_u \cap V_1$ makes a foliation of $M \cap V^1$, for some neighborhood $V_1 \subset\subset U^1$ of p^1 in M and such that $M^1_u \cap V_1$ is contained in $(M^1)^+$ for $u > 0$. In addition, we can assume that all the M^1_u coincide with M^1_0 in a neighborhood of ∂U_1.

Assume to simplify that the CR function f is of class $\mathcal{C}^1$ on $(M^1)^+ \cap U_1$, let $\tau \in \mathbb{R}$ with $\tau > 0$, fix $u > 0$, whence $M^1_u \cap V^1$ is contained in $(M^1)^+$, let $\widehat{z} \in (M^1)^+ \cap V^1$

be an arbitrary point and consider the following convolution integral of f with the Gaussian kernel:

$$G_\tau f(\widehat{z}) := \left(\frac{\tau}{\pi}\right)^{n/2} \int_{U_1 \cap M^1} e^{-\tau(z-\widehat{z})^2} f(z)\, dz, \tag{4.9}$$

where $(z-\widehat{z})^2 := (z_1-\widehat{z}_1)^2 + \cdots + (z_n - \widehat{z}_n)^2$ and $dz := dz_1 \wedge \cdots \wedge dz_n$. We claim that the value of $G_\tau f(\widehat{z})$ is the same if we replace integration on $U_1 \cap M^1$ by integration on $U_1 \cap M^1_{\widehat{u}}$, where $M^1_{\widehat{u}}$ is the unique maximally real leaf to which $\widehat{z}$ belong. Indeed, the region between M^1 and $M^1_{\widehat{u}}$ is an open diaphragm-like subset $\Sigma \subset M$ whose boundary $\partial\Sigma = M^1 - M^1_{\widehat{t}}$ is entirely contained in $(M^1)^+ \cap U_1$ and then Stokes' theorem gives:

(4.10)

$$\begin{aligned} G_\tau f(\widehat{z}) &= \left(\frac{\tau}{\pi}\right)^{n/2} \int_{U_1 \cap M_{\widehat{t}}} e^{-\tau(z-\widehat{z})^2} f(z)\, dz + \left(\frac{\tau}{\pi}\right)^{n/2} \int_\Sigma d\left(e^{-\tau(z-\widehat{z})^2} f(z)\, dz\right) \\ &= \left(\frac{\tau}{\pi}\right)^{n/2} \int_{U_1 \cap M_{\widehat{t}}} e^{-\tau(z-\widehat{z})^2} f(z)\, dz, \end{aligned}$$

where the second integral vanishes, because f and $e^{-(z-\widehat{z})^2}$ are $\mathcal{C}^1_{CR}$.

Analyzing the real and the imaginary part of the phase function $-\tau(z-\widehat{z})^2$ on $M^1_{\widehat{u}}$, one verifies ([BT1981]) that the integral over $U_1 \cap M^1_{\widehat{u}}$ tends to $f(\widehat{z})$ as τ tends to ∞, provided that the submanifold $U_1 \cap M_{\widehat{z}}$ is sufficiently close to the real plane $\mathbb{R}^n$ in $\mathcal{C}^1$ norm (Gauss' kernel is an approximation of Dirac's measure). Finally, developing in power series, truncating the exponential in the first expression (4.9) which defines $G_\tau f(\widehat{z})$ and integrating termwise, we get a sequence of polynomials $(P_\nu(z))_{\nu \in \mathbb{N}}$. □

4.11. A family of straightenings. Our main goal is to construct a semi-local half-wedge attached to a one-sided neighborhood $(M^1_\gamma)^+$ of M^1 in M along γ, which shall consist of analytic discs attached to $(M^1_\gamma)^+$. First of all, we need a convenient family of normalizations of the local geometries of M and of M^1 along the points $\gamma(s)$ of our characteristic curve γ, for all s with $-1 \leqslant s \leqslant 1$.

Let Ω be a thin neighborhood of $\gamma([-1,1])$ in $\mathbb{C}^n$, say a union of polydiscs of fixed radius centered at the points $\gamma(s)$. Then there exists n real valued $\mathcal{C}^{2,\alpha}$ functions $r_1(z,\bar{z}), \dots, r_n(z,\bar{z})$ defined in Ω such that $M \cap \Omega$ is given by the $(n-1)$ Cartesian equations $r_2(z,\bar{z}) = \cdots = r_n(z,\bar{z}) = 0$ and such that moreover, $M^1 \cap \Omega$ is given by the n Cartesian equations $r_1(z,\bar{z}) = r_2(z,\bar{z}) = \cdots = r_n(z,\bar{z}) = 0$. We first center the coordinates at $\gamma(s)$ by setting $z' := z - \gamma(s)$. Then the defining functions centered at $z' = 0$ become

$$r_j\left(z' + \gamma(s), \bar{z}' + \overline{\gamma(s)}\right) - r_j\left(\gamma(s), \overline{\gamma(s)}\right) =: r'_j(z', \bar{z}' : s), \tag{4.12}$$

for $j = 1, \dots, n$, and they are parametrized by $s \in [-1,1]$. Now, we drop the primes on coordinates and we denote by $r_j(z,\bar{z} : s)$, $j = 1, \dots, n$, the defining equations for the new M_s and M^1_s, which correspond to the old M and M^1 locally in a neighborhood of $\gamma(s)$. Next, we straighten the tangent planes by using the linear change of coordinates $z' = A_s \cdot z$, where the $n \times n$ matrix A_s is defined by $A_s := 2i\left(\frac{\partial r_j}{\partial z_k}(0,0:s)\right)_{1 \leqslant j,k \leqslant n}$. Then the defining equations for the two transformed M'_s

and for $M_s^{1'}$ are given by

$$r'_j(z', \bar{z}' : s) := r_j \left(A_s^{-1} \cdot z', \overline{A}_s^{-1} \cdot \overline{z}' : s \right), \tag{4.13}$$

and we check immediately that the matrix $\left(\frac{\partial r'_j}{\partial z_k}(0, 0 : s) \right)_{1 \leqslant j,k \leqslant n}$ is equal to $2i$ times the $n \times n$ identity matrix, whence $T_0 M'_s = \{y'_2 = \cdots = y'_n = 0\}$ and $T_0 {M'}_s^1 = \{y'_1 = y'_2 = \cdots = y'_n = 0\}$. It is important to notice that the matrix A_s is only $\mathcal{C}^{1,\alpha}$ with respect to s. Consequently, if we now drop the primes on coordinates, the defining equations for M_s and for M_s^1 are of class $\mathcal{C}^{2,\alpha}$ with respect to $(z, \bar{z})$ and only of class $\mathcal{C}^{1,\alpha}$ with respect to s.

Applying then the $\mathcal{C}^{2,\alpha}$ implicit function theorem, we deduce that there exist $(n-1)$ functions $\varphi_j(x, y_1 : s)$, $j = 2, \ldots, n$, which are all of class $\mathcal{C}^{2,\alpha}$ with respect to (x, y_1) in a real cube $\mathbb{I}_{n+1}(2\rho_1) := \big\{(x, y_1) \in \mathbb{R}^n \times \mathbb{R} : \, |x| < 2\rho_1, \, |y_1| < 2\rho_1\big\}$, for some $\rho_1 > 0$, which are uniformly bounded in $\mathcal{C}^{2,\alpha}$-norm as the parameter s varies in $[-1, 1]$, which are of class $\mathcal{C}^{1,\alpha}$ with respect to s, such that M_s may be represented in the polydisc $\Delta_n(\rho_1)$ by the $(n-1)$ graphed equations

$$y_2 = \varphi_2(x, y_1 : s), \ldots\ldots, \;\; y_n = \varphi_n(x, y_1 : s), \tag{4.14}$$

or more concisely by $y' = \varphi'(x, y_1 : s)$, if we denote the coordinates $(z_2, \ldots, z_n)$ simply by $z' = x' + iy'$. Here, by construction, we have the normalization conditions $\varphi_j(0 : s) = \partial_{x_k}\varphi_j(0 : s) = \partial_{y_1}\varphi_j(0 : s) = 0$, for $j = 2, \ldots, n$ and $k = 1, \ldots, n$. Sometimes in the sequel, we shall use the notation $\varphi_j(z_1, x' : s)$ instead of $\varphi_j(x, y_1 : s)$. Similarly, again by means of the implicit function theorem, we obtain n functions $h_k(x : s)$, for $k = 1, \ldots, n$, which are of class $\mathcal{C}^{2,\alpha}$ in the cube $\mathbb{I}_n(2\rho_1)$ (after possibly shrinking ρ_1) enjoying the same regularity property with respect to s, such that M_s^1 is represented in the polydisc $\Delta_n(\rho_1)$ by the n graphed equations

$$y_1 = h_1(x : s), \; y_2 = h_2(x : s), \; \ldots\ldots, \; y_n = h_n(x : s). \tag{4.15}$$

In addition, we can assume that

$$h_j(x : s) \equiv \varphi_j\left(x, h_1(x : s) : s\right), \qquad j = 2, \ldots, n. \tag{4.16}$$

Here, by construction, we have the normalization conditions $h_k(0 : s) = \partial_{x_l} h_k(0 : s) = 0$ for $k, l = 1, \ldots, n$.

In the sequel, we shall denote by $\widehat{z} = \Phi_s(z)$ the final change of coordinates which is centered at $\gamma(s)$ and which straightens simultaneously the tangent planes to M at $\gamma(s)$ and to M^1 at $\gamma(s)$ and we shall denote by M_s and by M_s^1 the transformations of M and of M^1.

Also, we must remind that the following regularity properties hold for the functions $\varphi_j(x, y_1 : s)$ and $h_k(x : s)$.

- **(a)** For fixed s, they are of class $\mathcal{C}^{2,\alpha}$ with respect to their principal variables, namely excluding the parameter s.
- **(b)** They are of class $\mathcal{C}^{1,\alpha}$ with respect to all their variables, including the parameter s.
- **(c)** Each of their first order partial derivative with respect to one of their principal variables is of class $\mathcal{C}^{1,\alpha}$ with respect to all their variables, including the parameter s.

Indeed, these properties are clearly satisfied for the functions (4.13) and they are inherited after the two applications of the implicit function theorem which yielded the functions $\varphi_j(x, y_1 : s)$ and $h_k(x : s)$.

4.17. Contact of a small "round" analytic disc with M^1. Let $r \in \mathbb{R}$ with $0 \leqslant r \leqslant r_1$, where r_1 is small in comparison with ρ_1. Then the "round" analytic disc $\overline{\Delta} \ni \zeta \to \widehat{Z}_{1;r}(\zeta) := ir(1-\zeta) \in \mathbb{C}$ with values in the complex plane equipped with the coordinate $z_1 = x_1 + iy_1$ is centered at the point ir of the y_1-axis, is of radius r and is contained in the open upper half plane $\{z_1 \in \mathbb{C} : y_1 > 0\}$, except its boundary point $\widehat{Z}_{1;r}(1) = 0$. In addition, the tangent direction $\frac{\partial}{\partial\theta}\widehat{Z}_{1;r}(1) = r$ is directed along the positive x_1-axis, *see* in advance Figure 9 below.

We denote by T_1 the Hilbert transform[5] on $\partial\Delta$ vanishing at 1, namely $(T_1 X)(1) = 0$, whence $T_1(T_1(X)) = -X + X(1)$. Thanks to a standard processus, we may lift this scalar disc $ir(1-\zeta)$ as disc attached to M of the form

$$\widehat{Z}_{r:s}(\zeta) = \big(ir(1-\zeta), \widehat{Z}'_{r:s}(\zeta)\big) \in \mathbb{C} \times \mathbb{C}^{n-1}, \tag{4.18}$$

where the real part $\widehat{X}'_{r:s}(\zeta)$ of $\widehat{Z}'_{r:s}(\zeta)$ satisfies the following Bishop-type equation on $\partial\Delta$

$$\widehat{X}'_{r:s}(\zeta) = -\Big[T_1\varphi'\big(\widehat{Z}_{1;r}(\cdot), \widehat{X}'_{r:s}(\cdot) : s\big)\Big](\zeta), \qquad \zeta \in \partial\Delta. \tag{4.19}$$

By [Tu1994, Tu1996, MP2006], if r_1 is sufficiently small, there exists a solution which is $\mathcal{C}^{2,\alpha-0}$ with respect to (r, ζ), but only $\mathcal{C}^{1,\alpha-0}$ with respect to (r, ζ, s). Notice that for $r = 0$, the disc $\widehat{Z}_{1;0}(e^{i\theta})$ is constant equal to 0 and by uniqueness of the solution of (4.19), it follows that $\widehat{Z}'_{0:s}\big(e^{i\theta}\big) \equiv 0$. It follows trivially that $\partial_\theta \widehat{X}_{0:s}\big(e^{i\theta}\big) \equiv 0$ and that $\partial_\theta\partial_\theta \widehat{X}_{0:s}\big(e^{i\theta}\big) \equiv 0$, which will be used in a while. Notice also that $\widehat{X}_{r:s}(1) = 0$ for all r and all s.

On the other hand, since by assumption, we have $h_1(0 : s) = 0$ and $\partial_{x_k} h_1(0 : s) = 0$ for $k = 1, \dots, n$, it follows from the chain rule that if we set

$$F(r, \theta : s) := h_1\big(\widehat{X}_{r:s}\big(e^{i\theta}\big) : s\big) \tag{4.20}$$

where θ satisfies $0 \leqslant |\theta| \leqslant \pi$, then the following four equations hold

$$F(0, \theta : s) \equiv 0, \qquad F(r, 0 : s) \equiv 0, \qquad \partial_\theta F(r, 0 : s) \equiv 0, \qquad \partial_\theta F(0, \theta : s) \equiv 0. \tag{4.21}$$

We deduce that there exists a constant $C > 0$ such that the following five inequalities hold for $0 \leqslant |\theta| \leqslant \pi$, for $0 \leqslant r \leqslant r_1$, for $s \in [-1, 1]$ and for $|x| \leqslant \rho_1$:

$$\begin{cases} \big|\widehat{X}_{r:s}\big(e^{i\theta}\big)\big| \leqslant C \cdot r, \\ \big|\partial_\theta \widehat{X}_{r:s}\big(e^{i\theta}\big)\big| \leqslant C \cdot r, \\ \big|\partial_\theta\partial_\theta \widehat{X}_{r:s}\big(e^{i\theta}\big)\big| \leqslant C \cdot r^{\frac{\alpha}{2}}, \\ \sum_{k=1}^{n} \big|\partial_{x_k} h_1(x)\big| \leqslant C \cdot |x|, \\ \sum_{k_1,k_2=1}^{n} \big|\partial_{x_{k_1}}\partial_{x_{k_2}} h_1(x)\big| \leqslant C. \end{cases} \tag{4.22}$$

[5] Complete, self-contained background is provided in [MP2006].

As in Lemma 6.4 (*see* below), the third inequality comes from $\partial_\theta \partial_\theta \widehat{X}_{0:s}\big(e^{i\theta}\big) \equiv 0$ and $\widehat{X}_{r:s}\big(e^{i\theta}\big) \in \mathcal{C}^{2,\alpha/2}$.

Computing now the second derivative of $F(r, \theta : s)$ with respect to θ, we obtain

(4.23)
$$\begin{aligned}\partial_\theta \partial_\theta F(r,\theta : s) = & \sum_{k=1}^n \partial_{x_k} h_1\big(\widehat{X}_{r:s}\big(e^{i\theta}\big) : s\big) \cdot \partial_\theta \partial_\theta \widehat{X}_{k;r:s}\big(e^{i\theta}\big) + \\ & + \sum_{k_1,k_2=1}^n \partial_{x_{k_1}} \partial_{x_{k_2}} h_1\big(\widehat{X}_{r:s}\big(e^{i\theta}\big)\big) \cdot \partial_\theta \widehat{X}_{k_1;r,s}\big(e^{i\theta}\big) \cdot \partial_\theta \widehat{X}_{k_2;r,s}\big(e^{i\theta}\big),\end{aligned}$$

and we may apply the majorations (4.22) to get

(4.24)
$$\begin{aligned}\big|\partial_\theta \partial_\theta F(r,\theta : s)\big| & \leqslant C \cdot \big|\widehat{X}_{r:s}(e^{i\theta})\big| \cdot C \cdot r^{\frac{\alpha}{2}} + C \cdot (C \cdot r)^2 \\ & \leqslant r \cdot C^3 \left[r^{\frac{\alpha}{2}} + r^2\right].\end{aligned}$$

Lemma 4.25. *If* $r_1 \leqslant \min\big(1, \ \big(\frac{1}{4C^3\pi^2}\big)^{\frac{2}{\alpha}}\big)$, *then* $\widehat{Z}_{r:s}\big(\partial\Delta\backslash\{1\}\big)$ *is contained in* $(M_s^1)^+$ *for all* r *with* $0 < r \leqslant r_1$ *and all* s *with* $-1 \leqslant s \leqslant 1$.

Proof. In the polydisc $\Delta_n(\rho_1)$, the positive half-side $(M_s^1)^+$ in M is represented by the single equation $y_1 > h_1(x : s)$, hence we have to check that $\widehat{Y}_{1;r}\big(e^{i\theta}\big) > \big|h_1\big(\widehat{X}_{r:s}\big(e^{i\theta}\big) : s\big)\big|$, for all θ with $0 < |\theta| \leqslant \pi$.

The y_1-component $\widehat{Y}_{1;r}\big(e^{i\theta}\big)$ of $\widehat{Z}_{r:s}\big(e^{i\theta}\big)$ is equal to $r\big(1 - \cos\theta\big)$. We have the elementary minoration $r(1-\cos\theta) \geqslant r \cdot \theta^2 \cdot \frac{1}{\pi^2}$, valuable for $0 \leqslant |\theta| \leqslant \pi$. Also, taking account of the second and of the fourth relations (4.21), Taylor's integral formula yields

(4.26)
$$F(r,\theta : s) = \int_0^\theta (\theta - \theta') \cdot \partial_\theta \partial_\theta F\big(r, \theta' : s\big) \cdot d\theta'.$$

Observing that $r^2 \leqslant r^{\frac{\alpha}{2}}$, since $0 < r \leqslant r_1 \leqslant 1$, and using the majoration (4.24), we may estimate, taking account of the assumption on r_1 written in the statement of the lemma:

(4.27)
$$\big|F(r,\theta : s)\big| \leqslant r \cdot \frac{\theta^2}{2} \cdot C^3 [2r^{\frac{\alpha}{2}}] \leqslant r \cdot \theta^2 \cdot \frac{1}{4\pi^2}.$$

This yields the desired inequality $r(1 - \cos\theta) > \big|F(r, \theta : s)\big|$. □

We now fix once for all a radius r_0 with $0 < r_0 \leqslant r_1$. In the remainder of the present Section 4, we shall deform the disc $\widehat{Z}_{r_0:s}(\zeta)$ by adding many more parameters. We notice that for all θ with $0 \leqslant |\theta| \leqslant \frac{\pi}{4}$, we have the trivial minoration $\partial_\theta \partial_\theta \widehat{Y}_{1;r_0}\big(e^{i\theta}\big) = r_0 \cos\theta \geqslant \frac{r_0}{\sqrt{2}}$. Also, by (4.24) and by the inequality on r_1 written in Lemma 4.25, we deduce $\big|\partial_\theta \partial_\theta h_1\big(\widehat{X}_{r_0:s}\big(e^{i\theta}\big)\big)\big| \leqslant \frac{r_0}{2\pi^2}$ for all θ with $0 \leqslant |\theta| \leqslant \pi$. Since we shall need a generalization of Lemma 4.25 in Lemma 4.51 below, let us remember these two interesting inequalities, valid for $0 \leqslant |\theta| \leqslant \frac{\pi}{4}$:

(4.28)
$$\big|\partial_\theta \partial_\theta h_1\big(\widehat{X}_{r_0:s}\big(e^{i\theta}\big)\big)\big| \leqslant \frac{r_0}{2\pi^2} \ < \ \frac{r_0}{\sqrt{2}} \leqslant \partial_\theta \partial_\theta \widehat{Y}_{1;r_0}\big(e^{i\theta}\big).$$

4.29. Normal deformations of the disc $\widehat{Z}_{r:s}(\zeta)$. So, we fix r_0 small with $0 < r_0 \leqslant r_1$ and we consider the disc $\widehat{Z}_{r_0:s}(\zeta)$ for $\zeta \in \overline{\Delta}$. Then the point $\widehat{Z}_{r_0:s}(-1)$ belongs to $(M_s^1)^+$ for each s and stays at a positive distance from M_s^1 as s varies in $[-1,1]$. It follows that we can choose a subneighborhood ω_s of $\widehat{Z}_{r_0:s}(-1)$ in $\mathbb{C}^n$ which is contained in Ω and whose diameter is uniformly bounded from below.

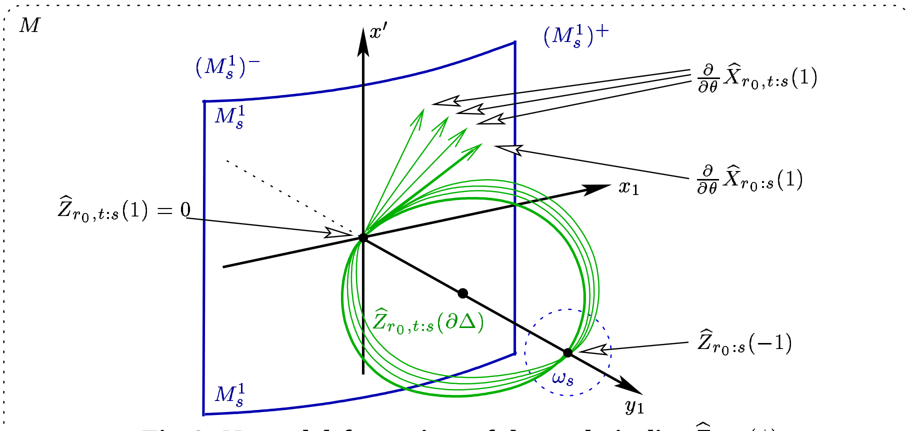

Fig. 9: Normal deformations of the analytic disc $\widehat{Z}_{r_0:s}(\zeta)$

Following [Tu1994, MP2006], we introduce *normal deformations* of the analytic discs $\widehat{Z}_{r_0:s}(\zeta)$. Let $\kappa : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a $\mathcal{C}^{2,\alpha}$ mapping fixing the origin and satisfying $\partial_{x_k}\kappa_j(0) = \delta_k^j$ (Kronecker's symbol). For $j = 2, \dots, n$, let $\eta_j = \eta_j(z_1, x' : s)$ be a real-valued $\mathcal{C}^{2,\alpha}$ function compactly supported in a neighborhood of the point of $\mathbb{R}^{n+1}$ with coordinates $\big(\widehat{Z}_{1;r_0:s}(-1), \widehat{X}'_{r_0:s}(-1)\big)$ and equal to 1 at this point. We then define the $\mathcal{C}^{2,\alpha}$ deformed generic submanifold $M_{s,t}$ of equations

$$\begin{aligned} y' &= \varphi'(z_1, x' : s) + \kappa(t) \cdot \eta'(z_1, x' : s) \\ &=: \Phi'(z_1, x', t : s). \end{aligned} \tag{4.30}$$

Notice that $M_{s,0} \equiv M_s$ and that $M_{s,t}$ coincides with M_s in a small neighborhood of the origin, for all t. If $\mu = \mu(e^{i\theta} : s)$ is a real-valued nonnegative $\mathcal{C}^{2,\alpha}$ function defined for $e^{i\theta} \in \partial\Delta$ and for $s \in [-1,1]$ whose support is concentrated near the segment $\{-1\} \times [-1,1]$, then ([Tu1996, MP2006]), for each fixed $s \in [-1,1]$, there exists a $\mathcal{C}^{2,\alpha-0}$ solution of the Bishop-type equation

$$\widehat{X}'_{r_0,t:s}(e^{i\theta}) = -\Big[T_1\Phi'\big(\widehat{Z}_{1;r_0:s}(\cdot), \widehat{X}'_{r_0,t:s}(\cdot), t\mu(\cdot : s) : s\big)\Big](e^{i\theta}), \tag{4.31}$$

which produces the family of analytic discs

$$\widehat{Z}_{r_0,t:s}(e^{i\theta}) := \Big(\widehat{Z}_{1;r_0:s}(e^{i\theta}), \widehat{X}'_{r_0,t:s}(e^{i\theta}) + iT_1\big[\widehat{X}'_{r_0,t:s}(\cdot)\big](e^{i\theta})\Big) \tag{4.32}$$

having boundaries contained in $M \cup \omega_s$. Taking account of the regularity properties **(a)**, **(b)** and **(c)** stated after (4.16), the general solution $\widehat{Z}_{r_0,t:s}(\zeta)$ enjoys similar regularity properties.

- **(a)** For fixed s, it is of class $\mathcal{C}^{2,\alpha-0}$ with respect to (t,ζ).
- **(b)** It is of class $\mathcal{C}^{1,\alpha-0}$ with respect to all the variables (t,ζ,s).
- **(c)** Each of its first order partial derivative with respect to the principal variables (t,ζ) is of class $\mathcal{C}^{1,\alpha-0}$ with respect to all the variables (t,ζ,s).

Since the solution is $\mathcal{C}^{1,\alpha-0}$ with respect to s, it crucially follows that the vector

$$v_{1:s} := -\frac{\partial \widehat{Z}_{r_0,t:s}}{\partial \rho}(1), \tag{4.33}$$

which points inside the analytic disc, varies continuously with respect to s. The next key proposition may be established as in [Tu1994, MP1999], taking account of the uniformity with respect to s.

Lemma 4.34. *There exists a real-valued nonnegative $\mathcal{C}^{2,\alpha}$ function $\mu = \mu(e^{i\theta} : s)$ defined for $e^{i\theta} \in \partial\Delta$ and $s \in [-1,1]$ whose support is concentrated near $\{-1\} \times [-1,1]$ such that the mapping*

$$\mathbb{R}^{n-1} \ni t \longmapsto \frac{\partial \widehat{X}'_{r_0,t:s}}{\partial \theta}\big(e^{i\theta}\big)\Big|_{\theta=0} \in \mathbb{R}^{n-1} \tag{4.35}$$

is maximal equal to $(n-1)$ at $t=0$.

Geometrically speaking, since the vector $\frac{\partial \widehat{X}_{1;r_0:s}}{\partial\theta}\big(e^{i\theta}\big)\big|_{\theta=0}$ is nonzero, it follows that when the parameter t varies, the set of lines generated by the vectors $\frac{\partial \widehat{X}_{r_0,t:s}}{\partial\theta}\big(e^{i\theta}\big)\big|_{\theta=0}$ covers an open cone in the space $T_{p_1}M^1 \equiv \mathbb{R}^n$ equipped with coordinates (x_1, x'), *see* again Figure 9 above for an illustration.

4.36. Adding pivoting and translation parameters. Let $\chi = (\chi_1, \chi') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\nu \in \mathbb{R}$ satisfying $|\chi| < \varepsilon$ and $|\nu| < \varepsilon$ for some small $\varepsilon > 0$. Then the mapping

$$\begin{aligned} \mathbb{R}^{n+1} \ni (\chi_1, \chi', \nu) \longmapsto \big(\, & \chi_1 + i[h_1(\chi : s) + \nu], \chi' + i\varphi'\big(\chi, h_1(\chi : s) + \nu : s\big) \,\big) \\ & =: \widehat{p}(\chi, \nu : s) \in M_s \end{aligned} \tag{4.37}$$

is a $\mathcal{C}^{2,\alpha}$ diffeomorphism onto a neighborhood of the origin in M_s with:

- **(a)** $\nu > 0$ if and only if $\widehat{p}(\chi, \nu : s) \in (M_s^1)^+$;
- **(b)** $\nu = 0$ if and only if $\widehat{p}(\chi, \nu : s) \in M_s^1$;
- **(c)** $\nu < 0$ if and only if $\widehat{p}(\chi, \nu : s) \in (M_s^1)^-$.

If $\tau \in \mathbb{R}$ with $|\tau| < \varepsilon$ is a supplementary parameter, we now define a crucial deformation of the first component $\widehat{Z}_{1;r_0:s}\big(e^{i\theta}\big)$ by setting

$$\widehat{Z}_{1;r_0,\tau,\chi,\nu:s}\big(e^{i\theta}\big) := ir_0\big(1 - e^{i\theta}\big)[1 + i\tau] + \chi_1 + i[h_1(\chi : s) + \nu]. \tag{4.38}$$

Of course, we have $\widehat{Z}_{1;r_0,0,0,0:s}\big(e^{i\theta}\big) \equiv \widehat{Z}_{1;r_0:s}\big(e^{i\theta}\big)$. Geometrically speaking, this perturbation corresponds to add firstly a small "rotation parameter" τ which rotates (and slightly dilates) the disc $ir_0\big(1 - e^{i\theta}\big)$ passing through the origin in $\mathbb{C}_{z_1}$, to add secondly a small "translation parameter" (χ_1, χ') which will enable to cover a neighborhood of the origin in M_s^1 and to add thirdly a small translation parameter ν along the y_1-axis. Consequently, with this first $\mathbb{C}$-valued component $\widehat{Z}_{1;r_0,\tau,\chi,\nu:s}\big(e^{i\theta}\big)$, we can construct a $\mathbb{C}^n$-valued analytic disc $\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}(\zeta)$ satisfying

$$\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}(1) = \widehat{p}(\chi, \nu : s), \tag{4.39}$$

simply by solving the perturbed Bishop-type equation which extends (4.31)

$$\widehat{X}'_{r_0,t,\tau,\chi,\nu:s}\big(e^{i\theta}\big) = -\Big[T_1\big(\Phi'\big(\widehat{Z}_{1;r_0,\tau,\chi,\nu:s}(\cdot), \widehat{X}'_{r_0,t,\tau,\chi,\nu:s}(\cdot), t\mu(\cdot : s) : s\big)\big)\Big]\,\big(e^{i\theta}\big). \tag{4.40}$$

Of course, thanks to the sympathetic stability of Bishop's equation under perturbation, the solution exists and satisfies smoothness properties entirely similar to the ones stated after (4.32). To summarize, we list the seven variables upon which our final family of analytic discs depends.

(4.41)
$$\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}(\zeta): \quad \begin{cases} r_0 = \text{approximate radius.} \\ t = \text{normal deformation parameter.} \\ \tau = \text{pivoting parameter.} \\ \chi = \text{parameter of translation along } M^1. \\ \nu = \text{parameter of translation in } M \text{ transversally to } M^1. \\ s = \text{parameter of the characteristic curve } \gamma. \\ \zeta = \text{unit disc variable.} \end{cases}$$

For every t and every χ, we now want to adjust the pivoting parameter τ in order that the disc boundary $\widehat{Z}_{r_0,t,\tau,\chi,0:s}(e^{i\theta})$ for $\nu = 0$ is tangent to M_s^1. This tangency condition will be useful in order to derive the crucial Lemma 4.51 below.

Lemma 4.42. *Shrinking ε if necessary, there exists a unique $\mathcal{C}^{1,\alpha-0}$ map $(t,\chi,s) \mapsto \tau(t,\chi:s)$ defined for $|t| < \varepsilon$, for $|\chi| < \varepsilon$ and for $s \in [-1,1]$ satisfying $\tau(0,0:s) = \partial_{t_j}\tau(0,0:s) = \partial_{\chi_k}\tau(0,0:s) = 0$ for $j = 1,\dots,n-1$ and $k = 1,\dots,n$, such that the vector*

$$\frac{\partial}{\partial\theta}\Big|_{\theta=0} \widehat{Z}_{r_0,t,\tau(t,\chi:s),\chi,0:s}(e^{i\theta}) \tag{4.43}$$

is tangent to M_s^1 at the point $\widehat{Z}_{r_0,t,\tau(t,\chi:s),\chi,0:s}(1) = \widehat{p}(\chi,0:s) \in M_s^1$.

Proof. We remind that M_s is represented by the $(n-1)$ scalar equations $y' = \varphi'(x,y_1:s)$ and that M_s^1 is represented by the n equations $y_1 = h_1(x:s)$ and $y' = \varphi'(x,h_1(x:s):s) \equiv h'(x':s)$. We can therefore compute the Cartesian equations of the tangent plane to M_s^1 at the point $\widehat{p}(\chi,0:s) = \chi + ih(\chi:s)$:

$$\begin{cases} \mathsf{Y}_1 - h_1(\chi:s) = \sum_{k=1}^n \partial_{x_k} h_1(\chi:s)\,[\mathsf{X}_x - \chi_k], \\ \mathsf{Y}' - \varphi'(\chi,h_1(\chi:s):s) = \sum_{k=1}^n \left(\partial_{x_k}\varphi' + \partial_{y_1}\varphi' \cdot \partial_{x_k} h_1\right)[\mathsf{X}_k - \chi_k]. \end{cases} \tag{4.44}$$

On the other hand, we observe that the tangent vector

$$\frac{\partial}{\partial\theta}\Big|_{\theta=0} \widehat{Z}_{r_0,t,\tau,\chi,0:s}(e^{i\theta}) = \left(r_0[1+i\tau],\ \frac{\partial}{\partial\theta}\Big|_{\theta=0} \widehat{Z}'_{r_0,t,\tau,\chi,0:s}(e^{i\theta})\right) \tag{4.45}$$

is already tangent to M_s at the point $\widehat{p}(\chi,0:s)$, because $M_{s,t} \equiv M_s$ in a neighborhood of the origin. More precisely, since $\Phi' \equiv \varphi'$ in a neighborhood of the origin, we may differentiate with respect to θ at $\theta = 0$ the relation

$$\widehat{Y}'_{r_0,t,\tau,\chi,0:s}(e^{i\theta}) \equiv \varphi'\left(\widehat{X}_{r_0,\tau,\chi,0:s}(e^{i\theta}), \widehat{Y}_{1;r_0,\tau,\chi,0:s}(e^{i\theta}) : s\right) \tag{4.46}$$

which is valid for $|\theta| \leqslant \frac{\pi}{2}$, noticing in advance that it follows immediately from (4.38) that

$$\frac{\partial}{\partial\theta}\Big|_{\theta=0} \widehat{X}_{1;r_0,\tau,\chi,0:s}(e^{i\theta}) = r_0 \quad \text{and} \quad \frac{\partial}{\partial\theta}\Big|_{\theta=0} \widehat{Y}_{1;r_0,\tau,\chi,0:s}(e^{i\theta}) = r_0\tau, \tag{4.47}$$

hence we obtain by a direct application of the chain rule

$$\begin{aligned}\frac{\partial}{\partial\theta}\Big|_{\theta=0}\widehat{Y}'_{r_0,t,\tau,\chi,0:s}\big(e^{i\theta}\big)=\partial_{y_1}\varphi'\cdot r_0\tau+\\+\sum_{k=1}^n\partial_{x_k}\varphi'\cdot\big(\frac{\partial}{\partial\theta}\Big|_{\theta=0}\widehat{X}_{k;r_0,t,\tau,\chi,0:s}\big(e^{i\theta}\big)\big).\end{aligned}\tag{4.48}$$

By (4.44), the vector (4.45) belongs to the tangent plane to M_s^1 if and only if

$$\left\{\begin{aligned}r_0\tau&=\sum_{k=1}^n\partial_{x_k}h_1(\chi:s)\left[\frac{\partial}{\partial\theta}\Big|_{\theta=0}\widehat{X}_{k;r_0,t,\tau,\chi,0:s}\big(e^{i\theta}\big)\right],\\\frac{\partial}{\partial\theta}\Big|_{\theta=0}\widehat{Y}'_{r_0,t,\tau,\chi,0:s}\big(e^{i\theta}\big)&=\sum_{k=1}^n\big(\partial_{x_k}\varphi'+\partial_{y_1}\varphi'\cdot\partial_{x_k}h_1\big)\cdot\left[\frac{\partial}{\partial\theta}\Big|_{\theta=0}\widehat{X}_{k;r_0,t,\tau,\chi,0:s}\big(e^{i\theta}\big)\right].\end{aligned}\right.\tag{4.49}$$

We observe that the first line of (4.49) together with the relation (4.48) already obtained implies the second line of (4.49) by an obvious linear combination. Consequently, the vector (4.45) belongs to the tangent plane to M_s^1 at $\widehat{p}(\chi,0:s)$ if and only if the first line of (4.49) is satisfied. As r_0 is nonzero, as the first order derivatives $\partial_{x_k}h_1(\chi:s)$ are of class $\mathcal{C}^{1,\alpha}$ and vanish at $x=0$ and as $\frac{\partial}{\partial\theta}\big|_{\theta=0}\widehat{X}_{k;r_0,t,\tau,\chi,0:s}\big(e^{i\theta}\big)$ is of class $\mathcal{C}^{1,\alpha-0}$ with respect to all variables (t,τ,χ,s), it follows from the implicit function theorem that there exists a unique solution $\tau=\tau(t,\chi:s)$ of the first line of (4.49) which satisfies in addition the normalization conditions $\tau(0,0:s)=\partial_{t_j}\tau(0,0:s)=\partial_{\chi_k}\tau(0,0:s)=0$ for $j=1,\dots,n-1$ and $k=1,\dots,n$. This completes the proof of Lemma 4.42. □

We now define the analytic disc

$$\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta):=\widehat{Z}_{r_0,t,\tau(t,\chi:s),\chi,\nu:s}(\zeta).\tag{4.50}$$

Lemma 4.51. *Shrinking ε if necessary, the following two properties are satisfied*:

- **(1)** $\widehat{\mathcal{Z}}_{t,\chi,0:s}(\partial\Delta\backslash\{1\})\subset(M_s^1)^+$ *for all t, χ, ν and s with $|t|<\varepsilon$, with $|\chi|<\varepsilon$, with $|\nu|<\varepsilon$ and with $-1\leqslant s\leqslant 1$.*
- **(2)** *If ν satisfies $0<\nu<\varepsilon$, then $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\partial\Delta)\subset(M_s^1)^+$ for all t, χ and s with $|t|<\varepsilon$, with $|\chi|<\varepsilon$ and with $-1\leqslant s\leqslant 1$.*

Proof. To establish **(1)**, we first observe that the disc $\widehat{\mathcal{Z}}_{0,0,0:s}\big(e^{i\theta}\big)$ identifies with the disc $\widehat{Z}_{r_0:s}(e^{i\theta})$ defined in §4.29. According to Lemma 4.25, we know that $\widehat{\mathcal{Z}}_{0,0,0:s}(\partial\Delta\backslash\{1\})$ is contained in $(M_s^1)^+$. By continuity, if ε is sufficiently small, we can assume that for all t with $|t|<\varepsilon$, for all χ with $|\chi|<\varepsilon$ and for all θ with $\frac{\pi}{4}\leqslant|\theta|\leqslant\pi$, the point $\widehat{\mathcal{Z}}_{t,\chi,0:s}\big(e^{i\theta}\big)$ is contained in $(M_s^1)^+$. It remains to control the part of $\partial\Delta$ which corresponds to $|\theta|\leqslant\frac{\pi}{4}$.

Since the disc $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}\big(e^{i\theta}\big)$ is of class $\mathcal{C}^2$ with respect to all its principal variables $\big(t,\chi,\nu,e^{i\theta}\big)$, if $|t|<\varepsilon$, if $|\chi|<\varepsilon$ and if $0\leqslant|\theta|\leqslant\frac{\pi}{4}$, for sufficiently small ε, then the inequalities (4.28) are just perturbed a little bit, so we can assume that

$$\partial_\theta\partial_\theta\widehat{\mathcal{Y}}_{1;t,\chi,0:s}\big(e^{i\theta}\big)\geqslant r_0>\frac{r_0}{2}\geqslant\big|\partial_\theta\partial_\theta h_1\big(\widehat{\mathcal{X}}_{t,\chi,0:s}\big(e^{i\theta}\big)\big)\big|.\tag{4.52}$$

We claim that the inequality

$$\widehat{\mathcal{Y}}_{1;t,\chi,0:s}\big(e^{i\theta}\big)>\big|h_1\big(\widehat{\mathcal{X}}_{t,\chi,0:s}\big(e^{i\theta}\big)\big)\big|\tag{4.53}$$

holds for all $0 < |\theta| \leqslant \frac{\pi}{4}$, which will complete the proof of property **(1)**.

Indeed, we first remind that the tangency to M_s^1 of the vector $\frac{\partial}{\partial\theta}\big|_{\theta=0} \widehat{\mathcal{Z}}_{t,\chi,0:s}(e^{i\theta})$ at the point $\widehat{p}(\chi, 0:s)$ is equivalent to the first relation (4.49), which may be rewritten in terms of the components of the disc $\widehat{\mathcal{Z}}_{t,\chi,0:s}(e^{i\theta})$ as follows

$$\partial_\theta \widehat{\mathcal{Y}}_{1;t,\chi,0:s}(1) = \sum_{k=1}^{n} [\partial_{x_k} h_1]\big(\widehat{\mathcal{X}}_{t,\chi,0:s}(1)\big) \cdot \partial_\theta \widehat{\mathcal{X}}_{k;t,\chi,0:s}(1). \tag{4.54}$$

Substracting this relation from (4.53) and substracting also the relation $\widehat{Y}_{1;t,\chi,0:s}(1) = h_1\big(\mathcal{X}_{t,\chi,0:s}(1)\big)$, we see that it suffices to establish that for all θ with $0 < |\theta| \leqslant \frac{\pi}{4}$, we have the strict inequality

$$\begin{aligned}&\widehat{\mathcal{Y}}_{1;t,\chi,0:s}(e^{i\theta}) - \widehat{\mathcal{Y}}_{1;t,\chi,0:s}(1) - \theta \cdot \partial_\theta \widehat{\mathcal{Y}}_{1;t,\chi,0:s}(1) > \\ &\quad > \left| h_1\big(\widehat{\mathcal{X}}_{t,\chi,0:s}(e^{i\theta})\big) - h_1\big(\widehat{\mathcal{X}}_{t,\chi,0:s}(1)\big) - \sum_{k=1}^{n} [\partial_{x_k} h_1]\big(\widehat{\mathcal{X}}_{t,\chi,0:s}(1)\big) \cdot \partial_\theta \widehat{\mathcal{X}}_{k;t,\chi,0:s}(1) \right|\end{aligned} \tag{4.55}$$

However, by means of Taylor's integral formula, this last inequality may be rewritten as

$$\int_0^\theta (\theta-\theta')\cdot\partial_\theta\partial_\theta \widehat{\mathcal{Y}}_{1;t,\chi,0:s}(e^{i\theta'})\cdot d\theta' > \left| \int_0^\theta (\theta-\theta')\cdot\partial_\theta\partial_\theta \left[h_1\big(\widehat{\mathcal{X}}_{t,\chi,0:s}(e^{i\theta'})\big)\right]\cdot d\theta' \right| \tag{4.56}$$

and it follows immediately by means of (4.52).

Secondly, to check property **(2)**, we observe that by the definition (4.37), the parameter ν corresponds to a translation of the z_1-component of the disc boundary $\widehat{\mathcal{Z}}_{t,\chi,0:s}(\partial\Delta)$ along the y_1 axis. More precisely, we have

$$\frac{\partial}{\partial\nu} \widehat{\mathcal{Y}}_{1;t,\chi,\nu:s}(\zeta) \equiv 1 \qquad \text{and} \qquad \frac{\partial}{\partial\nu} \widehat{\mathcal{X}}_{1;t,\chi,\nu:s}(\zeta) \equiv 0. \tag{4.57}$$

On the other hand, differentiating Bishop's equation (4.40), and using the smallness of the function Φ', it may be checked that

$$\left| \frac{\partial}{\partial\nu} \widehat{Z}'_{r_0,t,\tau,\chi,\nu:s}(e^{i\theta}) \right| \ll 1, \tag{4.58}$$

if r_0 and ε are sufficiently small. It follows that the disc boundary $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\partial\Delta)$ is globally moved in the direction of the y_1-axis as $\nu > 0$ increases, hence is contained in $(M_s^1)^+$. The proof of Lemma 4.51 is complete. □

4.59. Holomorphic extension to a semi-local attached half-wedge. As a consequence of Lemma 4.34, of (4.39) and of property **(2)** of Lemma 4.51, we conclude that for every $s \in [-1, 1]$, our discs $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)$ satisfy all the requirements **(i)**, **(ii)** and **(iii)** of §4.2 insuring that the set defined by

$$\mathcal{HW}_s^+ := \left\{ \widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\rho) : \, |t| < \varepsilon, \, |\chi| < \varepsilon, \, 0 < \nu < \varepsilon, \, 1 - \varepsilon < \rho < 1 \right\} \tag{4.60}$$

is a local half-wedge of edge $(M_s^1)^+$ at the origin in the $\widehat{z}$-coordinates, which corresponds to the point $\gamma(s)$ in the z-coordinates. Coming back to the coordinates $z = \Phi_s^{-1}(\widehat{z})$, we define the family of analytic discs

$$\mathcal{Z}_{t,\chi,\nu:s}(\zeta) := \Phi_s^{-1}\big(\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)\big). \tag{4.61}$$

Given an arbitrary $f \in \mathcal{O}(\Omega)$ as in Proposition 4.6, through the change of coordinates $\widehat{z} = \Phi_s(z)$ and by restriction to $(M_s^1)^+$, we get a CR function $\widehat{f_s} \in \mathcal{C}^0_{CR}\big((M_s^1)^+ \cap U_1\big)$, for some small neighborhood U_1 of the origin in $\mathbb{C}^n$, whose size is uniform with respect to s. Thanks to an obvious generalization of the approximation Lemma 4.8 with a supplementary parameter $s \in [-1,1]$, we know that there exists a second uniform neighborhood $V_1 \subset\subset U_1$ of the origin in $\mathbb{C}^n$ such that $\widehat{f_s}$ is uniformly approximable by polynomials on $(M_s^1)^+ \cap V_1$. Furthermore, choosing r_0 and ε sufficiently small, we can insure that all the discs $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)$ are attached to $(M_s^1)^+ \cap V_1$. As in [Trp1990, Tu1994, Me1994, Jö1996], it then follows from the maximum principle applied to the approximating sequence of polynomials that for each $s \in [-1,1]$, the function $\widehat{f_s}$ extends holomorphically to the half-wedge defined by (4.60). Finally, we deduce that the holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically to the semi-local half-wedge attached to the one-sided neighborhood $(M_\gamma^1)^+$ defined by

(4.62)
$$\mathcal{HW}_\gamma^+ := \big\{\mathcal{Z}_{t,\chi,\nu:s}(\rho) : \, |t| < \varepsilon, \, |\chi| < \varepsilon, \, 0 < \nu < \varepsilon, 1-\varepsilon < \rho < 1, \, -1 \leqslant s \leqslant s\big\}.$$

Without shrinking Ω near the points $\mathcal{Z}_{t,\chi,\nu:s}(-1)$ (otherwise, the crucial rank property of Lemma 4.34 would degenerate), we can shrink the open set Ω in a very thin neighborhood of the characteristic segment γ in M and we can shrink $\varepsilon > 0$ if necessary in order that the intersection $\Omega \cap \mathcal{HW}_\gamma^+$ is connected. By the principle of analytic continuation, this implies that there exists a well-defined holomorphic function $F \in \mathcal{O}\big(\Omega \cup \mathcal{HW}_\gamma^+\big)$ with $F|_\Omega = f$.

The proof of Proposition 4.6 is complete. □

§5. CHOICE OF A SPECIAL POINT OF $C_{\rm nr}$ TO BE REMOVED LOCALLY

5.1. Choice of a first supporting hypersurface. Continuing with the proof of Theorem 3.19, we now analyze the assumption that C is nontransversal to $\mathsf{F}^c_{M^1}$. We first construct a foliated support hypersurface H^1.

Lemma 5.2. *Under the assumptions of Theorem 3.19, there exists a $\mathcal{C}^{2,\alpha}$ embedded characteristic curve $\gamma : [-1,1] \to M^1$ with $\gamma(-1) \not\in C$, $\gamma(0) \in C$, $\gamma(1) \not\in C$, and there exists a $\mathcal{C}^{1,\alpha}$ hypersurface H^1 of M^1 with $\gamma \subset H^1$ which is foliated by characteristic segments close to γ, such that locally in a neighborhood of H^1, the closed subset C is contained in $\gamma \cup (H^1)^-$, where $(H^1)^-$ denotes an open one-sided neighborhood of H^1 in M^1.*

Proof. By the nontransversality assumption, there exists a first characteristic curve $\widetilde{\gamma} : [-1,1] \to M^1$ with $\widetilde{\gamma}(-1) \not\in C$, $\widetilde{\gamma}(0) \in C$ and $\widetilde{\gamma}(1) \not\in C$, there exists a neighborhood $V^1_{\widetilde{\gamma}}$ of $\widetilde{\gamma}$ in M^1 and there exists a local $(n-1)$-dimensional submanifold R^1 passing through $\widetilde{\gamma}(0)$ which is transversal to $\widetilde{\gamma}$ such that the semi-local projection $\pi_{\mathsf{F}^c_{M^1}} : V^1_{\widetilde{\gamma}} \to R^1$ parallel to the characteristic curves maps C onto the closed subset $\pi_{\mathsf{F}^c_{M^1}}(C)$ with the property that $\pi_{\mathsf{F}^c_{M^1}}(\widetilde{\gamma})$ lies on the *boundary of* $\pi_{\mathsf{F}^c_{M^1}}(C)$ *with respect to the topology of* R^1. This property is illustrated in the right hand side of the following figure.

However, we want in addition a foliated supporting hypersurface H^1, which does not necessarily exist in a neighborhood of $\widetilde{\gamma}$. To construct H^1, let us first straighten the characteristic lines in a neighborhood of $\widetilde{\gamma}$, getting a product $[-1,1] \times$

$[-\delta_1, \delta_1]^{n-1}$, for some $\delta_1 > 0$, equipped with coordinates $(s, \chi) = (s, \chi_2, \dots, \chi_n) \in \mathbb{R} \times \mathbb{R}^{n-1}$, so that level-set $\{\chi = \text{cst}\}$ correspond to characteristic lines. Such a straightening is only of class $\mathcal{C}^{1,\alpha}$, because the line distribution $T^c M|_{M^1} \cap TM^1$ is only of class $\mathcal{C}^{1,\alpha}$. Clearly, we may assume that δ_1 is so small that there exists s_1 with $0 < s_1 < 1$ such that the two cubes $[-1, -s_1] \times [-\delta_1, \delta_1]^{n-1}$ and $[s_1, 1] \times [-\delta_1, \delta_1]^{n-1}$ do not meet the singularity C.

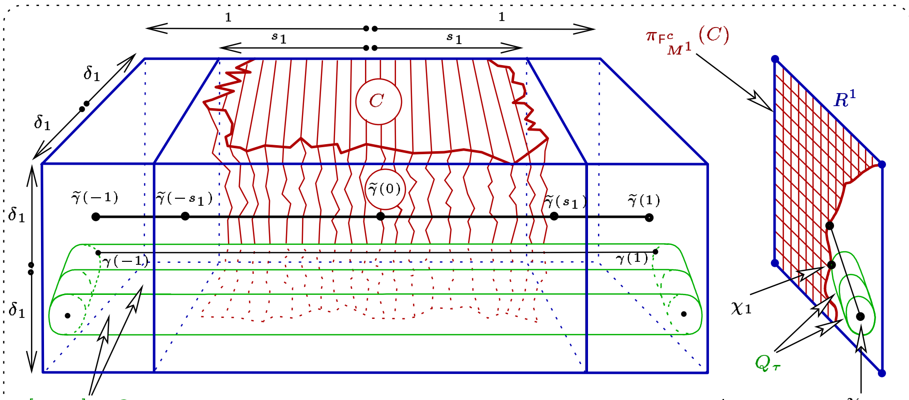

Fig. 10: Construction of a support hypersurface H^1

We may identify the transversal R^1 with $[-\delta_1, \delta_1]^{n-1}$; then the projection of $\widetilde{\gamma}$ is the origin of R^1. By assumption, $\pi_{\mathsf{F}^c_{M^1}}(C)$ is a proper closed subset of R^1 with the origin lying on its boundary. We can therefore choose a point χ_0 in the interior of R^1 lying outside $\pi_{\mathsf{F}^c_{M^1}}(C)$. Also, we can choose a small open $(n-1)$-dimensional ball Q_0 centered at this point which is contained in the complement $R^1 \backslash \pi_{\mathsf{F}^c_{M^1}}(C)$. Furtermore, we can include this ball in a one parameter family of $\mathcal{C}^{1,\alpha}$ domains $Q_\tau \subset R^1$, for $\tau \geqslant 0$, which are parts of ellipsoids stretched along the segment which joins the point χ_0 with the origin of R^1.

We then consider the tube domains $[-1, 1] \times Q_\tau$ in $[-1, 1] \times [-\delta_1, \delta_1]^{n-1}$. Clearly, there exists the smallest $\tau_1 > 0$ such that the tube $[-1, 1] \times Q_{\tau_1}$ meets the singularity C on its boundary $[-1, 1] \times \partial Q_{\tau_1}$. In particular, there exists a point $\chi_1 \in \partial Q_{\tau_1}$ such that the characteristic segment $[-1, 1] \times \{\chi_1\}$ intersects C. Increasing a little bit the curvature of ∂Q_{τ_1} in a neighborhood of χ_1 if necessary, we can assume that $\pi_{\mathsf{F}^c_{M^1}}(C) \cap \overline{Q_{\tau_1}} = \{\chi_1\}$ in a neighborhood of χ_1. Moreover, since by construction the two segments $[-1, -s_1] \times \{\chi_1\} \cup [s_1, 1] \times \{\chi_1\}$ do not meet C, we can reparametrize the characteristic segment $[-1, 1] \times \{\chi_1\}$ as $\gamma : [-1, 1] \to M^1$ with $\gamma(-1) \notin C$, $\gamma(0) \in C$ and $\gamma(1) \notin C$. Since all characteristic lines are $\mathcal{C}^{2,\alpha}$, we can choose the parametrization to be of class $\mathcal{C}^{2,\alpha}$. For the supporting hypersurface H^1, it suffices to choose a piece of $[-1, 1] \times \partial Q_{\tau_1}$ near $[-1, 1] \times \{\chi_1\}$. By construction, this supporting hypersurface is only of class $\mathcal{C}^{1,\alpha}$ and we have that C is contained in $\gamma \cup (H^1)^-$ semi-locally in a neighborhood of γ, as desired. □

5.3. Field of cones on M^1. With the characteristic segment γ constructed in Lemma 5.2, by an application of Proposition 4.6, we deduce that there exists a semi-local half-wedge $\mathcal{HW}_\gamma^+$ attached to $(M_\gamma^1)^+ \cap V_\gamma$, for some neighborhood V_γ of γ in M, to which $\mathcal{O}(\Omega)$ extends holomorphically.

Then, we remind that by (4.37), (4.39) and (4.50), for all t with $|t| < \varepsilon$, the point $\widehat{\mathcal{Z}}_{t,\chi,0:s}(1)$ identifies with the point $\widehat{p}(\chi, 0 : s) \in M_s^1$ defined in (4.37) (which is independent of t) and the mapping $\chi \mapsto \widehat{\mathcal{Z}}_{t,\chi,0:s}(1) \in M_s^1$ is a local diffeomorphism.

Sometimes in the sequel, we shall denote the disc $\mathcal{Z}_{t,\chi,\nu:s}(\zeta) \equiv \Phi_s^{-1}\left(\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)\right)$ defined in (4.61) by $\mathcal{Z}_{t,\chi_1,\chi',\nu:s}(\zeta)$, where $\chi' = (\chi_2, \dots, \chi_n) \in \mathbb{R}^{n-1}$. Since the characteristic curve is directed along the x_1-axis, which is transversal in $T_0 M_s^1$ to the space $\{(0, \chi')\}$, it follows that the mapping $(s, \chi') \longmapsto \mathcal{Z}_{t,0,\chi',0:s}(1) = \Phi_s^{-1}(\widehat{p}(0, \chi', 0 : s))$ is, independently of t, a diffeomorphism onto its image for $s \in [-1, 1]$ and for χ' close to the origin in $\mathbb{R}^{n-1}$. To fix ideas, we shall let χ' vary in the *closed* cube $[-\varepsilon, \varepsilon]^{n-1}$ (analogously to the fact that s runs in the *closed* interval $[-1, 1]$) and we shall denote by V_γ^1 the closed image of this diffeomorphism.

At every point $p := \mathcal{Z}_{t,0,\chi',0:s}(1) = \mathcal{Z}_{0,0,\chi',0:s}(1)$ of this neighborhood V_γ^1, we define an open infinite oriented cone contained in the n-dimensional linear space $T_p M^1$ by

$$\mathsf{C}_p := \mathbb{R}^+ \cdot \left\{ \frac{\partial \mathcal{Z}_{t,0,\chi',0:s}}{\partial \theta}(1) : \ |t| < \varepsilon \right\}. \tag{5.4}$$

The fact that C_p is indeed an open cone follows from Lemma 4.34, from (4.61) and from the fact that Φ_s^{-1} is a biholomorphism. This cone contains in its interior the nonzero vector

$$v_p^0 := \frac{\partial \mathcal{Z}_{0,0,\chi',0:s}}{\partial \theta}(1) \in \mathsf{C}_p \subset T_p M^1 \backslash \{0\}. \tag{5.5}$$

We shall say that C_p is the *cone created at* p *by the semi-local attached half-wedge* $\mathcal{HW}_\gamma^+$ (more precisely, by the family of analytic discs which covers this semi-local half-wedge).

As p varies, $p \mapsto \mathsf{C}_p$ constitutes a *field of cones over* V_γ^1, as illustrated by Figures 3 and 11.

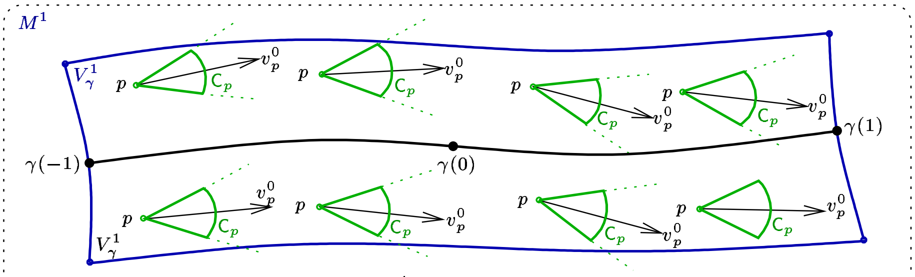

Fig. 11: Field of cones on TM^1 associated to the family $\mathcal{Z}_{t,0,\chi',0,s}(\zeta)$

The map $p \mapsto v_p^0$ is a $\mathcal{C}^{1,\alpha-0}$ vector field tangent to M^1, contained in the field of cones $p \mapsto \mathsf{C}_p$. Over V_γ^1, we also introduce a nowhere zero characteristic vector field X which satisfies $\exp(sX)(\gamma(0)) = \gamma(s)$ for all $s \in [-1, 1]$. As in Section 2, for every $p \in V_\gamma^1$, we define the *filled cone*

$$\mathsf{FC}_p := \mathbb{R}^+ \cdot \left\{ \lambda \cdot X_p + (1 - \lambda) \cdot v_p : \ 0 \leqslant \lambda < 1, \ v_p \in \mathsf{C}_p \right\}. \tag{5.6}$$

In $T_p M^1$ equipped with linear coordinates $(x_1, \dots, x_n)$ such that the characteristic direction $T_p^c M \cap T_p M^1$ isx the x_1-axis, we draw C_p, its filling FC_p and its projection $\pi'(\mathsf{C}_p)$ onto the $(x_2, \dots, x_n)$-space parallelly to the x_1-axis.

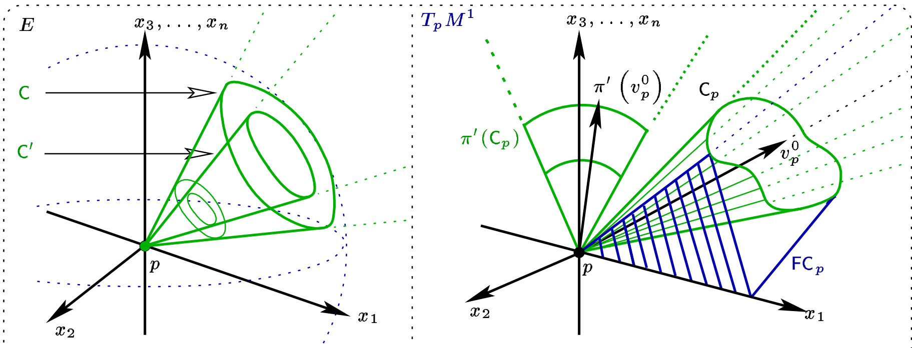

Fig. 12: A cone in T_pM^1 and its filling along the characteristic direction

For every nonzero vector $v_p \in \mathsf{C}_p$, it may be checked that a small neighborhood of the origin in the positive half-line $\mathbb{R}^+ \cdot Jv_p$ generated by Jv_p is contained in the attached half-wedge $\mathcal{HW}_\gamma^+$.

Lemma 5.7. *Fix a point $p \in V_\gamma^1$ and a vector v_p in the cone C_p created by the semi-local attached half-wedge $\mathcal{HW}_\gamma^+$ at p. Suppose that there exist two constants $c_1 > 0$ and $\Lambda_1 > 1$ such that for every c with $0 < c \leqslant c_1$, there exists a $\mathcal{C}^{2,\alpha-0}$ analytic disc $A_c(\zeta)$ with $A_c(\partial^+\Delta) \subset M^1$, such that*:

- **(i)** *the positive half-line generated by the boundary of A_c at $\zeta = 1$ coincides with the positive half-line generated by v_p, namely $\mathbb{R}^+ \cdot \frac{\partial A_c}{\partial \theta}(1) \equiv \mathbb{R}^+ \cdot v_p$;*
- **(ii)** $|A_c(\zeta)| \leqslant c^2 \cdot \Lambda_1$ *for all* $\zeta \in \overline{\Delta}$ *and* $c \cdot \frac{1}{\Lambda_1} \leqslant \left|\frac{\partial A_c}{\partial \theta}(1)\right| \leqslant c \cdot \Lambda_1$;
- **(iii)** $\left|\frac{\partial A_c}{\partial \theta}\left(\rho e^{i\theta}\right) - \frac{\partial A_c}{\partial \theta}(1)\right| \leqslant c^2 \cdot \Lambda_1$ *for all* $\zeta = \rho e^{i\theta} \in \overline{\Delta}$.

If c_1 is sufficiently small, then for every c with $0 < c \leqslant c_1$, the closed disc minus its half-boundary $A_c\left(\overline{\Delta}\backslash\partial^+\Delta\right)$ is contained in the semi-local half-wedge $\mathcal{HW}_\gamma^+$.

Furthermore, the same conclusion holds if the nonzero vector v_p belongs to the filled cone FC_p.

Details will be provided later. In fact, the reason why we introduce filled cones FC_p is because, for the selection of a special, locally removable point of C, we shall see that the corresponding direction $\mathbb{R}^+ \cdot \frac{\partial A_c}{\partial \theta}(1) \equiv \mathbb{R}^+ \cdot v_p$ of half-boundary $A(\partial^+\Delta) \subset M^1$ must unavoidably be almost parallel to the characteristic direction, and hopefully, vectors $v_p \in \mathsf{FC}_p$ may approach the characteristic direction arbitrarily. Indeed, not only we will have to assure that $A_c\left(\overline{\Delta}\backslash\partial^+\Delta\right) \subset \mathcal{HW}_\gamma^+$ (which works already for $v_p \in \mathsf{C}_p$), but also, we will have to insure that the disc A_c with $A_c(1) \in C$ satisfies $A_c\big(\partial^+\Delta\backslash\{1\}\big)$, as drawn in Figure 4, in order to be able to apply the continuity principle.

5.8. Choice of the special point $p_{\rm sp}$. We can now answer the question implicitly left inside Theorem 3.19: how to choose the special point $p_{\rm sp}$ to be removed locally?

Lemma 5.9. *Let γ be the characteristic segment constructed in Lemma 5.2, let $\mathcal{HW}_\gamma^+$ be the semi-local attached half-wedge of edge $(M_\gamma^1)^+ \cap V_\gamma$ constructed in Proposition 4.6, and let $p \mapsto \mathsf{FC}_p$ be the filled field of cones created by $\mathcal{HW}_\gamma^+$. Then there exists a special point $p_{\rm sp} \in C \cap V_\gamma^1$ such that*:

(i) *there exists a* $\mathcal{C}^{2,\alpha}$ *local supporting hypersurface* $H_{\rm sp}$ *of* M^1 *passing through* $p_{\rm sp}$ *such that, locally in a neighborhood of* $p_{\rm sp}$*, the closed subset* C *is contained in* $(H_{\rm sp})^- \cup \{p_{\rm sp}\}$*, where* $(H_{\rm sp})^-$ *denotes an open one-sided neighborhood of* $H_{\rm sp}$ *in* M^1*; and*:

(ii) *there exists a nonzero vector* $v_{\rm sp} \in T_{p_{\rm sp}} H_{\rm sp}$ *which belongs to the filled cone* $\mathsf{FC}_{p_{\rm sp}}$.

Proof. According to Lemma 5.2, the singularity C is contained in $\gamma \cup (H^1)^-$, where H^1 is a $\mathcal{C}^{1,\alpha}$ hypersurface containing γ which is foliated by characteristic segments. If $\lambda \in [0,1)$ is very close to 1, the vector field over V_γ^1 defined by

$$p \longmapsto v_p^\lambda := \lambda \cdot X_p + (1-\lambda) \cdot v_p \in T_p M^1 \tag{5.10}$$

is very close to the characteristic vector field X_p, so the integral curves of $p \mapsto v_p^\lambda$ are very close to the integral curves of $p \mapsto X_p$. If λ is sufficiently close to 1, we can choose a subneighborhood $V_\gamma^\lambda \subset V_\gamma^1$ of γ which is foliated by integral curves of $p \mapsto v_p^\lambda$. As in Lemma 5.2, let us fix an $(n-1)$-dimensional submanifold R^1 transversal to γ and passing through $\gamma(0)$. Since the vector field $p \mapsto v_p^\lambda$ is very close to the characteristic vector field, it follows that after projection onto R^1 parallelly to the integral curves of $p \mapsto v_p^\lambda$, the closed set $C \cap V_\gamma^\lambda$ is again a proper closed subset of R^1. We notice that, by its very definition, the vector v_p^λ belongs to the filled cone FC_p for all $p \in V_\gamma^\lambda$.

We can proceed exactly as in the proof of Lemma 5.2 with the foliation of V_γ^λ induced by the integral curves of the vector field $p \mapsto v_p^\lambda$, instead of the characteristic foliation, except that we want a supporting hypersurface $H_{\rm sp}$ which is of class $\mathcal{C}^{2,\alpha}$. Consequently, we first approximate the vector field $p \mapsto v_p^\lambda$ by a new vector field $p \mapsto \widetilde{v}_p^\lambda$ whose coefficients are of class $\mathcal{C}^{2,\alpha}$ (with respect to every local graphing function of M^1) and which is very close to the vector field $p \mapsto v_p^\lambda$ in $\mathcal{C}^{1,\alpha}$-norm. Again, we get a subneighborhood $\widetilde{V}_\gamma^\lambda \subset V_\gamma^\lambda$ of γ which is foliated by integral curves of $p \mapsto \widetilde{v}_p^\lambda$ and a projection of $C \cap \widetilde{V}_\gamma^\lambda$ which is a proper closed subset of R^1. Moreover, if the approximation is sufficiently sharp, we still have $\widetilde{v}_p^\lambda \in \mathsf{FC}_p$ for all $p \in \widetilde{V}_\gamma^\lambda$. Then by repeating the reasoning which yielded Lemma 5.2, we deduce that there exists an integral curve $\widetilde{\gamma}$ of the vector field $p \mapsto \widetilde{v}_p^\lambda$ satisfying (after reparametrization) $\widetilde{\gamma}(-1) \notin C$, $\widetilde{\gamma}(0) \in C$ and $\widetilde{\gamma}(1) \notin C$, together with a $\mathcal{C}^{2,\alpha}$ supporting hypersurface $\widetilde{H}$ of $\widetilde{V}_\gamma^\lambda$ which contains $\widetilde{\gamma}$ such that C is contained in $\widetilde{\gamma} \cup (\widetilde{H})^-$.

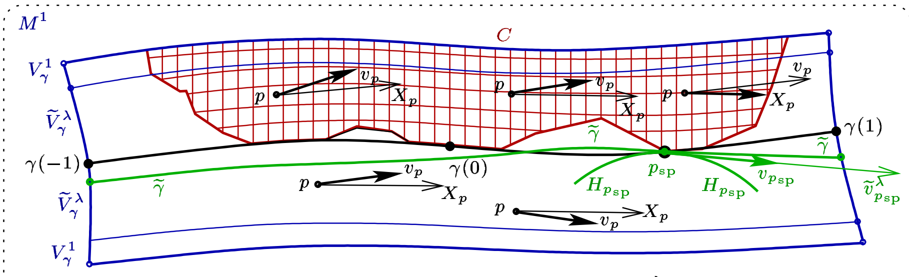

Fig. 13: Dotted integral curve of the vector field $p \mapsto v_p^\lambda$ **and choice of** $p_{\rm sp}$

To conclude the proof of Lemma 5.9, for the desired special point $p_{\rm sp}$, it suffices to choose $\widetilde{\gamma}(0)$. For the desired local supporting hypersurface $H_{p_{\rm sp}}$, we cannot choose directly a piece of $\widetilde{H}$ passing through $p_{\rm sp}$, because an open interval contained in $C \cap \widetilde{\gamma}$ may well be contained in $\widetilde{H}$. Fortunately, since we know that locally in a neighborhood of $p_{\rm sp}$, the closed subset C is contained in $(\widetilde{H})^- \cup \widetilde{\gamma}$, it suffices to choose for the desired supporting hypersurface $H_{p_{\rm sp}} \subset M^1$ a piece of a $\mathcal{C}^{2,\alpha}$ hypersurface passing through p_1, tangent to $\widetilde{H}$ at p_1 and satisfying $H_{p_{\rm sp}} \backslash \{p_{\rm sp}\} \subset (\widetilde{H})^+$ in a neighborhood of $p_{\rm sp}$. Finally, for the nonzero vector $v_{\rm sp}$, it suffices to choose any positive multiple of the vector $\widetilde{v^\lambda_{p_{\rm sp}}}$. This completes the proof of Lemma 5.9. □

5.11. Main removability proposition. We can now formulate the main removability proposition to which Theorem 3.19 is now fully reduced. We localize the situation at $p_{\rm sp}$, we denote this point simply by p_1, we denote its supporting hypersurface simply by H^1 and we denote its associated vector simply by $v_1 \in T_{p_1} H^1$. At p_1, we have a local half-wedge $\mathcal{HW}^+_{p_1} \subset \mathcal{HW}^+_\gamma$.

Proposition 5.12. *Let $M \subset \mathbb{C}^n$ be a $\mathcal{C}^{2,\alpha}$ generic submanifold of codimension $n-1 \geqslant 1$, hence of CR dimension 1, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$ one-codimensional submanifold which is maximally real in $\mathbb{C}^n$, let $p_1 \in M^1$, let $H^1 \subset M^1$ be a $\mathcal{C}^{2,\alpha}$ one-codimensional submanifold of M^1 passing through p_1 and let $(H^1)^-$ denote an open local one-sided neighborhood of H^1 in M^1. Let $C \subset M^1$ be a* nonempty *proper closed subset of M^1 with $p_1 \in C$ which is situated, locally in a neighborhood of p_1, only in one side of H^1, namely $C \subset (H^1)^- \cup \{p_1\}$. Let Ω be a neighborhood of $M \backslash C$ in $\mathbb{C}^n$, let $\mathcal{HW}^+_{p_1}$ be a local half-wedge of edge $(M^1)^+$ at p_1 generated by a family of analytic discs $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ satisfying the properties* **(i)**, **(ii)** *and* **(iii)** *of* §*4.2, let $\mathsf{C}_{p_1} \subset T_{p_1} M^1$ be the cone created by $\mathcal{HW}^+_{p_1}$ at p_1 and let FC_{p_1} be its filling.* As a main hypothesis, *assume that there exists a nonzero vector $v_1 \in T_{p_1} H^1$ which belongs to the filled cone FC_{p_1}.*

- **(I)** *If v_1 does not belong to $T^c_{p_1} M$, then there exists a local wedge $\mathcal{W}_{p_1}$ of edge M at (p_1, Jv_1) with $\mathcal{W}_{p_1} \cap \left[\Omega \cup \mathcal{HW}^+_{p_1}\right]$ connected (shrinking $\Omega \cup \mathcal{HW}^+_{p_1}$ if necessary) such that for every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}^+_{p_1}\right)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{W}_1 \cup \left[\Omega \cup \mathcal{HW}^+_{p_1}\right]\right)$ with $F\big|_{\Omega \cup \mathcal{HW}^+_{p_1}} = f$.*
- **(II)** *If v_1 belongs to $T^c_{p_1} M$, then there exists a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$ with $\omega_{p_1} \cap \left[\Omega \cup \mathcal{HW}^+_{p_1}\right]$ connected (shrinking $\Omega \cup \mathcal{HW}^+_{p_1}$ if necessary) such that for every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}^+_{p_1}\right)$, there exists a holomorphic function $F \in \mathcal{O}\left(\omega_{p_1} \cup \left[\Omega \cup \mathcal{HW}^+_{p_1}\right]\right)$ with $F\big|_{\Omega \cup \mathcal{HW}^+_{p_1}} = f$.*

The remainder of Section 5, and then Sections 6, 7, 8 and 9 are entirely devoted to the proof of this proposition.

5.13. A dichotomy. We shall indeed distinguish two cases:

- **(I)** the nonzero vector v_1 does not belong to the characteristic direction $T_{p_1} M^1 \cap T^c_{p_1} M$;
- **(II)** the nonzero vector v_1 belongs to the characteristic direction $T_{p_1} M^1 \cap T^c_{p_1} M$.

We must clarify the main assumption that v_1 belongs to the filling FC_{p_1} of the cone $\mathsf{C}_{p_1} \subset T_{p_1}M^1$ created by the local half-wedge $\mathcal{HW}^+_{p_1}$. As we have observed in §4.2, in the (generic) situation of Case **(I)**, a local half-wedge may be represented geometrically as the intersection of a (complete) local wedge of edge M at p_1, with a local one-sided neighborhood $(N^1)^+$ of a hypersurface N^1 passing through p_1, which is transversal to M and which satisfies $N^1 \cap M \equiv M^1$ in a neighborhood of p_1. The slope of the tangent space $T_{p_1}N^1$ to N^1 at p_1 with respect to the tangent space $T_{p_1}M$ to M at p_1 may be understood in terms of the cone C_{p_1}, as we will now explain. Afterwards, we shall consider Case **(II)** separately.

5.14. Cones, filled cones, subcones and local description of half-wedges in Case (I). For the sake of concreteness, it will be convenient to work in a holomorphic coordinate system $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ centered at p_1 in which $T_{p_1}M = \{y_2 = \cdots = y_n = 0\}$ and $T_{p_1}M^1 = \{y_1 = y_2 = \cdots = y_n = 0\}$ (the existence of such a coordinate system which straightens both $T_{p_1}M$ and $T_{p_1}M^1$ is a direct consequence of the considerations of §4.11). Let $\pi' : T_{p_1}M^1 \to T_{p_1}M^1/(T_{p_1}M^1 \cap T^c_{p_1}M)$ denote the canonical projection, namey $\pi'(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$. Sometimes, we shall denote the coordinates by $(z_1, z') = (x_1 + iy_1, x' + iy') \in \mathbb{C} \times \mathbb{C}^{n-1}$. In these coordinates, the characteristic direction at p_1 is the x_1-axis and we may assume that the tangent plane at p_1 of the one-sided neighborhood $(M^1)^+$ is given by $T_{p_1}(M^1)^+ = \{y' = 0, \ y_1 > 0\}$.

Let $\mathsf{C}_{p_1} \subset T_{p_1}M^1$ be the cone created by $\mathcal{HW}^+_{p_1}$ and let $\mathsf{C}'_{p_1} := \pi'(\mathsf{C}_{p_1})$ be its projection onto the x'-space, which yields an $(n-1)$-dimensional infinite cone in the x'-space, open in this space. Notice that, by the definition (5.6) of the filling (along the characteristic direction), the two projections $\pi'(\mathsf{C}_{p_1})$ and $\pi'(\mathsf{FC}_{p_1})$ are identical. We must now explain how these three cones C_{p_1}, FC_{p_1}, C'_{p_1} and the nonzero vector $v_1 \in \mathsf{FC}_{p_1}$ are disposed, geometrically.

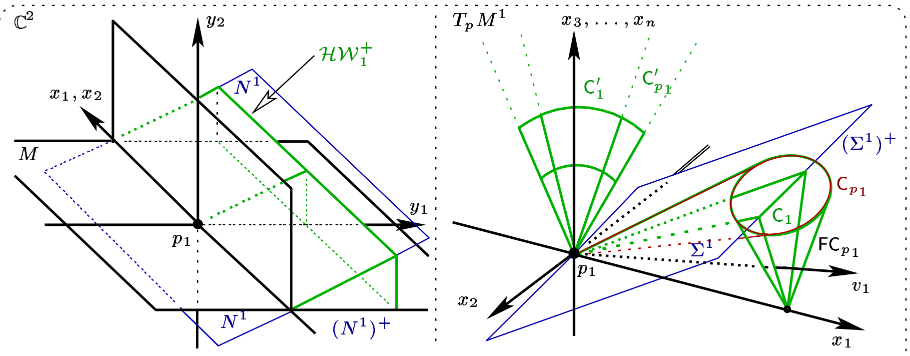

Fig. 14: Cone created by the half-wedge and its relation to the half-wedge

Because the discs $\mathcal{Z}_{t,\chi,\nu}$ of Proposition 5.12 (constructed in Section 4) are small, the tangent vector $\frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ is necessarily close to the complex tangent plane $T^c_{p_1}M$: this may be checked directly by differentiating Bishop's equation (4.40) with respect to θ, using the fact that the $\mathcal{C}^1$-norm of Φ' is small. Moreover, since this vector $\frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ also belongs to $T_{p_1}M^1$, it is in fact close to the positive x_1-axis. Furthermore, since the vector v_1 belongs to FC_{p_1} which contains the vector $\frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$, and since in the proof of Lemma 5.9 above we have chosen the special point, the supporting hypersurface and the vector v_1 with a parameter λ very close to 1, it follows that the vector $v_1 \equiv \widetilde{v}_{p_{\rm sp}}^\lambda$ is even closer to the positive x_1-axis. However, we suppose in Case **(I)** that v_1 is *not* directed along the x_1-axis, so v_1 has coordinates $(v_{1;1}, v_{2;1}, \dots, v_{n;1}) \in \mathbb{R}^n$ with $v_{1;1} > 0$, with $|v_{j;1}| << v_{1;1}$ for $j = 2, \dots, n$ and with at least one $v_{j;1}$ being nonzero.

We need some general terminology. Let C be an open infinite cone in a real linear subspace E of dimension $q \geqslant 1$. We say that C' is a *proper subcone* and we write $\mathsf{C}' \subset\subset \mathsf{C}$ (*see* the left hand side of Figure 12 above for an illustration) if the intersection of C' with the unit sphere of E is a relatively compact subset of the intersection of C with the unit sphere of E, this property being independent of the choice of a norm on E. We say that C is a *linear cone* if it may be defined by $\mathsf{C} = \{x \in E : \ \ell_1(x) > 0, \dots, \ell_q(x) > 0\}$ for some q linearly independent real linear forms $\ell_1, \dots, \ell_q$ on E.

In the x'-space, we now choose an open infinite strictly convex linear proper subcone $\mathsf{C}_1' \subset\subset \mathsf{C}_{p_1}'$ with the property that v_1 belongs to its filling FC_1', *cf.* Figure 14. Here, we may assume that C_1' is described by $(n-1)$ strict inequalities $\ell_1'(x') > 0, \dots, \ell_{n-1}'(x') > 0$, where the $\ell_j'(x')$ are linearly independent linear forms. It then follows that there exists a linear form $\sigma(x_1, x')$ of the form $\sigma(x_1, x') = x_1 + a_2 x_2 + \cdots + a_n x_n$ such that the original filled cone FC_{p_1} is contained in the linear cone

$$\mathsf{C}_1 := \left\{(x_1, x') \in \mathbb{R}^n : \ \ell_1'(x') > 0, \dots, \ell_{n-1}'(x') > 0, \ \sigma(x_1, x') > 0\right\}, \tag{5.15}$$

which contains the vector v_1. This cone is automatically filled, namely $\mathsf{C}_1 \equiv \mathsf{FC}_1$.

We remind that by genericity of M, the complex structure J of $T\mathbb{C}^n$ induces an isomorphim $T_{p_1}M/T_{p_1}^cM \to T_{p_1}\mathbb{C}^n/T_{p_1}M$. Hence $J\mathsf{C}_{p_1}'$ and $J\mathsf{C}_1'$ are open infinite strictly convex linear proper cones in $T_{p_1}\mathbb{C}^n/T_{p_1}M \cong \{(0,y') \in \mathbb{C}^n\}$. Since $J\mathsf{C}_1'$ is a proper subcone of $J\mathsf{C}_{p_1}'$ and since in the classical definition of a wedge, only the projection of the cone onto the quotient space $T_{p_1}M/T_{p_1}^cM$ has a contribution to the wedge, it then follows that the complete wedge $\mathcal{W}_{p_1}$ associated to the family $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ (*cf.* the paragraph after (4.5)) contains a wedge of the form

$$\mathcal{W}_1 := \left\{p + \mathsf{c}_1' : \ p \in M, \ \mathsf{c}_1' \in J\mathsf{C}_1'\right\} \cap \Delta_n(p_1, \delta_1), \tag{5.16}$$

for some δ_1 with $0 < \delta_1 < \varepsilon$, where ε is as in §4.2. Furthermore, as observed in §4.2, there exists a $\mathcal{C}^{2,\alpha}$ hypersurface N^1 of $\mathbb{C}^n$ passing through p_1 with the property that $N^1 \cap M \equiv M^1$ locally in a neighborhood of p_1 such that, shrinking $\delta_1 > 0$ if necessary, the local half-wedge $\mathcal{HW}_{p_1}^+$ contains a local half-wedge $\mathcal{HW}_1^+$ of edge $(M^1)^+$ at p_1 which is described as the geometric intersection of the complete wedge $\mathcal{W}_{p_1}$ with a one-sided neighborhood $(N^1)^+$, namely

$$\mathcal{HW}_1^+ := \mathcal{W}_1 \cap (N^1)^+. \tag{5.17}$$

An illustration for the case $n = 2$ where $M \subset \mathbb{C}^2$ is a hypesurface is provided in the left hand side of Figure 14. In addition, it follows from the definition of $\mathcal{HW}_{p_1}^+$ by means of the segments $\mathcal{Z}_{t,\chi,\nu}\big((1-\varepsilon, 1)\big)$ that we can assume that

$$T_{p_1}(N^1)^+ = T_{p_1}M \oplus J(\Sigma^1)^+, \tag{5.18}$$

where $(\Sigma_1)^+$ is the hyperplane one-sided neighborhood $\{(x_1, x') : \ \sigma(x_1, x') > 0\} \subset T_{p_1}M^1$. Equivalently, $T_{p_1}(N^1)^+$ is represented by the inequality $y_1 + a_2 y_2 + \cdots + a_n y_n > 0$. Consequently, there exists a $\mathcal{C}^{2,\alpha}$ function $\psi(x, y')$ with

$\psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0$ for $k = 1, \dots, n$ and $j = 2, \dots, n$ such that N^1 is represented by the equation $y_1 + a_2 y_2 + \cdots + a_n y_n = \psi(x, y')$ and $(N^1)^+$ by the inequation $y_1 + a_2 y_2 + \cdots + a_n y_n > \psi(x, y')$.

5.19. Cones, filled cones, subcones and local description of half-wedges in Case (II). Secondly, we assume that the nonzero vector v_1 of Proposition 5.12 belongs to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^c M$. In this case, as observed in §4.2, the half-wedge $\mathcal{HW}_{p_1}^+$ coincides with a local wedge of edge M^1 at (p_1, Jv_1). After a real dilation of the z_1-axis, we can assume that $v_1 = (1, 0, \dots, 0)$. Choosing an open infinite strictly convex linear proper subcone $\mathsf{C}_2 \subset\subset \mathsf{C}_{p_1} \subset T_{p_1}M^1 = \mathbb{R}_x^n$ defined by n strict inequalities $\ell_1(x) > 0, \dots, \ell_n(x) > 0$, where the $\ell_j(x)$ are linearly independent real linear forms — of course with C_2 containing the vector v_1 — it follows that there exists $\delta_1 > 0$ such that the half-wedge $\mathcal{HW}_{p_1}^+$ contains the following local wedge of edge M^1 at p_1:

$$\mathcal{W}_2 := \big\{p + \mathsf{c}_2 : \ p \in M^1, \ \mathsf{c}_2 \in J\mathsf{C}_2\big\} \cap \Delta_n(p_1, \delta_1). \tag{5.20}$$

We remind that it was observed in §4.2 (*cf.* especially the right hand side of Figure 7) that $\mathcal{W}_2$ contains $(M^1)^+$ locally in a neighborhood of p_1. In §5.22 below, we shall provide a more concrete representation of $\mathcal{W}_2$ in an appropriate system of coordinates.

5.21. A trichotomy. Let us pursue this discussion more concretely by introducing further normalizations. Our goal will now be to construct appropriate normalized coordinate systems. Analyzing further the dichotomy introduced in §5.13 by taking account of the presence of the one-codimensional submanifold $H^1 \subset M^1$, we shall distinguish three cases by dividing Case **(I)** in two subcases $(\mathbf{I_1})$ and $(\mathbf{I_2})$.

- $(\mathbf{I_1})$ The nonzero vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^c M$ and $\dim_{\mathbb{R}}\big(T_{p_1}H^1 \cap T_{p_1}^c M\big) = 0$.
- $(\mathbf{I_2})$ The nonzero vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^c M$ and $\dim_{\mathbb{R}}\big(T_{p_1}H^1 \cap T_{p_1}^c M\big) = 1$ (this possibility can only occur when $n \geqslant 3$).
- **(II)** The nonzero vector v_1 belongs to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^c M$.

In case $(\mathbf{I_1})$, we notice that the assumption $T_{p_1}H^1 \cap T_{p_1}^c M = \{0\}$ implies that v_1 does not belong to the characteristic direction, because $v_1 \in T_{p_1}H^1$. Also, in case **(II)**, we notice that $\dim_{\mathbb{R}}\big(T_{p_1}H^1 \cap T_{p_1}^c M\big) = 1$ because $v_1 \in T_{p_1}H^1$, because $T_{p_1}H^1 \subset T_{p_1}M^1$ and because the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^c M$ is one-dimensional.

In each of the above three cases, it will be convenient in Section 8 below to work with simultaneously normalized defining (in)equations for M, for M^1, for $(M^1)^+$, for H^1, for $(H^1)^+$, for C_1', for v_1, for C_1, for $(N^1)^+$ and for $\mathcal{HW}_{p_1}^+$, in a single coordinate system centered at p_1. In the next paragraphs, we shall set up further elementary normalization lemmas in a *common system of coordinates*, firstly for Case $(\mathbf{I_1})$, secondly for Case $(\mathbf{I_2})$ and thirdly for Case **(II)**. This technical work is unavoidable and it will be achieved rigorously.

First of all, in the above coordinate system (z_1, z') with $T_{p_1}M = \{y_2 = \cdots = y_n = 0\}$ and with $T_{p_1}M^1 = \{y_1 = y_2 = \cdots = y_n = 0\}$, by means of the implicit function theorem, we can represent locally M by $(n-1)$ graphed equations of the form $y_2 = \varphi_2(x, y_1), \dots, y_n = \varphi_n(x, y_1)$, where the φ_j are $\mathcal{C}^{2,\alpha}$ functions satisfying $\varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_1}\varphi_j(0) = 0$ for $j = 2, \dots, n$, $k = 1, \dots, n$ and we can represent M^1 by n graphed equations $y_1 = h_1(x), y_2 = h_2(x), \dots, y_n = h_n(x)$, where the h_j are $\mathcal{C}^{2,\alpha}$ functions satisfying $h_j(0) = \partial_{x_k} h_j(0) = 0$ for $j, k = 1, \dots, n$.

5.22. First order normalizations in Case $(\mathbf{I_1})$. Thus, let us deal first with Case $(\mathbf{I_1})$. After a possible permutation of coordinates, we can assume that $T_{p_1}H^1$, which is a one-codimensional subspace of $T_{p_1}M^1$, is given by the equations

$$x_1 = b_2x_2 + \cdots + b_nx_n, \quad y_1 = 0, \quad y' = 0, \tag{5.23}$$

for some real numbers $b_2, \dots, b_n$. If we define the linear invertible transformation $\widehat{z}_1 := z_1 - b_2z_2 - \cdots - b_nz_n$, $\widehat{z}' := z'$, then the plane $T_{p_1}H^1$ written in (5.23) clearly transforms to the plane $\widehat{x}_1 = \widehat{y}_1 = \widehat{y}' = 0$, and (fortunately) $T_{p_1}M$ and $T_{p_1}M^1$ are left unchanged, namely $T_{p_1}\widehat{M} = \{\widehat{y}' = 0\}$ and $T_{p_1}\widehat{M}^1 = \{\widehat{y}_1 = \widehat{y}' = 0\}$.

Dropping the hats on coordinates, we have $T_{p_1}M = \{y' = 0\}$, $T_{p_1}M^1 = \{y_1 = y' = 0\}$, $T_{p_1}H^1 = \{x_1 = y_1 = y' = 0\}$. Let $\mathsf{C}_1' \subset\subset \mathsf{C}_{p_1}'$ be the open infinite strictly convex linear cone introduced in §5.14, which is contained in the real $(n-1)$-dimensional space $\{(0, x')\}$ and which is defined by $(n-1)$ strict inequalities $\ell_1'(x') > 0, \dots, \ell_{n-1}'(x') > 0$. By means of a real linear invertible transformation of the form $\widehat{z}_1 := z_1$, $\widehat{z}' := A' \cdot z'$, where A' is an $(n-1) \times (n-1)$ real matrix, we can transform C_1' to a cone $\widehat{\mathsf{C}}_1'$ defined by the simpler inequalities $\widehat{x}_2 > 0, \dots, \widehat{x}_n > 0$. Fortunately, this transformation stabilizes $T_{p_1}M$, $T_{p_1}M^1$ and $T_{p_1}H^1$.

Dropping the hats on coordinates, we now have $T_{p_1}M = \{y' = 0\}$, $T_{p_1}M^1 = \{y_1 = y' = 0\}$, $T_{p_1}H^1 = \{x_1 = y_1 = y' = 0\}$ and $\mathsf{C}_1' = \{(0, x') : x_2 > 0, \dots, x_n > 0\}$. Then the nonzero vector $v_1 \in T_{p_1}H^1$ which belongs to C_1' has coordinates $v_1 = (0, v_{2;1}, \dots, v_{n;1}) \in \mathbb{R}^n$, where $v_{2;1} > 0, \dots, v_{n;1} > 0$. By means of real dilations or real contractions of the real axes $\mathbb{R}_{x_2}, \dots, \mathbb{R}_{x_n}$ (a transformation which does not perturb the previously achieved normalizations), we can arrange that $v_1 = (0, 1, \dots, 1)$ and that $T_{p_1}(M^1)^+ = \{y' = 0, y_1 > 0\}$, $T_{p_1}(H^1)^+ = \{y = 0, \ x_1 > 0\}$.

Finally, the linear one-codimensional subspace $\Sigma^1 \subset T_{p_1}M^1$ introduced in §5.14 which does not contain the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM \equiv \mathbb{R}_{x_1}$ may be represented by an equation of the form $\sigma(x_1, x') := x_1 + a_2x_2 + \cdots + a_nx_n = 0$, for some real numbers $a_2, \dots, a_n$. The vector v_1 belongs to the cone C_1 defined by (5.15), hence $a_2 + \cdots + a_n > 0$. After a dilation of the x_1-axis, we can even assume that $a_2 + \cdots + a_n = 1$. We remind that by (5.18), the half-space $T_{p_1}(N^1)^+$ is given by $y_1 + a_2y_2 + \cdots + a_ny_n > 0$, hence there exists a $\mathcal{C}^{2,\alpha}$ function $\psi(x, y')$ with $\psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0$ for $k = 1, \dots, n$ and $j = 2, \dots, n$ such that $(N^1)^+$ is represented by the inequation $y_1 + a_2y_2 + \cdots + a_ny_n > \psi(x, y')$. Consequently, in this coordinate system, we may represent concretely the local half-wedge $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ constructed in §5.14 as

$$\begin{cases} \mathcal{HW}_1^+ = \Big\{ (z_1, z') \in \mathbb{C}^n : |z_1| < \delta_1, \ |z'| < \delta_1, \\ \qquad\qquad y_1 + a_2y_2 + \cdots + a_ny_n - \psi(x, y') > 0, \\ \qquad\qquad y_2 - \varphi_2(x, y_1) > 0, \dots, \ y_n - \varphi_n(x, y_1) > 0 \Big\}. \end{cases} \tag{5.24}$$

For the continuation of the proof of Proposition 5.12, it will also be convenient to proceed to further second order normalizations of the totally real submanifolds M^1 and H^1. These normalizations will all be tangent to the identity tranformation, hence they will leave the previously achieved normalizations unchanged.

5.25. Second order normalizations in Case ($\mathbf{I_1}$). Let us then perform a second order Taylor development of the defining equations of M^1

$$y = h(x) = \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, x_{k_1} x_{k_2} + o(|x|^2), \tag{5.26}$$

where the $a_{k_1,k_2} = \frac{1}{2}\partial_{x_{k_1}}\partial_{x_{k_2}} h(0)$ are vectors of $\mathbb{R}^n$. If we define the quadratic invertible transformation

$$\widehat{z} := z - i \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, z_{k_1} z_{k_2} = \Phi(z), \tag{5.27}$$

which is tangent to the identity mapping at the origin, then for $x + iy = x + ih(x) \in M^1$, we have by replacing (5.26) in the imaginary part of $\widehat{z}$ given by (5.27)

$$\begin{aligned} \widehat{y} &= y - \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, x_{k_1} x_{k_2} + \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, y_{k_1} y_{k_2} \\ &= \mathrm{o}(|x|^2) \\ &= \mathrm{o}\big(\left|\mathrm{Re}\, \Phi^{-1}(\widehat{z})\right|^2 \big) = \mathrm{o}\big(\, |(\widehat{x}, \widehat{y})|^2 \, \big), \end{aligned} \tag{5.28}$$

whence by applying the $\mathcal{C}^{2,\alpha}$ implicit function theorem to solve (5.28) in terms of $\widehat{y}$, we find that $\widehat{M^1} := \Phi(M^1)$ may be represented by an equation of the form $\widehat{y} = \widehat{h}(\widehat{x})$, for some $\mathbb{R}^n$-valued local $\mathcal{C}^{2,\alpha}$ mapping $\widehat{h}$ which satisfies $\widehat{h}(\widehat{x}) = \mathrm{o}(|\widehat{x}|^2)$.

Dropping the hats on coordinates, we can assume that the functions $h_1, \dots, h_n$ vanish at the origin to second order. Since $T_{p_1} H^1 = \{y = 0, \ x_1 = 0\}$, there exists a $\mathcal{C}^{2,\alpha}$ function $g(x')$ with $g(0) = \partial_{x_k} g(0) = 0$ for $k = 2, \dots, n$ such that $(H^1)^+$ is given by the equation $x_1 > g(x')$. We want to normalize also the defining equation $x_1 = g(x')$ of H^1. Instead of requiring, similarly as for $h_1, \dots, h_n$, that g vanishes to second order at the origin (which would be possible), we shall normalize g in order that $g(x') = -x_1^2 - \cdots - x_n^2 + \mathrm{o}\big(|x'|^{2+\alpha}\big)$ (which will also be possible, thanks to the total reality of H^1).

The reason why we want $(H^1)^+ = \{x_1 > g(x')\}$ to be strictly concave is a trick to avoid having to construct discs half-attached to M^1 with prescribed second order jet, in order that their half-boundary does almost not touch the singularity C, which lies behing the wall $H^1 \subset M^1$, namely $C \subset p_1 \cup (H^1)^-$. In Section 8 below, we shall construct such discs whose half-boundary is almost tangent to $(H^1)^-$ at p_1, and by arranging in advance strong geometric convexity of $(H^1)^-$, it will suffice that the half boundaries are tangent to H^1. in Figure 18, the half boundaries are the vertical lines slightly rotated and indeed, they do not enter much $(H^1)^-$.

Thus, we perform a second order Taylor development of the defining equations of H^1

$$\text{(5.29)} \qquad \begin{cases} x_1 = g\,(x') = \sum_{k_1,k_2=2}^{n} b_{k_1,k_2}\, x_{k_1} x_{k_2} + o\big(|x'|^2\big), \\ y = h\big(g(x'),x'\big) =: k\big(x'\big) = \mathrm{o}\big(|x'|^2\big), \end{cases}$$

where the $b_{k_1,k_2} = \frac{1}{2}\partial_{x_{k_1}}\partial_{x_{k_2}} g(0)$ are real numbers. If we define the quadratic invertible transformation

$$\text{(5.30)} \qquad \begin{cases} \widehat{z}_1 := z_1 - \sum_{k_1,k_2=2}^{n} b_{k_1,k_2}\, z_{k_1} z_{k_2} - z_2^2 - \cdots - z_n^2, \\ \widehat{z}' := z', \end{cases}$$

which is tangent to the identity mapping, then for $\big(g(x') + ik_1(x'), x' + ik'(x')\big) \in H^1$, we have by replacing (5.29) in the real part of $\widehat{z}_1$, given by (5.30):
(5.31)

$$\begin{aligned} \widehat{x}_1 &= x_1 - \sum_{k_1,k_2=2}^{n} b_{k_1,k_2}\, x_{k_1} x_{k_2} + \sum_{k_1,k_2=2}^{n} b_{k_1,k_2}\, y_{k_1} y_{k_2} - \sum_{k=2}^{n} x_k^2 + \sum_{k=2}^{n} y_k^2, \\ &= -\, x_2^2 - \cdots - x_n^2 + \mathrm{o}\big(|x'|^2\big) \\ &= -\, \widehat{x}_2^2 - \cdots - \widehat{x}_n^2 + \mathrm{o}\big(|(\widehat{x},\widehat{y})|^2\big). \end{aligned}$$

Similarly (dropping the elementary computations), we may obtain for the imaginary part of $\widehat{z}_1$ and for the imaginary part of $\widehat{z}'$

$$\text{(5.32)} \qquad \widehat{y}_1 = \mathrm{o}\big(|(\widehat{x},\widehat{y})|^2\big) \quad \text{and} \quad \widehat{y}' = \mathrm{o}\big(|(\widehat{x},\widehat{y})|^2\big),$$

whence by applying the $\mathcal{C}^{2,\alpha}$ implicit function theorem to solve the system (5.31), (5.32) in terms of $\widehat{x}_1$, $\widehat{y}_1$ and $\widehat{y}'$, we find that $\widehat{H}^1 := \Phi(H^1)$ may be represented by equations of the form

$$\text{(5.33)} \qquad \begin{cases} \widehat{x}_1 = \widehat{g}\big(\widehat{x}'\big) = -\widehat{x}_2^2 - \cdots - \widehat{x}_n^2 + \mathrm{o}\big(|\widehat{x}'|^2\big), \\ \widehat{y} = \widehat{k}\big(\widehat{x}'\big) = o\big(|\widehat{x}'|^2\big). \end{cases}$$

It remains to check that the above transformation has not perturbed the previous second order normalizations of $h_1, \ldots, h_n$ (this is important), which is easy: replacing y by $h(x) = \mathrm{o}(|x|^2)$ in the imaginary parts of $\widehat{z}_1$ and of $\widehat{z}'$ defined by the transformation (5.30), we get firstly

$$\text{(5.34)} \qquad \begin{aligned} \widehat{y}_1 &= y_1 - \sum_{k_1,k_2}^{n} b_{k_1,k_2}\, \left(x_{k_1} y_{k_2} + y_{k_1} x_{k_2}\right) - 2\sum_{k=2}^{n} x_k y_k \\ &= \mathrm{o}\big(|x|^2\big) \\ &= \mathrm{o}\big(|\mathrm{Re}\,\Phi^{-1}(\widehat{z})|^2\big) = \mathrm{o}\big(|(\widehat{x y})|^2\big), \end{aligned}$$

and similarly

$$\text{(5.35)} \qquad \widehat{y}' = o\big(|(\widehat{x},\widehat{y})|^2\big),$$

whence by applying the $\mathcal{C}^{2,\alpha}$ implicit function theorem to solve the system (5.34), (5.35) in terms of $\widehat{y}$, we find that $\widehat{M}^1 := \Phi(M^1)$ may be represented by equations of the form $\widehat{y} = \widehat{h}\,(\widehat{x}) = \mathrm{o}\big(|\widehat{x}|^2\big)$. Thus, after dropping the hats on coordinates, all the desired normalizations are satisfied. We now summarize these normalizations and we formulate just afterwards the analogous normalizations for Cases $\mathbf{(I_2)}$ and **(II)**.

5.36. Simultaneous normalizations. In the following lemma, the final choice of sufficiently small radii $\rho_1 > 0$ and $\delta_1 > 0$ is made after that all the biholomorphic changes of coordinates are performed.

Lemma 5.37. *Let M, M^1, p_1, H^1, v_1, $(H^1)^+$, $\mathcal{HW}_{p_1}^+$, C_{p_1} and FC_{p_1} be as in Proposition 5.12. Then there exists a sub-half-wedge $\mathcal{HW}_1^+$ contained in $\mathcal{HW}_{p_1}^+$ such that the following normalizations hold in each of the three cases* $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ *and* **(II)**:

- $\mathbf{(I_1)}$ *If $\dim_{\mathbb{R}}\big(T_{p_1}H^1 \cap T_{p_1}^cM\big) = 0$ (whence $v_1 \not\in T_{p_1}^cM$), then there exists a system of holomorphic coordinates $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ vanishing at p_1 with the vector v_1 equal to $(0, 1, \dots, 1)$, there exists positive numbers ρ_1 and δ_1 with $0 < \delta_1 < \rho_1$, there exist $\mathcal{C}^{2,\alpha}$ functions $\varphi_2, \dots, \varphi_n, h_1, \dots, h_n, g, k_1, \dots, k_n, \psi$, all defined in real cubes of edge $2\rho_1$ and of the appropriate dimension, and there exist real numbers $a_1, \dots, a_n$ with $a_2 + \cdots + a_n = 1$, such that, if we denote $z' := (z_2, \dots, z_n) = x' + iy'$, then M, M^1, $(M^1)^+$, H^1, $(H^1)^+$ and N^1 are represented in the polydisc of radius ρ_1 centered at p_1 by the following graphed (in)equations and the sub-half-wedge $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ is represented in the polydisc of radius δ_1 centered at p_1 by the following inequations*

$$
(5.38)\qquad
\begin{cases}
M: & y_2 = \varphi_2(x, y_1), \dots\dots, \ y_n = \varphi_n(x, y_1), \\
M^1: & y_1 = h_1(x), y_2 = h_2(x), \dots\dots, \ y_n = h_n(x), \\
(M^1)^+: & y_1 > h_1(x), y_2 = \varphi_2(x, y_1), \dots\dots, \ y_n = \varphi_n(x, y_1), \\
H^1: & x_1 = g(x'), \ y_1 = k_1(x'), \dots\dots, \ y_n = k_n(x'), \\
(H^1)^+: & x_1 > g(x'), \ y_1 = h_1(x), y_2 = h_2(x), \dots\dots, \ y_n = h_n(x), \\
N^1: & y_1 + a_2y_2 + \cdots + a_ny_n = \psi(x, y'), \\
\mathcal{HW}_1^+: & y_1 + a_2y_2 + \cdots + a_ny_n > \psi(x, y'), \\
& \qquad y_2 > \varphi_2(x, y_1), \dots, y_n > \varphi_n(x, y_1),
\end{cases}
$$

where we can assume that M^1 coincides with the intersection $M \cap \{y_1 = h_1(x)\}$, that H^1 coincides with the intersection $M^1 \cap \{x_1 = g(x')\}$ and that N^1 contains M^1, which yields at the level of defining equations the following three collections of identities

$$
(5.39)\qquad
\begin{cases}
h_2(x) \equiv \varphi_2(x, h_1(x)), \dots\dots, h_n(x) \equiv \varphi_n(x, h_1(x)), \\
k_1(x') \equiv h_1(g(x'), x'), \dots\dots, k_n(x') \equiv h_n(g(x'), x'), \\
\psi(x, h'(x)) \equiv h_1(x) + a_2h_2(x) + \cdots + a_nh_n(x),
\end{cases}
$$

and where the following normalizations hold:

(5.40)
$$\begin{cases}\varphi_j(0)=\partial_{x_k}\varphi_j(0)=\partial_{y_1}\varphi_j(0)=0, & j=2,\dots,n,\ k=1,\dots,n,\\ h_j(0)=\partial_{x_k}h_j(0)=\partial_{x_{k_1}}\partial_{x_{k_2}}h_j(0)=0, & j,k,k_1,k_2=1,\dots,n,\\ g(0)=\partial_{x_k}g(0)=k_j(0)=\partial_{x_k}k_j(0)=0, & j=1,\dots,n,\ k=2,\dots,n,\\ \qquad\partial_{x_{k_1}}\partial_{x_{k_2}}g(0)=-\delta_{k_1}^{k_2}, & k_1,k_2=2,\dots,n,\\ \psi(0)=\partial_{x_k}\psi(0)=\partial_{y_j}\psi(0)=0, & k=1,\dots,n,\ j=2,\dots,n.\end{cases}$$

In other words, $T_0M=\{y'=0\}$ *(hence* T_0^cM *coincides with the complex* z_1*-axis),* $T_0N^1=\{y_1+a_2y_2+\cdots+a_ny_n=0\}$ *and the second order Taylor approximations of the defining equations of* M^1*, of* H^1 *and of* $(H^1)^+$ *are the quadrics*

$$\begin{cases}T_{p_1}^{(2)}M^1: & y_1=0,\dots\dots,\ y_n=0,\\ T_{p_1}^{(2)}H^1: & x_1=-x_2^2-\cdots-x_n^2,\ y_1=0,\dots\dots,\ y_n=0,\\ T_{p_1}^{(2)}(H^1)^+: & x_1>-x_2^2-\cdots-x_n^2,\ y_1=0,\dots\dots,\ y_n=0.\end{cases}\tag{5.41}$$

($\mathbf{I_2}$) *Similarly, if* $\dim_{\mathbb{R}}\left(T_{p_1}H^1\cap T_{p_1}^cM\right)=1$ *and if* $v_1\notin T_{p_1}^cM$ *(this possibility can only occur in the case* $n\geqslant 3$*), then there exists a system of holomorphic coordinates* $z=(z_1,\dots,z_n)=(x_1+iy_1,\dots,x_n+iy_n)$ *vanishing at* p_1 *with* v_1 *equal to* $(1,\dots,1,0)$*, there exists positive numbers* ρ_1 *and* δ_1 *with* $0<\delta_1<\rho_1$*, there exist* $\mathcal{C}^{2,\alpha}$*-smoooth functions* $\varphi_2,\dots,\varphi_n,h_1,\dots,h_n,g,k_1,\dots,k_n,\psi$ *all defined in real cubes of edge* $2\rho_1$ *and of the appropriate dimension, such that if we denote* $z'':=(z_1,\dots,z_{n-1})=x''+iy''$ *and* $z'=(z_2,\dots,z_n)=x'+iy'$*, then* M*,* M^1*,* $(M^1)^+$*,* H^1*,* $(H^1)^+$ *and* N^1 *are represented in the polydisc of radius* ρ_1 *centered at* p_1 *by the following graphed (in)equations and the sub-half-wedge* $\mathcal{HW}_1^+\subset\mathcal{HW}_{p_1}^+$ *is represented in the polydisc of radius* δ_1 *centered at* p_1 *by the following inequations*

$$\begin{cases}M: & y_2=\varphi_2(x,y_1),\dots\dots,\ y_n=\varphi_n(x,y_1),\\ M^1: & y_1=h_1(x),y_2=h_2(x),\dots\dots,\ y_n=h_n(x),\\ (M^1)^+: & y_1>h_1(x),y_2=\varphi_2(x,y_1),\dots\dots,\ y_n=\varphi_n(x,y_1),\\ H^1: & x_n=g(x''),\ y_1=k_1(x''),\dots\dots,\ y_n=k_n(x''),\\ (H^1)^+: & x_n>g(x''),\ y_1=h_1(x),y_2=h_2(x),\dots\dots,\ y_n=h_n(x),\\ N^1: & y_2+\cdots+y_{n-1}-y_n=\psi(x,y'),\\ \mathcal{HW}_1^+: & y_2+\cdots+y_{n-1}-y_n>\psi(x,y'),\\ & \qquad y_1>\varphi_1(x,y_1),\dots,y_{n-1}>\varphi_{n-1}(x,y_1),\end{cases}\tag{5.42}$$

where we can assume that M^1 *coincides with the intersection* $M\cap\{y_1=h_1(x)\}$*, that* H^1 *coincides with the intersection* $M^1\cap\{x_1=g(x')\}$ *and that* N^1 *contains* M^1*, which yields at the level of defining equations the following* *three collections of identities*

$$
\tag{5.43}
\left\{
\begin{aligned}
h_2(x) \equiv \varphi_2(x, h_1(x)), \dots\dots, h_n(x) \equiv \varphi_n(x, h_1(x)), \\
k_1(x'') \equiv h_1(x'', g(x'')), \dots\dots, k_n(x'') \equiv h_n(x'', g(x'')), \\
\psi(x, h'(x)) \equiv h_1(x) + h_2(x) + \cdots + h_{n-1}(x) - h_n(x),
\end{aligned}
\right.
$$

and where the following normalizations hold:

(5.44)

$$
\left\{
\begin{aligned}
& \varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_1}\varphi_j(0) = 0, \quad j = 2, \dots, n, \ k = 2, \dots, n, \\
& h_j(0) = \partial_{x_k} h_j(0) = \partial_{x_{k_1}}\partial_{x_{k_2}} h_j(0) = 0, \quad j, k, k_1, k_2 = 1, \dots, n, \\
& g(0) = \partial_{x_k} g(0) = k_j(0) = \partial_{x_k} k_j(0) = 0, \quad j = 1, \dots, n, \ k = 1, \dots, n-1, \\
& \qquad \partial_{x_{k_1}}\partial_{x_{k_2}} g(0) = -\delta_{k_1}^{k_2}, \quad k_1, k_2 = 1, \dots, n-1, \\
& \psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0, \quad k = 1, \dots, n, \ j = 2, \dots, n.
\end{aligned}
\right.
$$

In other words, $T_0 M = \{y' = 0\}$ *(hence* $T_0^c M$ *coincides with the complex* z_1*-axis),* $T_0 N^1 = \{y_1 + y_2 + \cdots + y_{n-1} - y_n = 0\}$ *and the second order Taylor approximations of the defining equations of* M^1*, of* H^1 *and of* $(H^1)^+$ *are the quadrics*

$$
\tag{5.45}
\left\{
\begin{aligned}
T_{p_1}^{(2)} M^1 : \quad & y_1 = 0, \dots\dots, \ y_n = 0, \\
T_{p_1}^{(2)} H^1 : \quad & x_n = -x_1^2 - \cdots - x_{n-1}^2, \ y_1 = 0, \dots\dots, \ y_n = 0, \\
T_{p_1}^{(2)} (H^1)^+ : \quad & x_n > -x_1^2 - \cdots - x_{n-1}^2, \ y_1 = 0, \dots\dots, \ y_n = 0.
\end{aligned}
\right.
$$

(II) *Finally, if* $\dim_{\mathbb{R}} \left(T_{p_1} H^1 \cap T_{p_1}^c M\right) = 1$ *and if* $v_1 \in T_{p_1}^c M$ *(this possibility can occur in all cases* $n \geqslant 2$*), then there exists a system of holomorphic coordinates* $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ *vanishing at* p_1 *with* v_1 *equal to* $(1, 0, \dots, 0)$*, there exist positive numbers* ρ_1 *and* δ_1 *with* $0 < \delta_1 < \rho_1$*, there exist* $\mathcal{C}^{2,\alpha}$*-smoooth functions* $\varphi_2, \dots, \varphi_n, h_1, \dots, h_n, g, k_1, \dots, k_n$ *all defined in real cubes of edge* $2\rho_1$ *and of the appropriate dimension, such that if we denote* $z'' := (z_1, \dots, z_{n-1}) = x'' + iy''$ *and* $z' = (z_2, \dots, z_n) = x' + iy'$*, then* M*,* M^1*,* $(M^1)^+$*,* H^1 *and* $(H^1)^+$ *are represented in the polydisc of radius* ρ_1 *centered at* p_1 *by the first five (in)equations of* (5.42) *together with the normalizations* (5.45) *and such that the local wedge* $\mathcal{W}_2 \subset \mathcal{HW}_{p_1}^+$ *of edge* M^1 *at* p_1 *is represented in the polydisc of radius* δ_1 *centered at* p_1 *by the following inequations*

(5.46)

$$
\left\{
\begin{aligned}
\mathcal{W}_2 : \quad & y_1 - h_1(x) > -\left[y_2 - h_2(x)\right], \dots\dots, \ y_1 - h_1(x) > -\left[y_n - h_n(x)\right], \\
& y_1 - h_1(x) > y_2 - h_2(x) + \cdots + y_n - h_n(x).
\end{aligned}
\right.
$$

5.47. Summarizing figure and proof of Lemma 5.37. To illustrate this technical lemma, by specifying the value $n = 3$, we draw the cones C_1 and C_2 together with the vector v_1, the tangent plane $T_{p_1} H^1$ and the hyperplane Σ^1 in the three cases **($\mathbf{I_1}$)**, **($\mathbf{I_2}$)** and **(II)**. In the left part of this figure, the cone C_1 is given by $x_2 > 0$, $x_3 > 0$, $x_1 > -\frac{1}{2}x_2 - \frac{1}{2}x_3$, namely we choose the values $a_2 = a_3 = \frac{1}{2}$ for the drawing; in the central part, the cone C_1 is given by $x_1 > 0$, $x_2 > 0$, $x_2 > x_3$; in the right part, the cone C_2 is given by $x_1 > -x_2$, $x_1 > -x_3$, $x_1 > x_2 + x_3$.

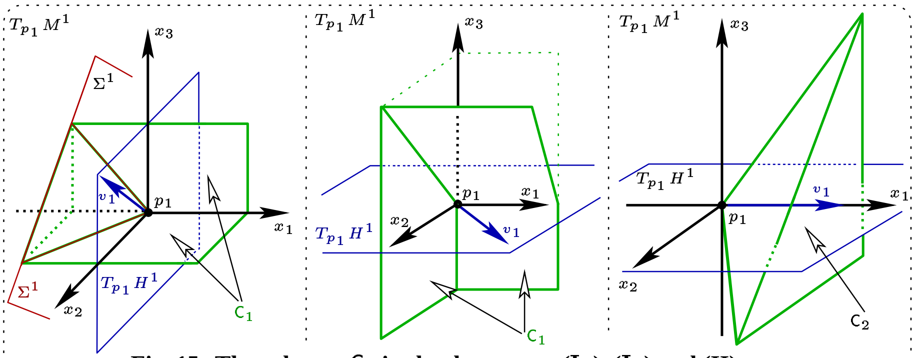

Fig. 15: The subcone C_1 in the three cases $(\mathbf{I_1})$, $(\mathbf{I_2})$ and (II)

Proof. Case $(\mathbf{I_1})$ has been completed before the statement of Lemma 5.37.

For Case $(\mathbf{I_2})$, we reason similarly, as follows. We start with the normalizations $T_{p_1}M = \{y' = 0\}$ and $T_{p_1}M^1 = \{y = 0\}$ as in the end of §5.21. By assumption, $T_{p_1}H^1$ contains the characteristic direction, which coincides with the x_1-axis. By means of an elementary real linear transformation of the form $\widehat{z}_1 := z_1$, $\widehat{z}' = A' \cdot z'$, we may first normalize $T_{p_1}H^1$ to be the hyperplane (after dropping the hats on coordinates) $\{x_n = 0, \ y = 0\}$. Similarly, we may normalize v_1 to be the vector $(1, 1, \dots, 1, 0)$. Let again $\pi' : (x_1, x') \mapsto x'$ denote the canonical projection on the x'-space. Then $\pi'(v_1) = (1, \dots, 1, 0)$. Using again a real linear transformation of the form $\widehat{z}_1 := z_1$, $\widehat{z}' = A' \cdot z'$, we can assume that the proper subcone $\mathsf{C}_1' \subset\subset \mathsf{C}_{p_1}' \equiv \pi'(\mathsf{C}_{p_1})$ which contains the vector v_1 is given (after dropping the hats on coordinates) by

$$\text{(5.48)} \qquad \mathsf{C}_1' : \quad x_2 > 0, \dots, x_{n-1} > 0, \ x_2 + \cdots + x_{n-1} > x_n.$$

Following §5.14 (*cf.* Figure 14), we choose a linear cone $\mathsf{C}_1 \subset\subset \mathsf{FC}_{p_1}$ defined by the $(n-1)$ inequations of C_1' plus one inequation of the form $x_1 > a_2x_2 + \cdots + a_nx_n$ with $1 > a_2 + \cdots + a_{n-1}$, since v_1 belongs to C_1. Then by means of a real linear transformation of the form $\widehat{z}_1 := z_1 + a_2z_2 + \cdots + a_nz_n$, $\widehat{z}' := z'$, which stabilizes $\pi'(v_1)$ and the inequations (5.48) of C_1', we can assume that the supplementary inequation for C_1, namely the inequation for $(\Sigma^1)^+$, is simply (after dropping the hats on coordinates) $x_1 > 0$. Then the vector v_1 is mapped to the vector of coordinates $(1 - a_2 - \cdots - a_n, 1, \dots, 1, 0)$, which we map to the vector of coordinates $(1, 1, \dots, 1, 0)$ by an obvious positive scaling of the x_1-axis. In conclusion, in the final system of coordinates, the cone C_1 is given by

$$\text{(5.49)} \qquad \mathsf{C}_1 : \quad x_1 > 0, x_2 > 0, \dots, x_{n-1} > 0, \ x_2 + \cdots + x_{n-1} - x_n > 0.$$

This implies that the half-wedge $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ may be represented by the inequations of the last two line of (5.42). To conclude the proof of Case $(\mathbf{I_2})$ of Lemma 5.37, it suffices to observe that, as in Case $(\mathbf{I_1})$, the further second order normalizations do not perturb the previously achieved first order normalizations, because the transformations are tangent to the identity mapping at the origin.

Finally, we treat Case **(II)** of Lemma 5.37, starting with the system of coordinates $(z_1, \dots, z_n)$ of the end of §5.21. After an elementary real linear transformation stabilizing the characteristic x_1-axis, we can assume that $v_1 = (1, 0, \dots, 0)$ and that the convex infinite linear cone C_2 introduced in §5.19 which contains v_1 is given by the inequations

$$x_1 > -x_2, \ldots\ldots, \; x_1 > -x_n, \; x_1 > x_2 + \cdots + x_n. \tag{5.50}$$

This implies that the local wedge $\mathcal{W}_2 \subset \mathcal{HW}_{p_1}^+$ of edge M^1 at p_1 introduced in §5.19 may be represented by the inequations (5.46). Finally, the second order normalizations, which are tangent to the identity mapping, are achieved as in the two previous cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$.

The proof of Lemma 5.37 is complete. □

§6. Three preparatory lemmas in Hölder spaces

We first collect a few very elementary lemmas that will be useful in our geometric construction of half-attached analytic discs (Section 7). The index notation g_{x_k} denotes partial derivative.

6.1. Local growth of $\mathcal{C}^{2,\alpha}$ mappings. Let $n \in \mathbb{N}$ with $n \geqslant 1$ and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We consider the norm $|x| := \max_{1 \leqslant k \leqslant n} |x_k|$. If $g = g(x)$ is an $\mathbb{R}^n$-valued $\mathcal{C}^1$ map defined in the real cube $\{x \in \mathbb{R}^n : \; |x| < 2\rho_1\}$, for some $\rho_1 > 0$, and if $|x'|, |x''| \leqslant \rho$, for some $\rho < 2\rho_1$, then for $j = 1, \ldots, n$, we have the *mean value inequality*

$$\big|g_j(x') - g_j(x'')\big| \leqslant \big|x' - x''\big| \cdot \Big(\sum_{k=1}^n \sup_{|x| \leqslant \rho} |g_{j,x_k}(x)|\Big). \tag{6.2}$$

By the definition of the norm $|\cdot|$, we deduce $\big|g(x') - g(x'')\big| \leqslant \|g\|_{\mathcal{C}^1} \cdot |x' - x''|$.

Let α with $0 < \alpha < 1$ and let $h = h(x) = \big(h_1(x), \ldots, h_n(x)\big)$ be an $\mathbb{R}^n$-valued $\mathcal{C}^{2,\alpha}$ map defined in $\{x \in \mathbb{R}^n : \; |x| < 2\rho_1\}$. For every $\rho < 2\rho_1$, we define:

$$\begin{aligned} \|h\|_{\mathcal{C}^{2,\alpha}(\{|x| \leqslant \rho\})} := \sup_{|x| \leqslant \rho} |h(x)| + \sum_{k=1}^n \sup_{|x| \leqslant \rho} \big|h_{x_k}(x)\big| + \sum_{k_1, k_2 = 1}^n \big|h_{x_{k_1} x_{k_2}}(x)\big| + \\ + \sum_{k_1, k_2 = 1}^n \sup_{|x'|, |x''| \leqslant \rho, \; x' \neq x''} \frac{\big|h_{x_{k_1} x_{k_2}}(x') - h_{x_{k_1} x_{k_2}}(x'')\big|}{|x' - x''|^\alpha} < \infty. \end{aligned} \tag{6.3}$$

Lemma 6.4. *Under the above assumptions, let*

$$K_1 := \|h\|_{\mathcal{C}^{2,\alpha}(\{|x| \leqslant \rho_1\})} < \infty \tag{6.5}$$

be the $\mathcal{C}^{2,\alpha}$ norm of h over $\{|x| \leqslant \rho_1\}$ and assume that $h_j(0) = 0$, $h_{j,x_k}(0) = 0$ and $h_{j,x_{k_1} x_{k_2}}(0) = 0$, for all $j, k, k_1, k_2 = 1, \ldots, n$. Then for $|x| \leqslant \rho_1$ we have:

$$\begin{cases} [\mathbf{1}]: \quad |h(x)| \leqslant |x|^{2+\alpha} \cdot K_1, \\ [\mathbf{2}]: \quad \sum_{k=1}^n |h_{x_k}(x)| \leqslant |x|^{1+\alpha} \cdot K_1, \\ [\mathbf{3}]: \quad \sum_{k_1, k_2 = 1}^n \Big|h_{x_{k_1} x_{k_2}}(x)\Big| \leqslant |x|^\alpha \cdot K_1. \end{cases} \tag{6.6}$$

6.7. A $\mathcal{C}^{1,\alpha}$ estimate for composition of mappings. Recall that Δ is the open unit disc in $\mathbb{C}$ and that $\partial\Delta$ is its boundary, namely the unit circle. We shall constantly denote the complex variable in $\overline{\Delta} := \Delta \cup \partial\Delta$ by $\zeta = \rho\, e^{i\theta}$, where $0 \leqslant \rho \leqslant 1$ and where $|\theta| \leqslant \pi$, except when we consider two points $\zeta' = e^{i\theta'}$, $\zeta'' = e^{i\theta''}$, in which case we may obviously choose $|\theta'|,\ |\theta''| \leqslant 2\pi$ with $0 \leqslant |\theta' - \theta''| \leqslant \pi$. Let now $X(\zeta) = \big(X_1(\zeta), \dots, X_n(\zeta)\big)$ be an $\mathbb{R}^n$-valued mapping which is $\mathcal{C}^{1,\alpha}$ on $\partial\Delta$. We define its $\mathcal{C}^{1,\alpha}$-norm precisely by

(6.8)
$$\|X\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} := \sup_{|\theta|\leqslant\pi} \left|X\left(e^{i\theta}\right)\right| + \sup_{|\theta|\leqslant\pi} \left|\frac{dX\left(e^{i\theta}\right)}{d\theta}\right| + \sup_{0<|\theta'-\theta''|\leqslant\pi} \frac{\left|\frac{dX(e^{i\theta'})}{d\theta} - \frac{dX(e^{i\theta''})}{d\theta}\right|}{|\theta'-\theta''|^\alpha},$$

and its $\mathcal{C}^1$-norm $\|X\|_{\mathcal{C}^1(\partial\Delta)}$ by keeping only the first two terms.

Lemma 6.9. *If h is as in Lemma 6.5, and if moreover $\left|X\left(e^{i\theta}\right)\right| \leqslant \rho$ for all θ with $|\theta| \leqslant \pi$, withe $\rho \leqslant \rho_1$, then we have the following composition norm estimates*:

$$\begin{aligned}\|h(X)\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant \sup_{|x|\leqslant\rho} |h(x)| &+ \Big(\sum_{k=1}^n \sup_{|x|\leqslant\rho} |h_{x_k}(x)|\Big) \cdot \|X\|_{\mathcal{C}^1(\partial\Delta)} + \\ &+ \Big(\sum_{k_1,k_2=1}^n \sup_{|x|\leqslant\rho} \left|h_{x_{k_1}x_{k_2}}(x)\right|\Big) \cdot \pi^{1-\alpha} \cdot \left[\|X\|_{\mathcal{C}^1(\partial\Delta)}\right]^2 + \\ &+ \Big(\sum_{k=1}^n \sup_{|x|\leqslant\rho} |h_{x_k}(x)|\Big) \cdot \|X\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)},\end{aligned}$$

(6.10)
$$\begin{aligned}\sum_{k=1}^n \|h_{x_k}(X)\|_{\mathcal{C}^\alpha(\partial\Delta)} &\leqslant \sum_{k=1} \sup_{|x|\leqslant\rho} |h_{x_k}(x)| + \\ &+ \Big(\sum_{k_1,k_2=1}^n \sup_{|x|\leqslant\rho} \left|h_{x_{k_1}x_{k_2}}(x)\right|\Big) \cdot \pi^{1-\alpha} \cdot \|X\|_{\mathcal{C}^1(\partial\Delta)},\end{aligned}$$

$$\begin{aligned}\sum_{k_1,k_2=1}^n \left\|h_{x_{k_1}x_{k_2}}(X)\right\|_{\mathcal{C}^\alpha(\partial\Delta)} &\leqslant \sum_{k_1,k_2=1}^n \sup_{|x|\leqslant\rho} \left|h_{x_{k_1}x_{k_2}}(x)\right| + \\ &+ \|h\|_{\mathcal{C}^{2,\alpha}(\{|x|\leqslant\rho\})} \cdot \left(\|X\|_{\mathcal{C}^1(\partial\Delta)}\right)^\alpha.\end{aligned}$$

Proof. We summarize the computations. Applying the definition (6.8), using the chain rule for the calculation of $dh\big(X(e^{i\theta})\big)/d\theta$, and using the trivial inequality $|a'b' - a''b''| \leqslant |a'| \cdot |b' - b''| + |b''| \cdot |a' - a''|$, we may majorize

(6.11)
$$\begin{aligned}\|h(X)\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant \sup_{|\theta|\leqslant\pi} \left|h(X(e^{i\theta}))\right| &+ \Big(\sum_{k=1}^n \sup_{|\theta|\leqslant\pi} \left|h_{x_k}(X(e^{i\theta}))\right|\Big) \cdot \max_{1\leqslant k\leqslant n} \sup_{|\theta|\leqslant\pi} \left|\frac{dX_k(e^{i\theta})}{d\theta}\right| + \\ &\sup_{0<|\theta'-\theta''|\leqslant\pi} \sum_{k=1}^n \frac{\left|h_{x_k}(X(e^{i\theta'})) - h_{x_k}(X(e^{i\theta''}))\right|}{|\theta'-\theta''|^\alpha} \cdot \max_{1\leqslant k\leqslant n} \sup_{|\theta'|\leqslant\pi} \left|\frac{dX_k(e^{i\theta'})}{d\theta}\right| + \\ &\Big(\sum_{k=1}^n \sup_{|\theta''|\leqslant\pi} \left|h_{x_k}(e^{i\theta''})\right|\Big) \cdot \Big(\max_{1\leqslant k\leqslant n} \sup_{0<|\theta'-\theta''|\leqslant\pi} \frac{\left|\frac{dX_k(e^{i\theta'})}{d\theta} - \frac{dX_k(e^{i\theta''})}{d\theta}\right|}{|\theta'-\theta''|^\alpha}\Big),\end{aligned}$$

which yields the first inequality of (6.10) after using (6.2) for the second line of (6.11) and the trivial majoration $|\theta' - \theta''|^{1-\alpha} \leqslant \pi^{1-\alpha}$. The second and the third inequalities of (6.10) are established similarly, which completes the proof. □

Lemma 6.12. *With h as in Lemma 6.4, suppose that there exist constants $c_1 > 0$, $K_2 > 0$ with $c_1 K_2 \leqslant \rho_1$ such that for each $c \in \mathbb{R}$ with $0 \leqslant c \leqslant c_1$, there exists $X_c \in \mathcal{C}^{1,\alpha}(\partial\Delta, \mathbb{R}^n)$ with $\|X_c\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c \cdot K_2$. Then there exists a constant $K_3 > 0$ such that the following three estimates hold*:

$$\tag{6.13} \left\{ \begin{aligned} \|h(X_c)\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} &\leqslant c^{2+\alpha} \cdot K_3, \\ \sum_{k=1}^n \|h_{x_k}(X_c)\|_{\mathcal{C}^{\alpha}(\partial\Delta)} &\leqslant c^{1+\alpha} \cdot K_3, \\ \sum_{k_1,k_2=1}^n \left\|h_{x_{k_1} x_{k_2}}(X_c)\right\|_{\mathcal{C}^{\alpha}(\partial\Delta)} &\leqslant c^{\alpha} \cdot K_3. \end{aligned} \right.$$

Proof. Applying Lemmas 6.4 and 6.9, we see that it suffices to choose

$$\tag{6.14} K_3 := \max\left(K_1 K_2^{2+\alpha}(3 + \pi^{1-\alpha}),\ K_1 K_2^{1+\alpha}(1 + \pi^{1-\alpha}),\ 2K_1 K_2^{\alpha}\right),$$

which completes the proof. □

Up to now, we have introduced three positive constants K_1, K_2, K_3. In Sections 7, 8 and 9 below, we shall introduce further positive constants K_4, K_5, K_6, K_7, K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, K_{16}, K_{17}, K_{18} and K_{19}, whose precise value will not be important.

§7. FAMILIES OF ANALYTIC DISCS HALF-ATTACHED TO MAXIMALLY REAL SUBMANIFOLDS

7.1. Preliminary. If $\partial^+\Delta := \{\zeta \in \partial\Delta : \mathrm{Re}\,\zeta \geqslant 0\}$ denotes the *positive half-boundary* of Δ, we say that an analytic disc $A \in \mathcal{O}(\Delta, \mathbb{C}^n) \cap \mathcal{C}^0(\overline{\Delta})$ is *half-attached* to a set $E \subset \mathbb{C}^n$ if $A(\partial^+\Delta) \subset E$.

We will construct local families of analytic discs $Z^1_{c,x,v}(\zeta) : \overline{\Delta} \to \mathbb{C}^n$, where $c \in \mathbb{R}^+$ is small, where $x \in \mathbb{R}^n$ is small and where $v \in \mathbb{R}^n$ is small, which are half-attached to a $\mathcal{C}^{2,\alpha}$ maximally real submanifold M^1 of $\mathbb{C}^n$, which satisfy $Z^1_{c,0,v}(1) \equiv p_1 \in M^1$, such that the boundary point $Z^1_{c,x,v}(1)$ covers a neighborhood of p_1 in M^1 when x varies (c and v being fixed) and such that the tangent vector $\frac{\partial Z^1_{c,0,v}}{\partial\theta}(1)$ at the fixed point p_1 covers a cone in $T_{p_1}M^1$ when v varies. With this choice, when x varies, v varies and ζ varies (but c is fixed), the set of points $Z^1_{c,x,v}(\zeta)$, covers a thin wedge of edge M^1 at p_1. By maximal reality of M^1, the tangent vector $\frac{\partial Z^1_{c,0,v}}{\partial\theta}(1) \in T_{p_1}M^1$ will be arbitrary, hence the associated wedge can have arbitrary orientation.

To summarize symbolically the structure of the desired family:

$$\tag{7.2} Z^1_{c,x,v}(\zeta) : \left\{ \begin{aligned} c &= \text{small scaling factor}, \\ x &= \text{translation parameter}, \\ v &= \text{rotation parameter}, \\ \zeta &= \text{unit disc variable}. \end{aligned} \right.$$

We begin our constructions in the "flat" case where the maximally real submanifold M^1 coincides with $\mathbb{R}^n$. Afterwards, we perform a pertubation argument, using the scaling parameter c in an essential way.

7.3. A family of analytic discs sweeping $\mathbb{R}^n \subset \mathbb{C}^n$ with prescribed first order jets. We denote the coordinates over $\mathbb{C}^n$ by $z = x + iy = (x_1 + iy_1, \dots, x_n + iy_n)$. Let $c \in \mathbb{R}$ with $c \geqslant 0$ be a "scaling factor", let $n \geqslant 2$, let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and consider the algebraically parametrized family of analytic discs defined by

$$B_{c,x,v}(s+it) := \big(x_1 + cv_1(s+it), \dots, x_n + cv_n(s+it)\big), \tag{7.4}$$

where $s + it \in \mathbb{C}$ is the holomorphic variable. For $c \neq 0$, the map $B_{c,x,v}$ embeds the complex line $\mathbb{C}$ into $\mathbb{C}^n$ and *sends* $\mathbb{R}$ *into* $\mathbb{R}^n$ *with arbitrary first order jet at* 0: center point $B_{c,x,v}(0) = x$ and tangent direction $\partial B_{c,x,v}(s)/\partial s|_{s=0} = cv$.

To localize our family of analytic discs, we restrict the map (7.4) to the following specific set of values: $0 \leqslant c \leqslant c_0$ for some $c_0 > 0$; $|x| \leqslant c$; $|v| \leqslant 2$; and $|s+it| \leqslant 4$. To localize $\mathbb{R}^n$, we shall denote $M^0 := \{x \in \mathbb{R}^n : |x| \leqslant \rho_0\}$, where $\rho_0 > 0$, and we notice that $B_{c,x,v}(\{|s+it| \leqslant 4\}) \subset M^0$ for all c, all x and all v provided that $c_0 \leqslant \rho_0/9$.

We then consider the mapping $(s+it) \longmapsto B_{c,x,v}(s+it)$ as a local (nonsmooth) analytic disc defined in the rectangle $\{s + it \in \mathbb{C} : |s| \leqslant 4, \ 0 \leqslant t \leqslant 4\}$ whose bottom boundary part $B_{c,x,v}([-4,4])$ is a small real segment contained in $\mathbb{R}^n$.

7.5. A useful conformal equivalence. To get rid of the corners of the rectangle, we proceed as follows. In the complex plane equipped with coordinates $s + it$, let $\mathcal{D}(i\sqrt{3}, 2)$ be the open disc of center $i\sqrt{3}$ and of radius 2. Let $\mu : (-2,2) \to [0,1]$ be an *even* $\mathcal{C}^\infty$ function satisfying $\mu(s) = 0$ for $0 \leqslant s \leqslant 1$; $\mu(s) > 0$ and $d\mu(s)/ds > 0$ for $1 < s < 2$; and $\mu(s) = \sqrt{3} - \sqrt{4 - s^2}$ for $\sqrt{3} \leqslant s < 2$. The simply connected domain $C^+ \subset \{t > 0\}$ which is represented in Figure 16 may be formally defined as

$$\begin{cases} C^+ \cap \{t \geqslant \sqrt{3} - 1\} := \mathcal{D}(i\sqrt{3}, 2) \cap \{t \geqslant \sqrt{3} - 1\}, \\ C^+ \cap \{0 < t < \sqrt{3} - 1\} := \{s + it \in \mathbb{C} : \ t > \mu(s)\}. \end{cases} \tag{7.6}$$

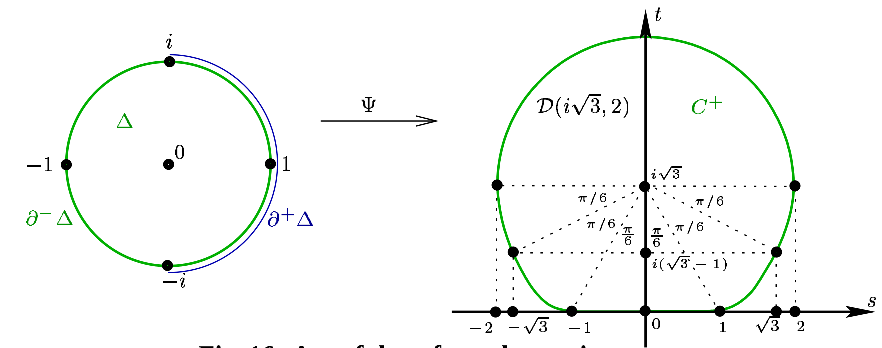

Fig. 16: A useful conformal mapping

Let $\Psi : \Delta \to C^+$ be a conformal equivalence (Riemann's theorem). Since the boundary ∂C^+ is $\mathcal{C}^\infty$, the mapping Ψ extends as a $\mathcal{C}^\infty$ diffeomorphism $\partial\Delta \to \partial C^+$. After a reparametrization of Δ, we can (and we shall) assume that $\Psi(\partial^+\Delta) = [-1, 1]$, $\Psi(1) = 0$ and $\Psi(\pm i) = \pm 1$. It follows that $d\Psi(e^{i\theta})/d\theta$ is a positive real number for all $e^{i\theta} \in \partial^+\Delta$.

7.7. Flat families of half-attached analytic discs. Thanks to Ψ, we can define a family of small analytic discs which are half-attached to the flat maximally real manifold $M^0 \equiv \{x \in \mathbb{R}^n : \ |x| \leqslant \rho_0\}$ as follows

$$Z^0_{c,x,v}(\zeta) := B_{c,x,v}\big(\Psi(\zeta)\big) = \big(x + cv\Psi(\zeta)\big). \tag{7.8}$$

We then have $Z^0_{c,x,v}(\partial^+\Delta) \subset M^0$ and $Z^0_{c,x,v}(1) = x$. Notice that every disc $Z_{c,x,v}\big(\overline{\Delta}\big)$ is contained in a single complex line. Starting with a maximally real submanifold of $\mathbb{C}^n$ as in Proposition 5.12, but dealing with the flat maximally real submanifold $M^0 \equiv \mathbb{R}^n$, we first construct a flat model of the desired family of analytic disc.

Lemma 7.9. *Let $p_0 \equiv 0 \in M^0$ denote the origin and let $v_0 \in T_{p_0}M^0$ be a tangent vector with $|v^0| = 1$. Then there exists a constant $\Lambda_0 > 0$ and there exists a $\mathcal{C}^\infty$ family $A^0_{c,x,v}(\zeta)$ of analytic discs defined for $c \in \mathbb{R}$ with $0 \leqslant c \leqslant c_0$ for some $c_0 > 0$ with $c_0 \leqslant \rho_0/9$, for $x \in \mathbb{R}^n$ with $|x| \leqslant c$ and for $v \in \mathbb{R}^n$ with $|v| \leqslant c$, which enjoys the following six properties*:

- **(1_0)** $A^0_{c,0,v}(1) = p_0 = 0$ *for all c and all v.*
- **(2_0)** $A^0_{c,x,v} : \overline{\Delta} \to \mathbb{C}^n$ *is an embedding and* $\big|A^0_{c,x,v}(\zeta)\big| \leqslant c \cdot \Lambda_0$ *for all c, all x, all v and all ζ.*
- **(3_0)** $A^0_{c,x,v}(\partial^+\Delta) \subset M^0$ *for all c, all x and all v.*
- **(4_0)** $\frac{\partial A^0_{c,0,0}}{\partial\theta}(1)$ *is a positive multiple of v_0 for all $c \neq 0$.*
- **(5_0)** *For all c, all v and all $e^{i\theta} \in \partial^+\Delta$, the mapping $x \longmapsto A^0_{c,x,v}\left(e^{i\theta}\right) \in M^0$ is of rank n.*
- **(6_0)** *For all $e^{i\theta} \in \partial^+\Delta$, all $c \neq 0$ and all x, the mapping $v \longmapsto \frac{\partial A^0_{c,x,v}}{\partial\theta}\left(e^{i\theta}\right)$ is of rank n at $v = 0$. Consequently, when v varies, the positive half-lines $\mathbb{R}^+ \cdot \frac{\partial A_{c,0,v}}{\partial\theta}(1)$ describe an open infinite cone containing v_0 with vertex p_0 in $T_{p_0}M^0$.*

Proof. Proceeding similarly as in the proof of Lemma 5.37, we can find a new complex affine coordinate system centered at p_0 and stabilizing $\mathbb{R}^n$, which we shall still denote by $(z_1, \dots, z_n)$, in which the vector v_0 has coordinates $(0, \dots, 0, 1)$. In this coordinate system, we then construct the family $Z^0_{c,x,v}(\zeta)$ as in (7.8) above and we define the desired family simply as follows:

$$A^0_{c,x,v}(\zeta) := Z^0_{c,x,v_0+v}(\zeta), \tag{7.10}$$

where we restrict the variations of the parameter v to $|v| \leqslant c$. Notice that every disc $A^0_{c,x,v}\big(\overline{\Delta}\big)$ is contained in a single complex line. All the properties are then elementary consequences of the explicit expression (7.8) of $Z^0_{c,x,v}(\zeta)$.

Finally, we notice that it follows from properties **(5_0)** and **(6_0)** that the set of points $A^0_{c,x,v}(\zeta)$, where $c > 0$ is fixed, where x varies, where v varies and where ζ varies covers a local wedge of edge M^0 at p_0. □

7.11. Curved families of half-attached analytic discs. Our main goal in this section is to obtain a statement similar to Lemma 7.9 after replacing the flat maximally real submanifold $M^0 \cong \mathbb{R}^n$ by a curved $\mathcal{C}^{2,\alpha}$ maximally real submanifold M^1. We set up a formulation which will be appropriate for the achievement of the proof of Proposition 5.12 (Sections 8 and 9).

We will first construct a family $Z^1_{c,x,v}(\zeta)$ as a perturbation of the family $Z^0_{c,x,v}(\zeta)$, and then shrink the domain of variation of x, requiring $|x| \leqslant c^2$, in order to insure small disc size $\leqslant c^2 \cdot \Lambda_1$ (instead of $\leqslant c \cdot \Lambda_1$, which would be the property analogous to $\mathbf{(2_0)}$). Then c will *not* be considered as a parameter, so we denote by $A^1_{x,v:c}(\zeta)$ the desired family, putting c after a semicolon. In fact, in our construction, we unavoidably loose the $\mathcal{C}^{2,\alpha-0}$-smoothness with respect to c, and the family degenerates to a constant for $c = 0$.

Lemma 7.12. *Let M^1 be $\mathcal{C}^{2,\alpha}$ maximally real submanifold of $\mathbb{C}^n$, let $p_1 \in M^1$ and let $v_1 \in T_{p_1}M^1$ be a tangent vector with $|v_1| = 1$. Then there exists a positive constant $\Lambda_1 > 0$ and there exists $c_1 \in \mathbb{R}$ with $c_1 > 0$ such that for every $c \in \mathbb{R}$ with $0 < c \leqslant c_1$, there exists a family $A^1_{x,v:c}(\zeta)$ of analytic discs defined for $x \in \mathbb{R}^n$ with $|x| \leqslant c^2$ and for $v \in \mathbb{R}^n$ with $|v| \leqslant c$ which is $\mathcal{C}^{2,\alpha-0}$ with respect to (x, v, ζ) and which enjoys the following six properties*:

- $\mathbf{(1_1)}$ $A^1_{0,v:c}(1) = p_1$ *for all* v.
- $\mathbf{(2_1)}$ $A^1_{x,v:c} : \overline{\Delta} \to \mathbb{C}^n$ *is an embedding and* $\left|A^1_{x,v:c}(\zeta)\right| \leqslant c^2 \cdot \Lambda_1$ *for all* x, *all* v *and all* ζ.
- $\mathbf{(3_1)}$ $A^1_{x,v:c}(\partial^+\Delta) \subset M^1$ *for all* x *and all* v.
- $\mathbf{(4_1)}$ $\frac{\partial A^1_{0,0:c}}{\partial \theta}(1)$ *is a positive multiple of* v_1.
- $\mathbf{(5_1)}$ *The mapping* $x \longmapsto A^1_{x,0:c}(1) \in M^1$ *is of rank* n.
- $\mathbf{(6_1)}$ *The mapping* $v \longmapsto \frac{\partial A^1_{0,v:c}}{\partial \theta}\left(e^{i\theta}\right)$ *is of rank* n *at* $v = 0$. *Consequently, as* v *varies, the positive half-lines* $\mathbb{R}^+ \cdot \frac{\partial A^1_{0,v:c}}{\partial \theta}(1)$ *describe an open infinite cone containing* v_1 *with vertex* p_1 *in* $T_{p_1}M^1$ *and the set of points* $A^1_{x,v:c}(\zeta)$, *as* $|x| \leqslant c^2$, $|v| \leqslant c$ *and* $\zeta \in \Delta$ *vary, covers a wedge of edge* M^1 *at* (p_1, Jv_1).

In Figure 18 below, we represent the cone property $\mathbf{(6_1)}$. The remainder of this Section 7 is entirely devoted to complete the proof of Proposition 7.12.

7.13. Perturbed family of analytic discs half-attached to a maximally real submanifold. Thus, let $M^1 \subset \mathbb{R}^n$ be a locally defined maximally real $\mathcal{C}^{2,\alpha}$ submanifold passing through the origin. We can assume it to be represented by n Cartesian equations

$$y_1 = h_1(x_1, \dots, x_n), \cdots\cdots, y_n = h_n(x_1, \dots, x_n), \tag{7.14}$$

where $|x| \leqslant \rho_1$ for some $\rho_1 > 0$, where $h = h(x)$ is of class $\mathcal{C}^{2,\alpha}$ in $\{|x| < 2\rho_1\}$, and where, importantly, $h_j(0) = h_{j,x_k}(0) = h_{j,x_{k_1}x_{k_2}}(0) = 0$, for all $j, k, k_1, k_2 = 1, \dots, n$. We set $K_1 := \|h\|_{\mathcal{C}^{2,\alpha}(\{|x| \leqslant \rho_1\})}$. Also, we can assume that $v_1 = (0, \dots, 0, 1)$.

Our first goal is to produce a $\mathcal{C}^{2,\alpha-0}$ family of analytic discs $Z^1_{c,x,v}(\zeta)$ which are half-attached to M^1 and which are $\mathcal{C}^2$-close to the original family $Z^0_{c,x,v}(\zeta)$. After having constructed the family $Z^1_{c,x,v}(\zeta)$, we shall define the desired family $A^1_{x,v:c}(\zeta)$.

Let $d \in \mathbb{R}$ with $0 \leqslant d \leqslant 1$ and let the maximally real submanifold M^d (like "M *d*eformed") be defined precisely as the set of $z = x + iy \in \mathbb{C}^n$ with $|x| \leqslant \rho_1$ which satisfy the n Cartesian equations

$$y_1 = d \cdot h_1(x_1, \dots, x_n), \cdots\cdots, \; y_n = d \cdot h_n(x_1, \dots, x_n). \tag{7.15}$$

Of course, $M^d\big|_{d=0} \equiv \{x \in \mathbb{R}^n : \, |x| \leqslant \rho_1\}$ contains the M^0 of Lemma 7.9 if we choose $\rho_0 \leqslant \rho_1$, and moreover, $M^d\big|_{d=1} \equiv M^1$. Adding $d \in [0,1]$ as a parameter, we will construct a family of analytic dics $Z^d_{c,x,v}(\zeta)$ half-attached to M^d which is of class $\mathcal{C}^{2,\alpha-0}$ with respect to all its variables (c, x, v, d, ζ).

The disc $Z^d_{c,x,v}(\zeta) =: X^d_{c,x,v}(\zeta) + iY^d_{c,x,v}(\zeta)$ is half-attached to M^d if and only if

$$Y^d_{c,x,v}(\zeta) = d \cdot h\left(X^d_{c,x,v}(\zeta)\right), \quad \text{for } \zeta \in \partial^+\Delta. \tag{7.16}$$

and in addition, $Y^d_{c,x,v}$ should be a harmonic conjugate of $X^d_{c,x,v}$. However, the condition (7.16) does not give any relation between $X^d_{c,x,v}$ and $Y^d_{c,x,v}$ on the negative part $\partial^-\Delta$ of the unit circle. To fix this point, we assign the following complete equation on the unit circle

$$Y^d_{c,x,v}(\zeta) = d \cdot h\big(X^d_{c,x,v}(\zeta)\big) + Y^0_{c,x,v}(\zeta), \quad \text{for } \textit{all} \ \ \zeta \in \partial\Delta, \tag{7.17}$$

which coincides with (7.16) for $\zeta \in \partial^+\Delta$, since we have $Z^0_{c,x,v}(\partial^+\Delta) \subset \mathbb{R}^n$ by construction. Also, we require that $X^d_{c,x,v}(1) = x$, whence $Y^d_{c,x,v}(1) = d \cdot h(x)$.

By a theorem due to Privalov (*see e.g.* [MP2006]), the Hilbert transform T_1 has bounded norm $\|T_1\|_{\kappa,\alpha} \simeq \frac{\mathrm{cst}}{\alpha(1-\alpha)}$ as a linear operator $\mathcal{C}^{\kappa,\alpha}(\partial\Delta, \mathbb{R}^n) \to \mathcal{C}^{\kappa,\alpha}(\partial\Delta, \mathbb{R}^n)$ for $\kappa \in \mathbb{N}$ and $0 < \alpha < 1$, where cst is an absolute constant.

Thus, the mapping $\zeta \mapsto Y^d_{c,x,v}(\zeta)$ should necessarily coincide with the harmonic conjugate $\zeta \mapsto \big[T_1 X^d_{c,x,v}\big](\zeta) + d \cdot h(x)$ (this property is already satisfied for $d = 0$) and we deduce that $X^d_{c,x,v}(\zeta)$ should satisfy the Bishop-type equation

$$X^d_{c,x,v}(\zeta) = -T_1\left[d \cdot h\left(X^d_{c,x,v}\right)\right](\zeta) + X^0_{c,x,v}(\zeta), \quad \text{for all } \zeta \in \partial\Delta. \tag{7.18}$$

Conversely, if $X^d_{c,x,v}$ is a solution of this functional equation, then setting $Y^d_{c,x,v}(\zeta) := T_1 X^d_{c,x,v}(\zeta) + d \cdot h(x)$, the analytic disc $Z^d_{c,x,v}(\zeta) := X^d_{c,x,v}(\zeta) + iY^d_{c,x,v}(\zeta)$ is half-attached to M^d and more precisely, it satisfies (7.17).

Thanks to the solvability of Bishop's equation ([Tu1996, MP2006]), if c satisfies $0 \leqslant c \leqslant c_1$ with $c_1 > 0$ sufficiently small, there exists a unique solution $X^d_{c,x,v}(\zeta)$ to (7.18) which is $\mathcal{C}^{2,\alpha}$ with respect to ζ and $\mathcal{C}^{2,\alpha-0}$ with respect to all the variables (c, x, v, d, ζ), where $0 \leqslant c \leqslant c_1$, $|x| \leqslant c$, $|v| \leqslant 2$ and $\zeta \in \overline{\Delta}$. We shall now estimate the difference $\|Z^d_{c,x,v} - Z^0_{c,x,v}\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)}$ and prove that it is bounded by a constant times $c^{2+\alpha}$. In particular, if c_1 is sufficiently small, this will imply that $Z^d_{c,x,v}$ is nonconstant.

7.19. Size of the solution $X^d_{c,x,v}(\zeta)$ in $\mathcal{C}^{1,\alpha}$-norm. Following the beginning of the proof of the existence theorem in [Tu1996, MP2006], we introduce the map

$$F: \ X(\zeta) \longmapsto X^0_{c,x,v}(\zeta) - T_1\big[d \cdot h(X)\big](\zeta) \tag{7.20}$$

from a neighborhood of 0 in $\mathcal{C}^{1,\alpha}(\partial\Delta, \mathbb{R}^n)$ to $\mathcal{C}^{1,\alpha}(\partial\Delta, \mathbb{R}^n)$, and then we perform a *Picard iteration scheme*, setting firstly $X\{0\}^d_{c,x,v}(\zeta) := X^0_{c,x,v}(\zeta)$ and then inductively

$$X\{\nu+1\}^d_{c,x,v}(\zeta) := F\left(X\{\nu\}^d_{c,x,v}(\zeta)\right), \tag{7.21}$$

for every integer $\nu \geqslant 0$. According to [Tu1996, MP2006], the sequence $\big(X\{\nu\}^d_{c,x,v}(\zeta)\big)_{\nu\in\mathbb{N}}$ converges in $\mathcal{C}^{1,\alpha}(\partial\Delta)$ towards the unique solution $X^d_{c,x,v}(\zeta)$ of (7.18). We want to extract the supplementary information that

$\left\| X^d_{c,x,v} \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c \cdot K_2$ for some positive constant K_2, which will play the rôle of the constant K_2 of Lemma 6.12.

By construction (*cf.* (7.8)) there exists a constant $K_4 > 0$ such that

$$\left\| X^0_{c,x,v} \right\|_{\mathcal{C}^{2,\alpha}(\partial\Delta)} \leqslant c \cdot K_4. \tag{7.22}$$

Lemma 7.23. *Setting* $K_5 := K_1(3 + \pi^{1-\alpha}) \| T_1 \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)}$, *if*

$$c_1 \leqslant \min\left(\frac{\rho_1}{2K_4}, \left(\frac{1}{2^{2+\alpha}\, K_4^{1+\alpha}\, K_5} \right)^{\frac{1}{1+\alpha}} \right), \tag{7.24}$$

then $X^d_{c,x,v}$ *satisfies* $\left| X^d_{c,x,v}\left(e^{i\theta}\right) \right| \leqslant \rho_1$ *for all* $e^{i\theta} \in \partial\Delta$ *and there exists* $K_2 > 0$ *such that*

$$\left\| X^d_{c,x,v} \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c \cdot K_2. \tag{7.25}$$

In fact, it suffices to choose $K_2 := 2K_4$.

Proof. Indeed, applying Lemmas 6.4 and 6.9, if $X \in \mathcal{C}^{1,\alpha}(\partial\Delta, \mathbb{R}^n)$ satisfies $\left| X(e^{i\theta}) \right| \leqslant \rho_1$ for all $e^{i\theta} \in \partial\Delta$ and $\| X \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c \cdot 2K_4$ for all $c \leqslant c_1$, where c_1 is as in (7.24), we may estimate (remind $0 \leqslant d \leqslant 1$):

$$\begin{aligned} \| F(X) \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} &\leqslant \left\| X^0_{c,x,v} \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} + \| T_1 \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \cdot \| h(X) \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \\ &\leqslant c \cdot K_4 + \| T_1 \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \cdot K_1 (c \cdot 2K_4)^{2+\alpha} (3 + \pi^{1-\alpha}) \\ &= c \cdot \left(K_4 + c^{1+\alpha} 2^{2+\alpha} K_4^{2+\alpha} K_5 \right) \\ &\leqslant c \cdot (K_4 + c_1^{1+\alpha} 2^{2+\alpha} K_4^{2+\alpha} K_5) \\ &\leqslant c \cdot 2K_4. \end{aligned} \tag{7.26}$$

From the last inequality, it also follows that $\left| F\left(X(e^{i\theta})\right) \right| \leqslant \rho_1$ for all $e^{i\theta} \in \partial\Delta$. Consequently, the iteration (7.21) is well defined for all $\nu \in \mathbb{N}$ and from the inequality (7.26), we deduce that the limit $X^d_{c,x,v}$ satisfies the desired estimate $\left\| X^d_{c,x,v} \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c \cdot 2K_4$. □

Corollary 7.27. *Under the above assumptions, there exists a constant* $K_6 > 0$ *such that*

$$\left\| X^d_{c,x,v} - X^0_{c,x,v} \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c^{2+\alpha} \cdot K_6. \tag{7.28}$$

Proof. We estimate

$$\begin{aligned} \left\| X^d_{c,x,v} - X^0_{c,x,v} \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} &\leqslant \| T_1 \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \cdot \left\| h\left(X^d_{c,x,v} \right) \right\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \\ &\leqslant \| T_1 \|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \cdot K_1 (c \cdot 2K_4)^{2+\alpha} (3 + \pi^{1-\alpha}) \\ &\leqslant c^{2+\alpha} \cdot K_5 (2K_4)^{2+\alpha}. \end{aligned} \tag{7.29}$$

so that it suffices to set $K_6 := K_5(2K_4)^{2+\alpha}$. □

7.30. Smallness of the deformation in $\mathcal{C}^2$-norm. As already mentioned, the solution $X^d_{c,x,v}(\zeta)$ is in fact $\mathcal{C}^{2,\alpha}$ with respect to ζ and $\mathcal{C}^{2,\alpha-0}$ with respect to all variables (d,c,x,v,ζ). We can therefore differentiate twice Bishop's equation (7.18). First of all, if $X \in \mathcal{C}^{2,\alpha-0}(\partial\Delta,\mathbb{R}^n)$, we remind the commutation relation $\frac{\partial}{\partial\theta}(TX) = T\big(\frac{\partial X}{\partial\theta}\big)$, whence

$$\frac{\partial}{\partial\theta}\left(T_1X\right) = T\Big(\frac{\partial X}{\partial\theta}\Big), \tag{7.31}$$

since $T_1X = TX - TX(1)$. We may then compute the first order derivative of (7.18):

$$\frac{\partial}{\partial\theta}X^d_{c,x,v}\left(e^{i\theta}\right) - \frac{\partial}{\partial\theta}X^0_{c,x,v}\left(e^{i\theta}\right) = -T\Big[d\cdot\sum_{l=1}^n \frac{\partial h}{\partial x_l}\big(X^d_{c,x,v}\big)\,\frac{\partial X^d_{l;c,x,v}}{\partial\theta}\Big]\left(e^{i\theta}\right). \tag{7.32}$$

and then its second order partial derivatives $\partial^2/\partial v_k\partial\theta$, for $k=1,\dots,n$:

$$\begin{aligned}\frac{\partial^2 X^d_{c,x,v}}{\partial v_k\partial\theta} - \frac{\partial^2 X^0_{c,x,v}}{\partial v_k\partial\theta} = & -T\Big[d\cdot\sum_{l_1,l_2=1}^n \frac{\partial^2 h}{\partial x_{l_1}\partial x_{l_2}}\big(X^d_{c,x,v}\big)\,\frac{\partial X^d_{l_1;c,x,v}}{\partial v_k}\,\frac{\partial X^d_{l_2;c,x,v}}{\partial\theta}+\\ & + d\cdot\sum_{l=1}^n \frac{\partial h_j}{\partial x_l}\big(X^d_{c,x,v}\big)\,\frac{\partial^2 X^d_{l;c,x,v}}{\partial v_k\partial\theta}\Big].\end{aligned} \tag{7.33}$$

Let now K_2 be as in (7.25) and let K_3 be as in Lemma 6.12, applied to $X^d_{c,x,v}(\zeta)$.

Lemma 7.34. *If in addition to the inequality* (7.24)*, the constant c_1 satisfies*

$$c_1 \leqslant \Big(\frac{1}{2K_3\,\|T\|_{\mathcal{C}^\alpha(\partial\Delta)}}\Big)^{\frac{1}{1+\alpha}}, \tag{7.35}$$

then there exists $K_7>0$ such that for all d, all c, all x, all v, and for $k=1,\dots,n$:

$$\begin{cases}\Big\|\dfrac{\partial^2 X^d_{c,x,v}}{\partial v_k\partial\theta} - \dfrac{\partial^2 X^0_{c,x,v}}{\partial v_k\partial\theta}\Big\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha}\cdot K_7,\\ \Big\|\dfrac{\partial^2 X^d_{c,x,v}}{\partial\theta^2} - \dfrac{\partial^2 X^0_{c,x,v}}{\partial\theta^2}\Big\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha}\cdot K_7.\end{cases} \tag{7.36}$$

Proof. We check only the first inequality, the second being similar. Introducing for the second line of (7.33) a new simplified notation $\mathcal{R} := -T\left[d\cdot\sum_{l_1,l_2=1}^n \frac{\partial^2 h}{\partial x_{l_1}\partial x_{l_2}}\big(X^d_{c,x,v}\big)\,\frac{\partial X^d_{l_1;c,x,v}}{\partial v_k}\,\frac{\partial X^d_{l_2;c,x,v}}{\partial\theta}\right]$ and setting further obvious simplifying changes of notation, we can rewrite (7.33) as

$$\mathcal{X}^d - \mathcal{X}^0 = \mathcal{R} - T\left[d\cdot\mathcal{H}\mathcal{X}^d\right]. \tag{7.37}$$

Thanks to the inequality $\big\|X^d_{c,x,v}\big\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leqslant c\cdot K_2$ already established in Lemma 7.23 and thanks to Lemma 6.12, we know that the vector $\mathcal{R}\in\mathcal{C}^\alpha(\partial\Delta,\mathbb{R}^n)$ and the matrix $\mathcal{H}\in\mathcal{C}^{1,\alpha}(\partial\Delta,\mathcal{M}_{n\times n}(\mathbb{R}))$ are small:

$$\begin{cases}\|\mathcal{R}\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha}\cdot\|T\|_{\mathcal{C}^\alpha(\partial\Delta)}K_3(K_2)^2\\ \|\mathcal{H}\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{1+\alpha}\cdot K_3.\end{cases} \tag{7.38}$$

We then rewrite (7.37) under the form

$$\mathcal{X}^d - \mathcal{X}^0 = \mathcal{S} - T\left[d\cdot\mathcal{H}(\mathcal{X}^d - \mathcal{X}^0)\right], \tag{7.39}$$

with $\mathcal{S} := \mathcal{R} - T\left[d \cdot \mathcal{H}\mathcal{X}^0\right]$. Using the inequality $\|\mathcal{X}^0\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c \cdot K_4$ which is a direct consequence of (7.22) and taking (7.38) into account, we deduce:

$$\|\mathcal{S}\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha} \cdot \|\!|T|\!\|_{\mathcal{C}^\alpha(\partial\Delta)} \left[K_3(K_2)^2 + K_3K_4\right]. \tag{7.40}$$

Taking the $\mathcal{C}^\alpha(\partial\Delta)$ norm of both sides of (7.39), we deduce the estimate

$$\begin{aligned} \left\|\mathcal{X}^d - \mathcal{X}^0\right\|_{\mathcal{C}^\alpha(\partial\Delta)} &\leqslant c^{2+\alpha} \cdot \frac{\|\!|T|\!\|_{\mathcal{C}^\alpha(\partial\Delta)}\left[K_3(K_2)^2 + K_3K_4\right]}{1 - c^{1+\alpha} \cdot \|\!|T|\!\|_{\mathcal{C}^\alpha(\partial\Delta)}K_3} \\ &\leqslant c^{2+\alpha} \cdot 2\|\!|T|\!\|_{\mathcal{C}^\alpha(\partial\Delta)}\left[K_3(K_2)^2 + K_3K_4\right], \end{aligned} \tag{7.41}$$

where we use (7.35). It suffices to set $K_7 := 2\|\!|T|\!\|_{\mathcal{C}^\alpha(\partial\Delta)}\left[K_3(K_2)^2 + K_3K_4\right]$. □

7.42. Adjustment of the tangent vector. Let $v_1 \in T_{p_1}M^1$ with $|v_1| = 1$, as in Lemma 7.12. Coming back to the first family $Z^0_{c,x,v}(\zeta)$ defined by (7.8), we observe that

$$\begin{cases} \dfrac{\partial Z^0_{j;c,0,v_1}}{\partial x_k}(1) = \delta^j_k, & j,k = 1,\dots,n, \\ \dfrac{\partial^2 Z^0_{j;c,0,v_1}}{\partial v_k \partial\theta}(1) = c\,\dfrac{\partial\Psi}{\partial\theta}\left(e^{i\theta}\right)\delta^j_k, & j,k = 1,\dots,n. \end{cases} \tag{7.43}$$

From now on, we shall set $d = 1$ and we shall only consider the family $Z^1_{c,x,v}(\zeta)$. Thanks to the estimates (7.28) and (7.36), we deduce that if c_1 is sufficiently small, then for all c with $0 < c \leqslant c_1$, the two Jacobian matrices

$$\left(\frac{\partial Z^1_{j;c,0,v_1}}{\partial x_k}(1)\right)_{1\leqslant j,k\leqslant n} \quad \text{and} \quad \left(\frac{\partial^2 Z^1_{j;c,0,v_1}}{\partial v_k\partial\theta}(1)\right)_{1\leqslant j,k\leqslant n} \tag{7.44}$$

are invertible. It would follow that if set $A^1_{x,v:c}(\zeta) := Z^1_{c,x,v_1+v}(\zeta)$, then the disc $A^1_{x,v:c}(\zeta)$ would satisfy the two rank properties **(5_1)** and **(6_1)** of Lemma 7.12. However, the tangency condition **(4_1)** would certainly not be satisfied, because as d varies from 0 to 1, the disc $Z^d_{c,x,v}(\zeta)$ undergoes a nontrivial deformation.

Consequently, for every c with $0 < c \leqslant c_1$, we have to adjust the "cone parameter" v in order to maintain the tangency condition.

Lemma 7.45. *For every c with $0 < c \leqslant c_1$, there exists a vector $v(c) \in \mathbb{R}^n$ such that*

$$\frac{\partial Z^1_{c,0,v_1+v(c)}}{\partial\theta}(1) = \frac{\partial Z^0_{c,0,v_1}}{\partial\theta}(1) = c \cdot \frac{\partial\Psi}{\partial\theta}(1) \cdot v_1. \tag{7.46}$$

Furthermore, there exists a constant $K_8 > 0$ such that $|v(c)| \leqslant c^{1+\alpha} \cdot K_8$.

Proof. Unfortunately, we cannot apply the implicit function theorem, because the mapping $Z^1_{c,x,v}$ is identically zero when $c = 0$, so we have to proceed differently. First, we set

$$C_1 := \frac{\partial\Psi}{\partial\theta}(1), \quad \text{and} \quad C_2 := \|\Psi\|_{\mathcal{C}^2(\overline{\Delta})}. \tag{7.47}$$

The constant C_2 will be used only in Section 8 below. Choose $K_8 \geqslant \frac{2K_6}{C_1}$. According to the explicit expression (7.8), the set of points

$$\left\{\frac{\partial X^0_{c,0,v_1+v}}{\partial\theta}(1) \in \mathbb{R}^n : \ |v| \leqslant c^{1+\alpha} \cdot K_8\right\} \tag{7.48}$$

covers a cube in $\mathbb{R}^n$ centered at the point $\frac{\partial X^0_{c,0,v_1}}{\partial\theta}(1)$ of radius $c^{2+\alpha}\cdot C_1K_8$. Thanks to the estimate (7.28), we deduce that the (deformed) set of points

$$\left\{\frac{\partial X^1_{c,0,v_1+v}}{\partial\theta}(1)\in\mathbb{R}^n:\ |v|\leqslant c^{1+\alpha}\cdot K_8\right\} \tag{7.49}$$

covers a cube in $\mathbb{R}^n$ centered at the same point $\frac{\partial X^0_{c,0,v_1}}{\partial\theta}(1)$, but of radius

$$c^{2+\alpha}\cdot C_1K_8 - c^{2+\alpha}\cdot K_6 \geqslant c^{2+\alpha}\cdot K_6. \tag{7.50}$$

Consequently, there exists at least one $v(c)\in\mathbb{R}^n$ with $|v(c)|\leqslant c^{1+\alpha}\cdot K_8$ such that (7.46) holds, which completes the proof. □

7.51. Construction of the family $A^1_{x,v:c}(\zeta)$. We can now complete the proof of the main Lemma 7.12. First of all, with $\Psi(\zeta)$ as in §7.5, we consider the composed conformal mapping

$$\zeta\longmapsto c\Psi(\zeta)\longmapsto\frac{i-c\Psi(\zeta)}{i+c\Psi(\zeta)}=:\Phi_c(\zeta). \tag{7.52}$$

The image $\Phi_c(\zeta)$ of the unit disc is a small domain contained in Δ and concentrated near 1.

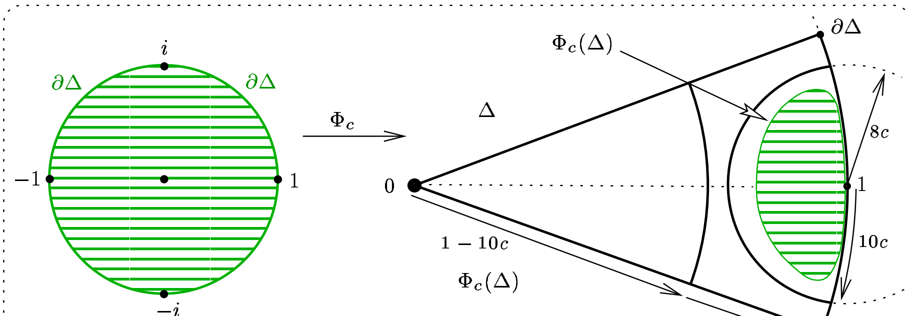

Fig. 17: A sub-half-disc

More precisely, assuming that c satifies $0<c\leqslant c_1$ with $c_1\ll 1$ as in the previous paragraphs, and taking account of the definition of $\Psi(\zeta)$, it can be checked easily that $\Phi_c(1)=1$, that $\Phi_c(\partial^+\Delta)$ is contained in $\{e^{i\theta}\in\partial^+\Delta:\ |\theta|<10c\}$, and that

$$\Phi_c\left(\overline{\Delta}\backslash\partial^+\Delta\right)\subset\{\zeta\in\Delta:\ |\zeta-1|<8c\}\subset\{\rho e^{i\theta}\in\Delta:\ |\theta|<10c,\ 1-10c<\rho<1\}. \tag{7.53}$$

the second inclusion being trivial.

We can finally define the desired family of analytic discs, writing the parameter c after a semi-colon, since we have lost the $\mathcal{C}^{2,\alpha-0}$-smoothness with respect to it after the application of Lemma 7.45, and since c will be fixed afterwards anyway:

$$A^1_{x,v:c}(\zeta):=Z^1_{c,x,v_1+v(c)+v}\left(\Phi_c(\zeta)\right). \tag{7.54}$$

We restrict the variation of the parameters x to $|x|\leqslant c^2$ and v to $|v|\leqslant c$. Property $\mathbf{(4_1)}$ holds immediately, thanks to the choice of $v(c)$. Properties $\mathbf{(1_1)}$, $\mathbf{(3_1)}$, $\mathbf{(5_1)}$ and $\mathbf{(6_1)}$ as well as the embedding property in $\mathbf{(2_1)}$ are direct consequences of the immersive properties (7.44) satisfied by $Z^1_{c,x,v_1+v(c)+v}(\zeta)$, using the chain rule and the nonvanishing of the partial derivative $\frac{\partial\Phi_c}{\partial\theta}(1)$. The size estimate in $\mathbf{(2_1)}$ follows from (7.25), from (7.28), from the restriction of the domains of variation of x and of v and from (7.53). This completes the proof of Lemma 7.12. □

§8. GEOMETRIC PROPERTIES OF FAMILIES OF HALF-ATTACHED ANALYTIC DISCS

8.1. Preliminary. By Lemma 7.12, for every c with $0 < c \leqslant c_1$, the family of half-attached analytic discs $A^1_{x,v:c}(\zeta)$ covers a local wedge of edge M^1 at p_1. However, not only we want the family $A^1_{x,v:c}$ to cover a local wedge of edge M^1 at p_1, but we certainly want to remove the point p_1 of Proposition 5.12 by means of the continuity principle. Consequently, in each one of the three geometric situations $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ and **(II)** which we have normalized in Lemma 5.37 above, we shall firstly deduce from the tangency condition $\mathbf{(4_1)}$ of Lemma 7.12 that the (excised) half-boundary $A^1_{0,0:c}\big(\partial^+\Delta\backslash\{1\}\big)$ is contained in the open side $(H^1)^+$ (this is why we have normalized in Lemma 5.37 the second order terms of the supporting hypersurface H^1 in order that $(H^1)^+$ is strictly concave; we also want that $A^1_{0,0:c}(\partial^+\Delta\backslash\{1\})$ is contained in $(H^1)^+$ in order to apply the continuity principle). Secondly, we shall show that for all x with $|x| \leqslant c^2$, the disc interior $A_{x,0:c}(\Delta)$ is contained in the local half-wedge $\mathcal{HW}_1^+$ in the cases $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ and is contained in the wedge $\mathcal{W}_2$ in case **(II)**.

8.2. Geometric disposition of the discs with respect to H^1 and to $\mathcal{HW}_1^+$ or to $\mathcal{W}_2$. We remember that the positive c_1 of Lemmas 7.12, 7.23 and 7.34 was shrunk explicitely, in terms of the constants $K_1, K_2, K_3, \dots$. In this section, we shall again shrink c_1 a finite number of times, but without mentioning all the similar explicit inequalities which will appear. The precise statement of the main lemma of this section, which is a continuation of Lemma 7.12, is as follows; whereas we can essentially gather the three cases in the formal statement of the lemma, it is necessary to treat them separately in the proof, because the normalizations of Lemma 5.37 differ.

Lemma 8.3. *Let M, let M^1, let p_1, let H^1, let v_1, let $(H^1)^+$, let $\mathcal{HW}_1^+$ (or let $\mathcal{HW}_2$) and let a coordinate system $z = (z_1, \dots, z_n)$ vanishing at p_1 be as in Case* $\mathbf{(I_1)}$*, as in Case* $\mathbf{(I_2)}$ *or as in Case* **(II)** *of Lemma 5.37. Aa a local one-dimensional submanifold $T^1 \subset M^1$ transversal to H^1 in M^1 and passing through p_1, choose $T_1 := \{(x_1, 0, \dots, 0) + ih(x_1, 0, \dots, 0))\}$ in Case* $\mathbf{(I_1)}$ *and $T_1 := \{(0, \dots, 0, x_n) + ih(0, \dots, 0, x_n))\}$ in Cases* $\mathbf{(I_2)}$ *and* **(II)**. *For every c with $0 < c \leqslant c_1$, let $A^1_{x,v:c}(\zeta)$ be the family of analytic discs satisfying properties* $\mathbf{(1_1)}$, $\mathbf{(2_1)}$, $\mathbf{(3_1)}$, $\mathbf{(4_1)}$, $\mathbf{(5_1)}$ *and* $\mathbf{(6_1)}$ *of Lemma 7.12. Shrinking c_1 if necessary, then for every c with $0 < c \leqslant c_1$, the following three further properties hold*:

- $\mathbf{(7_1)}$ $A^1_{0,0:c}\big(\partial^+\Delta\backslash\{1\}\big) \subset (H^1)^+$;
- $\mathbf{(8_1)}$ $A^1_{x,0:c}\big(\partial^+\Delta\big)$ *is contained in $(H^1)^+$ for all x such that the point $A^1_{x,0:c}(1)$ belongs to $T^1 \cap (H^1)^+$;*
- $\mathbf{(9_1)}$ $A^1_{x,v:c}\big(\overline{\Delta}\backslash\partial^+\Delta\big)$ *is contained in the half-wedge $\mathcal{HW}_1^+$ or in the wedge $\mathcal{W}_2$ for all x and all v.*

Proof. We treat only Case $\mathbf{(I_1)}$, the other two cases being similar. Figure 18 just below illustrates properties $\mathbf{(7_1)}$ and $\mathbf{(8_1)}$ and also properties $\mathbf{(1_1)}$, $\mathbf{(5_1)}$ and $\mathbf{(6_1)}$ of Lemma 7.12.

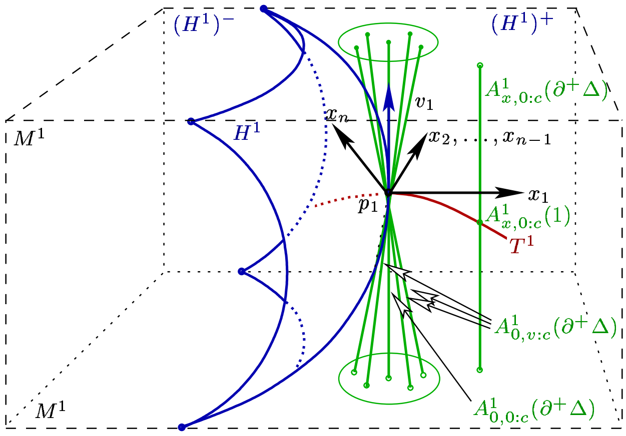

Fig. 18: View of the family of half boundaries in M^1

By construction of $A^1_{x,v:c}$, if the scaling parameter c_1 is small enough, the disc $A^1_{x,v:c}(\overline{\Delta})$ is only a slightly deformed small part of the straight complex line $\mathbb{C}\cdot(v_1 + Jv_1)$, where $v_1 \in T_{p_1}H^1$ is as in Lemma 7.12. Intuitively speaking, the reason why property **(7_1)** holds true then becomes clear: the open set $(H^1)^+$ is strictly concave and the small, almost straight curve $A^1_{0,0:c}(\partial^+\Delta)$ is tangent to H^1 at p_1. Concerning **(8_1)**, when x varies, the small segments $A^1_{x,0:c}(\partial^+\Delta)$ are essentially translated to the right (inside M^1) by the vector $x \in \mathbb{R}^n$. Also, **(9_1)** should hold because the half-wedge $\mathcal{HW}_1^+$ (or the wedge $\mathcal{W}_2$) is directed by $J\,v_1$. The next paragraphs will establish these properties rigorously.

Firstly, let us prove property **(7_1)** in Case **(I_1)**. According to Lemma 5.37, the vector v_1 is given by $(0, 1, \dots, 1)$ and the side $(H^1)^+ \subset M^1$ is defined by $x_1 > g(x') = -x_2^2 - \cdots - x_n^2 + \widehat{g}(x')$, with $\widehat{g}(x') = \mathrm{o}\big(|x'|^2\big)$ by (5.40)$_3$. According to Lemma 6.4, we then have $|\widehat{g}(x')| \leqslant K_9 \cdot \big(\sum_{j=2}^n x_j^2\big)^{\frac{\alpha+2}{2}}$, for some constant $K_9 > 0$. Since the strictly concave open subset $\big(\widetilde{H}^1\big)^+$ of M^1 with $\mathcal{C}^{2,\alpha}$ boundary defined by $x_1 > -x_1^2 - \cdots - x_n^2 + K_9 \cdot \big(\sum_{j=2}^n x_j^2\big)^{\frac{2+\alpha}{2}}$ is contained in $(H^1)^+$, it suffices to prove property **(7_1)** with $(H^1)^+$ replaced by $(\widetilde{H}^1)^+$.

By construction, the disc boundary $A_{0,0:c}(\partial\Delta)$ is tangent at p_1 to H^1, hence also to $\widetilde{H}^1$. Intuitively, it is again clear that the (excised) half-boundary $A_{0,0:c}\big(\partial^+\Delta\backslash\{1\}\big)$ should then be contained in the strictly concave side $(\widetilde{H}^1)^+$, *see* again Figure 18 above.

To proceed rigorously, we come back to the definition $A^1_{0,0:c}(\zeta) \equiv Z^1_{c,0,v_1+v(c)}\left(\Phi_c(\zeta)\right)$, with the tangency condition (7.46) satisfied. First of all, denoting $v(c) = (v_1(c), \dots, v_n(c))$, we compute the second order derivatives of the similar discs attached to M^0:

$$
\left\{
\begin{aligned}
&\frac{\partial^2 Z^0_{1;c,0,v_1+v(c)}}{\partial\theta^2}(1) = c\cdot\frac{\partial^2\Psi}{\partial\theta^2}\big(e^{i\theta}\big)\cdot v_1(c),\\
&\frac{\partial^2 Z^0_{j;c,0,v_1+v(c)}}{\partial\theta^2}(1) = c\cdot\frac{\partial^2\Psi}{\partial\theta^2}\big(e^{i\theta}\big)\cdot(1+v_j(c)), \qquad j = 2, \dots, n.
\end{aligned}
\right.
\tag{8.4}
$$

Using the definition (7.47), the inequality $|v(c)| \leqslant c^{1+\alpha} \cdot K_8$ and the second estimate (7.36), we deduce that

$$\text{(8.5)} \qquad \begin{cases} \left|\dfrac{\partial^2 Z^1_{1;c,0,v_1+v(c)}}{\partial\theta^2}(1)\right| \leqslant c^{2+\alpha}\cdot K_7 + c^{2+\alpha}\cdot C_2 K_8 =: c^{2+\alpha}\cdot 2K_{10} \\ \left|\dfrac{\partial^2 Z^1_{j;c,0,v_1+v(c)}}{\partial\theta^2}(1)\right| \leqslant c\cdot 2C_2, \qquad j=2,\dots,n. \end{cases}$$

Applying then Taylor's integral formula $F(\theta) = F(0) + \theta \cdot F'(0) + \int_0^\theta (\theta - \theta') \cdot \partial_\theta\partial_\theta F(\theta') \cdot d\theta'$ to $F(\theta) := X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right)$ and afterwards to $F(\theta) := X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right)$ for $j = 2,\dots,n$, taking account of the tangency conditions

$$\text{(8.6)} \quad \frac{\partial X^1_{1;c,0,v_1+v(c)}}{\partial\theta}(1) = 0, \qquad \frac{\partial X^1_{j;c,0,v_1+v(c)}}{\partial\theta}(1) = c\cdot C_1, \qquad j=2,\dots,n,$$

(a simple rephrasing of (7.46)) and using the inequalities (8.5), we deduce

$$\text{(8.7)} \qquad \begin{cases} \left|X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right| \leqslant \theta^2\cdot c^{2+\alpha}\cdot K_{10}, \\ \left|X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right) - \theta\cdot c\cdot C_1\right| \leqslant \theta^2\cdot c\cdot C_2, \quad j=2,\dots,n. \end{cases}$$

Recall the equation of $(\widetilde{H}^1)^+$:

$$\text{(8.8)} \qquad x_1 > \widetilde{g}(x') := -x_2^2 - \cdots - x_n^2 + K_9\Big(\sum_{j=2}^n x_j^2\Big)^{\frac{2+\alpha}{2}}$$

We now claim that if c_1 is sufficiently small, then for every θ with $0 < |\theta| < 10c$, we have

$$\text{(8.9)} \qquad X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right) > \widetilde{g}\big(X^1_{2;c,0,v_1+v(c)}(e^{i\theta}),\dots\dots, X^1_{n;c,0,v_1+v(c)}(e^{i\theta})\big).$$

Since $\Phi_c(\partial^+\Delta)$ is contained in $\{e^{i\theta} \in \partial^+\Delta : \ |\theta| < 10c\}$, this will imply the inclusion proving $\mathbf{(7_1)}$:

$$\text{(8.10)} \qquad \begin{aligned} A^1_{x,v:c}\big(\partial^+\Delta\backslash\{1\}\big) &= Z^1_{c,0,v_1+v(c)}\left(\Phi_c(\partial^+\Delta\backslash\{1\})\right) \subset \\ &\subset Z^1_{c,0,v_1+v(c)}\Big(\{e^{i\theta}\in\partial^+\Delta : \ 0<|\theta|\leqslant 10c\}\Big) \\ &\subset (\widetilde{H}^1)^+. \end{aligned}$$

To prove the claim, using (8.7), we get a minoration of the left hand side of (8.9):

$$\text{(8.11)} \qquad X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right) \geqslant -\theta^2\cdot c^{2+\alpha}\cdot K_{10}.$$

On the other hand, using two inequalities which are direct consequences of the second line of (8.7), provided that c_1 satisfies $10c_1 \cdot C_2 \leqslant \frac{C_1}{2}$, we have:

$$\text{(8.12)} \qquad \begin{aligned} \left|X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right| &\leqslant |\theta|\cdot c\cdot(C_1+|\theta|\cdot C_2) \leqslant |\theta|\cdot c\cdot\frac{3C_1}{2}, \\ \left[X^1_{j;c,0,v_1+v(c)}\right]^2 &\geqslant \theta^2\cdot c^2\cdot(C_1-|\theta|\cdot C_2)^2 \geqslant \theta^2\cdot c^2\cdot\frac{C_1^2}{4}, \end{aligned}$$

for $j = 2, \dots, n$. We deduce the following majoration of the right hand side of (8.9):

$$
\begin{aligned}
&\widetilde{g}\,\big(X^1_{2;c,0,v_1+v(c)}(e^{i\theta}), \dots\dots, X^1_{n;c,0,v_1+v(c)}(e^{i\theta})\big) = \\
&\qquad = -\sum_{j=2}^n \big[X^1_{j;c,0,v_1+v(c)}\big]^2 + K_9 \Big(\sum_{j=2}^n \big[X^1_{j;c,0,v_1+v(c)}(e^{i\theta})\big]^2\Big)^{\frac{2+\alpha}{2}} \\
&\qquad \leqslant -\theta^2 \cdot c^2 \cdot \frac{C_1^2}{4}(n-1) + |\theta|^{2+\alpha} \cdot c^{2+\alpha} \cdot \Big(\frac{(n-1)9C_1^2}{4}\Big)^{\frac{2+\alpha}{2}} K_9 \\
&\qquad \leqslant -\theta^2 \cdot c^2 \Big(\frac{C_1^2}{4}(n-1) - c^\alpha \cdot \Big(\frac{(n-1)9C_1^2}{4}\Big)^{\frac{2+\alpha}{2}} K_9\Big).
\end{aligned}
\tag{8.13}
$$

Thanks to the minoration (8.11) and to the majoration (8.13), in order that the inequality (8.9) holds for all θ with $0 < |\theta| \leqslant 10c$, it suffices that the right hand side of (8.11) be greater than the last line of (8.13). Writing this (strict) inequality and clearing the factor $\theta^2 \cdot c^2$, we see that it suffices that

$$
-K_{10} \cdot c^\alpha > -\Big(\frac{C_1^2}{4}(n-1) - c^\alpha \cdot \Big(\frac{(n-1)9C_1^2}{4}\Big)^{\frac{2+\alpha}{2}} K_9\Big),
\tag{8.14}
$$

or equivalently

$$
c_1 < \Big(\frac{\frac{C_1^2}{4}(n-1)}{K_{10} + \big(\frac{(n-1)9C_1^2}{4}\big)^{\frac{2+\alpha}{2}} K_9}\Big)^{\frac{1}{\alpha}}.
\tag{8.15}
$$

This completes the proof of property **($\mathbf{7_1}$)**.

Secondly, let us prove property **($\mathbf{8_1}$)** in Case **($\mathbf{I_1}$)**. As above, we come back to the definition $A^1_{x,0:c}(\zeta) := Z^1_{c,x,v_1+v(c)}(\Phi_c(\zeta))$ and we remind that $A^1_{x,0:c}(1) = Z^1_{c,x,v_1+v(c)}(1) = x + ih(x)$, which follows by putting $d = 1$ and $\zeta = 1$ in (7.18). Thanks to the inclusion $\Phi_c(\partial^+\Delta) \subset \{e^{i\theta} \in \partial^+\Delta : |\theta| < 10c\}$, it suffices to prove that the segment $Z_{c,x,v_1+v(c)}\big(\{e^{i\theta} : |\theta| < 10c\}\big)$ is contained in the open side $(\widetilde{H}^1)^+ \subset (H^1)^+$ defined by the inequation (8.8), if the point $x + ih(x)$ belongs to the transverse half-submanifold $T^1 \cap (H^1)^+$, namely if $x = (x_1, 0, \dots, 0)$ with $x_1 > 0$. In the sequel, we shall denote the disc $Z^1_{c,x,v_1+v(c)}(\zeta)$ by $Z^1_{c,x_1,x',v_1+v(c)}(\zeta)$, emphasizing the decomposition $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, and we shall also use the convenient notation

$$
Z'^1_{c,x_1,x',v_1+v(c)}\big(\rho e^{i\theta}\big) := \big(Z^1_{2;c,x_1,x',v_1+v(c)}(\rho e^{i\theta}), \dots\dots, Z^1_{n;c,x_1,x',v_1+v(c)}(\rho e^{i\theta})\big).
\tag{8.16}
$$

So, we have to show that for all c with $0 < c \leqslant c_1$, all x_1 with $0 < x_1 \leqslant c^2$ and all θ with $|\theta| < 10c$, then the following strict inequality holds true:

$$
X^1_{1;c,x_1,0,v_1+v(c)}(e^{i\theta}) > \widetilde{g}\big(X^{1'}_{c;x_1,0,v_1+v(c)}(e^{i\theta})\big).
\tag{8.17}
$$

First of all, coming back to the family of discs attached to M^0, we see by differentiating (7.8) twice with respect to x_1 that $\frac{\partial^2 Z^0_{c,x_1,0,v_1+v(c)}}{\partial x_1^2}(\zeta) \equiv 0$. Next, by differentiating twice Bishop's equation (7.18) with respect to x_1 and by reasoning as in Lemma 7.34, we get the estimate

$$
\Big\| \frac{\partial^2 Z^1_{c,x_1,0,v_1+v(c)}}{\partial x_1^2} \Big\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha} \cdot K_7,
\tag{8.18}
$$

with, say, the same constant $K_7 > 0$ as in Lemma 7.34, after enlarging it if necessary. Applying then Taylor's integral formula $F(x_1) = F(0) + x_1 \cdot \partial_{x_1} F(0) + \int_0^{x_1} (x_1 - \widetilde{x}_1) \cdot \partial_{x_1} \partial_{x_1} F(\widetilde{x}_1) \cdot d\widetilde{x}_1$ to $F(x_1) := X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right)$, we deduce the minoration

(8.19)
$$X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) \geqslant X^1_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) + x_1 \cdot \frac{\partial X^1_{1;c,0,0,v_1+v(c)}}{\partial x_1}\left(e^{i\theta}\right) - x_1^2 \cdot c^{2+\alpha} \cdot \frac{K_7}{2}.$$

On the other hand, by differentiating Bishop's equation (7.18) with respect to x_1 at $x = 0$, the derivative $\partial_{x_1} x$ yields the vector $(1, 0, \ldots, 0)$ and we obtain

(8.20)
$$\frac{\partial X_{c,0,0,v_1+v(c)}}{\partial x_1}\left(e^{i\theta}\right) = -T_1\Big[\sum_{l=1}^{n} \frac{\partial h}{\partial x_l}\left(X^1_{c,0,0,v_1+v(c)}(\cdot)\right) \frac{\partial X^1_{l;c,0,0,v_1+v(c)}}{\partial x_1}(\cdot)\Big](e^{i\theta}) + \\ + (1, 0, \ldots, 0).$$

Using then (6.13) $_2$ and (7.25), we deduce from (8.20)

(8.21)
$$\left\| \frac{\partial X^1_{1;c,0,0,v_1+v(c)}}{\partial x_1}(\cdot) - 1 \right\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha} \cdot |\!|\!| T_1 |\!|\!|_{\mathcal{C}^\alpha(\partial\Delta)} K_2 K_3,$$
$$\left\| \frac{\partial X^1_{j;c,0,0,v_1+v(c)}}{\partial x_1}(\cdot) \right\|_{\mathcal{C}^\alpha(\partial\Delta)} \leqslant c^{2+\alpha} \cdot |\!|\!| T_1 |\!|\!|_{\mathcal{C}^\alpha(\partial\Delta)} K_2 K_3.$$

Thanks to (8.21) $_1$, we can work out the minoration (8.19) by replacing the first order partial derivative $\frac{\partial X^1_{1;c,0,0,v_1+v(c)}}{\partial x_1}\left(e^{i\theta}\right) > 0$ in the right hand side of (8.19) by the constant 1, and applying trivial minoration $-x_1^2 \geqslant -x_1$, which yields:

(8.22)
$$X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) \geqslant X^1_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) + x_1 - x_1 \cdot c^{2+\alpha} \cdot K_{11},$$

for some constant $K_{11} > 0$. On the other hand, using the inequalities $|\partial_{x_j} \widetilde{g}(x')| \leqslant |x'| + K_9 \cdot |x'|^{1+\alpha} \cdot \left(1 + \frac{\alpha}{2}\right) (n-1)^{\frac{\alpha}{2}}$ for $j = 2, \ldots, n$, using the estimate (7.25) and using (6.2), we deduce an inequality of the form

(8.23)
$$\widetilde{g}\big(X'^1_{c,x_1,0,v_1+v(c)}(e^{i\theta})\big) \leqslant \widetilde{g}\big(X'^1_{c,0,0,v_1+v(c)}(e^{i\theta})\big) + x_1 \cdot c \cdot K_{12},$$

for some constant $K_{12} > 0$. Finally, putting together the two inequalities (8.22) and (8.23), and using the following immediate consequence of (8.9):

(8.24)
$$X_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) \geqslant \widetilde{g}\big(X'^1_{c,0,0,v_1+v(c)}(e^{i\theta})\big),$$

valuable for all θ with $|\theta| < 10c$, we deduce the desired inequality (8.17):

(8.25)
$$\begin{cases} X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) \geqslant X^1_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) + x_1 - x_1 \cdot c^{2+\alpha} \cdot K_{11} \\ \qquad \geqslant \widetilde{g}\big(X'^1_{c,0,0,v_1+v(c)}(e^{i\theta})\big) + x_1 - x_1 \cdot c^{2+\alpha} \cdot K_{11} \\ \qquad \geqslant \widetilde{g}\big(X'^1_{c,x_1,0,v_1+v(c)}(e^{i\theta})\big) + x_1 - x_1 \cdot c \cdot K_{11} - x_1 \cdot c \cdot K_{12} \\ \qquad > \widetilde{g}\big(X'^1_{c,x_1,0,v_1+v(c)}(e^{i\theta})\big), \end{cases}$$

for all x_1 with $0 < x_1 \leqslant c^2$, all θ with $|\theta| < 10c$ and all c with $0 < c \leqslant c_1$, provided

(8.26)
$$c_1 \leqslant \frac{1/2}{K_{11} + K_{12}}.$$

This completes the proof of property **($\mathbf{8_1}$)**.

Thirdly, let us prove property **($\mathbf{9_1}$)** in Case **($\mathbf{I_1}$)**. The half-wedge $\mathcal{HW}_1^+$ is defined by the n inequalities of the last two lines of (5.38), where $a_2 + \cdots + a_n = 1$. For notational convenience, we set $a_1 := 1$ and we write the first inequality defining $\mathcal{HW}_1^+$ simply as $\sum_{j=1}^n a_j y_j > \psi(x, y')$.

Because $\Phi_c\big(\overline{\Delta}\backslash\partial^+\Delta\big)$ is contained in the open sector $\big\{\rho e^{i\theta} \in \overline{\Delta} : \ |\theta| < 10c, \ 1 - 10c < \rho < 1\big\}$, taking account of the definition (7.54) of $A_{x,v:c}^1(\zeta)$, in order to check property **($\mathbf{9_1}$)**, it clearly suffices to show that $Z_{c,x,v_1+v(c)+v}^1\big(\big\{\rho e^{i\theta} \in \Delta : \ 1 - 10c < \rho < 1, \ |\theta| < 10c\big\}\big)$ is contained in $\mathcal{HW}_1^+$, which amounts to establish that for all x with $|x| \leqslant c^2$, all v with $|v| \leqslant c$, all $\rho e^{i\theta}$ with $1 - 10c < \rho < 1$ and with $|\theta| < 10c$, the following two collections of strict inequalities hold true

(8.27)
$$\begin{aligned}\sum_{k=1}^n a_k Y_{j;c,x,v_1+v(c)+v}^1\big(\rho e^{i\theta}\big) &> \psi\big(X_{c,x,v_1+v(c)+v}^1\big(\rho e^{i\theta}\big), \ Y'^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big), \\ Y_{j;c,x,v_1+v(c)+v}^1\big(\rho e^{i\theta}\big) &> \varphi_j\big(X_{c,x,v_1+v(c)+v}^1\big(\rho e^{i\theta}\big), \ Y_{1;c,x,v_1+v(c)+v}^1\big(\rho e^{i\theta}\big)\big),\end{aligned}$$

for $j = 2, \ldots, n$, provided c_1 is sufficiently small, where we use the notation (8.16).

We first treat the collection of $(n-1)$ strict inequalities in the second line of (8.27). First of all, by differentiating (7.8) twice with respect to θ, we obtain

$$\frac{\partial^2 Z_{c,x,v_1+v(c)+v}^0}{\partial\theta^2}\big(e^{i\theta}\big) = c \cdot \frac{\partial^2\Psi}{\partial\theta^2}\big(e^{i\theta}\big) \cdot [v_1 + v(c) + v]. \tag{8.28}$$

Using $(7.36)_2$, we deduce that there exists a constant $K_{13} > 0$ such that

$$\bigg|\frac{\partial^2 Z_{c,x,v_1+v(c)+v}^1}{\partial\theta^2}\big(e^{i\theta}\big)\bigg| \leqslant c \cdot K_{13}. \tag{8.29}$$

Using the inequality (6.2), using (8.29), and then taking account of the inequalities $|\theta| < 10c$, $|x| \leqslant c^2$ and $|v| < c$, we deduce the inequality

$$\begin{aligned}\bigg|\frac{\partial Z_{c,x,v_1+v(c)+v}^1}{\partial\theta}\big(e^{i\theta}\big) - \frac{\partial Z_{c,0,v_1+v(c)}^1}{\partial\theta}(1)\bigg| &\leqslant c \cdot \big(|\theta| + |x| + |v|\big) \\ &\leqslant c^2 \cdot K_{14},\end{aligned} \tag{8.30}$$

for some constant $K_{14} > 0$. On the other hand, differentiating (7.8) with respect to θ at $\theta = 0$ and applying the inequality (7.28), we obtain

$$\bigg|\frac{\partial Z_{c,0,v_1+v(c)}^1}{\partial\theta}(1) - c \cdot C_1 \cdot (0, 1, \ldots, 1)\bigg| \leqslant c^{2+\alpha} \cdot K_6, \tag{8.31}$$

where $C_1 = \frac{\partial\Psi}{\partial\theta}(1)$, as defined in (7.47). We remind that for every $\mathcal{C}^1$ function Z on $\overline{\Delta}$ which is holomorphic in Δ, we have $i\frac{\partial}{\partial\theta}Z(e^{i\theta}) = -\frac{\partial}{\partial\rho}Z(e^{i\theta})$. Consequently, we deduce from (8.30) the following first (among three) interesting inequality

$$\bigg| - \frac{\partial Z_{c,x,v_1+v(c)+v}^1}{\partial\rho}\big(e^{i\theta}\big) - c \cdot C_1 \cdot (0, i, \ldots, i)\bigg| \leqslant c^2 \cdot K_{15}, \tag{8.32}$$

for some $K_{15} > 0$. Next, according to the definition (7.8), we have

$$\frac{\partial^2 Z_{c,x,v_1+v(c)+v}^0}{\partial\rho^2}\big(\rho e^{i\theta}\big) = c \cdot \frac{\partial^2\Psi}{\partial\rho^2}\big(\rho e^{i\theta}\big) \cdot (v_1 + v(c) + v). \tag{8.33}$$

Reasoning as in the proof of Lemma 7.34, we may obtain an inequality similar to (7.36), with the second order partial derivative $\partial^2/\partial\theta^2$ replaced by the second order partial derivative $\partial^2/\partial\rho^2$. Putting this together with (8.33), we deduce that there exists a constant $K_{16}>0$ such that

$$\left|\frac{\partial^2 Z^1_{c,x,v_1+v(c)+v}}{\partial\rho^2}\big(\rho e^{i\theta}\big)\right| \leqslant c\cdot 2K_{16}. \tag{8.34}$$

Applying then Taylor's integral formula $F(\rho)=F(1)+(\rho-1)\cdot\partial_\rho F(1)+\int_1^\rho(\rho-\widetilde{\rho})\cdot\partial_\rho\partial_\rho F(\widetilde{\rho})\cdot d\widetilde{\rho}$ to the functions $F(\rho):=Y^1_{k;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)$ for $k=1,\dots,n$, we deduce the second interesting collection of inequalities, for $k=1,\dots,n$:

$$\begin{aligned}\Big|Y^1_{k;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)-Y^1_{k;c,x,v_1+v(c)+v}\big(e^{i\theta}\big)-&\\ -(\rho-1)\cdot\frac{\partial Y^1_{k;c,x,v_1+v(c)+v}}{\partial\rho}\big(e^{i\theta}\big)\Big|&\leqslant(1-\rho)^2\cdot c\cdot K_{16},\end{aligned} \tag{8.35}$$

On the other hand, thanks to the normalizations of the functions $\varphi_j(x,y_1)$ given in (5.40), we get (increasing possibly $K_1>0$) two inequalities:

(8.36)

$$\begin{aligned}&\sum_{k=1}^n|\varphi_{j,x_k}(x,y_1)|+|\varphi_{j,y_1}(x,y_1)|\leqslant(|x|+|y_1|)\cdot K_1,\\ &\big|\varphi_j(x,y_1)-\varphi_j(\widetilde{x},\widetilde{y}_1)\big|\leqslant\big(|x-\widetilde{x}|+|y_1-\widetilde{y}_1|\big)\cdot\Big(\sum_{k=1}^n\sup_{|x|,\,|y_1|\leqslant c\cdot K_2}|\varphi_{j,x_k}(x,y_1)|+\\ &\qquad+\sup_{|x|,\,|y_1|\leqslant c\cdot K_2}|\varphi_{j,y_1}(x,y_1)|\Big),\end{aligned}$$

for $j=2,\dots,n$, provided $|x|$, $|\widetilde{x}|$, $|y_1|$, $|\widetilde{y_1}|\leqslant c\cdot K_2$. On the other hand, computing $\frac{\partial Z^0_{c,x,v_1+v(c)+v}}{\partial\rho}\big(\rho e^{i\theta}\big)$ in (7.8), using (7.25), (7.28) and an inequality of the form (6.2), we deduce that there exists a constant $K_{17}>0$ such that

$$\Big|Z^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)-Z^1_{c,x,v_1+v(c)+v}\big(e^{i\theta}\big)\Big|\leqslant(1-\rho)\cdot c\cdot K_{17}. \tag{8.37}$$

Finally, using the inequality $|Z^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)|\leqslant c\cdot K_2$ obtained in (7.25), using the collection of inequalities (8.36) and using the inequality (8.37), we may deduce the third (and last) interesting inequality for $j=2,\dots,n$:

(8.38)

$$\begin{aligned}&\Big|\varphi_j\big(X^1_{c,x,v_1+v(c)+v}(\rho e^{i\theta}),\ Y^1_{1;c,x,v_1+v(c)+v}(\rho e^{i\theta})\big)-\\ &\qquad-\varphi_j\big(X^1_{c,x,v_1+v(c)+v}(e^{i\theta}),\ Y^1_{1;c,x,v_1+v(c)+v}(e^{i\theta})\big)\Big|\leqslant\\ &\Big(\Big|X^1_{c,x,v_1+v(c)+v}(\rho e^{i\theta})-X^1_{c,x,v_1+v(c)+v}(e^{i\theta})\Big|+\\ &\qquad+\Big|Y^1_{1;c,x,v_1+v(c)+v}(\rho e^{i\theta})-Y^1_{1;c,x,v_1+v(c)+v}(e^{i\theta})\Big|\Big)\cdot\\ &\qquad\cdot\Big(\sum_{k=1}^n\sup_{|x|,\,|y_1|\leqslant c\cdot K_2}\big|\varphi_{j,x_k}(x,y_1)\big|+\sup_{|x|,\,|y_1|\leqslant c\cdot K_2}\big|\varphi_{j,y_1}(x,y_1)\big|\Big)\leqslant\\ &\leqslant(1-\rho)\cdot c^2\cdot K_{18},\end{aligned}$$

for some constant $K_{18}>0$.

We can now complete the proof of the collection of inequalities in the second line of (8.27). As before, let c with $0<c\leqslant c_1$, let ρ with $10c<\rho<1$, let θ with $|\theta|<10c$, let x with $|x|\leqslant c^2$, let v with $|v|\leqslant c$ and let $j=2,\dots,n$. Starting with (8.35), using (8.32), using the fact that $Z^1_{c,x,v_1+v(c)+v}\big(\partial^+\Delta\big) \subset M^1 \subset M$ and using (8.38), we have

$$\begin{aligned}
& Y^1_{j;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big) \geqslant \\
& \quad \geqslant Y^1_{j;c,x,v_1+v(c)+v}\big(e^{i\theta}\big) + (\rho-1)\cdot \frac{\partial Y^1_{j;c,x,v_1+v(c)+v}}{\partial \rho}\big(e^{i\theta}\big) - (1-\rho)^2\cdot c\cdot K_{16} \geqslant \\
& \quad \geqslant Y^1_{j;c,x,v_1+v(c)+v}\big(e^{i\theta}\big) + (1-\rho)\cdot c\cdot C_1 - (1-\rho)\cdot c^2\cdot K_{15} - (1-\rho)^2\cdot c\cdot K_{16} \\
& \quad = \varphi_j\big(X^1_{1;c,x,v_1+v(c)+v}\big(e^{i\theta}\big),\ Y^1_{1;c,x,v_1+v(c)+v}\big(e^{i\theta}\big)\big) + (1-\rho)\cdot c\cdot C_1 - \\
& \qquad\qquad - (1-\rho)\cdot c^2\cdot K_{15} - (1-\rho)^2\cdot c\cdot K_{16} \\
& \quad \geqslant \varphi_j\big(X^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big) + (1-\rho)\cdot c\cdot C_1 - \\
& \qquad\qquad - (1-\rho)\cdot c^2\cdot K_{15} - (1-\rho)^2\cdot c\cdot K_{16} - (1-\rho)\cdot c^2\cdot K_{18} \\
& \quad \geqslant \varphi_j\big(X^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big) + (1-\rho)\cdot c\cdot [C_1 - \\
& \qquad\qquad - c\cdot K_{15} - 10c\cdot K_{16} - c\cdot K_{18}] \\
& \quad \geqslant \varphi_j\big(X^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big) + (1-\rho)\cdot c\cdot \frac{C_1}{2} \\
& \quad > \varphi_j\big(X^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y^1_{1;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big),
\end{aligned} \tag{8.39}$$

provided that

$$c_1 \leqslant \frac{C_1/2}{K_{15}+10K_{16}+K_{18}}. \tag{8.40}$$

This yields the collection of inequalities in the second line of (8.27).

For the first inequality (8.27), we proceed similarly. Recall that $v_1 = (0,1,\dots,1)$, that $a_1 = 1$ and that $a_2+\cdots+a_n = 1$. Since $Z^1_{c,x,v_1+v(c)+v}(\partial^+\Delta) \subset M^1 \subset N^1$, we have for all θ with $|\theta| \leqslant \frac{\pi}{2}$ the following relation

$$\sum_{k=1}^n a_k\, Y^1_{k;c,x,v_1+v(c)+v}\big(e^{i\theta}\big) = \psi\big(X^1_{c,x,v_1+v(c)+v}(e^{i\theta}),\ Y'^1_{c,x,v_1+v(c)+v}(e^{i\theta})\big). \tag{8.41}$$

Using that ψ vanishes to order one at the origin by the normalization conditions (5.40) and proceeding as in the previous paragraph concerning the functions φ_j, we obtain an inequality similar to (8.38):

$$\begin{aligned}
& \Big|\psi\big(X^1_{c,x,v_1+v(c)+v}(\rho e^{i\theta}),\ Y'^1_{c,x,v_1+v(c)+v}(\rho e^{i\theta})\big) - \\
& \qquad - \psi\big(X^1_{c,x,v_1+v(c)+v}\big(e^{i\theta}\big),\ Y'^1_{c,x,v_1+v(c)+v}(e^{i\theta})\big)\Big| \leqslant (1-\rho)\cdot c^2\cdot K_{19},
\end{aligned} \tag{8.42}$$

for some constant $K_{19} > 0$.

As before, let c with $0 < c \leqslant c_1$, let ρ with $10c < \rho < 1$, let θ with $|\theta| < 10c$, let x with $|x| \leqslant c^2$ and let v with $|v| \leqslant c$. Using then (8.35), (8.32), (8.41) and (8.42), we deduce the desired strict inequality

$$\begin{aligned}
\sum_{k=1}^n a_k\, Y^1_{k;c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big) &\geqslant \sum_{k=1}^n a_k\, Y^1_{k;c,x,v_1+v(c)+v}\big(e^{i\theta}\big)+\\
&+(1-\rho)\Big[\sum_{k=1}^n a_k\big(-\frac{\partial Y^1_{k;c,x,v_1+v(c)+v}}{\partial\rho}\big(e^{i\theta}\big)\big)\Big]-(1-\rho)^2\cdot c\cdot\big(\sum_{k=1}^n a_k\big)K_{16}\\
&\geqslant \sum_{k=1}^n a_k\, Y^1_{k;c,x,v_1+v(c)+v}\big(e^{i\theta}\big)+(1-\rho)\Big[\sum_{j=2}^n a_j\cdot c\cdot C_1-\sum_{k=1}^n a_k\cdot c^2\cdot K_{15}\Big]-\\
&\qquad -(1-\rho)^2\cdot c\cdot 2K_{16}
\end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{k=1}^n a_k\, Y^1_{k;c,x,v_1+v(c)+v}\big(e^{i\theta}\big)+(1-\rho)\cdot c\cdot C_1-\\
&\qquad -(1-\rho)\cdot c^2\cdot 2K_{15}-(1-\rho)^2\cdot c\cdot 2K_{16}\\
&=\psi\big(X^1_{c,x,v_1+v(c)+v}\big(e^{i\theta}\big),\ Y'^1_{c,x,v_1+v(c)+v}\big(e^{i\theta}\big)\big)+\\
&\qquad +(1-\rho)\cdot c\cdot[C_1-c\cdot 2K_{15}-10c\cdot 2K_{16}]
\end{aligned}\tag{8.43}$$

$$\begin{aligned}
&\geqslant \psi\big(X^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y'^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big)+\\
&\qquad +(1-\rho)\cdot c\cdot[C_1-c\cdot 2K_{15}-10c\cdot 2K_{16}-c\cdot K_{19}]\\
&\geqslant \psi\big(X^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y'^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big)+(1-\rho)\cdot c\cdot\frac{C_1}{2}\\
&> \psi\big(X^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big),\ Y'^1_{c,x,v_1+v(c)+v}\big(\rho e^{i\theta}\big)\big),
\end{aligned}$$

provided $c_1 \leqslant \frac{C_1/2}{2K_{15}+20K_{16}+K_{19}}$. This yields the first inequality of (8.27) and completes the proof of $(\mathbf{9_1})$ in Case $(\mathbf{I_1})$. □

§9. END OF PROOF OF PROPOSITION 1.13: APPLICATION OF THE CONTINUITY PRINCIPLE

9.1. Preliminary. In this section, we complete the proof of Proposition 5.12, hence the proof of Theorem 3.19, hence also the proof of the main Proposition 1.13 (at last!).

Translating M^1 inside M, we will introduce a supplementary small real parameter u, getting a family $A^1_{x,v,u:c}(\zeta)$ of analytic discs partially attached to the translate M^1_u. Applying the continuity principe to this family of discs, we shall show that, in Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$, there exists a local wedge $\mathcal{W}_{p_1}$ of edge M at p_1 to which $\mathcal{O}\big(\Omega\cup\mathcal{HW}_1^+\big)$ extends holomorphically; in Case **(II)**, there will exist a whole (small) neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$ to which $\mathcal{O}\big(\Omega\cup\mathcal{W}_2\big)$ extends holomorphically. To organize well this last main step of the proof of Proposition 5.12, we shall consider jointly Cases $(\mathbf{I_1})$, $(\mathbf{I_2})$ and then afterwards Case **(II)** separately.

9.2. Translations of M^1 in M. According to Lemma 5.37, in Case $(\mathbf{I_1})$, the one-codimensional submanifold $M^1\subset M$ is given by the equations $y'=\varphi'(x,y_1)$ and $x_1=g(x')$. If $u\in\mathbb{R}$ is a small real parameter, we may define a "translation" M^1_u of M^1 in M by the n equations

$$M^1_u:\quad y'=\varphi'(x,y_1),\quad x_1=g(x')+u.\tag{9.3}$$

Clearly, we have $M^1_u\subset(M^1)^+$ if $u>0$ and $M^1_u\subset(M^1)^-$ if $u<0$. We may perturb the family of analytic discs $Z^d_{c,x,v}(\zeta)$ half-attached to M^1 satisfying Bishop's equation (7.18) by requiring that it is attached to M_u^1. Thanks to the stability under perturbation of the solutions to Bishop's equation, we then obtain a new family of analytic discs $Z_{c,x,v,u}^d(\zeta)$ which is half-attached to M_u^1 and which is of class $\mathcal{C}^{2,\alpha-0}$ with respect to all variables (c,x,v,u,ζ). For $u=0$, this solution coincides with the family $Z_{c,x,v}^d(\zeta)$ constructed in §7.13. Using a similar definition as in (7.54), namely setting $A_{x,v,u:c}^1(\zeta) := Z_{c,x,v_1+v(c)+v,u}^1\big(\Phi_c(\zeta)\big)$, we obtain a new family of analytic discs which coincides, for $u=0$, with the family $A_{x,v:c}^1(\zeta)$ of Lemmas 7.12 and 8.3.

In Case $(\mathbf{I_2})$, taking account of the normalizations stated in Lemma 5.37, we may also construct a similar family of analytic discs $A_{x,v,u:c}^1(\zeta)$. From now on, we fix the scaling parameter c with $0 < c \leqslant c_1$, so that the nine properties $(\mathbf{1_1})$ to $(\mathbf{9_1})$ of Lemmas 7.12 and 8.3 are satisfied by $A_{x,v,0:c}^1(\zeta)$.

9.4. Definition of a local wedge of edge M at p_1 in Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$. First of all, in Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$, we restrict the variation of the parameter v to a certain $(n-2)$-dimensional linear subspace V_2 of $T_{p_1}\mathbb{R}^n \simeq \mathbb{R}^n$ as follows. By hypothesis, the vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^c M$, so the real vector space $(\mathbb{R}\cdot v_1) \oplus \big(T_{p_1}M^1 \cap T_{p_1}^c M\big) \subset T_{p_1}M^1$ is 2-dimensional. We choose an arbitrary $(n-2)$-dimensional real vector subspace $V_2 \subset T_{p_1}M^1$ which is a supplementary in $T_{p_1}M^1$ to $(\mathbb{R}\cdot v_1) \oplus \big(T_{p_1}M^1 \cap T_{p_1}^c M\big)$ and we shall let the parameter v vary only in V_2. Also, we choose a local $(n-1)$-dimensional submanifold $X_1 \subset M^1$ passing through p_1 with $\mathbb{R}\cdot v_1 \oplus T_{p_1}X_1 = T_{p_1}M^1$.

From the rank properties $(\mathbf{5_1})$ and $(\mathbf{6_1})$ of Lemma 7.12 and from the definitions of V_2 and of X_1, it may then be verifed (as in [MP1999, MP2002]) that, for $\varepsilon > 0$ small enough with $\varepsilon \ll c^2$, the mapping

$$(x,v,u,\rho,\theta) \longmapsto A_{x,v,u:c}^1\big(\rho\, e^{i\theta}\big) \tag{9.5}$$

is a *one-to-one immersion*[6] from the open set $\big\{(x,v,u,\rho) \in X_1 \times V_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : |x| < \varepsilon,\ |v| < \varepsilon,\ |u| < \varepsilon,\ 1-\varepsilon < \rho < 1,\ |\theta| < \varepsilon\big\}$ onto its image

(9.6)
$$\mathcal{W}_{p_1} := \Big\{ A_{x,v,u:c}^1\big(\rho e^{i\theta}\big) \in \mathbb{C}^n : (x,v,u,\rho,\theta) \in X_1 \times V_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ |x| < \varepsilon,\ |v| < \varepsilon,\ |u| < \varepsilon,\ 1-\varepsilon < \rho < 1,\ |\theta| < \varepsilon \Big\},$$

which is a local wedge of edge M at (p_1, Jv_1), with $\mathcal{W}_{p_1} \cap M = \emptyset$.

Let the singularity C with $p_1 \in C$ and $C\backslash\{p_1\} \subset (H^1)^-$, let the neighborhood Ω of $M\backslash C$ in $\mathbb{C}^n$, let the half-wedge $\mathcal{HW}_{p_1}^+$ be as in Proposition 5.12, and let the sub-half-wedge $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ be as in §5.14 and Lemma 5.37. In Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$, we shall prove that a (sufficiently thick) part of the envelope of holomorphy of $\Omega \cup \mathcal{HW}_1^+$ contains the wedge $\mathcal{W}_{p_1}$ and is schlicht over it.

9.7. Boundaries of analytic discs. Since we want to apply the continuity principle, we must verify that most discs $A_{x,v,u:c}^1(\zeta)$ have their boundaries in $\Omega \cup \mathcal{HW}_1^+$. To this aim, we decompose the boundary $\partial\Delta$ in three closed parts $\partial\Delta = \partial^1\Delta \cup \partial^2\Delta \cup \partial^3\Delta$,

[6]This property will be crucial to insure uniqueness of the holomorphic extension, when we apply the continuity principle in Lemma 9.16.

where

$$(9.8)\qquad \begin{cases} \partial^1\Delta := \left\{e^{i\theta}\in\partial\Delta:\ |\theta|\leqslant \pi/2-\varepsilon\right\}\subset\partial^+\Delta, \\ \partial^2\Delta := \left\{e^{i\theta}\in\partial\Delta:\ \pi/2+\varepsilon\leqslant|\theta|\leqslant\pi\right\}\subset\partial^-\Delta, \\ \partial^3\Delta := \left\{e^{i\theta}\in\partial\Delta:\ \pi/2-\varepsilon\leqslant|\theta|\leqslant\pi/2+\varepsilon\right\}\subset\partial\Delta, \end{cases}$$

where ε with $0<\varepsilon\ll c^2$ is as in §9.4 just above. This decomposition is illustrated in the left Figure 19 below. Next, we observe that the two points $A^1_{0,0,0:c}(i)$ and $A^1_{0,0,0:c}(-i)$ belong to $(H^1)^+\subset M\backslash C\subset\Omega$, hence there exists a fixed open neighborhood of these two points which is contained in Ω. We shall denote by ω^3 such a (disconnected) neighborhood, for instance the union of two small open polydiscs centered at these two points. To proceed further, we need a crucial geometric information about the boundaries of the analytic discs $A^1_{x,v,u:c}(\zeta)$ with $u\neq 0$.

Lemma 9.9. *In Cases* $(\mathbf{I_1})$ *and* $(\mathbf{I_2})$*, after shrinking $\varepsilon>0$ if necessary, then*

$$(9.10)\qquad A^1_{x,v,u:c}(\partial\Delta)\subset\Omega\cup\mathcal{HW}^+_1,$$

for all x with $|x|<\varepsilon$, for all v with $|v|<\varepsilon$ and for all nonzero $u\neq 0$ *with $|u|<\varepsilon$.*

Proof. Firstly, since $A^1_{0,0,0:c}(\pm i)\in\omega^3$, it follows just by continuity of the family $A^1_{x,v,u:c}(\zeta)$ that, after possibly shrinking $\varepsilon>0$, the closed arc $A^1_{x,v,u:c}(\partial^3\Delta)$ is contained in ω^3, for all x with $|x|<\varepsilon$, for all v with $|v|<\varepsilon$ and for all u with $|u|<\varepsilon$. Secondly, since $A^1_{0,0,0:c}(\partial^2\Delta)\subset A^1_{0,0,0:c}(\partial^-\Delta\backslash\{i,-i\})\subset\mathcal{HW}^+_1$, then by property $(\mathbf{9_1})$ of Lemma 8.3, it follows just thanks to continuity of the family $A^1_{x,v,u:c}(\zeta)$ that, after possibly shrinking $\varepsilon>0$, the closed arc $A^1_{x,v,u:c}(\partial^2\Delta)$ is contained in $\mathcal{HW}^+_1$, for all x with $|x|<\varepsilon$, for all v with $|v|<\varepsilon$ and for all u with $|u|<\varepsilon$. Thirdly, it follows from the inclusion $A^1_{x,v,u:c}(\partial^1\Delta)\subset A^1_{x,v,u:c}(\partial^+\Delta)\subset M^1_u$ and from the inclusion $M^1_u\subset\Omega$ for all $u\neq 0$ that, after possibly shrinking $\varepsilon>0$, the closed arc $A^1_{x,v,u:c}(\partial^1\Delta)$ is contained in Ω, for all x with $|x|<\varepsilon$, for all v with $|v|<\varepsilon$ and for all u with $|u|<\varepsilon$ and $u\neq 0$. □

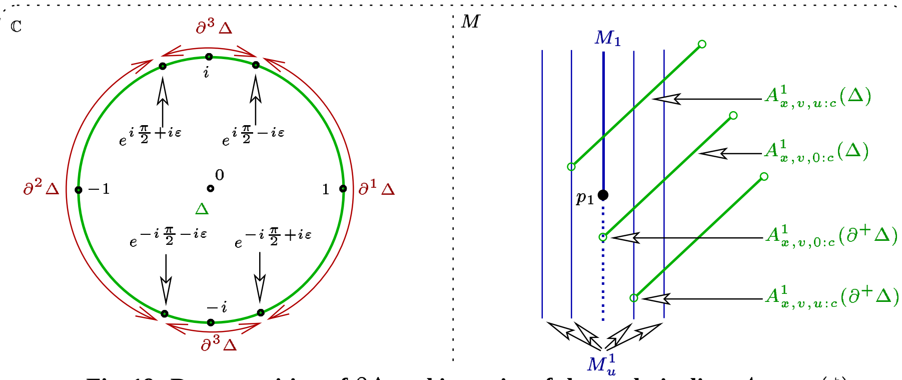

Fig. 19: Decomposition of $\partial\Delta$ and isotopies of the analytic discs $A_{x,v,u:c}(\zeta)$

9.11. Analytic isotopies. Following [Me1997], two analytic discs $A',A''\in\mathcal{O}(\Delta,\mathbb{C}^n)\cap\mathcal{C}^1(\overline{\Delta})$ which are both *embeddings* of $\overline{\Delta}$ into $\mathbb{C}^n$ are said to be *analytically isotopic* if there exists a $\mathcal{C}^1$ family of embedded analytic discs $A_\tau\in\mathcal{O}(\Delta,\mathbb{C}^n)\cap\mathcal{C}^1(\overline{\Delta})$, $\tau\in[0,1]$, with $A_0=A'$ and $A_1=A''$. If $\mathcal{D}\subset\mathbb{C}^n$ is a domain, a disc A' is *analytically isotopic to a point with boundaries inside* $\mathcal{D}$ if

$A''\big(\overline{\Delta}\big) \equiv p'' \in \mathcal{D}$ is a constant disc, if each A_τ is embedded, for $0 \leqslant \tau < 1$, and if $A_\tau(\partial\Delta) \subset \mathcal{D}$ for $0 \leqslant \tau \leqslant 1$.

In Case **($\mathbf{I_1}$)**, we fix some $x_0 = (x_{1;0}, 0, \dots, 0) \in \mathbb{R}^n$ with $0 < x_{1;0} < \varepsilon$. Then $A^1_{x_0,0,0:c}(1) = x_0 + ih(x_0)$ belongs to $T^1 \cap (H^1)^+$. Analogously, in Cases **($\mathbf{I_2}$)**, we fix some $x_0 = (0, \dots, 0, x_{n;0}) \in \mathbb{R}^n$ with $0 < x_{n;0} < \varepsilon$. Then in this second case, the point $A^1_{x_0,0,0:c}(1) = x_0 + ih(x_0)$ also belongs to $T^1 \cap (H^1)^+$. We fix this reference disc $A^1_{x_0,0,0:c}(\zeta)$, which satisfies $A_{x_0,0,0:c}(\partial^+\Delta) \subset (H^1)^+$.

Lemma 9.12. *In Cases* **($\mathbf{I_1}$)** *and* **($\mathbf{I_2}$)**, *every disc $A^1_{x,v,u:c}(\zeta)$ with $|x| < \varepsilon$, $|v| < \varepsilon$, $|u| < \varepsilon$ and $u \neq 0$ is analytically isotopic to the disc $A^1_{x_0,0,0:c}(\zeta)$, with the boundaries of the analytic discs of the isotopy being all contained in $\Omega \cup \mathcal{HW}^+_1$. The same property is enjoyed by every disc $A^1_{x,v,0:c}$ such that $A^1_{x,v,0:c}(\partial^+\Delta) \subset (H^1)^+$.*

Furthermore, $A^1_{x_0,0,0:c}\big(\overline{\Delta}\big) \subset \Omega \cup \mathcal{HW}^+_1$, hence $A^1_{x_0,0,0:c}$ is analytically isotopic to a point with boundaries inside $\Omega \cup \mathcal{HW}^+_1$ (just shrink its radius).

Consequently, all discs $A^1_{x,v,u:c}$ with $u \neq 0$ and all discs $A^1_{x,v,0:c}$ with $A^1_{x,v,0:c}(\partial^+\Delta) \subset (H^1)^+$ are analytically isotopic to a point with boundaries inside $\Omega \cup \mathcal{HW}^+_1$.

Proof. Since $\{u = 0\}$ is a hyperplane of the whole parameter space, there exists a $\mathcal{C}^{2,\alpha-0}$ curve $\tau \mapsto (x(\tau), v(\tau), u(\tau))$ in the parameter space which joins a given arbitrary point (x^*, v^*, u^*) with $u^* \neq 0$ to the point $(x_0, 0, 0)$ without meeting the hyperplane $\{u = 0\}$, except at its endpoint $(x_0, 0, 0)$. According to the previous Lemma 9.9, each boundary $A^1_{x(\tau),v(\tau),u(\tau):c}(\partial\Delta)$ is then automatically contained in $\Omega \cup \mathcal{HW}^+_1$.

Also, if $A^1_{x,v,0:c}(\partial^+\Delta) \subset (H^1)^+$, whence in particular $A^1_{x,v,0:c}(1) = x + ih(x) \in (H^1)^+$, we first isotope $A^1_{x,v,0:c}$ to $A^1_{x,0,0:c}$ just by moving in the v-parameter space along straight segment $[0, v]$. Thanks to the strong convexity of $(H^1)^-$ and to the almost straighteness of the half boundaries (Figure 18) which rotate slightly as $v' \in [0, v]$ varies, the half boundary $A^1_{x,v',0:c}(\partial^+\Delta)$ stays in $(H^1)^+$, while the remainder part of the boundary $A^1_{x,v',0:c}\big(\partial^-\Delta \backslash \{\pm i\}\big)$ stays in $\mathcal{HW}^+_1$. Then $A^1_{x,0,0:c}$ is trivially isotopic to $A^1_{x_0,0,0:c}$.

Finally, since $A^1_{x_0,0,0:c}(1)$ belongs to $T^1 \cap (H^1)^+$, property **($\mathbf{8_1}$)** of Lemma 8.3 insures that $A^1_{x_0,0,0:c}\big(\partial^+\Delta\big)$ is contained in $(H^1)^+$, hence in Ω. Then property **($\mathbf{9_1}$)** says that $A^1_{x_0,0,0:c}(1)\big(\overline{\Delta}\backslash\partial^+\Delta\big)$ is contained in $\mathcal{HW}^+_1$, which completes the proof. □

9.13. holomorphic extension to a local wedge of edge M at p_1. In Cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**, we define a $\mathcal{C}^{2,\alpha-0}$ connected hypersurface of $\mathcal{W}_{p_1}$:

$$\begin{aligned}\mathcal{M}_{p_1} := \Big\{ A^1_{x,v,0:c}\big(\rho e^{i\theta}\big) :\ & (x, v, \rho, \theta) \in X_1 \times V_2 \times \mathbb{R} \times \mathbb{R}, \\ & |x| < \varepsilon,\ |v| < \varepsilon,\ 1 - \varepsilon < \rho < \varepsilon,\ |\theta| < \varepsilon \Big\},\end{aligned} \tag{9.14}$$

together with a proper subset of $\mathcal{M}_{p_1}$:

(9.15)
$$\begin{aligned}\mathcal{C}_{p_1} := \Big\{ A^1_{x,v,0:c}\big(\rho e^{i\theta}\big) :\ & (x, v, \rho, \theta) \in \mathbb{R}^n \times V_2 \times \mathbb{R} \times \mathbb{R}, \\ & A^1_{x,v,0:c}\big(\partial^+\Delta\big) \not\subset (H^1)^+,\ |x| < \varepsilon,\ |v| < \varepsilon,\ 1 - \varepsilon < \rho < \varepsilon,\ |\theta| < \varepsilon \Big\}.\end{aligned}$$

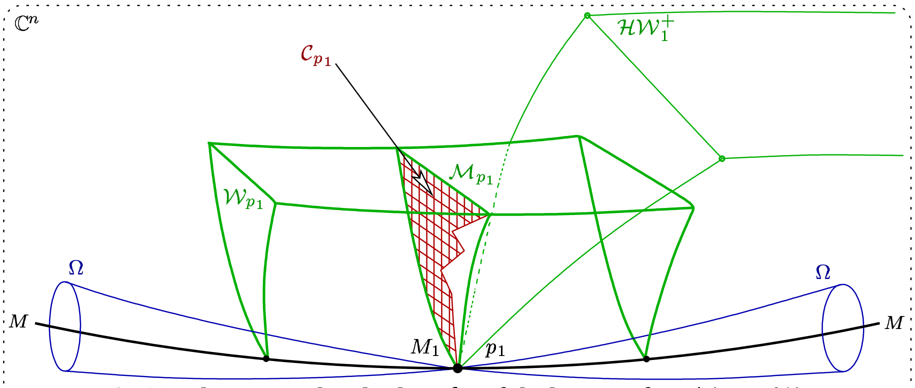

Fig. 20: The proper closed subset $\mathcal{C}_{p_1}$ of the hypersurface $\mathcal{M}_{p_1} \subset \mathcal{W}_{p_1}$

We can now state the main lemma of this section, completing the proof of Proposition 5.12.

Lemma 9.16. *In Cases* $(\mathbf{I_1})$ *and* $(\mathbf{I_2})$, *after possibly shrinking* Ω *in a small neighborhood of* p_1 *and after possibly shrinking* $\varepsilon > 0$, *the set* $\mathcal{W}_{p_1} \cap \big[\Omega \cup \mathcal{HW}_1^+\big]$ *is connected and for every holomorphic function* $f \in \mathcal{O}\big(\Omega \cup \mathcal{HW}_1^+\big)$, *there exists a holomorphic function* $F \in \mathcal{O}\big(\Omega \cup \mathcal{HW}_1^+ \cup \mathcal{W}_{p_1}\big)$ *such that* $F|_{\Omega \cup \mathcal{HW}_1^+} = f$.

Proof. Remind that $\varepsilon \ll c^2$ and remind that the wedge $\mathcal{W}_{p_1}$ with $\mathcal{W}_{p_1} \cap M = \emptyset$ in the two cases is of size $\mathrm{O}(\varepsilon)$. Since the singularity C is contained in $(H^1)^- \cup \{p_1\} \subset M^1$, its complement $M \backslash C$ is locally connected near p_1. The half-wedge $\mathcal{HW}_1^+$ defined in Lemma 5.37 by simple inequalities is of size $\mathrm{O}(\delta_1)$. If $\varepsilon << \delta_1$, after shrinking Ω if necessary in a smal neighborhood of p_1 whose size is $\mathrm{O}(\varepsilon)$, it follows that we can assume that $\mathcal{W}_{p_1} \cap \big[\Omega \cup \mathcal{HW}_1^+\big]$ is connected.

Let f be an arbitrary holomorphic function in $\mathcal{O}\big(\Omega \cup \mathcal{HW}_1^+\big)$. Thanks to the isotopy Lemma 9.12, applying the continuity principle ([Me1997]), we deduce that f extends holomorphically to a (very, very thin) neighborhood in $\mathbb{C}^n$ of every disc $A^1_{x,v,u:c}\big(\overline{\Delta}\big)$ with $u \neq 0$ and also, to neighborhood in $\mathbb{C}^n$ of every disc $A^1_{x,v,0:c}\big(\overline{\Delta}\big)$ such that $A^1_{x,v,0:c}(\partial^+\Delta) \subset (H^1)^+$.

Using the fact that the mapping (9.5) is one-to-one onto $\mathcal{W}_{p_1}$, we deduce that f extends uniquely at all such points $A^1_{x,v,u:c}\big(\rho e^{i\theta}\big) \in \mathcal{W}_{p_1}$ simply by means of Cauchy's formula:

$$f\big(A^1_{x,v,u:c}(\rho e^{i\theta})\big) := \int_{\partial\Delta} \frac{f\big(A^1_{x,v,u:c}(\widetilde{\zeta})\big)}{\widetilde{\zeta} - \rho e^{i\theta}}\, d\widetilde{\zeta}. \tag{9.17}$$

Consequently, f extends holomorphically and uniquely to the domain $\mathcal{W}_{p_1} \backslash \mathcal{C}_{p_1}$. Let $F \in \mathcal{O}\big(\mathcal{W}_{p_1} \backslash \mathcal{C}_{p_1}\big)$ denote this holomorphic extension. Since $\mathcal{W}_{p_1} \cap \big[\Omega \cup \mathcal{HW}_1^+\big]$ is connected, it follows that $[\mathcal{W}_{p_1} \backslash \mathcal{C}_{p_1}] \cap \big[\Omega \cup \mathcal{HW}_1^+\big]$ is also connected. From the principle of analytic continuation, we deduce that there exists a well-defined function, still denoted by F, which is holomorphic in $[\mathcal{W}_{p_1} \backslash \mathcal{C}_{p_1}] \cup \big[\Omega \cup \mathcal{HW}_1^+\big]$ and which extends f, namely $F|_{\Omega \cup \mathcal{HW}_1^+} = f$.

We remind that $A^1_{0,0,0:c}(\partial^+\Delta)$ is tangent to $(H^1)^-$ at p_1. By continuity, for small enough ε, it follows that $A^1_{x,v,u:c}(\partial^+\Delta) \cap C$ is contained in $A^1_{x,v,u:c}\big(\{e^{i\theta} : |\theta| \leqslant \frac{\pi}{4}\}\big)$ for all $|x| < \varepsilon$, $|v| < \varepsilon$, $|u| < \varepsilon$. Thus $A^1_{x,v,u:c}(\partial^+\Delta) \cap \Omega$ is always nonempty. The $\mathcal{C}^{2,\alpha-0}$ hypersurface $\mathcal{M}_{p_1} \subset \mathcal{W}_{p_1}$ is foliated by small pieces of analytic discs.

Each such piece is necessarily contained in a single CR orbit of $\mathcal{M}_{p_1}$. The residual singularity of the holomorphic function F can only be $\mathcal{C}'_{p_1} := \mathcal{C}_{p_1} \backslash \Omega$. Since $A^1_{x,v,u:c}\big(\partial^+ \Delta \backslash \{e^{i\theta} : |\theta| \leqslant \frac{\pi}{4}\}\big)$ is contained in Ω, it follows that $\mathcal{C}'_{p_1} \subset \mathcal{M}_{p_1}$ cannot contain any CR orbit of $\mathcal{M}_{p_1}$. According to Lemma 2.10 of [MP1999], F then extends holomorphically and uniquely through $\mathcal{C}'_{p_1}$.

The proofs of Lemma 9.16 and of Proposition 5.12 in Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$ are complete. □

9.18. End of proof of Proposition 5.12 in Case (II). According to Lemma 5.37, in Case **(II)**, the one-codimensional totally real submanifold $M^1 \subset M$ is given by the equations $y' = \varphi'(x, y_1)$ and $x_n = g(x'')$. If $u \in \mathbb{R}$ is a small real parameter, we may define a "translation" M^1_u of M^1 in M by the equations

$$y' = \varphi'(x, y_1), \qquad x_n = g\big(x''\big) + u. \tag{9.19}$$

Similarly as in §9.2, we may construct a family of analytic discs $A^1_{x,v,u:c}(\zeta)$ half-attached to M^1_u. We then we fix a small scaling parameter c with $0 < c \leqslant c_1$ so that properties $(\mathbf{1_1})$ to $(\mathbf{9_1})$ of Lemmas 7.12 and 8.3 hold true for $A^1_{x,v,0:c}$.

We restrict the variation of the parameter v to an arbitrary $(n-1)$-dimensional subspace V_1 of $T_{p_1}M^1 \simeq \mathbb{R}^n$ which is supplementary to the real line $\mathbb{R} \cdot v_1$ in $T_{p_1}M^1$ (this makes a difference with §9.4). Also, we choose a local $(n-1)$-dimensional submanifold $X_1 \subset M^1$ passing through p_1 with $\mathbb{R} \cdot v_1 \oplus T_{p_1}X_1 = T_{p_1}M^1$. If $\varepsilon > 0$ is small enough with $\varepsilon \ll c^2$, then for every fixed u, the mapping

$$(x, v, u, \rho) \longmapsto A^1_{x,v,u:c}(\rho) \tag{9.20}$$

is a *one-to-one immersion* from the open set $\{(x, v, \rho) \in \mathbb{R}^n \times V_1 \times \mathbb{R} \times \mathbb{R} : |x| < \varepsilon, |v| < \varepsilon, 1 - \varepsilon < \rho < 1\}$ into $\mathbb{C}^n$ onto its image

$$\begin{aligned} \mathcal{W}^1_u := \Big\{ A^1_{x,v,u:c}\big(\rho e^{i\theta}\big) \in \mathbb{C}^n : (x, v, \rho) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ |x| < \varepsilon, \ |v| < \varepsilon, \ 1 - \varepsilon < \rho < 1, \ |\theta| < \varepsilon \Big\} \end{aligned} \tag{9.21}$$

is a local wedge of edge M^1_u. Clearly, this wedge $\mathcal{W}^1_u$ is $\mathcal{C}^{2,\alpha-0}$ with respect to u.

Using the fact that in Case **(II)** we have

$$\frac{\partial A^1_{0,0,0:c}}{\partial \theta}(1) = v_1 = (1, 0, \dots, 0) \in T_{p_1}M^1 \cap T^c_{p_1}M, \tag{9.22}$$

one can prove that Lemma 9.9 holds true with $\mathcal{HW}^+_1$ replaced by $\mathcal{W}_2$ in (9.10) and also that Lemma 9.12 holds true, again with $\mathcal{HW}^+_1$ replaced by $\mathcal{W}_2$. Similarly as in the proof of Lemma 9.16, applying then the continuity principle and using the fact that the mapping (9.20) is one-to-one, after possibly shrinking Ω in a neighborhood of p_1, and shrinking $\varepsilon > 0$, we deduce that for each $u \neq 0$, there exists a holomorphic function $F \in \mathcal{O}\big(\Omega \cup \mathcal{W}_2 \cup \mathcal{W}^1_u\big)$ with $F\big|_{\Omega \cup \mathcal{W}_2} = f$.

To conclude, it suffices to observe that for every fixed small u with $-\varepsilon \ll u < 0$, the wedge $\mathcal{W}^1_u$ contains in fact a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$ (the reader may draw a figure).

The proofs of Proposition 5.12 and of Theorem 3.19 are complete now. □

9.23. End of proof of Proposition 1.13. In order to derive Proposition 1.13 from Theorem 3.19, we now remind the necessity of supplementary arguments about the stability of our constructions under deformation.

Coming back to the strategy developped in §3.16, we had a first wedge $\mathcal{W}_1$ attached to $M\backslash C_{\rm nr}$. Using a partition of unity, we introduce a one-parameter $\mathcal{C}^{2,\alpha}$ family of generic submanifolds M^d, $d \in \mathbb{R}$, $d \geqslant 0$, with $M^0 \equiv M$, with M^d containing $C_{\rm nr}$ and with $M^d\backslash C_{\rm nr}$ contained in $\mathcal{W}_1$. In the proof of Theorem 3.19, thanks to this deformation, the wedge $\mathcal{W}_1$ was replaced by a neighborhood Ω of $M\backslash C_{\rm nr}$ in $\mathbb{C}^n$.

In Sections 4 and 5, we constructed a semi-local half-wedge $(\mathcal{HW}_\gamma^+)^d$ attached to a one-sided neighborhood of $(M^1)^d$ in M^d along a characteristic segment γ^d of M^d. Now, we crucially claim that by arranging well this deformation M^d, we may achieve that the geometric extent of this semi-local half-wedge is uniform as $d > 0$ tends to zero, namely $(\mathcal{HW}_\gamma^+)^d$ tends to a nonvoid semi-local half-wedge $(\mathcal{HW}_\gamma^+)^0$ attached to a one-sided neighborhood of M^1 in M along γ, as d tends to zero. Indeed, in Section 4 we have constructed a family of analytic discs $(\mathcal{Z}_{t,\chi,\nu:s}(\zeta))^d$ (*cf.* (4.61)) which covers the half-wedge $(\mathcal{HW}_\gamma^+)^d$. Thanks to the stability of Bishop's equation under $\mathcal{C}^{2,\alpha}$ perturbations, the deformed family $(\mathcal{Z}_{t,\chi,\nu:s}(\zeta))^d =: \mathcal{Z}_{t,\chi,\nu:s}^d(\zeta)$ is also of class $\mathcal{C}^{2,\alpha-0}$ with respect to the parameter d. We remind that for every $d > 0$, the family $\mathcal{Z}_{t,\chi,\nu:s}^d(\zeta)$ was in fact constructed by means of a family $\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}^d(\zeta)$ obtained by solving Bishop's equation (4.40), where we now add the parameter d in the function Φ'. In order to construct the semi-local attached half-wedge, we have used the rank property stated in Lemma 4.34. This rank property relied on the possibility of deforming the disc $\widehat{Z}_{r_0,t:s}(\zeta)$ near the point $\widehat{Z}_{r_0,t:s}^d(-1)$ in the open neighborhood $\Phi_s(\Omega) \equiv \Phi_s\big(\mathcal{W}_1\big)$ of $\Phi_s\big(M^d\big)$. As $d > 0$ tends to zero, if M^d tends to M, the size of the neighborhood $\Phi_s\big(\mathcal{W}_1\big)$ shrinks to zero, hence it could seem that the we have no control on the semi-local attached half-wege $(\mathcal{HW}_\gamma^+)^d$ as $d > 0$ tends to zero. Fortunately, *since the points* $\widehat{Z}_{r_0,0:s}^d(-1)$ *in a neighborhood of which we introduce the deformations* (4.30) *stay at a uniformly positive distance* $\delta > 0$ *from the characteristic segment* γ, we may choose the deformation M^d of M to tend to M as d tends to zero only in some thin, elongated tubular neighborhood of γ, whose width is small in comparison to this distance δ. By smoothness with respect to d of the family $\mathcal{Z}_{t,\chi,\nu:s}^d(\zeta)$, we then deduce that the semi-local half-wedge $(\mathcal{HW}_\gamma^+)^d$ tends to a nontrivial semi-local half-wedge $(\mathcal{HW}_\gamma^+)^0$ as d tends to zero, which proves the claim.

Next, again thanks to the stability of Bishop's equation under perturbations, all the constructions of Sections 5, 6, 7, 8 and 9 above may be achieved to depend in a $\mathcal{C}^{2,\alpha-0}$ way with respect to d, hence uniformly. Importantly, we observe that if the deformation M^d is chosen so that M^d tends to M only in a small neighborhood of p_1 of size $\ll \varepsilon$, then the shrinking of ε which occurs in Lemma 9.9 may be achieved to be uniform as d tends to zero, because the part $A_{x,v,u:c}(\partial^3\Delta)$ stays in a uniform compact subset of Ω, as d tends to zero. At the end of the proof of Proposition 5.12, we then obtain univalent holomorphic extension to a local wedge $\mathcal{W}_{p_1}^d$ of edge M^d or to a neighborhood $\omega_{p_1}^d$ of M^d in $\mathbb{C}^n$, and they tend smoothly to a wedge $\mathcal{W}_{p_1}^0$ of edge M at p_1 or to a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$.

The proof of Proposition 1.13 is complete. □

§10 $\mathcal{W}$-REMOVABILITY IMPLIES L^{p}-REMOVABILITY

10.1. Preliminary. From [Me1994, Jö1996], we remind that if M' is a globally minimal $\mathcal{C}^{2,\alpha}$ generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geqslant 1$ and of codimension $d = n - m \geqslant 1$, there exists a wedge $\mathcal{W}'$ attached to M' constructed by means of analytic discs glued progressively to M' and to some intermediate conelike submanifolds attached to M'. Classically, one deduces that continuous CR functions on M' extend holomorphically to $\mathcal{W}'$, and continuously to $M' \cup \mathcal{W}'$.

For $L^{\mathsf{p}}_{loc,CR}$ functions, some supplementary, routine, though not straightforward, work has to be achieved. First of all, on a $\mathcal{C}^2$ generic submanifold M' of $\mathbb{C}^n$, the approximation theorem states that every $L^{\mathsf{p}}_{loc,CR}$ function on M' is locally the limit, in the L^{p} norm, of a sequence of polynomials (*cf.* Lemma 3.3 in [Jö1999b]). In the case where M' is a hypersurface, studied in [Jö1999b], the wedge is in fact a one-sided neighborhood of M', which we will denote by $\mathcal{S}'$. The theory of Hardy spaces on the unit disc transfers to parameterized families of small analytic discs glued to M', provided the boundaries of these discs foliate an open subset of M'. Using Carleson's imbedding theorem and the L^{p} approximation theorem, Jöricke established in [Jö1999b] that every $L^{\mathsf{p}}_{loc,CR}$ function defined in a globally minimal $\mathcal{C}^2$ hypersurface M' extends holomorphically in the Hardy space $H^{\mathsf{p}}(\mathcal{S}')$ of holomorphic functions defined in $\mathcal{S}'$ which enjoy L^{p} boundary values on M'. In [Po2000, Po2004], the theory was built in higher codimension, introducing and studying the Hardy space $H^{\mathsf{p}}(\mathcal{W}')$ (*see* also [MP2006]).

10.2. L^{p}-removability of nullsets. Let us say that a subset Φ of a $\mathcal{C}^{2,\alpha}$ generic submanifold is *stably $\mathcal{W}$-removable* if it is $\mathcal{W}$-removable with respect to every compactly supported sufficiently small $\mathcal{C}^{2,\alpha}$ deformation M^d of M leaving Φ fixed. Just by abstract nonsense, the singularity C of Proposition 1.13 (in which it only remains to show L^p-removability) is seen to be stably removable.

Proposition 10.3. *Let M be a $\mathcal{C}^{2,\alpha}$ generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geqslant 1$ and of codimension $d = n - m \geqslant 1$, hence of dimension $(2m + d)$, let $\Phi \subset M$ be a nonempty proper closed subset whose $(2m + d)$-dimensional Hausdorff measure is equal to zero. Assume that $M \backslash \Phi$ is globally minimal and let $\mathcal{W}$ be a wedge attached to $M \backslash \Phi$ such that every function in $L^{\mathsf{p}}_{loc}(M) \cap CR(M \backslash \Phi)$ extends holomorphically as a function in the Hardy space $H^{\mathsf{p}}(\mathcal{W})$. If Φ is stably $\mathcal{W}$-removable, then Φ is L^{p}-removable.*

Let us summarize informally the arguments. Fix $f \in L^{\mathsf{p}}_{loc}(M) \cap CR(M \backslash \Phi)$. As soon as wedge extension over points of Φ is known, we may deform M over Φ in the wedgelike domain, thus erasing the singularity Φ. We get a $L^{\mathsf{p}}_{loc,CR}$ function f^d on the deformed manifold M^d, without singularities anymore. As a crucial fact, when the deformation M^d tend to M, we shall have a uniform L^{p} control of the extension f^d, and this will insure that f^d tends to a CR extension of f through Φ.

Proof. We claim that Φ is L^{p}-removable for every p with $1 \leqslant \mathsf{p} \leqslant \infty$ if and only if Φ is L^1-removable. Indeed, suppose that for every function $f \in L^1_{loc}(M) \cap \mathrm{CR}(M \backslash \Phi)$, and every smooth $(n, m-1)$-form with compact support, we have $\int_M f \cdot \overline{\partial} \psi = 0$. Since L^{p}_{loc} is contained in L^1_{loc} (by Hölder's inequality), this property holds in particular for every $g \in L^{\mathsf{p}}_{loc}(M) \cap CR(M \backslash \Phi)$, hence Φ is L^{p}-removable, as claimed.

Let $f \in L^1_{loc,CR}(M \backslash \Phi) \cap L^1(M)$ be an arbitrary function. The goal is to show that f is in fact CR on Φ. Of course, it suffices to show that f is CR locally at every point of Φ. So, we fix an arbitrary point $q \in \Phi$. If ψ is an arbitrary $(n, m-1)$-form of class $\mathcal{C}^1$ supported in a sufficiently small neighborhood of q, we have to prove that $\int_M f \cdot \overline{\partial} \psi = 0$.

We also fix a small open polydisc $\mathcal{V}_q$ centered at q. We first claim that we can assume that the L^1_{loc} function f is holomorphic in a neighborhood of $(M \backslash \Phi) \cap \mathcal{V}_q$ in $\mathbb{C}^n$. Indeed, since $M \backslash \Phi$ is globally minimal, there exists a wedge $\mathcal{W}$ attached to $M \backslash \Phi$ such that every $L^1_{loc,CR}$ function on $M \backslash \Phi$, and in particular f, extends holomorphically as a function which belongs to the Hardy space $H^1(\mathcal{W})$. By slightly deforming $(M \backslash \Phi) \cap \mathcal{V}_q$ into $\mathcal{W}$ along Bishop discs glued to $M \backslash \Phi$, keeping Φ fixed, using the theory of Hardy spaces in wedges developed in [Po1997, MP1999, Po2000, Po2004, MP2006], we may obtain the following deformation result with L^1 control.

Proposition 10.4. *For every $\varepsilon > 0$, there exists a small $\mathcal{C}^{2,\alpha-0}$ deformation M^d of M with support contained in $\overline{\mathcal{V}}_q$ and there exists a function $f^d \in L^1_{loc}(M^d) \cap CR\left(M^d \backslash \Phi\right)$, such that*

(1) $M^d \cap \mathcal{V}_q \supset \Phi \cap \mathcal{V}_q \ni q$.
(2) $\left(M^d \backslash \Phi\right) \cap \mathcal{V}_q \subset \mathcal{W} \cap \mathcal{V}_q$.
(3) *f^d is holomorphic in the neighborhood $\mathcal{W} \cap \mathcal{V}_q$ of $(M^d \backslash \Phi) \cap \mathcal{V}_q$ in $\mathbb{C}^n$.*
(4) *$M \cap \mathcal{V}_q$ and $M^d \cap \mathcal{V}_q$ are graphed over the same $(2m+d)$ linear real subspace and $\left\| M^d \cap \mathcal{V}_q - M \cap \mathcal{V}_q \right\|_{\mathcal{C}^{2,\beta}} \leqslant \varepsilon$.*
(5) *The volume forms of $M \cap \mathcal{V}_q$ and of $M^d \cap \mathcal{V}_q$ may be identified and $\left| f - f^d \right|_{L^1(M \cap \mathcal{V}_q)} \leqslant \varepsilon$.*

Let us be more explicit about conditions **(4)** and **(5)**. Without loss of generality, we can assume that in coordinates $(z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ centered at q, we have $T_q M = \{v = 0\}$, hence the generic submanifolds M and M^d are represented locally by vectorial equations $v = \varphi(x, y, u)$ and $v = \varphi^d(x, y, u)$, where φ and φ^d are defined in the real cube $\mathbb{I}_{2m+d}(2\rho_1)$, for some small $\rho_1 > 0$ and that $\mathcal{V}_q$ is the polydisc $\Delta_n(\rho_1)$ of radius ρ_1. Then condition **(4)** simply means that $\|\varphi^d - \varphi\|_{\mathcal{C}^2(\mathbb{I}_{2m+d}(\rho_1))} \leqslant \varepsilon$ and condition **(5)** is clear if we choose $dxdydu$ as the volume form on M and on M^d.

Suppose that for every $\varepsilon > 0$ and for every deformation M^d, we can show that the function L^1_{loc} function f^d on M^d is in fact CR over $M^d \cap \Delta_n(\rho_1)$. Then we claim that f is CR in a neighborhood of q.

Indeed, to begin with, let us denote by $\overline{L}_1, \dots, \overline{L}_m$ a basis of $(0,1)$ vector fields tangent to M, having coefficients depending on the first order derivatives of φ. More precisely, in slightly abusive matrix notation, we can choose the basis $\overline{L} := \frac{\partial}{\partial \bar{z}} + 2(i - \varphi_u)^{-1} \varphi_{\bar{z}} \frac{\partial}{\partial \bar{w}}$. Let us denote this basis vectorially by $\overline{L} = \frac{\partial}{\partial \bar{z}} + A \frac{\partial}{\partial \bar{w}}$. To compute the formal adjoint of $\overline{L}$ with respect to the local Lebesgue measure $dxdydu$ on M, we choose two $\mathcal{C}^1$ functions ψ, χ of (x, y, u) with compact support in $\mathbb{I}_{2m+d}(\rho_1)$. Then the integration by part $\int \overline{L}(\psi) \cdot \chi \cdot dxdydu = \int \psi \cdot {}^T\overline{L}(\chi) \cdot dxdydu$ yields the explicit expression ${}^T\overline{L}(\chi) := -\overline{L}(\chi) - A_{\bar{w}} \cdot \chi$ of the formal adjoint of $\overline{L}$.

It follows immediately that if we denote by ${}^T(\overline{L}^d)$ the formal adjoint of the basis of CR vector fields tangent to M^d, then we have an estimate of the form $\|{}^T(\overline{L}^d) -$

${}^{T}(\overline{L})\|_{\mathcal{C}^1} \leqslant \mathrm{C} \cdot \varepsilon$, for some constant $C > 0$. Recall that f^d is assumed to be CR in $M^d \cap \Delta_n(\rho_1)$. Equivalently, we have $\int f^d \cdot {}^{T}(\overline{L}^d)(\psi) \cdot dxdydu = 0$ for every $\mathcal{C}^1$ function ψ with compact support in the cube $\mathbb{I}_{2m+d}(\rho_1)$. Then we deduce that (some explanation follows)

(10.5)

$$\begin{aligned}
\left| \int f \cdot {}^{T}\overline{L}(\psi) \cdot dxdydu \right| &= \left| \int \left[f \cdot {}^{T}\overline{L}(\psi) - f^d \cdot {}^{T}(\overline{L}^d)(\psi) \right] \cdot dxdydu \right| \\
&\leqslant \left| \int \left[f \cdot {}^{T}\overline{L}(\psi) - f \cdot {}^{T}(\overline{L}^d)(\psi) + f \cdot {}^{T}(\overline{L}^d)(\psi) - f^d \cdot {}^{T}(\overline{L}^d)(\psi) \right] \cdot dxdydu \right| \\
&\leqslant C_1(\psi) \cdot \varepsilon \cdot \int_{\mathbb{I}_{2m+d}(\rho_1)} |f| \cdot dxdydu + C_2(\psi) \cdot \int_{\mathbb{I}_{2m+d}(\rho_1)} |f - f^d| \cdot dxdydu \\
&\leqslant C(\psi, f, \rho_1) \cdot \varepsilon,
\end{aligned}$$

taking account of property **(5)** of Proposition 10.4 for the passage from the third to the fourth line, where $C(\psi, f, \rho_1)$ is a positive constant. As ε was arbitrarily small, it follows that $\int f \cdot {}^{T}\overline{L}(\psi) \cdot dxdydu = 0$ for every ψ, namely f is CR on $M \cap \Delta_n(\rho_1)$, as was claimed.

It remains to show that f^d is CR on $M^d \cap \Delta_n(\rho_1)$. For every deformation M^d stabilizing Φ as in Proposition 10.4, the wedge $\mathcal{W}$ attached to $M \backslash \Phi$ is still a wedge attached to $M^d \backslash \Phi$ and it contains a neighborhood of $\left(M^d \backslash \Phi\right) \cap \Delta_n(\rho_1)$ in $\mathbb{C}^n$. As Φ was supposed to be stably removable, it follows that there exists a wedge $\mathcal{W}_1$ attached to M^d (including points of Φ) to which holomorphic functions in $\mathcal{W}$ extend holomorphically.

Consequently, replacing $M^d \cap \Delta_n(\rho_1)$ by M, we are led to prove the following lemma, which, on the geometric side, is totally similar to Proposition 10.3, except that the wedge $\mathcal{W}$ attached to $M \backslash \Phi$ appearing in the formulation of Proposition 10.3 is now replaced by a neighborhood Ω of $M \backslash \Phi$ in $\mathbb{C}^n$.

Lemma 10.6. *Let M be a $\mathcal{C}^{2,\alpha}$ generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geqslant 1$ and of codimension $d = n - m \geqslant 1$, let $\Phi \subset M$ be a nonempty proper closed subset whose $(2m+d)$-dimensional Hausdorff measure is equal to zero. Let Ω be a neighborhood of $M \backslash \Phi$ in $\mathbb{C}^n$ and let $\mathcal{W}_1$ be a wedge attached to M, including points of Φ. Let $f \in L^1_{loc}(M)$ and assume that its restriction to $M \backslash \Phi$ extends as a holomorphic function $f' \in \mathcal{O}(\Omega \cup \mathcal{W}_1)$. Then f is CR all over M.*

Proof. It suffices to prove that f is CR at every point of Φ. Let $q \in \Phi$ be arbitrary and let $\mathcal{W}_q$ be a local wedge of edge M at q which is contained in $\mathcal{W}_1$. Without loss of generality, we can assume that in coordinates $(z, w) = (x + iy, u + iv) \in \mathbb{C}^m \times \mathbb{C}^d$ vanishing at q with $T_qM = \{v = 0\}$, the generic submanifold M is represented locally in the polydisc $\Delta_n(\rho_1)$ by $v = \varphi(x, y, u)$ for some $\mathcal{C}^{2,\alpha}$ $\mathbb{R}^d$-valued mapping φ defined on the real cube on $\mathbb{I}_{2m+d}(\rho_1)$. First of all, we construct a family of analytic discs half attached to M whose interior is contained in the local wedge $\mathcal{W}_q \subset \mathcal{W}_1$.

Lemma 10.7. *There exists a family of analytic discs $A_s(\zeta)$, with $s \in \mathbb{R}^{2m+d-1}$, $|s| \leqslant 2\delta$ for some $\delta > 0$, and $\zeta \in \overline{\Delta}$, which is of class $\mathcal{C}^{2,\alpha-0}$ with respect to all variables, such that*

(1) $A_0(1) = q$.

(2) $A_s(\overline{\Delta}) \subset \Delta_n(\rho_1)$.

(3) $A_s(\Delta) \subset \mathcal{W}_q \cap \Delta_n(\rho_1)$.

(4) $A_s(\partial^+\Delta) \subset M$.

(5) $A_s(i) \in M\backslash\Phi$ *and* $A_s(-i) \in M\backslash\Phi$ *for all* s.

(6) *The mapping* $[-2\delta, 2\delta]^{2m+d-1} \times [-\pi/2, \pi/2] \ni (s, \theta) \longmapsto A_s(e^{i\theta}) \in M$ *is an embedding onto a neighborhood of* q *in* M.

(7) *There exists* $\rho_2 > 0$ *such that the image of* $[-\delta, \delta]^{2m+d-1} \times [-\pi/4, \pi/4]$ *through this mapping contains* $M \cap \Delta_n(\rho_2)$.

Proof. Let M^1 be a $\mathcal{C}^{2,\alpha}$ maximally real submanifold of M passing through q such that $M^1 \cap \Phi$ is of zero measure with respect to the Lebesgue measure of M^1. Let $t \in \mathbb{R}^d$ and include M^1 in a parametrized family of maximally real submanifolds M_t^1 which foliates a neighborhood of q in M. Starting with a family of analytic discs $A_{c,x,v}^1(\zeta)$ which are half-attached to M^1 as constructed in Lemma 7.12 above, we first choose the rotation parameter v_0 and a sufficiently small scaling factor c_0 in order that $A_{c_0,0,v_0}^1(\pm i)$ does not belong to Φ. In fact, this can be done for almost every (c_0, v_0), because the mapping $(c, v) \mapsto A_{c,0,v}^1(\pm i)$ is of rank n at every point (c, v) with $c \neq 0$ and $v \neq 0$. In addition, we adjust the rotation parameter v_0 in order that the vector Jv_0 points inside a proper subcone of the cone which defines the wedge $\mathcal{W}_q$. If the scaling parameter c is sufficiently small, this implies that $A_{c_0,0,v_0}^1(\Delta)$ is contained in $\mathcal{W}_q \cap \Delta_n(\rho_1)$, as in Lemma 8.3 above. The translation parameter x runs in $\mathbb{R}^n$ and we may select a $(n-1)$-dimensional parameter subspace x' which is transversal in M^1 to the half boundary $A_{c_0,0,v_0}^1(\partial^+\Delta)$. With such a choice, there exists $\delta > 0$ such that the mapping $[-2\delta, 2\delta]^{n-1} \times [-\pi/2, \pi/2] \ni (x', \theta) \longmapsto A_{c_0,x',v_0}^1(e^{i\theta})$ is a diffeomorphism onto a neighborhood of q in M^1. Finally, using the stability of Bishop's equation under perturbations, we can deform this family of discs by requiring that it is half attached to M_t^1, thus obtaining a family $A_s(\zeta) := A_{c_0,x',v_0,t}^1(\zeta)$ with $s := (x', t) \in \mathbb{R}^{2m+d-1}$. Shrinking δ if necessary, we can check as in the proof of Lemma 8.3 **(9_1)** that property **(3)** holds. This completes the proof. □

Let now $f \in L_{loc}^1(M)$ and let $f' \in \mathcal{O}(\Omega \cup \mathcal{W}_1)$. Thanks to the foliation propery **(6)** of Lemma 10.7, it follows from Fubini's theorem that for almost every translation parameter s, the mapping $e^{i\theta} \mapsto f\left(A_s(e^{i\theta})\right)$ defines a L^1 function on $\partial^+\Delta$. In addition, the restriction of the function $f' \in \mathcal{O}(\Omega \cup \mathcal{W}_1)$ to the disc $A_s(\Delta) \subset \mathcal{W}_q \subset \mathcal{W}_1$ yields a holomorphic function $f'(A_s(\zeta))$ in Δ.

Lemma 10.8. *For almost every* s *with* $|s| \leqslant 2\delta$, *the function* $f'(A_s(\zeta))$ *belongs to the Hardy space* $H^1(\Delta)$.

Proof. Indeed, for almost every s, the intersection $\Phi \cap A_s(\partial^+\Delta)$ is of zero one-dimensional measure. By the assumption of Lemma 10.6, the restriction of $f \circ A_s$ and of $f' \circ A_s$ to $\partial^+\Delta\backslash\Phi$ coincide. Recall that $\partial^-\Delta = \{\zeta \in \partial\Delta : \operatorname{Re}\zeta \leqslant 0\}$. Since $A_s(\pm i)$ does not belong to Φ and since $A_s\left(e^{i\theta}\right)$ belongs to $\mathcal{W}_q$ for all θ with $\pi/2 < |\theta| \leqslant \pi$, it follows that $f \circ A_s|_{\partial^+\Delta}$ and $f' \circ A_s|_{\partial^-\Delta}$ (which is holomorphic in a neighborhood of $\partial^-\Delta$ in $\mathbb{C}$) match together in a function which is L^1 on $\partial\Delta$. Let us denote this function by f_s. Furthermore, f_s extends holomorphically to Δ as $f' \circ A_s|_\Delta$. Consequently, $f' \circ A_s|_\Delta$ belongs to the Hardy space $H^1(\Delta)$. □

Since the boundary value of f' on $M\backslash\Phi$ along the family of discs $A_s(\zeta)$ coincides with f, we can now denote both functions by the same letter f.

For $\varepsilon \geqslant 0$ small, let now $\chi_\varepsilon\left(s, e^{i\theta}\right)$ be a $\mathcal{C}^2$ function on $[-2\delta, 2\delta] \times \partial\Delta$ which equals ε for $|s| \leqslant \delta$ and for $\theta \in [-\pi/4, \pi/4]$ and which equals 0 if either $\pi/2 \leqslant |\theta| \leqslant \pi$ or $|s| \geqslant 2\delta/3$. We may require in addition that $\|\chi_\varepsilon\|_{\mathcal{C}^2} \leqslant \varepsilon$. We define a deformation M^ε of M compactly supported in a neighborhood of q by pushing M inside $\mathcal{W}_q$ along the family of discs $A_s(\zeta)$ as follows:

$$M^\varepsilon := \left\{ A_s\big([1-\chi_\varepsilon(s,e^{i\theta})]\, e^{i\theta}\big) : |\theta| \leqslant \pi/2, \ |s| \leqslant 2\delta \right\}. \tag{10.9}$$

Notice that M^ε coincides with M outside a small neighborhood of q. Then we have $\|M^\varepsilon - M\|_{\mathcal{C}^2} \leqslant C \cdot \varepsilon$, for some constant $C > 0$ which depends only on the $\mathcal{C}^2$ norms of $A_s(\zeta)$ and of $\chi_\varepsilon(s, e^{i\theta})$. If the radius ρ_2 is as in Property **(7)** of Lemma 10.7 above, the deformation $M^\varepsilon \cap \Delta_n(\rho_2)$ is entirely contained in $\mathcal{W}_q$ and since f is holomorphic in $\mathcal{W}_q$, its restriction to $M^\varepsilon \cap \Delta_n(\rho_2)$ is obviously CR.

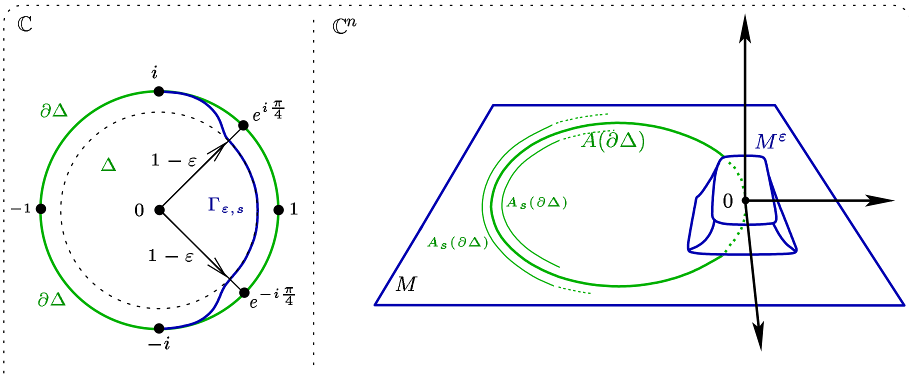

Fig. 21: The arc $\Gamma_{\varepsilon,s}$ and the deformation M^ε

As in [Jö1999b, MP1999, Po2000], we notice that for every s and every ε, the one-dimensional Lebesgue measure on the arc

$$\Gamma_{\varepsilon,s} := \left\{[1-\chi_\varepsilon(s,e^{i\theta})]e^{i\theta} \in \Delta : |\theta| \leqslant \pi\right\} \tag{10.10}$$

is a Carleson measure. Thanks to the geometric uniformity of these arcs $\Gamma_{\varepsilon,s}$, it follows from an inspection of the proof of Carleson's imbedding theorem that there exists a (uniform) constant C such that for all s with $|s| \leqslant 2\delta$ and all ε, one has the estimate

$$\int_{\Gamma_{\varepsilon,s}} \left| f\left(A_s\big([1-\chi_\varepsilon(s,e^{i\theta})]\, e^{i\theta}\big)\right)\right| \cdot d\theta \leqslant C \int_{\partial\Delta} |f| \cdot d\theta. \tag{10.11}$$

We are now ready to complete the proof of Lemma 10.6. Let $\pi_{x,y,u}$ denote the projection parallel to the v-space from $\mathbb{C}^n$ onto the (x,y,u)-space. The mapping $(s,\theta) \mapsto \pi_{x,y,u}(A_s(\theta))$ may be used to define new coordinates in a neighborhood of the origin in $\mathbb{C}^m \times \mathbb{R}^d$, an open subset above which M and M^ε are graphed. We shall now work with these coordinates. With respect to the coordinates (s,θ), on M and on M^ε, we have formal adjoints ${}^T\overline{L}$ and ${}^T(\overline{L}^\varepsilon)$ of the basis of CR vector fields with an estimation of the form $\left\|{}^T(\overline{L}^\varepsilon) - {}^T\overline{L}\right\|_{\mathcal{C}^1} \leqslant C \cdot \varepsilon$, for some constant $C > 0$. Let now $\psi = \psi(s,\theta)$ be $\mathcal{C}^1$ function with compact support in the set $\{|s| < \delta, \ |\theta| \leqslant \pi/4\}$. By construction, the subpart of M^ε defined by $\widetilde{M}^\varepsilon := \{A_s\left([1-\chi_\varepsilon(s,e^{i\theta})]\, e^{i\theta}\right) : |\theta| \leqslant \pi/4, \ |s| \leqslant \delta\}$ is contained in the wedge $\mathcal{W}_q$, hence the restriction of the holomorphic function $f \in \mathcal{W}_q$ to $\widetilde{M}^\varepsilon$ is obviously CR on $\widetilde{M}^\varepsilon$.

For simplicity of notation, we shall denote $f\left(A_s(e^{i\theta})\right)$ by $f_s(\theta)$ and $f\left(A_s\left(\left[1-\chi_\varepsilon(s,e^{i\theta})\right]\, e^{i\theta}\right)\right)$ by $f_s^\varepsilon(\theta)$. Since by construction for every $\varepsilon>0$, the L^1 function $(s,\theta)\mapsto f_s^\varepsilon(\theta)$ is annihilated in the distributional sense by the CR vector fields $\overline{L}^\varepsilon$ on $\widetilde{M}^\varepsilon$, we may compute (not writing the arguments (s,θ) of ψ)

$$
\begin{aligned}
&\left|\int_{|s|\leqslant\delta}\int_{|\theta|\leqslant\pi/4} f_s(\theta)\cdot {}^T\overline{L}(\psi)\cdot ds d\theta\right| = \\
&= \left|\int_{|s|\leqslant\delta}\int_{|\theta|\leqslant\pi/4}\left[f_s(\theta)\cdot {}^T\overline{L}(\psi) - f_s^\varepsilon(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi)\right]\cdot ds d\theta\right| \\
&\leqslant \left|\int_{|s|\leqslant\delta}\left(\int_{|\theta|\leqslant\pi/4}\left[f_s(\theta)\cdot {}^T\overline{L}(\psi) - f_s(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi) + \right.\right.\right. \\
&\qquad \left.\left.\left. + f_s(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi) - f_s^\varepsilon(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi)\right]\cdot d\theta\right)\cdot ds\right| \qquad (10.12) \\
&\leqslant C_1(\psi)\cdot\varepsilon\cdot\int_{|s|\leqslant\delta}\int_{|\theta|\leqslant\pi/4}|f_s(\theta)|\cdot ds d\theta + \\
&\qquad + C_2(\psi)\cdot\int_{|s|\leqslant\delta}\int_{|\theta|\leqslant\pi/4}|f_s(\theta)-f_s^\varepsilon(\theta)|\cdot ds d\theta \\
&\leqslant C_1(\psi,f,\delta)\cdot\varepsilon + C_2(\psi,\delta)\cdot\max_{|s|\leqslant\delta}\int_{|\theta|\leqslant\pi/4}|f_s(\theta)-f_s^\varepsilon(\theta)|\cdot ds d\theta.
\end{aligned}
$$

Thanks to the estimate (10.11) and thanks to Lebesgue's dominated convergence theorem, the last integral tends to zero as ε tends to zero. It follows that the integral in the first line can be made arbitrarily small, hence it vanishes. This proves that f is CR in a neighborhood of q and completes the proof of Lemma 10.6. □

The proof of Proposition 10.3 is complete. □

§11. Proofs of Theorem 1.2 and of Corollary 1.5

11.1. Tree of separatrices linking hyperbolic points. Let $M\subset\mathbb{C}^2$ be a globally minimal $\mathcal{C}^{2,\alpha}$ hypersurface, let $S\subset M$ be a $\mathcal{C}^{2,\alpha}$ surface and let $K\subset S$ be a proper compact subset of S. Assume that S is totally real outside a discrete subset of complex tangencies which are hyperbolic in the sense of Bishop. Since we aim to remove the compact subset K of S, we can shrink the open surface S around K in order that S contains *only finitely many* such hyperbolic complex tangencies, which we shall denote by $\{h_1,\dots,h_\lambda\}$, where λ is some integer, possibly zero. Furthermore, we can assume that ∂S is $\mathcal{C}^{2,\alpha}$. As a corollary of the qualitative theory of planar vector fields, due to Poincaré-Bendixson ([HS1974]), we know that

- **(i)** the hyperbolic points $h_1,\dots,h_\lambda$ are singularities of the characteristic foliation F_S^c;
- **(ii)** through every hyperbolic point $h_1,\dots,h_\lambda$, there are exactly four $\mathcal{C}^{2,\alpha}$ open *separatrices*;
- **(iii)** after perturbing slightly the boundary ∂S if necessary, these separatrices are all transversal to ∂S and the union of all separatrices together with all hyperbolic points makes a *finite tree without cycles in* S.

Precisely, by an (open) *separatrix*, we mean a $\mathcal{C}^{2,\alpha}$ curve $\tau : (0,1) \to S$ with $\frac{d\tau}{ds}(s) \in T_{\tau(s)}S \cap T^c_{\tau(s)}M \backslash \{0\}$ for every $s \in (0,1)$, namely its tangent vectors are all nonzero and characteristic, such that one limit point, say $\lim_{s\to 0} \tau(s)$ is a hyperbolic point, and the other $\lim_{s\to 1} \tau(s)$ either belong to the boundary ∂S or is a second hyperbolic point.

From the local study of saddle phase diagrams ([Ha1982]), we get in addition:

(iv) there exists $\varepsilon > 0$ and for every $l = 1, \dots, \lambda$, there exist two curves $\gamma_l^1, \gamma_l^2 : (-\varepsilon, \varepsilon) \to S$ which are of class $\mathcal{C}^{1,\alpha}$, *not more*, with $\gamma_l^i(0) = h_l$ and $\frac{d\gamma_l^i}{dt}(s) \in T_{\gamma_l^i(s)}S \cap T^c_{\gamma_l^i(s)}M \backslash \{0\}$ for every $s \in (-\varepsilon, \varepsilon)$ and for $i = 1, 2$, such that the four open segments $\gamma_l^1(-\varepsilon, 0)$, $\gamma_l^1(0, \varepsilon)$, $\gamma_l^2(-\varepsilon, 0)$ and $\gamma_l^2(0, \varepsilon)$ cover the four pieces of open separatrices incoming at h_l.

Let $\tau_1, \dots, \tau_\mu : (0,1) \to S$ denote all the separatrices of S, where μ is some integer, possibly equal to zero. By the *finite hyperbolic tree* T_S *of* S, we mean:

$$T_S := \{h_1, \dots, h_\lambda\} \bigcup_{1 \leqslant k \leqslant \mu} \tau_k(0,1). \tag{11.2}$$

We say that T_S has *no cycle* if it does not contain any subset homeomorphic to the unit circle. For instance, in the case where $S \equiv D$ is diffeomorphic to a real disc (as in the assumptions of Proposition 1.4), its hyperbolic tree T_D necessarily has no cycle. However, in the case where S is an annulus (for instance), there is a trivial example of a characteristic foliation with one (or two, or more) hyperbolic point(s) and a circle in the hyperbolic tree.

11.3. Hyperbolic decomposition in the disc case. Let the real disc D and the compact subset $K \subset D$ be as in Theorem 1.2. We shrink D slightly and smooth out its boundary, so that its hyperbolic tree T_D is finite and has no cycle. We may decompose D as the disjoint union

$$D = T_D \cup D_o, \tag{11.4}$$

where the complement of the hyperbolic tree $D_o := D \backslash T_D$ is an open subset of D entirely contained in the totally real part of D. Then D_o has finitely many connected components $D_1, \dots, D_\nu$, the *hyperbolic sectors of* D. Also, for $j = 1, \dots, \nu$, we define the proper closed subsets $C_j := D_j \cap K$ of D_j.

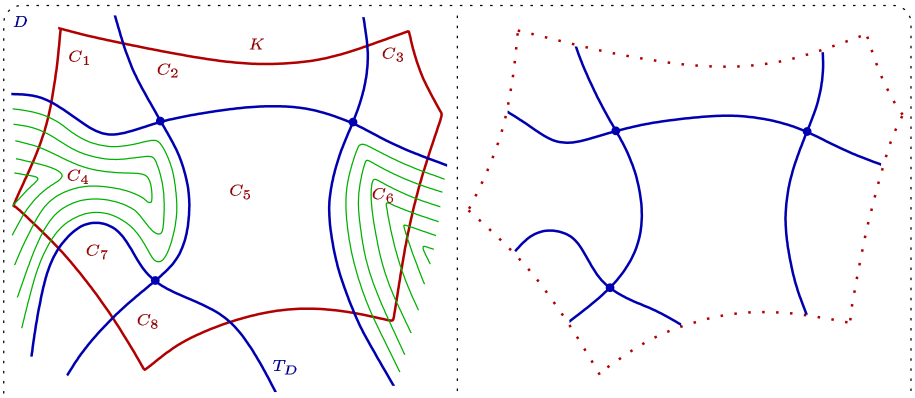

Fig. 22: Removal of hyperbolic sectors and removal of a tree without cycles

Again from the Poincaré-Bendixson theory, we know that for every component D_j (in which the characteristic foliation is nonsingular), the proper closed subset C_j is nontransversal to $\mathsf{F}^c_{D_j}$. In the figure, we have drawn the characteristic curves only in the two sectors D_4 and D_6.

11.5. Global minimality of some complements. We state a generalization of Lemma 3.5 to the case where some hyperbolic complex tangencies are allowed. Its proof is not immediate.

Proposition 11.6. *Let M be a $\mathcal{C}^{2,\alpha}$ hypersurface in $\mathbb{C}^2$ and let $S \subset M$ be $\mathcal{C}^{2,\alpha}$ surface which is totally real outside a discrete subset of hyperbolic complex tangencies. Assume that the hyperbolic tree T_S of S has no cycle. Then for every compact subset $K \subset S$ and for every point $p \in M \backslash K$:*

$$\mathcal{O}_{CR}(M\backslash K, p) = \mathcal{O}_{CR}(M, p)\backslash K. \tag{11.7}$$

As a corollary, $M\backslash K$ is globally minimal if M is so.

Proof. As above, we may assume that S coincides with the shrinking of a slightly larger surface and has finitely many hyperbolic points $\{h_1, \dots, h_\lambda\}$. Let $K_{T_S} := K \cap T_S$ be the track of K on the hyperbolic tree T_S. Since K_{T_S} may in general coincide with any arbitrary closed (*e.g.* Cantor) subset of T_S, in order to fix ideas, it will be convenient to deal with an enlargement $\underline{K}$ of K_{T_S}, simply defined by filling the possible holes of K_{T_S} in T_S: more precisely, $\underline{K}$ should contain all hyperbolic points of S together with all separatrices joining them and for every separatrix $\tau_k(0,1)$ with right limit point $\lim_{s\to 1} \tau_k(s)$ belonging to the boundary of S, we require that $\underline{K}$ contains the segment $\tau_k[0, r_1]$, where $r_1 < 1$ is close enough to 1 in order that $\underline{K}$ effectively contains K_{T_S}. Equivalently, $\underline{K}$ is a small shrinking of T_S, still compact in S.

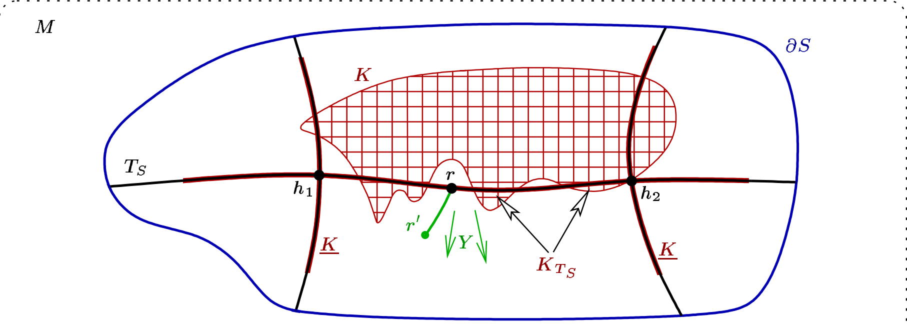

Fig. 23: the disposition of K, K_{T_S} and $\overline{K}$

The main step in the proof of Proposition 11.6 is as follows.

Lemma 11.8. *We have $\mathcal{O}_{CR}\big(M\backslash \underline{K}, q\big) = \mathcal{O}_{CR}(M, q)\backslash \underline{K}$, for every $q \in M\backslash \underline{K}$.*

Before pursuing, we establish the proposition. The inclusion $\mathcal{O}_{CR}(M\backslash K, p) \subset \mathcal{O}_{CR}(M, p)\backslash K$ is trivial. Reversely, for $q \in \mathcal{O}_{CR}(M, p)\backslash K$ arbitrary, we must find a piecewise smooth complex-tangential curve joining q to p and running entirely in $M\backslash K$. We first join p to some $p' \in M\backslash S$ and q to some $q' \in M\backslash S$ as follows.

Lemma 11.9. *The CR orbit $\mathcal{O}_{CR}(M\backslash K, r)$ of every $r \in M\backslash K$ contains points $r' \in M\backslash S$ arbitrarily close to r.*

Proof. If $r \in M \backslash S$, the claim is gratuitous. If $r \in S \backslash \{h_1, \dots, h_\lambda\}$, whence S is totally real in a neighborhood of r, we just choose a local section Y of $T^c M$ defined near r which is transversal to S at r and we follow the integral curve of Y to escape from S, as shown by the figure.

If $r = h_l$ is a hyperbolic point, we may use one of the four separatrices to join r to some point $r'' \in T_S$ close to r and $\neq h_l$, hence in the totally real part of S. Then we join r'' to some $r' \in M \backslash S$ as above by means of some vector field Y'' transversal to S at r''. □

Necessarily, both p' and q' belong to $\mathcal{O}_{CR}(M, p) \backslash K$ and hence, in order to get (11.7), it suffices to produce a piecewise smooth complex-tangential curve joining q' to p' which runs in $M \backslash K$.

Taking for granted Lemma 11.8, we first get a piecewise smooth complex-tangential curve joining q' to p' and running in $M' := M \backslash \underline{K}$. Equivalently, $q' \in \mathcal{O}_{CR}(M', p')$. The set $C' := K \cap M'$ is closed in M', is closed in $S' := S \backslash \underline{K}$ and is nontransversal to $\mathsf{F}^c_{S'}$. Lemmas 3.3 and 3.5 showed that $\mathcal{O}_{CR}\big(M' \backslash C', r'\big) = \mathcal{O}_{CR}(M', r') \backslash C'$, for every $r' \in M' \backslash C'$. Consequently, there exists a piecewise smooth complex-tangential curve joining q' to p' which runs in $M' \backslash C'$, hence in $M \backslash K$. Thus $q' \in \mathcal{O}_{CR}(M \backslash K, p')$ and hence in conclusion, $q \in \mathcal{O}_{CR}(M \backslash K, p)$, which completes the proof of Proposition 11.6. □

It remains to establish Lemma 11.8. As M is a hypersurface of $\mathbb{C}^2$, its CR orbits are of dimension either 2 or 3. We state an analog to Lemma 3.7.

Lemma 11.10. *Let M, S, T_S and $\underline{K} \subset T_S$ be as above. There exists a connected embedded submanifold $\Omega \subset M$ containing the hyperbolic tree T_S such that*:

- **(1)** *Ω is a $T^c M$-integral manifold, namely $T^c_p M \subset T_p \Omega$ for every $p \in \Omega$;*
- **(2)** *Ω is contained in a single CR orbit of M;*
- **(3)** *$\Omega \backslash \underline{K}$ is also contained in a single CR orbit of $M \backslash \underline{K}$.*

More precisely, Ω is an open neighborhood of T_S if it is of real dimension 3 and a complex curve surrounding T_S if it is of dimension 2.

Reasoning as in Lemma 3.10, to get Lemma 11.8, starting with a piecewise smooth complex tangential curve joining q to some arbitrary $r \in \mathcal{O}_{CR}(M, q) \backslash \underline{K}$, if it meets $\underline{K}$, we can modify the trajectory by running only inside $\Omega \backslash \underline{K}$ (surrounding the obstacle), whence $r \in \mathcal{O}_{CR}\big(M \backslash \underline{K}, q\big)$. This completes the proof of Lemma 11.8, granted Lemma 11.10, which we now prove. □

Proof. We shall construct Ω by means of a complex-tangential flowing procedure, stretching and enlarging local pieces of it.

Fix any point $p_0 \in \underline{K} \backslash \{h_1, \dots, h_\lambda\}$. Since S is totally real near p_0, there exists a locally defined $T^c M$-tangent vector field Y which is transversal to S at p_0. Consequently, for $\delta > 0$ small enough, the small curve $I_0 := \{\exp(sY)(p_0) : -\delta < s < \delta\}$ is transversal to S at p_0 and moreover, the two half-curves

$$I_0^{\pm} := \big\{ \exp(sY)(p_0) : 0 < \pm s < \delta \big\} \tag{11.11}$$

lie in $M \backslash S$.

Since $p_0 \in \underline{K}$ belongs to some open separatrix $\tau_k(0, 1)$, there exists a $\mathcal{C}^{1,\alpha}$ complex-tangential vector field X defined in a neighborhood of p_0 in M which is tangent to S and whose integral curve passing through p_0 is a piece of $\tau_k(0,1)$. Since Y is transversal to S at p_0, it follows that the set

$$\omega_0 := \big\{\exp(s_2X)\big(\exp(s_1Y)(p_0)\big) : -\delta < s_1,\, s_2 < \delta\big\} \tag{11.12}$$

is a small $\mathcal{C}^{1,\alpha}$ one-codimensional submanifold of M passing through p_0 which is transversal to S at p_0. Clearly, $T_{p_0}\omega_0 = T^c_{p_0}M$. Thanks to the fact that the flow of X stabilizes S, we see that the integral curves $s_2 \mapsto \exp(s_2X)(\exp(s_1Y)(p_0))$ are contained in $M\backslash S$ for every starting point $\exp(s_1Y)(p_0) \in I_0$ with $s_1 \neq 0$, namely for all $s_1 \neq 0$. We deduce that each one of the two open halves

$$\omega_0^\pm := \big\{\exp(s_2X)(\exp(s_1Y)(p_0)) : 0 < \pm s_1 < \delta,\ -\delta < s_2 < \delta\big\} \tag{11.13}$$

is contained in a *single* CR orbit of $M\backslash \underline{K}$.

To pursue, abandoning the consideration of ω_0^-, we shall assume that the piece of CR orbit ω_0^+ is of real dimension 2, whence it is a complex curve. Afterwards, we shall treat the (simpler) case where ω_0^+ is 3-dimensional.

By the *S-boundary* of a set $E \subset M\backslash S$, we shall mean the intersection $\partial E \cap S$ of the boundary of E in M with S. Thus, the S-boundary of ω_0^+ is just the piece of characteristic curve $\big\{\exp(s_2X) : -\delta \leqslant s_2 \leqslant \delta\big\}$, contained in $\tau_k(0,1)$.

Since $\tau_k(0,1)$ is an embedded segment, we may suppose from the beginning that the vector field X is defined in a neighborhood of $\tau_k(0,1)$ in M. Using then the flow of X, we may prolong the small piece ω_0^+ to get a semi-local $\mathcal{C}^{1,\alpha}$ submanifold ω_k^+ stretched along $\tau_k(0,1)$. Again, this piece ω_k^+ is (by construction) contained in a single CR orbit of $M\backslash \underline{K}$. By the fundamental stability property of CR orbits under flows, we deduce that ω_k^+ is in fact a long thin complex curve in $M\backslash \underline{K}$ whose S-boundary contains $\tau_k(0,1)$.

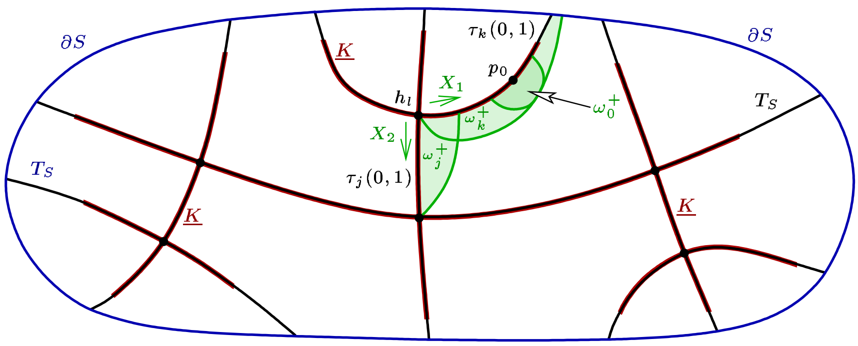

Fig. 24: Orbit neighborhood of the compact set K_{T_S}

Remind that by definition of separatrices, the point $\tau_k(0)$ is always a hyperbolic point. There is a dichotomy: either $\tau_k(1)$ is also a hyperbolic point or it lies in ∂S. If $\tau_k(1)$ is a hyperbolic point, then by the definition of $\underline{K}$, the complete closed separatrix $\tau_k[0,1]$ is contained in $\underline{K}$, hence it may *not* be crossed by means of a CR curve running in $M\backslash \underline{K}$.

So we must again prolong ω_k^+ and in the neighborhood of the hyperbolic point $h_l = \tau_k(0)$, the geometric situation is different. As in the figure, let $\tau_j(0,1)$ be the separatrix issued from h_l next to $\tau_l[0,1]$.

Lemma 11.14. *There exists a long thin complex curve ω_j^+ whose S-boundary contains $\tau_j(0,1)$ which is contained in the same CR orbit as ω_k^+ and which matches up* *with ω_k^+ near h_l. Geometrically, ω_k^+ and ω_j^+ coincide near h_l and constitute a piece of cornered complex curve.*

Proof. We introduce local holomorphic coordinates $(z,w) = (x+iy, u+iv) \in \mathbb{C}^2$ vanishing at h_l in which the hypersurface M is given as the graph $v = \varphi(x,y,u)$, where φ is a $\mathcal{C}^{2,\alpha}$ function. Since M contains the complex curve ω_k^+, it is Levi degenerate at $h_l \in \overline{\omega}_k^+$. Thus, we may assume that $\big|\varphi(x,y,u)\big| \leqslant C \cdot \big(|x| + |y| + |u|\big)^{2+\alpha}$.

The surface S, as a subset of M, is represented by one supplementary $\mathcal{C}^{2,\alpha}$ equation of the form $u = h(x,y)$. According to Bishop ([Bi1965]), a suitable change of holomorphic coordinates normalizes

$$
\begin{aligned}
h(x,y) &= z\bar{z} + \gamma\,(z^2 + \bar{z}^2) + \mathrm{O}\left(|z|^{2+\alpha}\right) \\
&= (2\gamma+1)\,x^2 - (2\gamma-1)\,y^2 + \mathrm{O}\left(|z|^{2+\alpha}\right),
\end{aligned} \tag{11.15}
$$

where $\gamma \in \mathbb{R}^+$ is a biholomorphic invariant satisfying $\gamma > \frac{1}{2}$ by the hyperbolicity assumption. Then the tangents at h_l to the two half-separatrices τ_k and τ_j are given respectively by the linear (in)equations $x > 0$, $y = -\frac{2\gamma+1}{2\gamma-1}\,x$, $u = 0$ and $x < 0$, $y = -\frac{2\gamma+1}{2\gamma-1}\,x$, $u = 0$. In the figure below, where we do not draw the axes, the u-axis is vertical, the y axis points behind h_l and the x-axis is horizontal, from the left to the right.

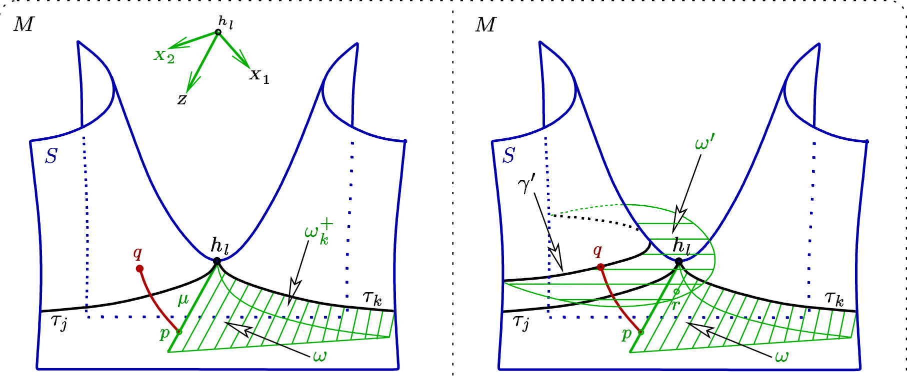

Fig. 25: Behavior of the complex curve orbit around a hyperbolic point

The saddle-looking surface S is represented in the 3-dimensional space M ; the horizontal plane passing through h_l is thought to be the complex tangent plane $T^c_{h_l}M$.

We introduce two T^cM-tangent vector fields X_1 and X_2 defined in a neighborhood of $h_l = \tau_k(0) = \tau_j(0)$ with $X_1(h_l)$ directed along τ_k in the sense of increasing s and $X_2(h_l)$ directed along τ_j in the sense of increasing s, defined by

$$
\left\{
\begin{aligned}
X_1 &= \frac{\partial}{\partial x} - \left(\frac{2\gamma+1}{2\gamma-1}\right)\frac{\partial}{\partial y} + A_1(x,y,u)\,\frac{\partial}{\partial u}, \\
X_2 &= -\frac{\partial}{\partial x} - \left(\frac{2\gamma+1}{2\gamma-1}\right)\frac{\partial}{\partial y} + A_2(x,y,u)\,\frac{\partial}{\partial u},
\end{aligned}
\right. \tag{11.16}
$$

with A_1 and A_2 being certain rational functions of the first order jet of φ. Since φ vanishes to second order at h_l, the two $\mathcal{C}^{1,\alpha}$ coefficients A_1 and A_2 satisfy

$$
|A_1, A_2(x,y,u)| < C \cdot \big(|x| + |y| + |u|\big)^{1+\alpha}. \tag{11.17}
$$

Let Z denote the vector field $X_1 + X_2$, as shown in the top of the left Figure 25. Using the flow of Z we begin by extending the banana-looking piece ω_k^+ of complex curve by introducing the submanifold ω consisting of points

$$\exp(s_2 Z)(\tau_k(s_1)), \tag{11.18}$$

where $0 < s_1 < \delta$ and $0 < s_2 < \delta$, for some small $\delta > 0$. One checks that all these points stay in $M \backslash S$, hence are contained in the same CR orbit as ω_k^+ in $M \backslash \underline{K}$. By the stability property of CR orbits, it follows that ω is a piece of complex curve contained in $M \backslash \underline{K}$. Roughly, the (horizontal) projection of ω onto $T_{h_l}^c M$ covers $\sim \frac{1}{8}$ of a neighborhood of h_l in $T_{h_l}^c M$.

For $0 < s < \delta$, let $\mu(s) := \exp(sZ)(h_l)$ denote the CR curve lying "between" τ_k and τ_j and which constitutes a part of the boundary of ω. Let p be an arbitrary point of this curve, close to h_l.

Lemma 11.19. *The integral curve $s \mapsto \exp(-sX_1)(p)$ of $-X_1$ issued from p necessarily intersects S at a point q close to h_l and close to τ_j (cf. Figure 25).*

Proof. This integral curve is contained in the real 2-surface passing through h_l defined by

$$\Sigma := \big\{ \exp(-s_2 X_1)(\exp(s_1 Z)(h_l)) : \, -\delta < s_1, s_2 < \delta \big\}, \tag{11.20}$$

for some $\delta > 0$. Because the vector fields X_1, X_2 and $Z = X_1 + X_2$ have $\mathcal{C}^{1,\alpha}$ coefficients, the surface Σ is only $\mathcal{C}^{1,\alpha}$ in general. In M equipped with the three real coordinates (x, y, u), we may parametrize Σ by a mapping of the form

$$(s_1, s_2) \longmapsto \left(s_2 - 2s_1 \Big(\frac{2\gamma + 1}{2\gamma - 1} \Big), \; s_2 \Big(\frac{2\gamma + 1}{2\gamma - 1} \Big), \; u(s_1, s_2) \right), \tag{11.21}$$

where u is of class $\mathcal{C}^{1,\alpha}$. It is clear that $u(0) = u_{s_1}(0) = u_{s_2}(0) = 0$, so that there is a constant C such that

$$|u(s_1, \, s_2)| < C \cdot \big(|s_1| + |s_2| \big)^{1+\alpha}. \tag{11.22}$$

Furthermore, we claim that u satisfies the better estimate

$$|u(s_1, s_2)| < C \cdot \big(|s_1| + |s_2| \big)^{2+\alpha}, \tag{11.23}$$

for some constant $C > 0$. In other words, Σ osculates the complex tangent plane $T_{h_l}^c M$ to second order at h_l : Σ is more flat than S at h_l. Since the second order terms $z\bar{z} + \gamma(z^2 + \bar{z}^2)$ of the graphing function h of S are nonvanishing, the curve $s \mapsto \exp(-sX_1)(p)$ must necessarily intersect the saddle S, whence Lemma 11.19 follows.

To verify the remaining claim, we will reason with the two linear combinations $L_1 := \frac{\partial}{\partial x} + B_1(x, y, u) \frac{\partial}{\partial u}$ and $L_2 := \frac{\partial}{\partial y} + B_2(x, y, u) \frac{\partial}{\partial u}$ of X_1 and X_2. This will lighten the computations (with X_1 and X_2, the principle is the same).

Here, B_1 and B_2 are $\mathcal{C}^{1,\alpha}$ and satisfy (11.17). Denote by $s_1 \longmapsto \big(s_1, \lambda(s_1), \mu(s_1) \big)$ the integral curve of L_1 passing through the origin. It is $\mathcal{C}^{2,\alpha}$ and we have

$$|\lambda(s_1)| < C \cdot |s_1|^{2+\alpha} \qquad \text{and} \qquad |\mu(s_1)| < C \cdot |s_1|^{2+\alpha}, \tag{11.24}$$

for some $C > 0$. In the definition of Σ, we replace X_1 and X_2 by L_1 and L_2 (with X_1 and X_2, the principle is the same, although the obtained graphing function $u^L(s_1, s_2)$

differs). Considering the composition of flows $\exp(s_2L_2)\big(\exp(s_1L_1)(0)\big)$, we have to solve the system of ordinary differential equations

$$\frac{dx}{ds_2} = 0, \qquad \frac{dy}{ds_2} = 1, \qquad \frac{du}{ds_2} = B_2(x, y, u) \tag{11.25}$$

with initial conditions

$$x(0) = s_1, \qquad y(0) = \lambda(s_1), \qquad u(0) = \mu(s_1). \tag{11.26}$$

This yields $x(s_1, s_2) = s_1$, $y(s_1, s_2) = \lambda(s_1) + s_2$ and the integral equation

$$u^L(s_1, s_2) = \mu(s_1) + \int_0^{s_2} B_2\big(s_1, s_2' + \lambda(s_1), u^L(s_1, s_2')\big)\, ds_2'. \tag{11.27}$$

Since u^L already satisfies (11.22), inserting the estimate (11.17) satisfied by B_2 and integrating, it is now elementary to obtain $\big|u^L(s_1, s_2)\big| \leqslant C \cdot \big|s_1| + |s_2|\big)^{2+\alpha}$. □

We can now achieve the proof of Lemma 11.14. So, for various points $p = \mu(s)$ close to h_l the intersection points $q \in S$ exist. If all points q belong to τ_j, we are done: the piece ω extends as a cornered (roughly $\frac{1}{4}$) piece of complex curve with S-boundary $\tau_k \cup \tau_j$ near h_l.

Assume therefore that one such point q does not belong to τ_j, as drawn in the left hand side of Figure 25. Suppose that q lies above τ_j, the case where q lies under τ_j being similar. The characteristic curve $\gamma' \subset S$ passing through q stays above τ_j and is nonsingular. Prolongating the complex curve ω in $M \backslash \underline{K}$ by means of the flow of $-X_1$, we deduce that there exists at q a local piece ω_q^+ of complex curve with S-boundary contained in γ' which is contained in the same CR orbit as ω. Using then the flow of a CR vector having γ' as an integral curve, we can prolong ω_q^+ along γ', which yields a long thin banana-looking complex curve with boundary in γ'. However, this piece may remain too thin. Fortunately, thanks to the flow of $X_1 - X_2$, we can extend it as a piece ω' of complex curve with boundary γ' which then goes over h_l, with respect to a complex projection onto $T_{h_l}^c M$, as illustrated in Figure 25 above. We claim that this yields a contradiction.

Indeed, as ω and ω' are complex curves, they are locally defined as graphs of holomorphic functions g and g' defined in domains D and D' in the complex line $T_{h_l}^c M$. By construction, there exists a point in $r \in D \cap D'$ at which the values of g and g' are distinct. However, since by construction g and g' coincide in a neighborhood of the CR curve joining p to q, they must coincide at r because of the principle of analytic continuation: this is a contradiction. In conclusion, the CR orbit passes through the hyperbolic point h_l, in a neighborhood of which it consists of a cornered complex curve with boundary $\tau_k \cup \tau_j$. This completes the proof of Lemma 11.14. □

We can now conclude Lemma 11.10. Again, we may prolong ω_j^+ all along $\tau_j(0,1)$. If $\tau_j(1)$ is a new hyperbolic point h_m, again we prolong, *etc.*

Since the hyperbolic tree T_S does not contain any cycle, after some steps, an end-point $\tau_k(1)$ will not be a hyperbolic point, hence belong to ∂S. But we arranged at the beginning that $\underline{K} \cap \tau_k(0,1) = \tau_k(0, r_1]$, where $r_1 < 1$. It is then crucial that when a limit point $\tau_k(1)$ belongs to ∂S, we escape from $\underline{K}$ and using a local CR vector field Y transversal to S, *we may cross the separatrix* $\tau_k(0,1)$ *at some point* $\tau_k(r_2)$ where r_2 satisfies $r_1 < r_2 < 1$. Hence, we pass to the other side of S in M and then, by means of a further prolongation, we turn around to the other side of $\tau_k(0,1)$. Also, the two pieces in either side of $\tau_k(0,1)$ match up at least in a $\mathcal{C}^{1,\alpha}$ way. Then thanks to the stability property of orbits under flows, we deduce that these two pieces match up as a piece of complex curve containing $\tau_k(0,1)$ in its interior.

Continuing the prolongation, we construct the complex curve Ω surrounding T_S, which is obviously contained in a single CR orbit of M. Furthermore, by construction, $\Omega\backslash \underline{K}$ is contained in a single CR orbit of $M\backslash\underline{K}$. Thus, we have established Lemma 3.12 under the assumption that the initial CR orbit of q_0^+ is two-dimensional.

Assume finally that the CR orbit of q_0^+ is 3-dimensional. By a similar (and in fact easier) propagation procedure, we may construct a neighborhood Ω in M of the hyperbolic tree satisfying conditions **(1)**, **(2)** and **(3)** of Lemma 11.10. □

11.28. Proofs of Theorem 1.2 and of Corollary 1.5. We treat directly the more general Corollary 1.5. As already known, it suffices to establish the $\mathcal{W}$-removability of K.

Let ω_1 be a one-sided neighborhood of $M\backslash K$ in $\mathbb{C}^2$. Let $\underline{K} \subset T_S$ be a filling of $K_{T_S} = K \cap S$, as in Lemma 11.8. By this lemma, $M\backslash\underline{K}$ is globally minimal. Because $K \cap (S\backslash T_S)$ is nontransversal to $\mathsf{F}^c_{S\backslash T_S}$ by assumption, we may apply Proposition 1.4 to the totally real surface $S\backslash T_S$ in the globally minimal $M\backslash\underline{K}$ to remove $K \cap (S\backslash T_S)$. We deduce that there exists a one-sided neighborhood ω_2 of $M\backslash\underline{K}$ in $\mathbb{C}^2$ such that (after shrinking ω_1 if necessary), holomorphic functions in ω_1 extend holomorphically to ω_2. Then we slightly deform M inside ω_2 over points of $K \cap (S\backslash T_S)$. We obtain a $\mathcal{C}^{2,\alpha}$ hypersurface M^d with $M^d\backslash\underline{K} \subset \omega_2$. Also, by stability of global minimality under small perturbations, we can assume that M^d is also globally minimal.

Since M and M^d are of codimension 1, the union of a one-sided neighborhood ω^d of M^d in $\mathbb{C}^2$ together with ω_2 constitutes a complete one-sided neighborhood of M in $\mathbb{C}^2$. To conclude the proof of Corollary 1.5, it suffices therefore to show the following.

Lemma 11.29. *$\underline{K}$ is CR-, $\mathcal{W}$- and L^{p}-removable.*

Proof. Let $K_{\rm nr} \subset \underline{K}$ denote the smallest nonremovable subset. Reasoning by contradiction, assume $K_{\rm nr}$ is nonempty.

Let T' be a connected component of the minimal subtree of T containing $K_{\rm nr}$. By a *subtree* of a tree T defined as in (11.2) above, we mean of course a finite union of some of the separatrices $\tau_1(0,1)$ together with all hyperbolic points which are endpoints of separatrices. Since T' does not contain any cycle, there exists at least one extremal branch of T', say $\tau_1(0,1)$ after renumbering.

At first, suppose to simplify that the subtree T' consists only of a single branch $\tau_1[0,1]$. Thanks to properties **(iii)** and **(iv)** of §11.1, we can enlarge a little bit this branch by prolongating the curve $\tau_1(0,1)$ as an open $\mathcal{C}^{2,\alpha}$ Jordan arc $\tau_1[0,1+\varepsilon)$, for some $\varepsilon > 0$. But then by [Me1997, MP1999], every proper closed subset of $\tau_1[0,1+\varepsilon)$ is $\mathcal{W}$-removable, hence $K_{\rm nr}$ is removable, a contradiction.

If T' consists of at least two branches, with $\tau_1(1)$ being an extremal point, since the hyperbolic point $\tau_1(0)$ belongs to another separatrix $\tau_k[0,1] \subset T'$, it follows from the assumption that T' is the smallest subtree containing $K_{\rm nr}$ that $K_{\rm nr} \cap \tau_1(0,1]$ must be nonempty. But then since we may prolong τ_1 to $(0,1+\varepsilon]$, by [Me1997, MP1999] again, $K_{\rm nr} \cap \tau_1(0,1]$ is $\mathcal{W}$-removable, a contradiction to its definition. □

The proofs of Theorem 1.2 and of Corollary 1.5 are complete. □

§12. Polynomial convexity of certain real 2-discs

12.1. Convexity and removability. Let $K \Subset D \subset \partial\Omega \Subset \mathbb{C}^2$ be as in Corollary 1.3. Recall that K is polynomially convex if it coincides with its *polynomial hull*

$$\widehat{K} := \Big\{ z \in \mathbb{C}^2 : \, |p(z)| \leqslant \max_{w \in K} |p(w)|, \ \text{ for every } \ p \in \mathbb{C}[z_1, z_2] \Big\}. \tag{12.2}$$

In complex dimension $n = 2$ (only), removability is closely related to convexity properties ([Jö1988, FS1991, Stu1993, Du1993]). Indeed, for strictly pseudoconvex domains $\Omega \Subset \mathbb{C}^2$, a structural result due to Stout shows that a compact set $K \subset \partial\Omega$ is removable if and only if it is $\mathcal{O}\big(\overline{\Omega}\big)$-convex[7]. If in addition $\overline{\Omega}$ is polynomially convex, removability of K holds if and only if K is *polynomially* convex. In such a situation, removability yields information about polynomial convexity as a byproduct.

Corollary 1.3 improves these results, passing to weakly pseudoconvex boundaries. For totally real discs in *convex* boundaries, it was already shown in [Po2004]. So far, it seems to be the best available insight into polynomial convexity of discs in $\mathbb{C}^2$ (more generally, one may also consider surfaces S as in Corollary 1.5 instead of discs as in Theorem 1.2). In fact, the general question of characterizing polynomial convexity of arbitrary surfaces (*not* contained in pseudoconvex boundaries), even for totally real discs in $\mathbb{C}^2$, is still mostly open ([HN1994], pp. 353–355).

Proof of Corollary 1.3. Since $\overline{\Omega}$ (containing K) is assumed to be polynomially convex, we deduce:

$$\widehat{K} \cap \big(\mathbb{C}^2 \backslash \overline{\Omega}\big) = \emptyset, \quad \text{ or equivalently: } \quad \widehat{K} \subset \overline{\Omega}. \tag{12.3}$$

Firstly, a general fact of independent interest will yield $\widehat{K} \cap \partial\Omega = K$.

Lemma 12.4. *Let $\Omega \Subset \mathbb{C}^2$ be a domain with $\mathcal{C}^{2,\alpha}$ boundary whose closure is polynomially convex and let $K \subset \overline{\Omega}$ be an arbitrary compact set. Then $\widehat{K} \cap \big(\partial\Omega \backslash K\big)$ coincides with the union of all the complex-curve CR orbits of $\partial\Omega \backslash K$.*

Indeed, in the situation of Corollary 1.3, $K \Subset D$ lies in a globally minimal boundary $\partial\Omega$ and we already verified in Proposition 11.6 that $\partial\Omega \backslash K$ is also globally minimal, namely it contains no complex-curve CR orbit, whence $\widehat{K} \cap \partial\Omega = K$.

Secondly, we will control $\widehat{K} \cap \Omega$ thanks to the pseudoconvexity of $\Omega \big\backslash \widehat{K}$.

Lemma 12.5. *Let $\Omega \Subset \mathbb{C}^2$ be an arbitrary pseudoconvex domain. Then for any compact set $K \Subset \mathbb{C}^2 \backslash \Omega$, the open set $\Omega \big\backslash \widehat{K}$ is a union of pseudoconvex domains.*

Granted these two lemmas and Theorem 1.2, we may conclude the proof of Corollary 1.3. Indeed, since $\widehat{K}$ does not meet $\partial\Omega \backslash K$, there is a domain contained in $\Omega \big\backslash \widehat{K}$ whose closure contains the connected hypersurface $\partial\Omega \backslash K$, namely a one-sided neighborhood $\mathcal{V}(\partial\Omega \backslash K)$. The $\mathcal{W}$-removability of K in Theorem 1.2 yields univalent holomorphic extension from $\mathcal{V}(\partial\Omega \backslash K)$ to Ω. Thus, there can exist only one pseudoconvex component of $\Omega \big\backslash \widehat{K}$, the domain Ω itself ! Thus $\Omega \big\backslash \widehat{K} = \Omega$, which, together with $\widehat{K} \subset \overline{\Omega}$ and $\widehat{K} \cap \partial\Omega = K$, gives $\widehat{K} = K$. □

[7] To define the $\mathcal{O}\big(\overline{\Omega}\big)$-hull, simply replace polynomials by functions holomorphic in some (unspecified) neighborhood of $\overline{\Omega}$.

Proof of Lemma 12.4. We assume $K \neq \emptyset$ throughout.

Let $\mathcal{O}$ be a complex-curve CR orbit of $\partial\Omega\backslash K$. Its closure $\overline{\mathcal{O}}^{\partial\Omega\backslash K}$ in $\partial\Omega\backslash K$ is a relatively closed subset of $\partial\Omega\backslash K$ laminated by complex curves, in which $\mathcal{O}$ (as well as every other maximal connected complex curve) is dense, for the topology induced from $\partial\Omega\backslash K$ (*see* [Jö1999a, MP2006]). The full closure $\overline{\mathcal{O}}^{\partial\Omega} \supset \overline{\mathcal{O}}^{\partial\Omega\backslash K}$ is compact and the complement $\overline{\mathcal{O}}^{\partial\Omega}\backslash\overline{\mathcal{O}}^{\partial\Omega\backslash K}$ is contained in K. Since $\mathbb{C}^2$ cannot contain any compact set laminated by complex curves ([Jö1999a, MP2006]) and since $K \neq \emptyset$, this complement $\overline{\mathcal{O}}^{\partial\Omega}\backslash\overline{\mathcal{O}}^{\partial\Omega\backslash K} \subset K$ must be nonempty.

Pick $z \in \mathcal{O}$ and let p be an arbitrary holomorphic polynomial. To verify that $|p(z)| \leqslant \max_K |p|$, two cases occur.

- **(a)** The maximum of $|p|$ on the compact set $\overline{\mathcal{O}}^{\partial\Omega}$ is attained at some point $w \in \overline{\mathcal{O}}^{\partial\Omega}\backslash\overline{\mathcal{O}}^{\partial\Omega\backslash K}$; then $w \in K$, whence obviously $|p(z)| \leqslant |p(w)| \leqslant \max_K |p|$.
- **(b)** The maximum of $|p|$ on $\overline{\mathcal{O}}^{\partial\Omega}$ is attained at some point $w \in \overline{\mathcal{O}}^{\partial\Omega\backslash K}$; then by the lamination property, w belongs to (the interior of) some immersed complex curve $\mathcal{O}'$, again dense in the sense that $\overline{\mathcal{O}'}^{\partial\Omega\backslash K} = \overline{\mathcal{O}}^{\partial\Omega\backslash K}$. The maximum principle entails that $p|_{\mathcal{O}'}$ is equal to a constant $\sigma \in \mathbb{C}$, whence by continuity $p\big|_{\overline{\mathcal{O}'}^{\partial\Omega\backslash K}}$, $p\big|_{\overline{\mathcal{O}}^{\partial\Omega\backslash K}}$ and $p\big|_{\overline{\mathcal{O}}^{\partial\Omega}}$ are all equal to the same constant σ and since $\overline{\mathcal{O}}^{\partial\Omega} \cap K \neq \emptyset$, we conclude that $|p(z)| \leqslant \max_{\overline{\mathcal{O}}^{\partial\Omega}} |p| = |p(w)| = |\sigma| \leqslant \max_K |p|$.

Thus, it remains to show that every point $p \in \partial\Omega\backslash K$ which belongs to a 3-dimensional $\big($hence open in $\partial\Omega\backslash K\big)$ CR orbit does not belong to $\widehat{K}$. The local Oka criterion ([Stu1971]) is suitable for that purpose. We state an adapted and simplified version, using discs.

If $K \subset \mathbb{C}^2$ is a compact set, a point $p_0 \in \mathbb{C}^2$ does *not* belong to $\widehat{K}$ provided one can construct a continuous one-parameter family $\{A_t\}_{0\leqslant t\leqslant 1}$ of analytic discs $A_t : \Delta \to \mathbb{C}^2$ such that

- $A_t(\overline{\Delta}) \cap K = \emptyset$ for every t with $0 \leqslant t \leqslant 1$;
- $A_t(\partial\Delta) \cap \widehat{K} = \emptyset$ for every t with $0 \leqslant t \leqslant 1$;
- $A_1(\overline{\Delta}) \cap \widehat{K} = \emptyset$;
- $p_0 \in A_0(\Delta)$.

Since everything is biholomorphically invariant, we observe a direct analogy with the continuity principle.

Thus, let $\mathcal{O}$ be an open, 3-dimensional CR orbit of $\partial\Omega\backslash K$. Since $\mathcal{O}$ has CR dimension 1, it contains some strongly pseudoconvex points (otherwise $T^c\mathcal{O}$ would be Frobenius-integrale, hence $\mathcal{O}$ would be a complex-curve CR orbit). To establish $\mathcal{O} \cap \widehat{K} = \emptyset$, it is thus sufficient to show:

- **(i)** no strictly pseudoconvex point $p_0 \in \partial\Omega\backslash K$ can be contained in $\widehat{K}$; and:
- **(ii)** the property $q \notin \widehat{K}$ propagates along CR curves running inside $\mathcal{O}$.

To show **(i)**, we choose local holomorphic coordinates centered at p_0 in which Ω corresponds to

$$y_2 > |z_1|^2 + \mathrm{O}\big(|z_1| + |x_2|\big)^{2+\alpha}. \tag{12.6}$$

For small $\varepsilon > 0$, the family $A_t(\zeta) := \big(\varepsilon\zeta, \, -\varepsilon \, i \, t\big)$, which translates downwards a small piece of the complex line $T_{p_0}^c \partial\Omega$, does satisfy the four items above, whence $p_0 \notin \widehat{K}$.

Gratuitously, we deduce that all points p_0' belonging to some neighborhood U_{p_0} of p_0 in $\partial\Omega \backslash K$ also avoid $\widehat{K}$.

To show **(ii)**, we recall from [Tu1994, MP2006] that for every $q \in \mathcal{O} = \mathcal{O}_{CR}\big(\partial\Omega\backslash K, p_0\big)$ and for every small $\varepsilon > 0$, there exist $\ell \in \mathbb{N}$ with $\ell = \mathrm{O}(1/\varepsilon)$ and a chain of $\mathcal{C}^{2,\alpha-0}$ analytic discs $A^1, A^2, \dots, A^{\ell-1}, A^\ell$ attached to M with the properties:

- $A^1(-1) =: p_0' \in U_{p_0}$;
- $A^1(1) = A^2(-1)$, $A^2(1) = A^3(-1), \dots, A^{\ell-1}(1) = A^\ell(-1)$;
- $A^\ell(1) = q$;
- $\|A^k\|_{\mathcal{C}^1(\overline{\Delta})} \leqslant \varepsilon$, for $k = 1, 2, \dots, \ell$;
- each A^k is an embedding $\overline{\Delta} \to \mathbb{C}^2$.

By construction ([Tu1994, MP2006]), the projections onto $T_{A^k(1)}^c \partial\Omega$ of each $A^k(\zeta)$ are round discs $\Delta \ni \zeta \mapsto \lambda(1-\zeta) \in \mathbb{C}$, for some appropriate $\lambda \in \mathbb{C}$ satisfying $|\lambda| = \mathrm{O}(\varepsilon)$. Hence we are reduced to proving for a small round disc the implication $A(-1) \notin \widehat{K} \implies A(1) \notin \widehat{K}$. Roundness is useful to control the geometry.

We first consider the case where A is transverse to $\partial\Omega$ at $A(1)$, namely $-\frac{\partial A}{\partial \rho}(1) \notin T_{A(1)}\partial\Omega$. Since Ω is pseudoconvex, this vector $-\frac{\partial A}{\partial \rho}(1)$ points inside Ω. We choose coordinates centered at $A(1)$ in which Ω is represented by $y_2 > \varphi(x_1, y_1, x_2)$, with $\varphi(0) = 0$ and $d\varphi(0) = 0$. Perturbing slightly the base point $A(1)$ and solving an appropriate Bishop-type equation, we may as in [Tu1994, MP1999, MP2002] construct a 2-parameter family of analytic discs A_{s_1,s_2} attached to $\partial\Omega\backslash K$ with $A_{0,0} = A$ whose boundaries $A_{s_1,s_2}(\partial\Delta)$ foliate a neighborhood of $A(1)$ in $\partial\Omega\backslash K$. Then the interiors $A_{s_1,s_2}(\Delta)$ foliate the pseudoconvex side of $\partial\Omega\backslash K$ near $A(1)$. Fixing a very small $\delta > 0$, there exists a unique $A_{s_1',s_2'}$ such that the point $A(1) = 0$ is contained in the image of pushed-down disc $A_{s_1',s_2'} + (0, -i\delta)$. For an $\eta > 0$ which is approximately equal to twice the diameter of $A_{s_1',s_2'}(\overline{\Delta})$, we look at the family $A_t := A_{s_1',s_2'} + (0, -it)$, where $\delta \leqslant t \leqslant \eta$. Clearly the final disc $A_\eta(\overline{\Delta})$ lies in $\mathbb{C}^2\backslash\overline{\Omega}$, hence it does not meet $\widehat{K} \subset \overline{\Omega}$. Also, the boundaries $A_t(\partial\Delta)$ do not intersect $\widehat{K}$ since they lie in $\mathbb{C}^2\backslash\overline{\Omega}$ for all t with $\delta \leqslant t \leqslant \eta$. Since the A_t are round discs graphed over the z_1-axis, we find holomorphic defining functions as required in the Oka criterion. Hence we deduce that $\bigcup_{\delta \leqslant t \leqslant \eta} A_t\big(\overline{\Delta}\big) \ni A(1)$ does not meet $\widehat{K}$.

The second case where A is tangent to M at $A(1)$ can be reduced to the above arguments by means of a preliminary slight normal deformation of the disc near the point $A(-1)$ ([Tu1994, MP1999, MP2002]) which produces a new disc A^d nontangent to $\partial\Omega$ at $A(1)$. □

Proof of Lemma 12.5. We verify that $\Omega\backslash\widehat{K}$ does satisfy the Kontinuitätssatz. Let Φ be an injective holomorphic map sending a neighborhood of $[0,1] \times \{|z_2| \leqslant 1\}$ in $\mathbb{C}^2$ into $\mathbb{C}^2$. Assuming that Φ maps

$$\big(\{0\} \times \{|z_2| \leqslant 1\}\big) \bigcup \big([0,1] \times \{|z_2| = 1\}\big) \tag{12.7}$$

into $\Omega\backslash\widehat{K}$, we have to show $\Phi\big([0,1]\times\{|z_2|\leqslant 1\}\big)\subset\Omega\backslash\widehat{K}$ also.

Assume on the contrary that there exists a smallest $t^*\in(0,1]$ such that $\Phi\big(\{t^*\}\times\{|z_2|<1\}\big)\not\subset\Omega\backslash\widehat{K}$. Then the open analytic disc $\Phi\big(\{t^*\}\times\{|z_2|<1\}\big)$ contains a point of $\partial\Omega$ or a point of $\widehat{K}\backslash\partial\Omega$. In the first case, we would contradict the pseudoconvexity of Ω and in the second case, we would contradict the Oka criterion for $\widehat{K}$. □

§13. Proof of Theorem 1.9

13.1. The geometric recipe. We first construct the 2-torus $K=T^2$, then construct the maximally real M^1 and finally define M as a certain thickening of M^1. The argument to insure global minimality of M involves computations with Lie brackets and is postponed to the end.

Firstly, in $\mathbb{R}^3=\mathbb{R}^3\oplus i\{0\}\subset\mathbb{C}^3$ equipped with the coordinates (x_1,x_2,x_3), where $x_j=\operatorname{Re} z_j$ for $j=1,2,3$, pick the "standard" 2-dimensional torus T^2 of Cartesian equation

$$\left(\sqrt{x_1^2+x_2^2}-2\right)^2+x_3^2=1. \tag{13.2}$$

This torus is stable under the rotations directed by the x_3-axis; its intersection with the (x_1,x_3)-plane consists of two circles of radius 1 centered at the points $x_1=2$ and $x_1=-2$; it bounds a three-dimensional open "full" torus T^3; both T^2 and T^3 are contained in the ball B^3 of radius 5 centered at the origin.

It is better to drop the square root: one checks that the equations of T^2 and T^3 are equally given by $T^2:=\{\rho=0\}$ and $T^3:=\{\rho<0\}$, by means of the *polynomial* defining function

$$\rho(x_1,x_2,x_3):=(x_1^2+x_2^2+x_3^2+3)^2-16\,(x_1^2+x_2^2), \tag{13.3}$$

which has nonvanishing differential at every point of T^2. Consequently, the extrinsinc complexification of T^2, namely the complex hypersurface defined by

$$\Sigma:=\{(z_1,z_2,z_3)\in\mathbb{C}^3:\ \rho(z_1,z_2,z_3)=0\} \tag{13.4}$$

cuts $\mathbb{R}^3$ along T^2 with the transversality property $T_x\mathbb{R}^3\cap T_x\Sigma=T_xT^2$ for every point $x\in T^2$.

Secondly, according to Reeb ([CLN1985]) *see* also the figures there), by considering the space $\mathbb{R}^3\equiv S^3\backslash\{\infty\}$ as a punctured three-dimensional sphere S^3, one may glue a second three-dimensional full torus $\widetilde{T}^3$ to T^3 along T^2 with $\infty\in\widetilde{T}^3$ and then construct a foliation of S^3 by 2-dimensional surfaces all of whose leaves, except one, are diffeomorphic to $\mathbb{R}^2$, are contained in either T^3 or in $\widetilde{T}^3$ and are accumulating on T^2, and finally, whose single compact leaf is the above 2-torus T^2. This yields the so-called *Reeb foliation* of S^3, which is $\mathcal{C}^\infty$ and orientable. Consequently, there exists a $\mathcal{C}^\infty$ smooth vector field $L=a_1(x)\,\partial_{x_1}+a_2(x)\,\partial_{x_2}+a_3(x)\,\partial_{x_3}$ of norm 1, namely $a_1(x)^2+a_2(x)^2+a_3(x)^2=1$ for every $x\in\mathbb{R}^3$, which is everywhere orthogonal (with respect to the standard Euclidean structure) to the leaves of the Reeb foliation. Geometrically, the integral curves of L accumulate asymptotically on the two nodal (central) circles of T^3 and of $\widetilde{T}^3$.

The open ball $B^3\subset\mathbb{R}^3$ of radius 5 centered at the origin will be our maximally real submanifold M^1. The two-dimensionaly torus T^2 will be our nonremovable compact set K. The integral curves of the vector field L will be our characteristic lines. Since L is orthogonal to T^2, these characteristic lines will of course be everywhere transverse to K, so that $K = T^2$ is nontransversal to the integral curves of L.

Thirdly, it remains to construct the generic submanifold M of CR dimension 1 containing M^1 and to check that K will be nonremovable.

First of all, we notice that L provides the characteristic directions of M^1 if and only if $T_xM = T_p\mathbb{R}^3 \oplus \mathbb{R} J\, L(x)$ for every point $x \in M^1 \equiv B^3$. Consequently, all submanifolds $M \subset \mathbb{C}^3$ obtained by slightly thickening M^1 in the direction of $J\, L(x)$ will be convenient; in other words, only the first jet of M along M^1 is prescribed by our choice of the characterisctic vector field L. Notice that all such thin strips M along M^1 will be diffeomorphic to a real 4-ball.

The fact that K is nonremovable for all such generic submanifolds M is now clear: the hypersurface $\Sigma = \{z \in \mathbb{C}^3 : \rho(z) = 0\}$ satisfying $T_x\Sigma = T_xT^2 \oplus \mathbb{R} J\, T_xT^2$ for all $x \in T^2$ and L being transversal to T^2, we easily deduce the transversality property $T_x\Sigma + T_xM = T_x\mathbb{C}^3$ for all $x \in T^2$, a geometric property which insures that the holomorphic function $1/\rho(z)$, which is CR on $M\backslash K$, does not extend holomorphically to any wedge of edge M at any point of K. Intuitively, $T_x\Sigma/T_xM$ absorbs all the normal space $T_x\mathbb{C}^3/T_xM$ at every point $x \in T^2 = K$, leaving no room for any open cone.

Finally, to fulfill all the hypotheses of Proposition 1.13 (except of course nontransversality of K to $\mathsf{F}^c_{M^1}$), we have to insure that M is globally minimal. We claim that by bending strongly the second and the fourth order jet of M along M^1 (without modifying the first order jet which must be prescribed by $J\, L$), one may insure that M is of type 4 in the sense of Definition 4.22(III) in [MP2006] at every point of M^1; since being of finite type is an open property, it follows that M is finite type at every point provided that, as a strip, M is sufficiently thin along M^1. As is known, finite-typeness at every point implies local minimality at every point which in turn implies global minimality. This completes the recipe.

13.5. Finite-typisation. To complete the arguments of Theorem 1.9, it remains to construct a generic submanifold $M \subset \mathbb{C}^3$ of CR dimension 1 satisfying $T_xM = T_xM^1 \oplus \mathbb{R} J\, L(x)$ for every $x \in M^1$, which is of *type* 4 *at every point* $x \in M^1$.

First of all, let us denote by $L = a_1(x)\, \partial_{x_1} + a_2(x)\, \partial_{x_2} + a_3(x)\, \partial_{x_3}$ the unit vector field which was constructed as a field orthogonal to the Reeb foliation: it is defined over $\mathbb{R}^3$ and has $\mathcal{C}^\infty$ coefficients satisfying $a_1(x)^2 + a_2(x)^2 + a_3(x)^2 = 1$ for all $x \in \mathbb{R}^2$. The two-dimensional quotient vector bundle $T\mathbb{R}^3/(\mathbb{R}L)$ with contractible base being necessarily trivial, it follows that we can complete L by two other $\mathcal{C}^\infty$ unit vector fields K^1 and K^2 defined over $\mathbb{R}^3$ such that the triple $(L(x), K^1(x), K^2(x))$ forms a direct orthonormal frame at every point $x \in \mathbb{R}^3$. Let us denote the coefficients of K^1 and of K^2 by

$$\begin{aligned} K^1 &= \rho_1\, \partial_{x_1} + \rho_2\, \partial_{x_2} + \rho_3\, \partial_{x_3}, \\ K^2 &= r_1\, \partial_{x_1} + r_2\, \partial_{x_2} + r_3\, \partial_{x_3}, \end{aligned} \tag{13.6}$$

where ρ_j and r_j for $j = 1, 2, 3$ are $\mathcal{C}^\infty$ functions of $x \in \mathbb{R}^3$ satisfying $\rho_1^2 + \rho_2^2 + \rho_3^2 = 1$ and $r_1^2 + r_2^2 + r_3^2 = 1$. In our case, K^1 and K^2 may even be constructed directly by means of a trivialization of the bundle tangent to the Reeb foliation.

Let $P > 0$ be a constant, which will be chosen later to be large. Since by construction we have the two orthogonality relations $a_1\rho_1 + a_2\rho_2 + a_3\rho_3 = 0$ and

$a_1r_1+a_2r_2+a_3r_3=0$, it follows that every generic submanifold $M_P\subset\mathbb{C}^3$ defined by the two Cartesian equations

$$\begin{aligned}0 &= \rho = y_1\,\rho_1(x)+y_2\,\rho_2(x)+y_3\,\rho_3(x)+P\left[y_1^2+y_2^2+y_3^2\right],\\ 0 &= r = y_1\,r_1(x)+y_2\,r_2(x)+y_3\,r_3(x)+P^3\left[y_1^4+y_2^4+y_3^4\right]\end{aligned}\tag{13.7}$$

enjoys the property that the vector field $JL(x)=a_1(x)\,\partial_{y_1}+a_2(x)\,\partial_{x_2}+a_3(x)\,\partial_{x_3}$ is tangent to M_P at every $x\in\mathbb{R}^3$. As desired, we deduce that $T_x^cM=\mathbb{R}L(x)\oplus J\mathbb{R}L(x)$ for every $x\in\mathbb{R}^3$, a property which insures that $\mathbb{R}L(x)$ is the characteristic direction of M^1 in M_P, independently of P.

To complete the final minimalization argument for the construction of a nonremovable compact set $C:=T^2\subset M^1\subset M$ which appears in the Introduction, it suffices now to apply the following lemma with $R=5$. Though calculatory, its proof is totally elementary.

Lemma 13.8. *For every $R>0$, there exist $P>0$ sufficiently large such that M_P is of type 4 at every point $x\in\mathbb{R}^3$ with $x_1^2+x_2^2+x_3^2\leqslant R^2$.*

Proof. As above, let $M_P=\{z\in\mathbb{C}^3:\rho=r=0\}$. By writing the tangency condition, one checks immediately that the one-dimensional complex vector bundle $T^{1,0}M_P$ is generated over $\mathbb{C}$ by the vector field $\mathbb{L}:=A_1\,\partial_{z_1}+A_2\,\partial_{z_2}+A_3\,\partial_{z_3}$, with the explicit expressions

$$\begin{aligned}A_1&:=4\rho_{z_3}r_{z_2}-4\rho_{z_2}r_{z_3},\\ A_2&:=4\rho_{z_1}r_{z_3}-4\rho_{z_3}r_{z_1},\\ A_3&:=4\rho_{z_2}r_{z_1}-4\rho_{z_1}r_{z_2}.\end{aligned}\tag{13.9}$$

Using the expressions (13.7) for ρ and r, we see that these three components restrict on $\{y=0\}$ as the Plücker coordinates of the bivector (K^1,K^2), namely

$$\begin{aligned}A_1|_{y=0}&=\rho_2r_3-\rho_3r_2=:\Delta_{2,3},\\ A_2|_{y=0}&=\rho_3r_1-\rho_1r_3=:\Delta_{3,1},\\ A_3|_{y=0}&=\rho_1r_2-\rho_2r_1=:\Delta_{1,2}.\end{aligned}\tag{13.10}$$

As K^1 and K^2 are of norm 1 and orthogonal at every point, it follows by direct computation that $\Delta_{2,3}^2+\Delta_{3,1}^2+\Delta_{1,2}^2=1$ and that the vector of coordinates $(\Delta_{2,3},\Delta_{3,1},\Delta_{1,2})$ is orthogonal to both K^1 and K^2. Moreover, as the orthonormal trihedron $(L(x),K^1(x),K^2(x))$ is direct at every point, we deduce that necessarily

$$\Delta_{2,3}\equiv a_1,\qquad \Delta_{3,1}\equiv a_2,\qquad \Delta_{1,2}\equiv a_3.\tag{13.11}$$

Next, we compute in length A_1, A_2 and A_3 using (13.7). As their complete explicit development will not be crucial for the sequel and as we shall perform with them differentiations and linear combinations yielding relatively complicated expressions, let us adopt the following notation: by $\mathcal{R}^0$, we denote various expressions which are polynomials in the jets of the functions ρ_1,ρ_2,ρ_3 and r_1,r_2,r_3. Similarly, by $\mathcal{R}^I$, by $\mathcal{R}^{II}$, by $\mathcal{R}^{III}$ and by $\mathcal{R}^{IV}$, we denote polynomials in the transverse variables (y_1,y_2,y_3) which are homogeneous of degree 1, 2, 3 and 4 and have as coefficients various expressions $\mathcal{R}^0$.

Importantly, we make the convention that such expressions $\mathcal{R}^0$, $\mathcal{R}^I$, $\mathcal{R}^{II}$, $\mathcal{R}^{III}$ and $\mathcal{R}^{IV}$ should be totally independent of the constant P. Consequently, if P appears somehow, we shall write it as a factor, as for instance in $P\,\mathcal{R}^I$ or in $P^3\,\mathcal{R}^{III}$.

With this convention at hand, we may develope (13.9) using the expressions (13.7) by writing out only the terms which will be useful in the sequel and by treating the rest as controlled remainders. Let us detail the computation of A_1:

(13.12)

$$\begin{aligned} A_1 \ &= 4 \left[-\frac{i}{2}\rho_3 - iPy_3 + \mathcal{R}^I \right] \left[-\frac{i}{2} r_2 - 2iP^3 y_2^3 + \mathcal{R}^I \right] - \\ &\quad - 4 \left[-\frac{i}{2}\rho_2 - iPy_2 + \mathcal{R}^I \right] \left[-\frac{i}{2} r_3 - 2iP^3 y_3^3 + \mathcal{R}^I \right] \\ &= -\rho_3 r_2 - 4P^3\rho_3 y_2^3 + \mathcal{R}^I - 2Pr_2y_3 + P^4\mathcal{R}^{IV} + P\mathcal{R}^I + \mathcal{R}^I + P^3\mathcal{R}^{IV} + \mathcal{R}^{II} \\ &\quad + \rho_2 r_3 + 4P^3\rho_2 y_3^3 + \mathcal{R}^I + 2Pr_3y_2 + P^4\mathcal{R}^{IV} + P\mathcal{R}^I + \mathcal{R}^I + P^3\mathcal{R}^{IV} + \mathcal{R}^{II} \\ &= \rho_2 r_3 - \rho_3 r_2 + 2Pr_3y_2 - 2Pr_2y_3 + 4P^3\rho_2 y_3^3 - 4P^3\rho_3 y_2^3 + \\ &\quad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}. \end{aligned}$$

In the development, before simplification, we firstly write out in lines 3 and 4 all the 9×2 terms of the two product: for instance, the third term of the first product, namely $4(-\frac{i}{2}\rho_3)(\mathcal{R}^I)$, yields a term $\mathcal{R}^I$ whereas the fifth term $4(-iPy_3)(-2iP^3y_2^3)$ yields a term $P^4\mathcal{R}^{IV}$; secondly, we simplify the obtained sum: by our convention, $\mathcal{R}^I + \mathcal{R}^I = \mathcal{R}^I$, whereas $\mathcal{R}^I + P\mathcal{R}^I$ cannot be simplified, since the large constant P will be chosen later. With these technical explanations at hand, we shall not provide any intermediate detail for the further computations, whose rules are totally analogous. For A_1, A_2 and A_3, we obtain

$$\text{(13.13)} \quad \begin{cases} A_1 \ = \rho_2 r_3 - \rho_3 r_2 + 2Pr_3y_2 - 2Pr_2y_3 + 4P^3\rho_2 y_3^3 - 4P^3\rho_3 y_2^3 + \\ \qquad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}, \\ A_2 \ = \rho_3 r_1 - \rho_1 r_3 + 2Pr_1y_3 - 2Pr_3y_1 + 4P^3\rho_3 y_1^3 - 4P^3\rho_1 y_3^3 + \\ \qquad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}, \\ A_3 \ = \rho_1 r_2 - \rho_2 r_1 + 2Pr_2y_1 - 2Pr_1y_2 + 4P^3\rho_1 y_2^3 - 4P^3\rho_2 y_1^3 + \\ \qquad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}. \end{cases}$$

Now that we have written the complex vector field $\mathbb{L}$ and its coefficients A_1, A_2 and A_3, in order to establish Lemma 13.8, it suffices to choose $P > 0$ sufficiently large in order that the four complex vector fields

$$\text{(13.14)} \qquad \overline{\mathbb{L}}\,|_{y=0}, \quad \mathbb{L}|_{y=0}, \quad \big[\overline{\mathbb{L}}, \mathbb{L}\big]\,|_{y=0}, \quad \big[\overline{\mathbb{L}}, \big[\overline{\mathbb{L}}, \big[\overline{\mathbb{L}}, \mathbb{L}\big]\big]\big]\,|_{y=0}$$

are linearly independent at every point $x \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \leqslant R^2$. At the end of the proof, we shall explain why we cannot insure type 3 at every point, namely why the consideration of $\big[\overline{\mathbb{L}}, \big[\overline{\mathbb{L}}, \mathbb{L}\big]\big]\,|_{y=0}$ instead of the length four last Lie bracket in (13.14) would fail.

As promised, we shall now summarize all the subsequent computations. As we aim to restrict the last Lie bracket to $\{y = 0\}$ which is of length four and whose coefficients involve derivatives of order at most three of the coefficients A_1, A_2 and A_3, we can already neglect the last two remainders $P^3\mathcal{R}^{IV}$ and $P^4\mathcal{R}^{IV}$ in (13.13). In other words, we can consider A^1, A^2 and $A^3 \operatorname{mod}(IV)$. Similarly, in the computation of the Lie bracket

$$\text{(13.14)} \qquad \big[\overline{\mathbb{L}}, \mathbb{L}\big] =: C_1\,\partial_{z_1} + C_2\,\partial_{z_2} + C_3\,\partial_{z_3} - \overline{C_1}\,\partial_{\bar{z}_1} - \overline{C_2}\,\partial_{\bar{z}_2} - \overline{C_3}\,\partial_{\bar{z}_3},$$

before restriction to $\{y=0\}$, we can restrict our task to developing the coefficients

$$\begin{aligned} C_1 &:= \overline{A_1} A_{1,\bar z_1} + \overline{A_2} A_{1,\bar z_2} + \overline{A_3} A_{1,\bar z_3}, \\ C_2 &:= \overline{A_1} A_{2,\bar z_1} + \overline{A_2} A_{2,\bar z_2} + \overline{A_3} A_{2,\bar z_3}, \\ C_3 &:= \overline{A_1} A_{3,\bar z_1} + \overline{A_2} A_{3,\bar z_2} + \overline{A_3} A_{3,\bar z_3} \end{aligned} \tag{13.16}$$

only modulo order (III), which yields by means of the expressions (13.13)

$$\begin{aligned} C_1 \bmod (III) &\equiv -iP\rho_1 + 6iP^3 a_3 \rho_2 y_3^2 - 6iP^3 a_2 \rho_3 y_2^2 + \mathcal{R}^0 + \mathcal{R}^I + \\ &\quad + P\mathcal{R}^I + P^2\mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^2\mathcal{R}^{II}, \\ C_2 \bmod (III) &\equiv -iP\rho_2 + 6iP^3 a_1 \rho_3 y_1^2 - 6iP^3 a_3 \rho_1 y_3^2 + \mathcal{R}^0 + \mathcal{R}^I + \\ &\quad + P\mathcal{R}^I + P^2\mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^2\mathcal{R}^{II}, \\ C_3 \bmod (III) &\equiv -iP\rho_3 + 6iP^3 a_2 \rho_1 y_2^2 - 6iP^3 a_1 \rho_2 y_1^2 + \mathcal{R}^0 + \mathcal{R}^I + \\ &\quad + P\mathcal{R}^I + P^2\mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^2\mathcal{R}^{II}. \end{aligned} \tag{13.17}$$

We must mention the use of natural rule hold for computing the partial derivatives $A_{j,\bar z_k}$: we have for instance $\partial_{\bar z_k}\left(\mathcal{R}^{II}\right) = \mathcal{R}^I + \mathcal{R}^{II}$. Also, we have used the hypothesis that $(L(x), K^1(x), K^2(x))$ provides a direct orthonormal frame at every $x \in \mathbb{R}^3$, which yields in particular the three relations

$$a_2 r_3 - a_3 r_2 = -\rho_1, \qquad a_3 r_1 - a_1 r_3 = -\rho_2, \qquad a_1 r_2 - a_2 r_1 = -\rho_3. \tag{13.18}$$

After mild computation, the coefficients F_1, F_2 and F_3 of the length four Lie bracket

$$\left[\mathbb{L}, \left[\mathbb{L}, \left[\mathbb{L}, \mathbb{L}\right]\right]\right] = F_1\,\partial_{z_1} + F_2\,\partial_{z_2} + F_3\,\partial_{z_3} + G_1\,\partial_{\bar z_1} + G_2\,\partial_{\bar z_2} + G_3\,\partial_{\bar z_3} \tag{13.19}$$

are given, after restriction to $\{y=0\}$, by

$$\begin{aligned} F_1|_{y=0} &= 3iP^3 a_2^3 \rho_3 - 3iP^3 a_3^3 \rho_2 + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0, \\ F_2|_{y=0} &= 3iP^3 a_3^3 \rho_1 - 3iP^3 a_1^3 \rho_3 + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0, \\ F_3|_{y=0} &= 3iP^3 a_1^3 \rho_2 - 3iP^3 a_2^3 \rho_1 + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0, \end{aligned} \tag{13.20}$$

We can now complete the proof of Lemma 13.8. In the basis $(\partial_{z_1}, \partial_{z_2}, \partial_{z_3}, \partial_{\bar z_1}, \partial_{\bar z_2}, \partial_{\bar z_3})$, the 4×6 matrix associated with the four vector fields (13.14) (without mentioning $|_{y=0}$)

$$\begin{pmatrix} 0 & 0 & 0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 \\ C_1 & C_2 & C_3 & -\overline{C_1} & -\overline{C_2} & -\overline{C_3} \\ F_1 & F_2 & F_3 & G_1 & G_2 & G_3 \end{pmatrix} \tag{13.21}$$

has rank four at a point $x \in \mathbb{R}^3$ if and only if the 3×3 determinant in the left low corner is nonvanishing, namely if and only if the developped expression

(13.22)

$$\begin{aligned} &\left| \begin{matrix} a_1 & a_2 & a_3 \\ -iP\rho_1 + \mathcal{R}^0 & -iP\rho_2 + \mathcal{R}^0 & -iP\rho_3 + \mathcal{R}^0 \\ 3iP^3 a_2^3 \rho_3 - 3iP^3 a_3^3 \rho_2 + & 3iP^3 a_3^3 \rho_1 - 3iP^3 a_1^3 \rho_3 + & 3iP^3 a_1^3 \rho_2 - 3iP^3 a_2^3 \rho_1 + \\ +\mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 & +\mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 & +\mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 \end{matrix} \right| \\ &= 3P^4 \left(r_3[a_1^3\rho_2 - a_2^3\rho_1] + r_2[a_3^3\rho_1 - a_1^3\rho_3] + r_1[a_2^3\rho_3 - a_3^3\rho_2] \right) + \\ &\quad + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 + P^3\mathcal{R}^0 + P^4\mathcal{R}^0 \\ &= 3P^4 \left(a_1^4 + a_2^4 + a_3^4 \right) + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 + P^3\mathcal{R}^0 + P^4\mathcal{R}^0 \end{aligned}$$

is nonvanishing.

At this point, the conclusion of the lemma is now an immediate consequence of the following trivial assertion: *Let a_1, a_2 and a_3 be $\mathcal{C}^\infty$ functions on $\mathbb{R}^3$ satisfying $a_1(x)^2 + a_2(x)^2 + a_3(x)^2 = 1$ for all $x \in \mathbb{R}^3$ and let $\mathcal{R}_0^0$, $\mathcal{R}_1^0$, $\mathcal{R}_2^0$, $\mathcal{R}_3^0$ and $\mathcal{R}_4^0$ be $\mathcal{C}^\infty$ functions on $\mathbb{R}^3$. For every $R > 0$, there exists a constant $P > 0$ large enough so that the function*

$$3P^4 \left(a_1^4 + a_2^4 + a_3^4 \right) + \mathcal{R}_0^0 + P\mathcal{R}_1^0 + P^2\mathcal{R}_2^0 + P^3\mathcal{R}_3^0 + P^4\mathcal{R}_4^0 \tag{13.23}$$

is positive at every $x \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \leqslant R^2$.

If we had put $y_1^3 + y_2^3 + y_3^3$ instead of $y_1^4 + y_2^4 + y_3^4$ in the second equation (13.7), we would have considered the length three Lie bracket $\left[\mathbb{L}, \left[\mathbb{L}, \mathbb{L}\right]\right] |_{y=0}$ instead of the length four Lie bracket in (13.14), and hence instead of the quartic $a_1^4 + a_2^4 + a_3^4$ in (13.23), we would have obtained the cubic $a_1^3 + a_2^3 + a_3^3$, a function which (unfortunately) vanishes, for instance if $a_1(x) = \frac{1}{\sqrt{2}}$, $a_2(x) = -\frac{1}{\sqrt{2}}$ and $a_3(x) = 0$. We notice that in our example, this value of (a_1, a_2, a_3) is indeed attained at the point $x \in T^2$ of coordinates $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 0)$, whence the necessity of passing to type 4. The proof of Lemma 13.8 is complete. □

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