UNRESTRICTED PREFERENTIAL STRUCTURES

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Abstract

One of the major open problems of preferential structures was perhaps to give an unrestricted representation result. We give a solution to the problem in this short, technical paper. Up to now - to the author's knowledge - all representation results for preferential structures were subject to some restriction: definability preservation (the author's terminology, fullness in Lehmann's terminology) or some kind of finiteness. The results presented here are valid without any restrictions.

1 INTRODUCTION

Definability preservation was a major restriction for representation results for preferential models considered so far. See [8], [5] for discussions. We use the ideas of [8] and [10] for the constructions, and were incited by the work on [1] to reconsider definability preservation. As a matter of fact, the results and proofs presented here are a direct adaptation of (some of) those in [8] and [10].

The basic problem with lack of definability preservation is that the tool at our disposal, definable sets of models, is too coarse to fully analyze the situation. This problem was solved, in another context, in [1] by an inductive construction. This solution cannot be applied here to unrestricted preferential structures, as one model may have multiple copies, and a minimal set of models which minimizes all copies of one model may itself not be definable.

The paper is technically self-contained. To keep it short, it contains no motivations or general discussions of preferential structures. The reader is referred e.g. to [8] or [9] for such considerations, and e.g. to [4] and [3] for the work by Lehmann and his co-authors.

As in our other articles, we first show an algebraic representation result, which is then applied to logic. This separation of the algebraic and the logical part has the advantage that the main (algebraic) work can be reused in other contexts. Moreover, it clearly brings to light the problems involved in both parts.

Definition 1.1

A *preferential structure* $\mathcal{Z}$ is a pair $< \mathcal{X}, \prec >$, where $\mathcal{X}$ is a set of pairs, and $\prec$ a binary relation on $\mathcal{X}$. We say that $\mathcal{Z}$ is over $\{x : \exists i. < x, i > \in \mathcal{X}\}$.

No postulates about $\prec$ are made - this contrasts with some authors, like Lehmann, who assume e.g. smoothness. We use pairs $< x, i >$, where i is an index. The use of labelling functions is an equivalent technique.

Given such $\mathcal{Z}$ and arbitrary U, we define the $\mathcal{Z}$-minimal elements of U as follows: $\mu_{\mathcal{Z}}(U)$ $:= \{ x \in U : \exists < x, i > \in \mathcal{X}. \neg \exists < x', i' > \in \mathcal{X}. (x' \in U \wedge < x, i > \succ < x', i' >) \}$. Note that U need not be a subset of X, if $\mathcal{Z}$ is over X.

We use sometimes $\mathcal{Z} \lceil U$ for $\{< x, i > \in \mathcal{X} : x \in U\}$.

A preferential structure $\mathcal{Z}$ over X is called a classical preferential model iff X consists of classical models of some fixed language $\mathcal{L}$.

Remark 1.2

One of the referees suggested the following conceptually very interesting reformulation:

A preferential model is a pair $(X, \prec)$, where X is an arbitrary set, and $\prec$ a binary relation over X. Let now any function $\phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ ($\mathcal{P}(X)$ the powerset of X) be given, and $f : X \rightarrow Z$ be an arbitrary function into an arbitrary set Z. We can define $\phi_f : \mathcal{P}(Z) \rightarrow \mathcal{P}(Z)$ by $\phi_f(U) := f[\phi(f^{-1}[U])]$ - where $g[V] := \{g(x) : x \in V\}$. If X is a set of pairs, Z the set of first elements of pairs in X, f the projection to the first coordinate, ϕ the function which chooses in $V \subseteq X$ those pairs $< x, i >$ which are $\prec$-minimal in V, then $\phi_f(U) = \mu_{(X,\prec)}(U)$ for any $U \subseteq Z$. This reformulation gives a clear algebraic picture.

We do not use this idea in the sequel, however, as we will work with pairs in our constructions, and it seems easier to follow the somewhat complicated techniques by using the pair notation immediately.

Definition 1.3

Given a propositional language $\mathcal{L}$, $M_{\mathcal{L}}$ will be the set of its models, and $M(T)$ will be the set of models of an $\mathcal{L}$-theory T. (A theory will be any set of formulas.) $\boldsymbol{D}_{\mathcal{L}}$ will be the set of definable sets of $\mathcal{L}$-models, $\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T$ an $\mathcal{L}$ -theory $\}$. Note that $\boldsymbol{D}_{\mathcal{L}}$ contains $M_{\mathcal{L}}$ and is closed under infinite intersections (and finite unions). Conversely, if $X \subseteq M_{\mathcal{L}}$, $Th(X) := \{\phi : \forall m \in X.m \models \phi\}$. $\vdash$ will denote the classical consequence relation, $\mid\!\sim$ any (other) consequence relation, and $\overline{T} := \{\phi : T \vdash \phi\}$, $\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$. A consequence relation will here be an arbitrary binary relation between sets of formulas on the left, and formulas on the right.

Definition 1.4

A preferential structure $\mathcal{Z}$ is called *definability preserving* (for some fixed language $\mathcal{L}$) iff for all $\mathcal{L}$-theories T $\mu_{\mathcal{Z}}(M(T)) = M(T')$ for some $\mathcal{L}$-theory T'.

Notation 1.5

$\prec^*$ will be the transitive closure of a binary relation $\prec$.

Π will be the cartesian product, and $\mathcal{P}$ the power set operator.

$ran(f)$ will be the range of the function f.

Definition 1.6

Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be given.

(1) For $B \in \mathcal{Y}$, we call $A \subseteq B$ a *small* subset of B (and $B - A$ *dense* in B) iff there is no $X \in \mathcal{Y}$ s.t. $B - A \subseteq X \subsetneqq B$.

(2) If $\mathcal{Y}$ is closed under arbitrary intersections, $Z \in \mathcal{Y}$, and $A \subseteq Z$, $\widetilde{A}$ will be the smallest $X \in \mathcal{Y}$ with $A \subseteq X$.

Intuitively, $\mathcal{Y}$ is $\boldsymbol{D}_{\mathcal{L}}$.

To put our present results in perspective, we now quote some of our earlier results on definability preserving preferential structures (see [8] and [10]). Proposition 1.7 gives an algebraic characterization of preferential structures, which is translated into logic in Proposition 1.8. The latter, however, holds only for definability preserving structures.

Proposition 1.7

Let Z be a set, $\mathcal{Y}$ a subset of $\mathcal{P}(Z)$, μ a function $\mu : \mathcal{Y} \to \mathcal{Y}$.

μ is representable by a transitive preferential structure iff μ satisfies

$(\mu 1)$ $\mu(X) \subseteq X$,

$(\mu 2')$ $X \subseteq Y \to \mu(Y) \cap X \subseteq \mu(X)$.

Proposition 1.8

Let $\mid\sim$ be a logic for $\mathcal{L}$.

Then there is a (transitive) definability preserving classical preferential model $\mathcal{M}$ s.t. $\overline{\overline{T}} = Th(\mu_{\mathcal{M}}(M(T)))$ iff

($\mid\sim$ 1) $\overline{T} = \overline{T'} \rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$,

($\mid\sim$ 2) $\overline{\overline{T}}$ is classically closed,

($\mid\sim$ 3) $T \subseteq \overline{\overline{T}}$,

($\mid\sim$ 4′) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$

for all $T, T' \subseteq \mathcal{L}$.

Condition ($\mu2'$) may be called the central condition of general preferential structures, it is very strong, and expresses the locality (or absoluteness) of choice done in preferential models: If $y \in Y$ is among the best elements of Y, and $X \subseteq Y$, and $y \in X$, then y must also be among the best elements of X. If "best" means that there is no better one, the property is immediate, any $z \in X$ better than y must also be better in Y. If the choice of "best" were made in a "context dependent" manner, this would not necessarily be so. Daniel Lehmann (see [5]) pointed out to me that condition ($\mu2'$) was considered as early as 1954 by authors working on rational choice (and is sometimes called "*coherence*"), see [2], [11], [7].

In the present article, we will use condition

($\mu2$) $\bigcup\{Y' - \mu(Y') : Y' \in \mathcal{Y}, Y' \subseteq Y\} \cap \mu(Y)$ is a small subset of $\mu(Y)$ for $Y \in \mathcal{Y}$.

Any $Y' \subseteq Y$ s.t. $(Y' - \mu(Y')) \cap \mu(Y) \neq \emptyset$ violates in a certain way condition ($\mu2'$) : If $y' \in Y' - \mu(Y')$, then there is $y'' \in Y'$, $y'' \prec y'$, and as $Y' \subseteq Y$, such y' cannot be in $\mu(Y)$. ($\mu2$) says that such exceptions are rare in $\mu(Y)$. For this reason, we cannot expect any $\mu_{\mathcal{Z}}$ to be exactly μ, but only approximately so, and we find $\mathcal{Z}$ s.t. for all $Y \in \mathcal{Y}$ $\underline{\mu(Y) = \mu_{\mathcal{Z}}(Y)}$ - see Proposition 2.1 below.

If we set $\mathcal{Y} := \boldsymbol{D}_{\mathcal{L}}$, this will not be felt on the logics side, as we cannot differentiate between $\mu(Y)$ and $\mu_{\mathcal{Z}}(Y)$ by logical means - see the discussion below - so this approximation disappears in Proposition 3.1.

Remark 1.9

If $\mathcal{Y} = \mathcal{P}(Z)$, then $(\mu 2)$ is equivalent to $(\mu 2')$.

Proof:

If $\mathcal{Y} = \mathcal{P}(Z)$, then $A \subseteq B$ is a small subset of B iff $A = \emptyset$ - this holds even if $B = \emptyset$. $\bigcup\{Y' - \mu(Y') : Y' \in \mathcal{Y}, Y' \subseteq Y\} \cap \mu(Y) = \emptyset$ iff for all $Y' \subseteq Y$ $(Y' - \mu(Y')) \cap \mu(Y) = \emptyset$ iff for all $Y' \subseteq Y$ $\mu(Y) \cap Y' \subseteq \mu(Y')$. $\Box$

The logical condition $(\mid\sim 4')$ is called infinite conditionalization, one half of the deduction theorem is a consequence of it: Set $T' := \{\phi\}$, if $T \cup \{\phi\} \mid\sim \psi$, then by $(\mid\sim 4')$ $\overline{\overline{T}} \cup \{\phi\} \vdash \psi$, so by the deduction theorem of classical propositional logic $\overline{\overline{T}} \vdash \phi \rightarrow \psi$, so $T \mid\sim \phi \rightarrow \psi$ (using deductive closure of $\overline{\overline{T}}$).

If $T' \vdash T$, then $T \cup T' = T'$, so by $(\mid\sim 4')$ $M(\overline{\overline{T'}}) \supseteq M(\overline{\overline{T}} \cup T') = M(\overline{\overline{T}}) \cap M(T')$ - which is the precise translation of $(\mu 2')$. In other words, if $T' \vdash T$, then $(M(T') - M(\overline{\overline{T'}})) \cap M(\overline{\overline{T}}) = \emptyset$.

The logical condition used in this article

$(\mid\sim 4)$ Let T, T_i, $i \in I$ be theories s.t. $\forall i$ $T_i \vdash T$, then there is no ϕ s.t. $\phi \notin \overline{\overline{T}}$ and $M(\overline{\overline{T}} \cup \{\neg\phi\}) \subseteq \bigcup\{M(T_i) - M(\overline{\overline{T_i}}) : i \in I\}$

expresses the same fact approximately - all T' s.t. $T' \vdash T$ and $(M(T') - M(\overline{\overline{T'}})) \cap M(\overline{\overline{T}}) \neq \emptyset$ cannot "nibble away" too much from $M(\overline{\overline{T}})$: it must not be detectable by any formula ϕ. This is a natural condition: Logics (in the infinite case) is a relatively coarse instrument, it does not permit to describe all sets of models, many sets of models are different, yet define the same theory. This gives us room to do (almost) illegal things. Definability preserving preferential models would detect such T' - therefore they are not permitted: It was shown in [8] that the properties used there to characterize definability preserving preferential structures do not necessarily hold in not definability preserving preferential structures (and that there are not definability preserving preferential structures where they hold none the less). On the other hand, if $\mathcal{Y} = \mathcal{P}(Z)$, then we can distinguish everything, and $(\mu 2)$ is equivalent to $(\mu 2')$.

2 THE COMBINATORIAL RESULTS

2.1 The general, not necessarily transitive case

Let Z be an arbitrary set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $\mu : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y}$ closed under arbitrary intersections, and $Z \in \mathcal{Y}$.

Consider the conditions

$(\mu 1)$ $\mu(U) \subseteq U$,

$(\mu 2)$ $\bigcup\{Y' - \mu(Y') : Y' \in \mathcal{Y}, Y' \subseteq U\} \cap \mu(U)$ is a small subset of $\mu(U)$ for $U \in \mathcal{Y}$.

Proposition 2.1

Let Z be an arbitrary set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $\mu : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y}$ closed under arbitrary intersections, and $Z \in \mathcal{Y}$.

μ satisfies $(\mu 1)$ and $(\mu 2)$ iff there is a preferential structure $\mathcal{Z}$ over Z s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \mu_{\widetilde{\mathcal{Z}}}(U)$.

Proof:

"$\rightarrow$"

Definition 2.2

For $x \in Z$, let $\mathcal{Y}_x := \{Y \in \mathcal{Y} : x \in Y - \mu(Y)\}$, and $\Pi_x := \Pi \mathcal{Y}_x$.

Construction 2.3

Let $\mathcal{Z} := < \{< x, f >: x \in Z,\ f \in \Pi_x\},\ \{< x, f > \succ < x', f' >: x' \in ran(f)\} >$

This construction is the basis of Construction 2.5, too. Suppose $x \in X - \mu(X)$. If μ is to be represented by a preferential structure $\mathcal{Z}$, x, (or, more precisely, all copies of x) cannot be minimal in the structure. Thus, for each copy $< x, i >$ of x, there must be some $< x', i' >$ s.t. $x' \in X$, $< x', i' > \prec < x, i >$, i.e. which "kills" $< x, i >$. An element $x \in X$ might be non-minimal in X and X'. At the same time, X might need two elements, say y and y', to kill (all copies of) x, and X' might need two elements, say z and z', to kill x. But we do not know which copy is killed by which elements. If these are all the possibilities to kill the copies of x, any Y which kills x has to contain $\{y, y'\}$ or $\{z, z'\}$. But this is equivalent to the fact that the range of any choice function in the product $\{y, y'\} \times \{z, z'\}$ has non-empty intersection with Y.

Note that for each x there is some $f \in \mathcal{Y}_x$ (by the Axiom of Choice, as the Y's considered are not empty - if $\mathcal{Y}_x = \emptyset$, then $\emptyset \in \Pi_x$).

Let now $U \in \mathcal{Y}$.

(1) $\mu_{\mathcal{Z}}(U) \subseteq \mu(U)$: $\mu_{\mathcal{Z}}(U) \subseteq U$ is trivial. Let now $x \in U - \mu(U)$, then $U \in \mathcal{Y}_x$, so for all $< x, f > \in \mathcal{Z}$ $ran(f) \cap U \neq \emptyset$, so $x \notin \mu_{\mathcal{Z}}(U)$.

(2) $\mu_{\mathcal{Z}}(U)$ is dense in $\mu(U)$: Let $U' := \{x \in \mu(U) : \neg \exists Y \in \mathcal{Y}(Y \subseteq U,\ x \in Y - \mu(Y))\}$. By prerequisite, U' is a dense subset of $\mu(U)$. For $x \in U'$, we can find $f \in \Pi_x$ with $ran(f) \cap U = \emptyset$, so such $< x, f >$ will be minimal in $\mathcal{Z} \lceil U$. (This obviously also holds if $\mathcal{Y}_x = f = \emptyset$.) Thus $U' \subseteq \mu_{\mathcal{Z}}(U)$. As $\mu(U) \in \mathcal{Y}$, U' is dense in $\mu(U)$, and $U' \subseteq \mu_{\mathcal{Z}}(U) \subseteq \mu(U)$, $\mu(U) = \widetilde{\mu_{\mathcal{Z}}(U)}$.

" $\Leftarrow$ "

($\mu 1$) $\mu_{\mathcal{Z}}(U) \subseteq U$, so by $U \in \mathcal{Y}$ $\mu(U) = \widetilde{\mu_{\mathcal{Z}}(U)} \subseteq U$.

($\mu 2$) If ($\mu 2$) is false, there is $U \in \mathcal{Y}$ s.t. $U' := \bigcup \{Y' - \mu(Y') : Y' \in \mathcal{Y}, Y' \subseteq U\} \cap \mu(U)$ is not a small subset of $\mu(U)$. Thus $\widetilde{\mu(U) - U'} \subsetneq \mu(U)$. By $\mu_{\mathcal{Z}}(Y') \subseteq \mu(Y')$, $Y' - \mu(Y') \subseteq Y' - \mu_{\mathcal{Z}}(Y')$. No copy of any $x \in Y' - \mu_{\mathcal{Z}}(Y')$ with $Y' \subseteq U$ can be minimal in $\mathcal{Z} \lceil U$. Thus, by $\mu_{\mathcal{Z}}(U) \subseteq \mu(U)$, $\mu_{\mathcal{Z}}(U) \subseteq \mu(U) - U'$, so $\widetilde{\mu_{\mathcal{Z}}(U)} \subseteq \widetilde{\mu(U) - U'} \subsetneq \mu(U)$, contradiction. $\square$

2.2 The transitive case

We now show that the relation can be chosen transitive, using the techniques developped in [10].

Construction 2.3 cannot be made transitive as it is, see [10] for details. The following construction avoids a certain excess in the relation $\prec$ of Construction 2.3: There, too many elements $< y, g >$ are smaller than some $< x, f >$, as the relation is independent of g. This excess prevents transitivity, as we argue now.

Transitivity permits substitution in the following sense: If (the two copies of) x is killed by y_1 and y_2 together, and y_1 is killed by z_1 and z_2 together, then x should be killed by z_1, z_2, and y_2 together.

But the old construction substitutes too much: In the old construction, we considered elements $< x, f >$, where $f \in \Pi_x$, with $< y, g > \prec < x, f >$ iff $y \in ran(f)$, independent of g. The construction can, in general, not be made transitive.

The new construction avoids this, as it "looks ahead", and not all elements $< y_1, t_{y_1} >$ are smaller than $< x, t_x >$, where y_1 is a child of x in t_x (or $y_1 \in ran(f)$). This "looking ahead" is coded by trees. The new construction is basically the same as Construction 2.3, but avoids to make too many copies smaller than the copy to be killed.

We need no new properties of μ to achieve transitivity here, as a killed element x might (partially) "commit suicide", i.e. for some $i, i' < x, i > \prec < x, i' >$, so we cannot substitute x by any set which does not contain x : In this simple situation, if $x \in X - \mu(X)$, we cannot find out whether all copies of x are killed by some $y \neq x$, $y \in X$. We can assume without loss of generality that there is an infinite descending chain of x-copies, which are not killed by other elements. Thus, we cannot replace any y_i as above by any set which does not contain y_i, but then substitution becomes trivial, as any set substituting y_i has to contain y_i. Thus, we need no new properties to achieve transitivity.

Proposition 2.4

Let Z be an arbitrary set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $\mu : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y}$ closed under arbitrary intersections, and $Z \in \mathcal{Y}$.

μ satisfies $(\mu 1)$ and $(\widetilde{\mu 2})$ iff there is a transitive preferential structure $\mathcal{Z}$ over Z s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \mu_{\mathcal{Z}}(U)$.

Proof:

The easy direction follows from the easy direction in Proposition 2.1.

We turn to the other direction. The preferential structure is defined in Construction 2.5, Claim 2.7 shows representation for the simple structure, Claim 2.8 representation for the transitive closure of the structure.

Construction 2.5

(1) For $x \in Z$, let T_x be the set of trees t_x s.t. (a) all nodes are elements of Z, (b) the root of t_x is x, (c) $height(t_x) \leq \omega$, (d) if y is an element in t_x, then there is $f \in \Pi_y$ s.t. the set of children of y is $ran(f)$.

(2) For $x, y \in Z$, $t_x \in T_x$, $t_y \in T_y$, set $t_x \rhd t_y$ iff y is a (direct) child of the root x in t_x, and t_y is the subtree of t_x beginning at y.

(3) Let $\mathcal{Z} := < \{< x, t_x >: x \in Z,\, t_x \in T_x\}$, $\{< x, t_x > \succ < y, t_y >: t_x \rhd t_y\} >$.

Fact 2.6

(1) The construction ends at some y iff $\mathcal{Y}_y = \emptyset$, consequently $T_x = \{x\}$ iff $\mathcal{Y}_x = \emptyset$. (We identify the tree of height 1 with its root.)

(2) If $\mathcal{Y}_x \neq \emptyset$, tc_x, the totally ordered tree of height ω, branching with $card = 1$, and with all elements equal to x is an element of T_x. Thus, with (1), $T_x \neq \emptyset$ for any x.

(3) If $f \in \Pi_x$, $f \neq \emptyset$, then the tree tf_x with root x and otherwise composed of the subtrees t_y for $y \in ran(f)$, where $t_y := y$ iff $\mathcal{Y}_y = \emptyset$, and $t_y := tc_y$ iff $\mathcal{Y}_y \neq \emptyset$, is an element of T_x. (Level 0 of tf_x has x as element, the $t'_y s$ begin at level 1.)

(4) If y is an element in t_x and t_y the subtree of t_x starting at y, then $t_y \in T_y$.

(5) $< x, t_x > \succ < y, t_y >$ implies $y \in ran(f)$ for some $f \in \Pi_x$. $\Box$

Claim 2.7

$\forall U \in \mathcal{Y}.\mu(U) = \widetilde{\mu_{\mathcal{Z}}(U)}$.

Proof:

(1) $\mu_{\mathcal{Z}}(U) \subseteq \mu(U)$: $\mu_{\mathcal{Z}}(U) \subseteq U$ is trivial. $x \in \mu_{\mathcal{Z}}(U) \rightarrow$ there is $< x, t_x >$ minimal in $\mathcal{Z}\lceil U$, thus $x \in U$ and there is no $< y, t_y > \in \mathcal{Z}$, $< y, t_y > \prec < x, t_x >$, $y \in U$. Let f define the set of children of the root x in t_x. If $ran(f) \cap U \neq \emptyset$, if $y \in U$ is a child of x in t_x, and if t_y is the subtree of t_x starting at y, then $t_y \in T_y$ and $< y, t_y > \prec < x, t_x >$, so $< x, t_x >$ is not minimal in $\mathcal{Z}\lceil U$. But if $x \in U - \mu(U)$, then $U \in \mathcal{Y}_x$, so $ran(f) \cap U \neq \emptyset$.

(2) $\mu_{\mathcal{Z}}(U)$ is dense in $\mu(U)$: Let $U' := \{x \in \mu(U) : \neg\exists Y \in \mathcal{Y}(Y \subseteq U,\ x \in Y - \mu(Y))\}$. By prerequisite, U' is a dense subset of $\mu(U)$. Let $x \in U'$, then we can find $f \in \Pi_x$ with $ran(f) \cap U = \emptyset$. If $\mathcal{Y}_x = \emptyset$, then the tree x has no $\rhd$-successors, and $< x, x >$ is $\succ$-minimal in $\mathcal{Z}$. If $\mathcal{Y}_x \neq \emptyset$ and $f \in \Pi_x$ s.t. $ran(f) \cap U = \emptyset$, then $< x, tf_x >$ is $\succ$-minimal in $\mathcal{Z}\lceil U$. As $\mu(U) \in \mathcal{Y}$, U' is dense in $\mu(U)$, and $U' \subseteq \mu_{\mathcal{Z}}(U) \subseteq \mu(U)$, $\mu(U) = \widetilde{\mu_{\mathcal{Z}}(U)}$. $\Box$

We consider now the transitive closure of $\mathcal{Z}$.

Claim 2.8

Let $\mathcal{Z}' := < \{< x, t_x >: x \in Z,\ t_x \in T_x\}, < x, t_x > \succ' < y, t_y >$ iff $t_x \rhd^* t_y >$. Then $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$.

Proof:

Suppose there is $U \in \mathcal{Y}$, $x \in U$, $x \in \mu_{\mathcal{Z}}(U)$, $x \notin \mu_{\mathcal{Z}'}(U)$. Then there must be an element $< x, t_x > \in \mathcal{Z}$ with no $< x, t_x > \succ < y, t_y >$ for any $y \in U$. Let $f \in \Pi_x$ determine the set of children of x in t_x, then $ran(f) \cap U = \emptyset$, and consider tf_x. As all elements $\neq x$ of tf_x are already in $ran(f)$, no element of tf_x is in U. Thus there is no $< z, t_z > \prec^* < x, tf_x >$ in $\mathcal{Z}$ with $z \in U$, so $< x, tf_x >$ is minimal in $\mathcal{Z}'\lceil U$, contradiction.

$\Box$ (Claim 2.8 and Proposition 2.4)

Note that the relations can be made acyclic, as was shown in [8], preserving transitivity.

By the way, reading above results (or those of [10]) differently shows the following.

Corollary 2.9

Let $\mathcal{Z}$ be any preferential structure over Z. Then there is a transitive structure $\mathcal{Z}'$ s.t. for all U $\mu_{\mathcal{Z}}(U) = \mu_{\mathcal{Z}'}(U)$.

Proof:

$\mathcal{Z}$ defines $\mu_{\mathcal{Z}}(U)$ for all $U \in \mathcal{P}(Z)$ (even for all sets U, but knowing $\mu_{\mathcal{Z}}(U)$ for all $U \subseteq Z$ suffices). By Proposition 2.1, $(\mu 1)$ and $(\mu 2)$ hold for $\mu_{\mathcal{Z}}$ and $\mathcal{P}(Z)$. By Proposition 2.4, there is transitive $\mathcal{Z}'$ representing $\mu_{\mathcal{Z}}$, i.e. $\mu_{\mathcal{Z}}(U) = \widetilde{\mu_{\mathcal{Z}'}}(U) = \mu_{\mathcal{Z}'}(U)$ (the latter as $\mathcal{Y} = \mathcal{P}(Z)$) for all $U \in \mathcal{P}(Z)$. $\Box$

It is natural to try now to characterize unrestricted smooth preferential structures. We do not have a solution, and it might be difficult to find a nice one. The techniques of [9] and [10] show in principle a way how to construct such structures. But, and this may be an inherent problem, in the construction one has to work with $\mu(X)$, and not with X itself - but $\mu(X)$ is only "roughly" known.

3 THE LOGICAL RESULTS

We work in a fixed propositional language $\mathcal{L}$, T etc. will be $\mathcal{L}$-theories, ϕ etc. will be $\mathcal{L}$-formulas.

Consider the conditions

$(\mid\sim 1)$ $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$,

$(\mid\sim 2)$ $\overline{\overline{T}}$ is classically closed,

$(\mid\sim 3)$ $T \subseteq \overline{\overline{T}}$,

$(\mid\sim 4)$ Let T, T_i, $i \in I$ be theories s.t. $\forall i$ $T_i \vdash T$, then there is no ϕ s.t. $\phi \notin \overline{\overline{T}}$ and $M(\overline{\overline{T}} \cup \{\neg\phi\}) \subseteq \bigcup\{M(T_i) - M(\overline{\overline{T_i}}) : i \in I\}$

for all T, T', T_i.

Proposition 3.1

Let $\mid\sim$ be a logic for $\mathcal{L}$.

Then there is a transitive classical preferential model $\mathcal{M}$ s.t. $\overline{\overline{T}} = Th(\mu_{\mathcal{M}}(M(T)))$ for all T iff $(\mid\sim 1) - (\mid\sim 4)$ hold.

The proof is an easy consequence of Propositions 2.4 and 3.2 and will be shown after the proof of the latter. Proposition 3.2 and its proof are largely analogous to results shown e.g. in [8].

Proposition 3.2

(a) If $\mu : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ satisfies $(\mu 1) - (\mu 2)$ (for $\mathcal{Y} = \boldsymbol{D}_{\mathcal{L}}$), then $\mid\sim$ defined by $\overline{\overline{T}} := Th(\mu(M(T)))$ satisfies $(\mid\sim 1) - (\mid\sim 4)$.

(b) If $\mid\sim$ satisfies $(\mid\sim 1) - (\mid\sim 4)$, then there is $\mu : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ s.t. $\overline{\overline{T}} = Th(\mu(M(T)))$ for all T and μ satisfies $(\mu 1) - (\mu 2)$ (for $\mathcal{Y} = \boldsymbol{D}_{\mathcal{L}}$).

Proof:

(a)

Suppose $\overline{\overline{T}} = Th(\mu(M(T)))$ for some such μ, and all T. Thus $\mu(M(T)) = M(\overline{\overline{T}})$. $(\mid\sim 1)$: If $\overline{T} = \overline{T'}$, then $M(T) = M(T')$, so $\mu(M(T)) = \mu(M(T'))$, and $\overline{\overline{T}} = \overline{\overline{T'}}$. $(\mid\sim 2)$ is trivial by definition, and $(\mid\sim 3)$ is trivial by $\mu(U) \subseteq U$. We show $(\mid\sim 4)$: Suppose there were theories T, T_i, $i \in I$ s.t. $\forall i\ T_i \vdash T$, and some ϕ s.t. $\phi \notin \overline{\overline{T}}$ and $M(\overline{\overline{T}} \cup \{\neg\phi\}) \subseteq X := \bigcup\{M(T_i) - M(\overline{\overline{T_i}}) : i \in I\}$. Then $M(\overline{\overline{T}}) - X \subseteq M(\overline{\overline{T}} \cup \{\phi\}) \overset{\subseteq}{\neq} M(\overline{\overline{T}})$, but by $\mu(M(S)) = M(\overline{\overline{S}})$ $\mu(M(T)) - \bigcup\{M(T') - \mu(M(T')) : T' \vdash T\} \subseteq M(\overline{\overline{T}}) - \bigcup\{M(T') - M(\overline{\overline{T'}}) : T' \vdash T\}$ $\subseteq M(\overline{\overline{T}}) - X$, contradicting $(\mu 2)$.

(b)

Let $\mid\sim$ satisfy $(\mid\sim 1) - (\mid\sim 4)$. We define $\mu : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$, show $\overline{\overline{T}} = Th(\mu(M(T)))$, and that $(\mu 1)$ and $(\mu 2)$ hold. Set $\mu(M(T)) := M(\overline{\overline{T}})$ for any T. If $M(T) = M(T')$, then $\overline{T} = \overline{T'}$, thus $\overline{\overline{T}} = \overline{\overline{T'}}$ by $(\mid\sim 1)$, so $M(\overline{\overline{T}}) = M(\overline{\overline{T'}})$, and μ is well-defined. Moreover, by $(\mid\sim 3)$, $\mu(U) \subseteq U$. We now show $\overline{\overline{T}} = Th(\mu(M(T)))$. $\phi \in Th(\mu(M(T))) :\leftrightarrow \forall m \in \mu(M(T)).m \models \phi \leftrightarrow \forall m \in M(\overline{\overline{T}}).m \models \phi \leftrightarrow \overline{\overline{T}} \vdash \phi \leftrightarrow \phi \in \overline{\overline{T}}$ (as $\overline{\overline{T}}$ is classically closed). It remains to show that μ satisfies $(\mu 2)$: Suppose there were $U \in \boldsymbol{D}_{\mathcal{L}}$ s.t. $\bigcup\{Y' - \mu(Y') : Y' \in \boldsymbol{D}_{\mathcal{L}}, Y' \subseteq U\} \cap \mu(U)$ is not a small (in $\boldsymbol{D}_{\mathcal{L}}$) subset of $\mu(U)$. Let $U = M(T)$, $Y' = M(T')$. So $\mu(U) = M(\overline{\overline{T}})$. Then there must be some $X \in \boldsymbol{D}_{\mathcal{L}}$ with $\mu(U) - \bigcup\{Y' - \mu(Y') : Y' \in \boldsymbol{D}_{\mathcal{L}}, Y' \subseteq U\} \subseteq X \overset{\subseteq}{\neq} \mu(U)$. So there must be some $\phi \notin \overline{\overline{T}}$ with $X \subseteq M(\overline{\overline{T}} \cup \{\phi\})$, so $M(\overline{\overline{T}} \cup \{\neg\phi\}) \subseteq \bigcup\{M(T') - M(\overline{\overline{T'}}) : T' \vdash T\}$, contradicting $(\mid\sim 4)$. □

Proof of Proposition 3.1:

"$\rightarrow$"

Let for some transitive classical preferential model $\mathcal{M}$ $\overline{\overline{T}} = Th(\mu_{\mathcal{M}}(M(T)))$ for all T. For the function $\mu : \boldsymbol{D}_{\mathcal{L}} \rightarrow \boldsymbol{D}_{\mathcal{L}}$ defined by $\mu(M(T)) := M(\overline{\overline{T}})$ $\mu_{\mathcal{M}}(\widetilde{M(T)}) = \mu(M(T))$ (in $\boldsymbol{D}_{\mathcal{L}}$) holds, so by Proposition 2.4 μ satisfies ($\mu 1$), ($\mu 2$). Thus, by Proposition 3.2, $\mid\sim'$ defined by $\overline{\overline{T}}' := Th(\mu(M(T)))$ satisfies ($\mid\sim$ 1) – ($\mid\sim$ 4), but $\overline{\overline{T}}' = Th(\mu(M(T))) = Th(M(\overline{\overline{T}})) = \overline{\overline{T}}$.

" $\leftarrow$ "

Let $\mid\sim$ satisfy ($\mid\sim$ 1) – ($\mid\sim$ 4). By Proposition 3.2, there is $\mu : \boldsymbol{D}_{\mathcal{L}} \rightarrow \boldsymbol{D}_{\mathcal{L}}$ s.t. $\overline{\overline{T}} = Th(\mu(M(T)))$ and μ satisfies ($\mu 1$), ($\mu 2$). So by Proposition 2.4, there is a transitive classical preferential model $\mathcal{M}$ s.t. $\mu(M(T)) = \mu_{\mathcal{M}}(\widetilde{M(T)})$ (in $\boldsymbol{D}_{\mathcal{L}}$), so $\overline{\overline{T}} = Th(\mu(M(T))) = Th(\mu_{\mathcal{M}}(M(T)))$. □

4 CONCLUSION

We have characterized unrestricted preferential structures, i.e. preferential structures without any definability condition. The case of smooth unrestricted preferential structures remains open - see the discussion at the end of Section 2. It may also be interesting to try to apply our techniques to other characterzation results in the infinite case appealing to definability preservation, like those in [6].

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