A TWO-STAGE APPROACH TO FIRST ORDER DEFAULT REASONING

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Abstract

Our subject is the representation and analysis of simple first-order default statements of ordinary language, such as "normally, birds fly". There are, among other approaches, two kinds of analysis, both semantic in style. One interprets "normally, birds fly" along the lines of "for every item x in the domain of discourse, the most normal models of "x is a bird" are models of "x flies". This is the *preferential models* approach, first outlined by Bossu/Siegel and Shoham, and studied by Kraus, Lehmann, Magidor and others. The other interprets "normally, birds fly" along the lines of "there is an important subset of the birds, all of whose elements fly". This is the *generalized quantifier* approach, formulated and developed by the author. The purpose of the present paper is to show how the two approaches may usefully be combined into a single two-stage approach, and how such a combination provides an elegant account of certain problematic examples.

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1 INTRODUCTION

1.1 Overview

Reiter's approach to open normal defaults, via Skolem constants (see [Rei80]) was criticized by F.Baader and B.Hollunder in [BH92] (see also [BS93]) for its sensitivity to syntax. We follow the terminology of [Rei80], merely reminding the reader that open normal defaults without prerequisites have the form $\frac{:\phi(x)}{\phi(x)}$, whilst those with prerequisites have the form $\frac{\psi(x):\phi(x)}{\phi(x)}$ $. In\ [BS93],\ we\ have\ suggested\ a$ 'limit preferential approach' (as discussed already in [Lew73], and introduced into the context of nonmonotonic reasoning by G.Bossu and P.Siegel in [BS85]), which, because of its semantical basis, avoids the problem of syntax dependence. It seems to capture well the "dynamic" or "inference-greedy" aspect of defaults, to make true as many instances of the defaults as possible.
In [Sch95-1], we have taken a totally different approach in order to capture the "static" or "normality" aspect of defaults. There, a default expresses a fact about the world rather than a strategy of the reasoner. Roughly, a default "normally, birds fly", expresses something like "most" birds fly, or the "interesting" birds do. If you say that normally birds fly, you have to be able to show that a substantial subset of the set of birds must be able to fly. If we accept that these are minimal conditions justifying a sentence like "normally $\phi(x)''$, then we conclude that "normally $\phi(x)''$ and "normally $\neg\phi(x)''$ are impossible - on the premise that two substantial subsets (of one base set) must have a non-empty intersection. In our opinion, the minimal condition that $\phi(x)$ has to hold for a substantial subset to justify the sentence "normally $\phi(x)''$, leads naturally to a formalization of "normally" via "important" or "large" subsets: "normally $\phi(x)''$ holds iff $\phi(x)$ holds for all x in some important subset of the set under consideration.
In [Sch95-1] we interpreted open normal defaults as generalized quantifiers, between the classical existential and universal quantifier in strength. More precisely, we introduced a new quantifier "almost all" into first-order logic (FOL), interpreted semantically by "large" subsets. Large subsets were characterized by a weak filter, added to a standard first-order logic model, forming a more complex structure. A sound and complete axiomatization for our semantics was given, extending classical FOL.

This approach had the advantage of giving not only a notion of consistency of default theories - e.g. $\{\frac{:\phi(x)}{\phi(x)}, \frac{:\neg\phi(x)}{\neg\phi(x)}\}$ was ruled out as inconsistent - but also of allowing the use of negated and nested defaults, giving such formulas a clear meaning. Yet, it treated only the minimal, static aspect of defaults, not the dynamic, inference-greedy one.
Considering defaults as inference rules has a dynamic aspect, an "inference pressure", which the formalism of [Sch95-1] does not have: Apply the default as much as you can!
In this article, we marry the approaches in a two-stage process, starting with the static aspect. On the basis of this, we make as many instances true as possible. This is expressed by a preference relation on the extended FOL models of the default theory. Given a theory of classical and default information, we consider the set of extended FOL models of that theory - if it is consistent under our logic. We choose the "good" models among those by a preference relation, which reflects the "inference-greedy" aspect: as many instances as possible will be made to hold.
The fact that we choose only among the extended FOL models guarantees that the minimal, static requirements are met, i.e. that the default theory "means something objectively", that really the default properties hold for a "large" subset. The preference relation makes the properties hold for as many elements as possible. For example, for the theory $\{\frac{:\phi(x)}{\phi(x)}\}$ $\phi(x)$ will not only be true for a large subset - sufficient for the interpretation as generalized quantifiers - but for all elements of the universe, as there is nothing which contradicts this extension. $\{\frac{:\phi(x)}{\phi(x)}, \frac{:\neg\phi(x)}{\neg\phi(x)}\}$, however, will be ruled out immediately as inconsistent.
Examples and general considerations lead us to take the limit preferential structure approach for the preference relation. In section 3, we give a restricted completeness result for such structures, passing through an algebraic characterization.
In section 2.5, we present several examples, and show that this approach not only captures some abstract intuitions about defaults, but can solve some concrete and interesting cases in an intuitively appealing manner as well.
To our knowledge, this two-stage approach, covering both the static and dynamic aspects of defaults, is new, and so is the restricted completeness result for limit preferential structures. (The results by Boutelier ([Bou90a], [Bou90b], [Bou92]) have some other limitations, see section 1.2 for details.)
Before going into details, we discuss in the rest of the Introduction preferential structures and our interpretation of defaults as generalized quantifiers. The formal definitions and properties will be given in sections 2.1 and 2.2. Most of this material has been published in [Sch95-1]. A more detailed presentation of the two-stage approach is given in section 1.4, it is formalized in section 2.3, with its logical properties discussed in section 2.4.

1.2 Recalling Preferential Structures

The Intuitive Background:
The basic idea is to interpret a primitive notion of "importance" or "value", introduced

into a given language and logic, by a function which chooses a subset of "important" models of a theory or formula of that language.
In other words, we work on a set of "possible worlds", i.e. models of the underlying base logic, but do not accord the same importance or value to all such models. Given a theory T of the base language and logic, we determine the semantical consequences of T in a structure $\mathcal{M}$ by considering only the set of "important" models of $T : T \models_{\mathcal{M}} \phi$ iff ϕ holds in all important models of T in our structure. More formally, such a structure $\mathcal{M}$ will then consist of a set M of models or possible worlds for the base logic, and a choice function f on $\mathcal{P}(M)$ - the power set of M. If $M(T)$ is the set of all base models of a theory T in $\mathcal{M}$, f singles out the set $f(M(T)) \subseteq M(T)$ of important models of T in the structure $\mathcal{M}$. We thus define $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$. (We simplify here for the moment. Later, we shall consider a more general case, where logically equivalent models may occur several times.)
A refinement of the idea is to work not with one subset of "maximally important" models, but with many subsets of important models, perhaps of increasing importance. We then have several choice functions f_i in the structure $\mathcal{M}$, and we define $T \models_{\mathcal{M}} \phi$ iff there is some f_i such that ϕ holds in all $m \in f_i(M(T))$. This structure captures the intuition that we may not dispose of ideal models, but can approximate them in the limit: Suppose for simplicity $f_i(M(T)) \supseteq f_{i+1}(M(T))$ for all $i \in \omega$, with $f_i(M(T))$ the set of T-models of importance i. Then, even if each $f_i(M(T))$ is non-empty, $\bigcap\{f_i(M(T)) : i \in \omega\}$ may be empty. In that case, there are no maximally important models, but we have non-empty sets of ever-better ones, which approximate the ideal. I shall call this approach the limit case, and the previous variant, for historical reasons, the minimal case.
Already this very abstract description makes it plausible that representation theorems for the limit case are harder to obtain than for the minimal case. In the limit case, we have to handle a possibly infinite set of choice functions, and there need not be a global f such that for all ϕ $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$. In other words, we do not always have a set of "joint witnesses" for all consequences of a theory, a set of models where all consequences hold.

Logical Consequences:

It is evident that such consequence relations will be well-behaved with respect to the base logic - provided the latter is sound and complete for the models we have chosen as possible worlds. So, if T and T' are equivalent with respect to the base logic, they will have the same set of semantic consequences, and, if $T \models_{\mathcal{M}} \phi$, and ϕ implies ψ in the base logic, then also $T \models_{\mathcal{M}} \psi$. Moreover, if ϕ is a consequence of T in the base logic, then $T \models_{\mathcal{M}} \phi$, as the choice functions will choose a subset of $M(T)$. These facts hold in both the limit and the minimal version.

Preferential Structures:

Preferential structures are a special case of the above, the choice is made *locally* by a binary relation $\prec$ on the set M of base models, where $m \prec m'$ iff m is considered to be more important than m'. They are thus very similar to Kripke structures, but the relation $\prec$ will be used differently.
In the minimal case, we define f by means of $\prec$ by $f(A) := \{a \in A : \neg\exists b \in A.b \prec a\}$.
In the limit case, the natural definition is to consider initial segments of A $(A \neq \emptyset)$: $\delta_A \subseteq A$ is called an initial segment of A iff
$(\delta 1)$ there is some $b \in \delta_A$ below each $a \in A$: $\forall a \in A \exists b \in \delta_A(b = a \text{ or } b \prec a)$,
$(\delta 2)$ δ_A is downward closed: $\forall a \in A \forall b \in \delta_A(a \prec b \to a \in \delta_A)$.
Each f_i corresponds then to the choice of one such δ_A for each $A \subseteq M$.
We thus have in the minimal case $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in \mu(T)$ - the set of $\prec -minimal$ models of T in $\mathcal{M}$. If, for instance, $M(T)$ consists of infinite descending chains, then $\mu(T) = \emptyset$, and $T \models_{\mathcal{M}} \phi$ for any ϕ, $\bot$ $(= false)$ included. On the other hand, any $m \in \mu(T)$ will be a "witness" of *all* $\models_{\mathcal{M}} -consequences$ of T, in the sense that all ϕ with $T \models_{\mathcal{M}} \phi$ will hold in any such m.
In the limit case, we have $T \models_{\mathcal{M}} \phi$ iff there is some $\delta_{T,\phi} \subseteq M(T)$ which satisfies $(\delta 1)$ and $(\delta 2)$ with respect to $M(T)$ and such that ϕ holds in all $m \in \delta_{T,\phi}$. Thus, in the limit case, $\mu(T)$ may be empty, but if $M(T) \neq \emptyset$, we will still not have $T \models_{\mathcal{M}} \bot$, as all $\delta_{T,\phi}$ are then non-empty. It is easily seen, that if $T \models_{\mathcal{M}} \phi$ and $T \models_{\mathcal{M}} \phi'$, and $\prec$ is transitive, then also $T \models_{\mathcal{M}} \phi \wedge \phi'$: if $\delta_{T,\phi}$ and $\delta_{T,\phi'}$ are suitable, then $\delta_{T,\phi} \cap \delta_{T,\phi'}$ will be a suitable $\delta_{T,\phi\wedge\phi'}$. Moreover, if $T \models_{\mathcal{M}} \phi$, and $M(T \cup \{\phi\}) \subseteq M(T') \subseteq M(T)$, then also $T' \models_{\mathcal{M}} \phi$.
An immediate consequence of the locality of the definition of f is a kind of upward absoluteness in the minimal case. An element, which is not minimal in A, *can't* be minimal in any B with $A \subseteq B$:
(1) $A \subseteq B \to f(B) \cap A \subseteq f(A)$.
In contrast, in the general case of arbitrary f, there need not be any interdependence between $f(A)$ and $f(B)$, even if $A \subseteq B$.
It turns out that (1) is *the* crucial property for minimal preferential structures. Any choice function which obeys (1) and the trivial property
(0) $f(A) \subseteq A$
can be represented by a preferential structure, i.e. by such a binary relation of preference (see [Sch92], Proposition 3.3). This is a very general "algebraic" characterization, the underlying set M need not consist of models, it may be just any arbitrary set.
A similar result for limit preferential structures seems to be missing up to now, see [Bou90a], [Bou90b], [Bou92] and section 3 for restricted cases. Boutilier's results are restricted in the sense that they treat finitely axiomatisable theories only, but such theories correspond exactly to clopen (= *closed* and open) sets in the standard topology (see below in this section). Yet clopen sets can neither be entered nor left by approximation: If x_i is a sequence of elements in a clopen set, then its limit (if it has any) is in X, as X is closed. If x_i is a sequence of elements not in X, then its limit is outside X, as the complement of X is closed. Thus, this restriction seems to go somewhat against the spirit

of the limit approach. (See below for details.) In the end, one might also criticize the fact that Boutilier lets the modal operators $\Diamond$ and $\Box$ do all the "nasty" work, which turns out so unpleasant in an attempt of a direct construction: From the semantical point of view, $\Diamond$ and $\Box$ are quantifiers over possible worlds which cooperate with the relation $\prec$, and it is precisely the interplay of both quantifiers which presents difficulties. But, it is always easy to criticize in hindsight ....

Interpretation:

We have so far deliberately left open the base logic and its models in M, as well as the intuition behind the "importance" of models of the base logic.

Non-monotonic Logic:

This "importance" may be read as "normality" in the case of non-monotonic logics: We are primarily interested in reasoning about the normal cases, and the preferred models are the most normal ones - where birds can fly, houses have doors etc.

As a matter of fact, preferential structures in their various forms provide an important and relatively well-studied kind of semantics for non-monotonic logics. They have been a powerful tool for investigation, providing - via additional properties of the relation $\prec$ - a technique for constructing semantics of logical systems of different strengths. Limit preferential structures for non-monotonic logics were introduced by G.Bossu and P.Siegel in [BS85]. The minimal case was first examined by Y.Shoham ([Sho87]) as a generalization of the minimal model Semantics for Circumscription. More or less general cases of preferential structures are characterized by soundness and completeness theorems in [KLM90], [LM92], [Sch92], [Sch96-1] for the minimal case, in [Bou90a], [Bou90b], [Bou92], and in this paper for the limit case. For an overview, see also [Mak94].

Deontic Logic:

Deontic logic reasons about the normatively acceptable situations, and about what ought to be done (by humans, robots etc.). Reasoning about normatively acceptable actions can be split into two subquestions: Reasoning about the normatively acceptable states, and reasoning about the problem of acting in a way conducive to those states. The latter question can be considered separately, at least in a first approximation.

In this framework, the preferred or more important models are those which are normatively more acceptable. Thus, preferential structures also provide a natural semantics for deontic logic, and, in fact, were examined as such before the advent of non-monotonic logics [Han69]. This was pointed out by D.Makinson in [Mak93].

In hindsight, it is no surprise that a local preference by a binary relation tends to emerge, when we examine choice functions which single out some states as more important or interesting than others. Such local preferences seem to correspond well to intuitions, and simplify the situation by making the choice context-independent.

In [Mak93], still other natural applications of preferential structures are discussed.

An Example:

Before we proceed, we give a simple example which shows that the relation $\models_{\mathcal{M}}$ defined by a preferential structure may indeed be a non-monotonic consequence relation.
Let $\mathcal{L}$ be the propositional language with two varibles p,q, let M consist of two (classical) models, $m \models p \wedge q$, $m' \models \neg p \wedge \neg q$, and let $m' \prec m$. Then $\emptyset \models_{\mathcal{M}} \neg q$, but $p \models_{\mathcal{M}} q$ in both the minimal and the limit definition.
As is the case already in our example, not all classical models for a given language $\mathcal{L}$ need occur in the base set M of a preferential structure $\mathcal{M}$ (e.g., in our example, an m'' with $m'' \models p \wedge \neg q$ is missing). Moreover, some classical models might occur several times, even infinitely often. Take for example $\mathcal{L}$ with one propositional variable p and consider the structure $\mathcal{M} :=< \{< m, i >: i < \omega\}, \prec>$, m a classical model with $m \models p$, and $< m, i > \prec < m, j >$ iff $j < i$. By abuse of language, we shall also write $< m, i > \models p$, etc. Then $\mu(M) = \emptyset$, so $true \models_{\mathcal{M}} \bot$ in the minimal reading, but $true \models_{\mathcal{M}} \phi$ iff ϕ is a classical consequence of p, in the limit reading. More details and examples of logics which require several copies of classical models to be representable by preferential models can be found in [Sch96-1].

Strengthenings of the Conditions for the Relation $\prec$:
Various additional conditions for the relation $\prec$ have been introduced and examined for minimal preferential structures.
The most natural one is perhaps transitivity.
An important condition, which results in nice properties of the semantic consequence relation $\models_{\mathcal{M}}$ is smoothness (terminology of D.Lehmann and his co-authors) or stopperedness (terminology of D.Makinson): Given a theory T, and a non-minimal model m of T, there is $m' \prec m$, which is a minimal model of T. Consequently, if $M(T) \neq \emptyset$, then $\mu(T) \neq \emptyset$. "Smoothness" or "stopperedness" can be violated in essentially two kinds of situation. First, suppose X consists of an infinite descending chain of elements x_i. Then, no $x_i \in X$ is minimal in X, and no x_i has a minimal $y \in X$ below it. Second, suppose that $X := \{x, y, z\}$, with $x \prec y \prec z$, but not $x \prec z$. Then z is not minimal in X, and there is no $a \in X$ below z which is minimal in X.
The counterpart for the consequence relation $\models_{\mathcal{M}}$ is cumulativity (see [KLM90] and [Gab85]) which says that two theories T, T' with $T \subseteq T' \subseteq \{\phi : T \models_{\mathcal{M}} \phi\}$ have the same consequences: $T \models_{\mathcal{M}} \phi$ iff $T' \models_{\mathcal{M}} \phi$. We may read this as "normal use of lemmas": If we have already deduced the "Lemma" ϕ from T, we neither lose nor win in terms of possible deductions by starting from $T \cup \{\phi\}$.
As a matter af fact, again a very general algebraic representation result can be obtained: A choice function f can be represented by a smooth minimal preferential structure iff it satisfies the conditions (0), (1) and
(2) $f(A) \subseteq B \subseteq A \rightarrow f(A) = f(B)$
and if its domain satisfies closure under finite intersections and unions (see [Sch96-1], Theorem 1).

Another strengthening of $\prec$ is rankedness, which may be seen as the existence of a "rotating scale with fixed origin": $\prec$ is called ranked (on M), iff there is an order-preserving (in both directions) function $f : (M, \prec) \rightarrow (X, \prec \bullet)$, where $\prec \bullet$ is a total order on X. Then two $\prec$ $-$*incomparable* elements $m, m' \in M$ behave exactly the same way with respect to $\prec$: $n \prec m$ iff $n \prec m'$, and $m \prec n$ iff $m' \prec n$. The corresponding property of $\models_{\mathcal{M}}$ is Rational Monotony: If $\alpha \models_{\mathcal{M}} \gamma$, then $\alpha \wedge \beta \models_{\mathcal{M}} \gamma$ or $\alpha \models_{\mathcal{M}} \neg\beta$ (see [LM92]). General representation results are again to be found in [Sch96-1].

Generalizations:

We can consider choice functions on arbitrary sets, which need not be sets of models.

We have already seen above two characterizations of such functions defined by minimal preferential structures.

The strength of these algebraic representation results lies in their generality. We can more or less easily obtain soundness and completeness results as corollaries for non-monotonic logics with classical propositional logic as background, see [Sch92], [Sch96-1], and section 3. But, these results can also be read with classical predicate logic in the background, giving representation theorems for preferential structures over models for predicate logic. Using these algebraic representation results, we have also obtained completeness and incompleteness results for Plausibility Logic. This is a sequent calculus for a very poor language without connectives, introduced by D.Lehmann, see [Leh92a], [Leh92b], [Sch96-3].

The Stalnaker/Lewis semantics for counterfactual conditionals presents essentially the same algebraic ideas as preferential structures. Thus our algebraic representation results are easily adapted to counterfactual conditionals in [Sch96-4].

But these representation results (or at least their ideas) can be used in still more general situations, where we do not compare single models, but whole "threads" of development in dynamic situations. This is done in [Sch95-3], where we consider preferences on developments.

In the first part (Theorem 2.8 there), we show that a deontic choice function of "good" developments defined in [Tho84] can be represented by a ranked, stoppered relation on all developments. Thus, we do not compare single models, but developments in a branching time structure. Again, we leave open the question of acting "morally". We do not discuss ways to achieve such desirable developments, or to continue them and not to go "astray". We only discuss - as R.Thomason does in [Tho84] - the "quality" of the developments, and show that again a local choice by a binary preference relation suffices. The preference relation can even be assumed to have very nice and strong properties.

In the second part (Theorem 3.4 there), we give a characterization of coupled logics which can be obtained from a preference relation on developments: Assume the information S and T at time point s and t respectively about a development to be given, and a preference over developments. We examine the theories S' and T' determined by the end-points (i.e. models) of the preferred developments among those which pass through S- and T-models.

This defines a pair of coupled logics, $< S, T > \mid\!\sim < S', T' >$.

Finitely Axiomatisable Theories and the Standard Topology

The standard topology says that a model m is in the closure of a set of models X iff for any formula ϕ with $m \models \phi$, there is $m' \in X$ with $m' \models \phi$ too. We shall see now that in this topology, the clopen sets are exactly the sets of models of a single formula.

Notation 1.1

Fix a propositional language $\mathcal{L}$, let $M_{\mathcal{L}}$ be the set of its classical models. Variables ϕ, ψ etc. will range over $\mathcal{L}-formulas$, S, T etc. over $\mathcal{L}-theories$, i.e. sets of $\mathcal{L}-formulas$. $M(T) \subseteq M_{\mathcal{L}}$ will be the set of the (classical) models of T. For an $\mathcal{L}-model\ m$, $Th(m)$ is defined as $\{\phi : m \models \phi\}$, for $X \subseteq M_{\mathcal{L}}$, $Th(X) := \{\phi : \forall m \in X.m \models \phi\}$, and $T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$. Con will stand for classical consistency, $\mathcal{P}$ will be the power set operator, and Π shall denote the cartesian product.

The Standard Topology

For more details see e.g. [Eng77], [Kel75].

Definition 1.1

The set $\mathcal{A} := \{M(\{\phi\}) : \phi \text{ an } \mathcal{L}-formula\} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is closed under finite unions. Therefore, $\mathcal{A}$ forms a basis of closed sets for the topology $\mathcal{T}$ defined by $O \in \mathcal{T}$ iff $A_O = M_{\mathcal{L}} - O$ is an (arbitrary) intersection of elements from $\mathcal{A}$, i.e. iff $A_O = M(T)$ for some $\mathcal{L}-theory\ T$. So the closed subsets of $M_{\mathcal{L}}$ are the sets $M(T)$ for some T. Let for $X \subseteq M_{\mathcal{L}}$ $\widehat{X}$ be the closure of X in $\mathcal{T}$, i.e. the intersection of the closed sets containing X. (The notation $-$ is reserved for other purposes.)

Fact 1.1

(a) $m \in \widehat{X}$ iff $\forall\phi(m \models \phi \rightarrow \exists m' \in X.m' \models \phi)$,
(b) for $X \subseteq Y \subseteq \widehat{X}$ $Th(X) = Th(Y)$,
(c) $\widehat{X} = M(Th(X))$,
(d) $X \subseteq M_{\mathcal{L}}$ clopen (= closed and open) iff $X = M(\{\phi\})$ for some formula ϕ.

Proof:

(a) $'' \rightarrow''$: Suppose there is ϕ such that $m \models \phi$, but for no $m' \in X$ $m' \models \phi$. Then $M(\{\neg\phi\})$ is closed, includes X, but not m, so $m \notin \widehat{X}$. $'' \leftarrow''$: If $m \notin \widehat{X}$, there is a theory T with $X \subseteq M(T)$, $m \notin M(T)$. So there is $\phi \in T$, $m \models \neg\phi$, but for all $m' \in X$, $m' \models \phi$.
(b) $'' \supseteq''$ is trivial. $'' \subseteq''$: If there were $\phi \in Th(X) - Th(Y)$, then there would be $m \in Y - X$, $m \models \neg\phi$, but for all $m' \in X$, $m' \models \phi$, contradicting (a).
(c) $M(Th(X))$ is the least closed set containing X.
(d) $'' \leftarrow''$: $M(\{\phi\})$ and $M(\{\neg\phi\})$ are both closed. $'' \rightarrow''$: Let X be clopen, so by definition, $X = M(T)$, $M_{\mathcal{L}} - X = M(T')$ for two theories T,T'. Wlog., T is deductively

closed. Suppose for no $\phi \in T$ $X = M(\{\phi\})$. So for all $\phi \in T$, $M(T) \overset{\subset}{\neq} M(\{\phi\})$, so $M(\{\phi\}) \cap M(T') \neq \emptyset$, so $Con(\{\phi\} \cup T')$. But by $M(T) \cap M(T') = \emptyset$, not $Con(T \cup T')$, so by compactness of classical logic and closure of T under classical consequence, there is $\phi \in T$ such that not $Con(\{\phi\} \cup T')$, contradiction. □

1.3 Recalling Defaults as Generalized Quantifiers

In [Sch95-1], we gave open normal defaults an interpretation as generalized quantifiers. This will be recalled informally in this section, the formal definitions and results are repeated in section 2.1.

"Normally, birds fly" is translated into "Most birds fly" or "The elements of a large or important subset of the set of birds fly". This is written $\nabla x bird(x) : fly(x)$, in an extension of the language of first-order logic. (We use "large" and "important" etc. interchangeably - what really matters will be the formal definition.) This is given a clear semantics in the form of a system of "important" subsets of the set of birds. The system is defined to be almost a filter. We add an additional structure, in the form of such a system, to first-order models to provide a semantics for the new quantifier. In such an enlarged structure, the default "Normally, birds fly" will hold, iff there is an "important" subset of the set of birds, all of whose elements fly.

With this approach, we are working in the full (indeed, enlarged) first-order framework, with defaults in the object language. So not only negated defaults have a clear meaning, but so do arbitrary boolean combinations of defaults, nested defaults, arbitrary combinations of defaults with classical quantifiers etc. In our "birds" example, a negated default means that there is no important subset of birds all of whose elements fly.

In [Sch95-1] we extended first-order logic to a sound and complete axiomatization for our semantics. We can thus not only derive defaults - e.g., if "normally, $\phi(x)''$ is true, and $\phi(x)$ implies $\psi(x)$ classically, then also "normally, $\psi(x)''$ is true - but also have a notion of consistency of default theories. For instance, a theory containing the axioms "normally, $\phi(x)''$, and "normally, $\neg\phi(x)''$, is ruled out as inconsistent, as we think it should be. Thus, we also have an explicit criterion of consistency, any logic should have.

From a more philosophical point of view, we might say that our formalism allows us to detect default inconsistencies of the form "normally $\phi(x)''$ and "normally $\neg\phi(x)''$ finer than the inconsistencies of classical logic. $\forall x\phi(x)$ and $\neg\phi(a)$ are already classically inconsistent. As we shall see in the formal part, $\nabla x\phi(x)$ and $\phi(a)$ are quite compatible, as exceptions are tolerated. On the other hand, $\nabla x\phi(x)$ and $\nabla x\neg\phi(x)$ are inconsistent. A classical interpretation of ∇ as $\forall$ gives too many inconsistencies, precisely because it is too restrictive, "normally" is not necessarily "all". Our weaker reading, in which we interpret "normally" not as "all", but as something like a qualitative "most", penetrates through the outer layer of classical or pseudo-conflicts (pseudo in our context) to those

which may exist on the level of normality itself.
We have deliberately chosen a very weak formalization of the concept of "large" subsets, weaker than a filter (see Definition 2.1 below): If B is the base set, then any system $\mathcal{N}(B)$ of important or large subsets of B has to satisfy (1) B is large: $B \in \mathcal{N}(B)$,
(2) supersets of large subsets are large: $A \subseteq A' \subseteq B$, $A \in \mathcal{N}(B) \to A' \in \mathcal{N}(B)$,
(3) the intersection of two large subsets is non-empty: $A, A' \in \mathcal{N}(B) \to A \cap A' \neq \emptyset$.
The third property is the most interesting one. A filter would require the intersection of two large subsets to be large itself, we content ourselves with non-empty intersection. This permits e.g. a simple probabilistic interpretation of $A \in \mathcal{N}(B)$ as "more than half the elements of B are in A''.
In the intended interpretation it is property (3) that permits us to deduce that "normally $\phi(x)''$ and "normally $\neg\phi(x)''$ together are impossible: there is no element which satisfies both $\phi(x)$ and $\neg\phi(x)$.
We do not think that any weaker system could still capture the notion of "important" or "large" subsets. In particular, (3) may be strengthened to the filter property $A, A' \in \mathcal{N}(B)$ $\to A \cap A' \in \mathcal{N}(B)$, but this would already preclude the abovementioned probabilistic reading. Note however, that our properties capture e.g. a "prototypical" reading: if $b \in B$ is any element of B, then $\{A \subseteq B : b \in A\}$ is a system as required.
Choosing a very weak system has several advantages:
First, it is easier to extend a weak system by adding axioms on the proof theoretical side, and additional properties on the semantic side, preserving soundness and completeness, than to retract some axioms and properties. Second, we thus give a broad interpretation, which can easily be modified to suit special domain purposes or personal tastes. Third, we are not forced to make many "philosophical" commitments, but provide a tool kit to incorporate the desired postulates. Such extensions will however be discussed in section 2.2.
In summary, we might say that we have tried to cover the minimal properties of the static part of "normally". We have given "normally" an interpretation in an extended FOL model, where $\mathcal{M} \models'' \textit{normally } \phi(x)''$ iff there is a large subset A of $\mathcal{M}'s$ universe such that for all $a \in A$, $\mathcal{M} \models \phi(a)$. This is a "local" interpretation in one model, not a "global" interpretation as in preferential structures. We give an interpretation of "normally" in *one* model, not by an external preference relation on a set of *many* models. This is also an expression of the static aspect we are trying to cover: In *this* extended FOL model, the minimal requirements permitting us to say "normally $\phi(x)''$ hold.
In the formal development, we first treat normal open defaults without prerequisites, i.e. of the form $\frac{:\phi(x)}{\phi(x)}$, written in our notation as $\nabla x \phi(x)$. We then extend this approach to normal open defaults with prerequisites, $\frac{\psi(x):\phi(x)}{\phi(x)}$, interpreted with the relativized quantifier as $\nabla x \in \{y : \psi(y)\} : \phi(x)$, abbreviated $\nabla x \psi(x) : \phi(x)$. This means that most of the x's that satisfy ψ will satisfy ϕ too. Of course, this is very different from $\nabla x(\psi(x) \to \phi(x))$ which means that most of the x's in the domain either fail ψ or satisfy ϕ. All essential

techniques and ideas are present in the case without prerequisite, the extension to defaults with prerequisites is straightforward and works by relativizing the quantifier.
(Defaults of the form $\frac{:\phi}{\phi}$, where ϕ has no free variables, are not treated. Intuitively, we would read them as implicitely quantifying over all models under consideration.)
Formally, we introduce the new (unbounded) quantifier ∇ into the language (and its dual $\clubsuit$ for proof theoretical purposes), see Definition 2.2. The crucial definition linking language and semantics is Definition 2.3 (see section 2.1), where we define an $\mathcal{N}-model$ and validity of a $\nabla-formula$ in an additional inductive step: An $\mathcal{N}-model$ is a classical first-order structure, with an $\mathcal{N}-system$ of large subsets over its universe M. $\nabla x\phi(x)$ is defined to hold in the $\mathcal{N}-model$, iff there is a large subset $A \subseteq M$ such that for all $a \in A$, $\phi(a)$ holds. As ϕ is not necessarily a classical formula, we can treat nested $\nabla's$. Moreover, the new quantifier is fully embedded into the classical setting. Thus, we can form boolean combinations, quantify classically over defaults, e.g. $\exists x\nabla y\phi(x,y)$, and give these formulas a precise meaning, preserving the constructive spirit of FOL. This seems to us one advantage over "global" Kripke style semantics where we have to "look elsewhere" for the interpretation of normality. Here, we can say "look, you see it holds $inthisuniverse$ on a large subset, so it normally holds here".
A corresponding axiomatization follows in Definition 2.4, of which the most interesting parts are:
1. $\nabla x\phi(x) \wedge \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \nabla x\psi(x)$,
2. $\nabla x\phi(x) \rightarrow \neg\nabla x\neg\phi(x)$,
3. $\forall x\phi(x) \rightarrow \nabla x\phi(x) \rightarrow \exists x\phi(x)$.
The first axiom says that implication preserves normality: If $\phi(x)$ normally holds, and $\phi(x)$ implies $\psi(x)$, then $\psi(x)$ normally holds. This corresponds to the second property of $\mathcal{N}-systems$. The second axiom says that "normally ϕ'' and "normally $\neg\phi''$ contradict each other. We have discussed this above. The third says that the $\nabla-quantifier$ is between the two classical ones. This corresponds to the fact that M is a large subset of itself. The second half of the axiom could be derived by $\neg\exists x\phi(x) \rightarrow \forall x\neg\phi(x) \rightarrow \nabla x\neg\phi(x) \rightarrow \neg\nabla x\phi(x)$, but we prefer to state it explicitly.
Lemma 2.1 states some basic and trivial consequences of the axioms. The soundness and completeness Theorem 2.2 is the central result of [Sch95-1].
As said above, the extension to normal open defaults with prerequisites is straightforward: $\nabla x\phi(x) : \psi(x)$ ($''if\ \phi(x)$, then normally $\psi(x)''$) is interpreted as a generalized quantifier relativized to $\{x : \phi(x)\}$, i.e. we consider an $\mathcal{N}-system$ not over the whole universe, but only over the subset where $\phi(x)$ holds. Again, we deliberately keep our system very weak. We do not demand any connections between the $\mathcal{N}-systems$ over the different $\{x : \phi(x)\}$ and $\{x : \phi'(x)\}$, apart from the trivial ones when e.g. $\{x : \phi(x)\} = \{x : \phi'(x)\}$. We even do not postulate $A \in \mathcal{N}(B) \wedge A \subseteq B' \subseteq B \rightarrow A \in \mathcal{N}(B')$, which a purely quantitative reading would justify: A large subset of B is a fortiori large in B', when $A \subseteq B' \subseteq B$. In general, we want to leave open as many intended developments and strengthenings as possible. The formal definitions and results are given in Definitions 2.5-2.7, Lemma 2.3,

Theorem 2.4.

Some remarks on the historical background:
The word "generalized quantifier" is perhaps not very well chosen: Traditionally, the interpretation of a generalized quantifier is by a subset of the universe which is large by cardinality. But if the universe U has infinite cardinality κ, then there are disjoint subsets of U of size κ. So, if "large" is read as "cardinality κ'', this contradicts our main axiom of normality, $A, B \in \mathcal{N}(U) \rightarrow A \cap B \neq \emptyset$, expressing that "normally, ϕ'' and "normally, $\neg\phi''$ together are incoherent. So there is a profound difference between the traditional concept of a generalized quantifier, and our idea. They are designed to speak about very different things.
The concept of a generalized quantifier was introduced by Mostowski in [Mos57], completeness for the interpretation as "there exist uncountably many" was studied by Keisler in [Kei70]. Standard models" for "there exist many" were investigated and corresponding completeness shown by Krivine and McAloon in [KM73]. Very short discussions can be found in the Handbook of Mathematical Logic, in contributions by Barwise [Bar77], p.44-45, and Keisler [Kei77], p.100-101.

1.4 A Two-Stage Approach

As already indicated in section 1.1, we start with the static aspect, and then choose among the extended FOL models of a default theory T with defaults interpreted as generalized quantifiers, the models that are "good" under a suitable preferential relation. The formal definition is given in section 2.3.
The first step assures us that the defaults have an objective meaning, the second will capture the "inference-greedy" aspect by making true as many instances of the defaults as possible.
Basically, a model M is considered better than a model M' for the theory T, iff
(1) for all defaults in T, M satisfies all positive instances of the defaults that M' satisfies (where c is a positive instance for the default $\frac{:\phi(x)}{\phi(x)}$, iff $\phi(c)$ holds in that model - the case for defaults with prerequisites is analogous)
(2) for no default in T, there is a negative instance which holds in M, but not in M' (where c is a negative instance for the default $\frac{:\phi(x)}{\phi(x)}$, iff $\neg\phi(c)$ holds in that model - the case for defaults with prerequisites is again analogous)
(3) there is at least one default in T, with one positive instance c in M, which is not positive in M'.
This definition is essentially taken from [BS93].
Note that enlarging the universe might introduce new negative instances without destroying old positive ones. This is the justification of the condition (2).
As our formalism with generalized quantifiers allows us to deduce defaults, e.g. by the axiom scheme $\nabla x\phi(x) \wedge \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \nabla x\psi(x)$, it is not clear which defaults

enter into the definition of the relation "M is better than M'". But consider a theory of two defaults $\nabla x\phi(x)$ and $\nabla x\psi(x)$; then we can deduce the defaults $\nabla x(\phi(x) \vee \neg\psi(x))$ and $\nabla x(\psi(x) \vee \neg\phi(x))$, which could block an attempt to extend the positive instances: Suppose that $\nabla x\phi(x)$ and $\nabla x\psi(x)$ hold in M', as well as $\neg\phi(a) \wedge \neg\psi(a)$ and $\neg\phi(b) \wedge \neg\psi(b)$. Suppose further that there is model M, which agrees with M' with the exception that $\phi(a)$ and $\psi(b)$ hold, too. Obviously, M is a better model, by the above definition, for the two defaults $\nabla x\phi(x)$ and $\nabla x\psi(x)$, and this corresponds to intuition. If we apply the definition to the derived defaults, however, we see that M is no longer better: Changing $\neg\phi(a)$ to $\phi(a)$ introduces a negative instance for the default $\nabla x(\psi(x) \vee \neg\phi(x))$. (Making also $\psi(a)$ true might be forbidden.) We therefore consider either only the defaults present in the original theory, or, alternatively, a set of defaults mentioned explicitly.
The following example justifies a limit preferential structure approach, beyond the abstract consideration that the existence of ideal situations is perhaps not always guaranteed. Consider a language with one unary predicate $P(x)$, the default $\nabla xP(x)$, and the classical theory which says that we have infinitely many instances of $\neg P(x)$. All models will be infinite. We have no best models, as we can always make another element a positive instance. The theory is consistent, and we would like to deduce at least that we have infinitely many positive instances too. So not all models will have the same intuitive value, but none is the best. So we have infinite descending chains, leading to an inconsistent set of conclusions in the minimal preferential structure approach: the set of minimal models is empty, and semantical entailment is defined by universal quantification over the set of minimal models. This will not happen in the limit preferential approach.

2 DEFINITION AND EXAMPLES

2.1 Repetition of the Basic Definitions and Results on Defaults as Generalized Quantifiers

Defaults as Generalized Quantifiers)
For proofs and further details, the reader is referred to [Sch95-1].

2.1.1 SEMANTICS:

Definition 2.1
We denote by $\mathcal{P}(X)$ the powerset of X.
$\mathcal{N}(M) \subseteq \mathcal{P}(M)$ is an $\mathcal{N}-system$ over M iff
a. $M \in \mathcal{N}(M)$,
b. $A \in \mathcal{N}(M)$, $A \subseteq B \subseteq M \rightarrow B \in \mathcal{N}(M)$,
c. $A, B \in \mathcal{N}(M) \rightarrow A \cap B \neq \emptyset$ if $M \neq \emptyset$ (thus, $\emptyset \notin \mathcal{N}(M)$, if $M \neq \emptyset$) (Note that this is weaker than the corresponding axiom for filters.)

Remark: Our semantics covers the two extremes:
- fix one element a of the universe U, then $\{A \subseteq U\colon\ a \in A\}$ will be an $\mathcal{N}-system$
- let some probability measure be given on U, then $\{A \subseteq U\colon\ p(A) > 0.5\}$ will be an $\mathcal{N}-system$.
(Note, however, that the former can also be expressed by a suitable point measure on U.)
We can thus cover both the "prototypical" and the "average" case.
To facilitate proofs and permit normal forms, we have introduced a complementary quantifier, $\clubsuit$ with the meaning $\clubsuit x\phi(x) :\longleftrightarrow \neg\nabla x\neg\phi(x)$. The intuitive reading of $\clubsuit x\phi(x)$ is thus roughly: "for at least a few x, $\phi(x)$ holds".

Definition 2.2
We augment a language $\mathcal{L}$ of first-order logic by the new quantifiers:
If ϕ and ψ are formulas, then so are $\nabla x\phi(x)$, $\clubsuit x\phi(x)$, $\nabla x\phi(x) : \psi(x)$, $\clubsuit x\phi(x) : \psi(x)$ for any variable x.
We call a formula of the extended language, containing ∇ or $\clubsuit$ a $\nabla\ \mathcal{L}-formula$.

Definition 2.3
$(\mathcal{N}-Model)$.
Let $\mathcal{L}$ be a first-order language, M be an $\mathcal{L}-structure$ and $\mathcal{N}(M)$ an $\mathcal{N}-system$ over M. Define $< M, \mathcal{N}(M) > \models \phi$ for any $\nabla\ \mathcal{L}-formula\ \phi$ inductively as usual, with two additional induction steps:
$< M, \mathcal{N}(M) > \models \nabla x\phi(x)$ iff $\{a \in M\colon < M, \mathcal{N}(M) > \models \phi[a]\} \in \mathcal{N}(M)$,
$< M, \mathcal{N}(M) > \models \clubsuit x\phi(x)$ iff $\{a \in M\colon < M, \mathcal{N}(M) > \models \neg\phi[a]\} \notin \mathcal{N}(M)$.

2.1.2 PROOF THEORY:

Definition 2.4
Let any axiomatization of predicate calculus be given. Augment this with the axiom schemata
1. $\nabla x\phi(x) \wedge \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \nabla x\psi(x)$,
2. $\nabla x\phi(x) \rightarrow \neg\nabla x\neg\phi(x)$,
3. $\forall x\phi(x) \rightarrow \nabla x\phi(x) \rightarrow \exists x\phi(x)$,
4. $\clubsuit x\phi(x) :\longleftrightarrow \neg\nabla x\neg\phi(x)$,
5. $\nabla x\phi(x) \longleftrightarrow \nabla y\phi(y)$ if x does not occur free in $\phi(y)$ and y does not occur free in $\phi(x)$ (for all ϕ, ψ).
We also denote the corresponding notion of derivability by $\vdash_\nabla$.

Lemma 2.1
The following formulas are derivable:
a. $\nabla x\phi(x) \wedge \nabla x\psi(x) \rightarrow \exists x(\phi \wedge \psi)(x)$,
b. $\nabla x\phi(x) \wedge \neg\nabla x\psi(x) \rightarrow \exists x(\phi \wedge \neg\psi)(x)$,
c. $\neg\nabla x\neg\phi(x) \rightarrow \exists x\phi(x)$,

d. $♣x\phi(x) \rightarrow \exists x\phi(x)$,
e. $\nabla x\phi(x) \wedge ♣x\psi(x) \rightarrow \exists x(\phi \wedge \psi)(x)$,
f. $\forall x(\phi(x) \longleftrightarrow \psi(x)) \rightarrow (\nabla x\phi(x) \longleftrightarrow \nabla x\psi(x)) \wedge (♣x\phi(x) \longleftrightarrow ♣x\psi(x))$,
g. $\forall x\phi(x) \rightarrow ♣x\phi(x)$.

Note that in general the following is *not* derivable: $♣x\phi(x) \wedge ♣x\psi(x) \rightarrow \exists x(\phi \wedge \psi)(x)$.

Theorem 2.2
The set of axioms given in Definition 2.4 is sound and complete for the semantics of Definition 2.1 and 2.3.

2.1.3 EXTENSION TO NORMAL DEFAULTS WITH PREREQUISITES

Definition 2.5
$\mathcal{N}^+(M) =< \mathcal{N}(N) : N \subseteq M >$ is an $\mathcal{N}^+ - system$ over M iff for each $N \subseteq M$ $\mathcal{N}(N)$ is an $\mathcal{N} - system$ over N.

(It suffices to consider the definable subsets N of M, i.e. those such that there is a formula $\phi(x)$ with $N = \{a \in M :< M, \mathcal{N}(M) >\models \phi(a)\}$.)

Definition 2.6
Extend the logic of first-order predicate calculus by adding the axiom schemata:
1. a. $\nabla x\phi(x) \longleftrightarrow \nabla x(x = x) : \phi(x)$,
b. $\forall x(\sigma(x) \longleftrightarrow \tau(x)) \wedge \nabla x\sigma(x) : \phi(x) \rightarrow \nabla x\tau(x) : \phi(x)$,
2. $\nabla x\phi(x) : \psi(x) \wedge \forall x(\phi(x) \wedge \psi(x) \rightarrow \vartheta(x)) \rightarrow \nabla x\phi(x) : \vartheta(x)$,
3. $\exists x\phi(x) \wedge \nabla x\phi(x) : \psi(x) \rightarrow \neg\nabla x\phi(x) : \neg\psi(x)$,
4. $\forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \nabla x\phi(x) : \psi(x) \rightarrow [\exists x\phi(x) \rightarrow \exists x(\phi(x) \wedge \psi(x))]$,
5. $♣x\phi(x) : \psi(x) \longleftrightarrow \neg\nabla x\phi(x) : \neg\psi(x)$,
6. $\nabla x\phi(x) : \psi(x) \longleftrightarrow \nabla y\phi(y) : \psi(y)$ (under the usual caveat for substitution.)
(for all ϕ, ψ, ϑ, σ, τ).

Lemma 2.3
The following are derivable:
a) the axioms of Definition 2.4, and the formulas of Lemma 2.1
b) the relativized versions of the formulas of Lemma 2.1, where the existential statements have to be weakened by an existential assumption (as, for instance, $\forall x(\phi(x) \rightarrow \psi(x))$ does not imply that there is x with $\psi(x)$).

Definition 2.7
Let $\mathcal{L}$ be a first-order language, and M an $\mathcal{L} - structure$. Let $\mathcal{N}^+(M)$ be an $\mathcal{N}^+ - system$ over M. For any formula ϕ define $< M, \mathcal{N}^+(M) >\models \phi$ inductively as usual, with the additional induction steps:

1. $< M, \mathcal{N}^+(M) > \models \nabla x \phi(x)$ iff $\{a \in M$: $< M, \mathcal{N}^+(M) > \models \phi[a]\} \in \mathcal{N}(M)$.
2. $< M, \mathcal{N}^+(M) > \models \clubsuit x \phi(x)$ iff $\{a \in M$: $< M, \mathcal{N}^+(M) > \models \neg\phi[a]\} \notin \mathcal{N}(M)$.
3. $< M, \mathcal{N}^+(M) > \models \nabla x \phi(x) : \psi(x)$ iff there is $A \in \mathcal{N}(\{x :< M, \mathcal{N}^+(M) > \models \phi(x)\})$ such that $\forall a \in A$ $(< M, \mathcal{N}^+(M) > \models \psi[a])$
4. $< M, \mathcal{N}^+(M) > \models \clubsuit x \phi(x) : \psi(x)$ iff $\{a \in M$: $< M, \mathcal{N}^+(M) > \models \phi[a] \wedge \neg\psi[a]\} \notin \mathcal{N}(\{x :< M, \mathcal{N}^+(M) > \models \phi(x)\})$.

Theorem 2.4
The set of axioms of Definition 2.6 is sound and complete for the $\mathcal{N}^+ - semantics$ as defined in Definitions 2.5 and 2.7.

2.2 Strengthening the Axioms of Normality

The common use of defaults, as well as systems presented in the literature (see e.g. [KLM90], [LM90]), motivate various extensions of our base system, in both the semantic and axiomatic presentations. On the semantic side, these extensions consist of adding properties to the single $\mathcal{N} - systems$. Of particular interest are the strengthenings of $\mathcal{N} - systems$ to filters (corresponding to simple "iterability", see Definition 2.8), and the addition of coherence properties between the $\mathcal{N} - systems$ over different subsets of the universe for the relativized case. The latter correspond e.g. to "iterability" with prerequisites, see Definition 2.9.
On the proof theoretic side, we can either add these strengthenings as general axiom schemata, or, more cautiously, only in certain cases, allowing us to derive new defaults. As the corresponding strengthenings on the semantic side are mostly evident from those of the proof theory, the reader is referred to [Sch95-1] for details.
The ease with which we can incorporate these strengthenings is due to the fact that our system was chosen at the outset to be very weak, while making the language very expressive.
The common use of defaults seems to presuppose strong assumptions on the structure of the universe (or, better, of our description thereof). E.g. applying *both* defaults "normally $\phi(x)''$ and "normally $\psi(x)''$ seems to presuppose that not only "normally $\phi(x)''$ and "normally $\psi(x)''$, but also "normally $\phi(x) \wedge \psi(x)''$. Even though our base system is too weak to validate such reasoning, we can easily do so by strengthening the system of "important" subsets to a filter, and introducing a corresponding axiom scheme as follows:

Definition 2.8
(Iterability for defaults without prerequisites).
$\nabla x \phi(x) \wedge \nabla x \psi(x) \rightarrow \nabla x(\phi \wedge \psi)(x)$.

In the presence of defaults with prerequisites, iterability takes a slightly extended form.

Definition 2.9
(Iterability for defaults with prerequisites).
$\nabla x\phi(x) \wedge \nabla x\sigma(x) : \psi(x) \rightarrow \nabla x\sigma(x) : (\phi \wedge \psi)(x)$,
$\nabla x\sigma(x) : \psi(x) \wedge \nabla x\sigma'(x) : \psi'(x) \rightarrow \nabla x(\sigma \wedge \sigma')(x) : (\psi \wedge \psi')(x)$.

In addition, "normal use" of defaults sanctions still another axiom scheme, which we call chaining.

Definition 2.10
(Chaining).
$\nabla x\phi(x) \wedge \nabla x\phi(x) : \psi(x) \rightarrow \nabla x\psi(x)$,
$\nabla x\sigma(x) : \psi(x) \wedge \nabla x\psi(x) : \psi'(x) \rightarrow \nabla x\sigma(x) : \psi'(x)$.

Furthermore, formal reasoning in Reiter's approach with the default $\nabla x fly(x)$ presupposes that logically consistent definable subsets are "probabilistically consistent" too. I.e., if things normally fly in the universe of birds, so do the elements of all subsets of this universe which are definable by some predicate, and which might consistently contain some flyer. In other words, the conventional formal use of the default $\nabla x fly(x)$ presupposes that every definable subset $\{x : \phi(x)\}$ of the universe which is consistent with the flyer subset $\{x : fly(x)\}$ behaves just like the universe itself. We may also say that ϕ is an irrelevant property for $\nabla x fly(x)$. This can be expressed by

Definition 2.11
(Homogeneity).
$\nabla x\phi(x) \wedge \exists x(\phi(x) \wedge \sigma(x)) \rightarrow \nabla x\sigma(x) : \phi(x)$.

The desire to have $\nabla x(\nabla y\phi(x,y)) \leftrightarrow \nabla y(\nabla x\phi(x,y))$ in analogy to $\forall x(\forall y\phi(x,y)) \leftrightarrow \forall y(\forall x\phi(x,y))$ motivates the following:

Definition 2.12
Let $\mathcal{N}$ be an $\mathcal{N}-$ *system* over U, $A \in \mathcal{N}$, and let $B_x \subseteq U$ for all $x \in A$. Define $\otimes\{B_x : x \in A\} := \{y \in U : \exists A_y \in \mathcal{N} \forall x \in A_y . y \in B_x\} = \{y \in U : \{x : y \in B_x\} \in \mathcal{N}\}$. $\mathcal{N}$ is closed under weak intersections, iff for all $A \in \mathcal{N}$ and all families $\{B_x \in \mathcal{N} : x \in A\}$, $\otimes\{B_x : x \in A\} \in \mathcal{N}$ holds.

Lemma 2.5
(a) $\mathcal{N}$ is closed under weak intersections iff there is $E \subseteq U$ s.t. $A \in \mathcal{N} \leftrightarrow E \subseteq A$.
(b) Let $\mathcal{M} =< M, \mathcal{N} >$. Then $\mathcal{M} \models \nabla x(\nabla y\phi(x,y))$ is equivalent to $\mathcal{M} \models \nabla y(\nabla x\phi(x,y))$ iff $\mathcal{N}$ is closed under weak intersections.

(a) and its proof are due to M.Magidor, Jerusalem.

Proof:
(a)
We show first
(1) If $\mathcal{N}$ is closed under weak intersections, then $\mathcal{N}$ is closed under finite intersections.
Proof of (1): Suppose not, let $A, B \in \mathcal{N}$, $A \cap B \notin \mathcal{N}$. Define $B_x := A$ iff $x \in A - B$, and $B_x := B$ iff $x \in A \cap B$. By prerequisite, $\otimes\{B_x : x \in A\} \in \mathcal{N}$. But $A - B \notin \mathcal{N}$, as $B \in \mathcal{N}$, and $A \cap B \notin \mathcal{N}$. So $\{x : y \in B_x\} \in \mathcal{N}$ iff $x \in A \cap B$, and $A \cap B \in \mathcal{N}$. *contradiction*.
We now define
(2) $A \subseteq U$ is positive iff $U - A \notin \mathcal{N}$. Thus, if $A \in \mathcal{N}$, then A is positive.
We next show
(3) Let $\mathcal{N}$ be closed under weak intersections, then A is positive iff there is $a \in A$ s.t. $\{a\}$ is positive.
Proof of (3): Let $\mathcal{N}$ be closed under weak intersections. $'' \leftarrow''$: trivial. $'' \rightarrow''$: Let A be positive, and $B \subseteq A$ be positive, and no $X \subseteq B$, $card(X) < card(B)$ be positive. Case 1: $card(B) < \omega$. So $U - B \notin \mathcal{N}$. If $card(B) = 1$, we are done. If not, there are B', B'' s.t. $B = B' \cup B''$, $B', B'' \neq \emptyset$, and by minimality $U - B' \in \mathcal{N}$, $U - B'' \in \mathcal{N}$, so by (1) $U - B \in \mathcal{N}$. *contradiction* Case 2: $card(B) \geq \omega$. We will derive a contradiction. Well-order B by $<$. If $B' \subseteq B$ is an initial segment of B under $<$, $B' \notin \mathcal{N}$. Define $B_x := \{y \in B : y > x\} \cup (U - B)$ for $x \in B$, and $B_x := U$ for $x \in U - B$. $B_x \in \mathcal{N}$: If $B_x \notin \mathcal{N}$, the initial segment $B - B_x$ would be positive, contradicting minimality. By closure under weak intersection, $\otimes\{B_x : x \in U\} \in \mathcal{N}$. But for $y \in B$ $\{x : y \in B_x\} = \{x : x < y\} \cup (U - B) \notin \mathcal{N}$: otherwise, by $(U - B) \cup \{x : x > y\} \in \mathcal{N}$, $U - B \in \mathcal{N}$, but B was assumed positive. Consequently $\otimes\{B_x : x \in U\} = U - B \notin \mathcal{N}$. *contradiction*
Finally, we show (a) of the Lemma: Let $E := \{x : \{x\} \text{ is } positive\}$. $'' \rightarrow''$: Let $A \in \mathcal{N}$, $E \not\subseteq A$, $e \in E - A$, so $U - \{e\} \notin \mathcal{N}$, so $A \notin \mathcal{N}$. *contradiction* $'' \leftarrow''$: Let $E \subseteq A$, $A \notin \mathcal{N}$, so U-A is positive, so by (3), there is $x \in U - A$, $x \in E$. *contradiction*
(b) $'' \leftarrow''$: Assume $\mathcal{M} \models \nabla x(\nabla y \phi(x, y))$, so there is an $A \in \mathcal{N}$ such that for all $a \in A$, $\mathcal{M} \models \nabla y \phi(a, y)$, so that for all $a \in A$, there is $B_a \in \mathcal{N}$ such that for all $b \in B_a$, $\mathcal{M} \models \phi(a, b)$. By closure under weak intersection, $B := \otimes\{B_a : a \in A\} := \{b \in M : \exists A_b \in \mathcal{N} \forall a \in A_b . b \in B_a\} \in \mathcal{N}$. But for $b \in B_a$ $\mathcal{M} \models \phi(a, b)$, so for $b \in B$ $\mathcal{M} \models \nabla x \phi(x, b)$, so by $B \in \mathcal{N}$, $\mathcal{M} \models \nabla y(\nabla x \phi(x, y))$. $'' \rightarrow''$: Assume there is a family $\{B_x : x \in A\} \subseteq \mathcal{N}$, $A \in \mathcal{N}$, but $B' := \otimes\{B_x : x \in A\} \notin \mathcal{N}$, $\mathcal{N}$ over M. Define the following structure $< M, \mathcal{N} >$ for the language $\mathcal{L}$ with the binary predicate $P(x, y)$ by interpreting P by $P(x, y) :\longleftrightarrow x \in A \wedge y \in B_x$. Then $\mathcal{M} \models \nabla x(\nabla y P(x, y))$. If, however, $\mathcal{M} \models \nabla y(\nabla x P(x, y))$ were to hold, then there would be some $B \in \mathcal{N}$ such that $\forall b \in B$ $\mathcal{M} \models \nabla x P(x, b)$, so for all $b \in B$, there would be some $A_b \in \mathcal{N}$ such that $\forall a \in A_b$ $\mathcal{M} \models P(a, b)$. So, by the interpretation of P, there is $B \in \mathcal{N}$ such that for all $b \in B$, $\exists A_b \in \mathcal{N} \forall a \in A_b (a \in A \wedge b \in B_a)$, but then $B \subseteq B'$, so $B' \in \mathcal{N}$, a contradiction. $\Box$

The following set of inference rules is discussed in depth with completeness results in

[KLM90], [LM92]:

Definition 2.13
Right Weakening: $\vdash \alpha \to \beta$, $\gamma |\sim \alpha \Rightarrow \gamma |\sim \beta$,
Reflexivity: $\alpha |\sim \alpha$,
And: $\alpha |\sim \beta$, $\alpha |\sim \gamma \Rightarrow \alpha |\sim \beta \wedge \gamma$,
Or: $\alpha |\sim \gamma$, $\beta |\sim \gamma \Rightarrow \alpha \vee \beta |\sim \gamma$,
Left Logical Equivalence: $\vdash \alpha \leftrightarrow \beta$, $\beta |\sim \gamma \Rightarrow \alpha |\sim \gamma$,
Cautious Monotony: $\alpha |\sim \beta$, $\alpha |\sim \gamma \Rightarrow \alpha \wedge \beta |\sim \gamma$,
Rational Monotony: $\alpha |\sim \beta \Rightarrow \alpha |\sim \neg\gamma$ or $\alpha \wedge \gamma |\sim \beta$,
One half of the Deduction Theorem: $\alpha |\sim \beta \Rightarrow |\sim \alpha \to \beta$.

Translating these rules into our formalism by writing $\phi(x) |\sim \psi(x)$ as $\nabla\phi(x) : \psi(x)$, gives another strengthening of our basic system. It and its semantical counterpart are discussed in [Sch95-1].

2.3 Formal Definition of the Two Stage Approach

We start with the static aspect, and choose among the extended first-order models of the theory T, where defaults are interpreted as generalized quantifiers, the models that are "good" under a preferential relation.

Notation 2.1
We use the following abbreviations: $\phi_n[\psi] := \exists x_1 \ldots x_n(\psi(x_1) \wedge \ldots \wedge \psi(x_n) \wedge x_1 \neq x_2 \wedge \ldots \wedge x_{n-1} \neq x_n \wedge \forall y(\psi(y) \to y = x_1 \vee \ldots \vee y = x_n))$ says that the extension of $\psi(x)$ has exactly n elements, $\phi_{\leq n}[\psi] := \exists x_1 \ldots x_n(\psi(x_1) \wedge \ldots \wedge \psi(x_n) \wedge \forall y(\psi(y) \to y = x_1 \vee \ldots \vee y = x_n))$ says that the extension of $\psi(x)$ has at most n elements, $\phi_{\geq n}[\psi] := \exists x_1 \ldots x_n(\psi(x_1) \wedge \ldots \wedge \psi(x_n)$ $\wedge\ x_1 \neq x_2 \wedge \ldots \wedge x_{n-1} \neq x_n)$ says that the extension of $\psi(x)$ has at least n elements, and we abbreviate $\phi_n[true]$ by ϕ_n - where $true(x)$ is, say, $x = x$.

The first stage:
Let T be a consistent theory in the extended first-order language. (If so desired, the axioms of normality may be strengthened.) Let $M(T)$ be the set of its models.

The second stage:

Definition 2.14
(This definition will be slightly refined in section 3, where we allow the possibility of several copies of models.)
Given a set M and a binary relation $\prec$ on M, we call $X \subseteq M$ a minimizing segment of M, iff

(a) for all $m \in M$, there is $x \in X$ such that $m = x$ or $x \prec m$,
(b) for all $m \in M$ and $x \in X$, if $m \prec x$, then $m \in X$ - i.e. X is downward closed.
Given a language $\mathcal{L}$, a set of $\mathcal{L}-models$ M, a relation $\prec$ on M, an $\mathcal{L}-theory$ T, an $\mathcal{L}-formula$ ϕ, we define $T \models_{M,\prec} \phi$ iff there is a minimizing segment X of $M(T)$ such that $\forall m \in X.m \models \phi$.

Definition 2.15
Let m, m' be models (of classical or extended FOL).
(a) For any open normal default $d := \nabla x\phi(x) : \psi(x)$ (defaults without prerequisites may be rewritten equivalently as $\nabla x true(x) : \psi(x)$), and any element c in the universe of m, we call c a positive instance for d in m, iff $m \models \phi(c) \wedge \psi(c)$, and a negative instance for d in m iff $m \models \phi(c) \wedge \neg\psi(c)$. The set of positive (negative) instances for d in m will be denoted $P(d,m)$ $(N(d,m))$.
(b) Let D be a set of open normal defaults. Define $m \prec m' :\longleftrightarrow m \prec_D m'$ iff
(1) $\forall d \in D\ P(d,m') \subseteq P(d,m)$,
(2) $\forall d \in D\ N(d,m) \subseteq N(d,m')$,
(3) $\exists d \in D\ P(d,m) \cap N(d,m') \neq \emptyset$.
Note that $\prec$ is transitive, and by $P(d,m) \cap N(d,m) = \emptyset$, irreflexive.
(c) Again let D be a set of open normal defaults and let $d \in D$ be fixed. Define $m \prec_d m'$ iff
(1) $P(d,m') \subseteq P(d,m)$,
(2) $N(d,m) \subseteq N(d,m')$,
(3) $P(d,m) \cap N(d,m') \neq \emptyset$.
We will denote by $\prec' := \prec'_D$ the transitive closure of the union of $\prec_d$ over all $d \in D$.

$\prec'$ was the order considered in [BS93]. In hindsight, $\prec$ may perhaps appear as the "cleaner" approach, and is chosen here. We first show that the two relations do not lead to the same consequence relation.

Example 2.1
Consider the language with the unary predicates P and Q, and the two constants a and b, and let $T := \{\phi_2, \neg\exists x(P(x) \wedge Q(x))\}$, and $D := \{d = \frac{:P(x)}{P(x)}, d' = \frac{:Q(x)}{Q(x)}\}$. T has 9 models, $M(T)$, which we may denote by
$< P = Q = \emptyset >$, $< P = \emptyset, Q = \{a\} >$, $< P = \emptyset, Q = \{b\} >$, $< P = \emptyset, Q = \{a,b\} >$, $< P = \{a\}, Q = \emptyset >$, $< P = \{a\}, Q = \{b\} >$, $< P = \{b\}, Q = \emptyset >$, $< P = \{b\}, Q = \{a\} >$, $< P = \{a,b\}, Q = \emptyset >$.
In the relation $\prec$, the set of models $X := \{< P = \emptyset, Q = \{a,b\} >, < P = \{a\}, Q = \{b\} >, < P = \{b\}, Q = \{a\} >, < P = \{a,b\}, Q = \emptyset >\}$ forms an $\prec -initial$ segment, which is contained in all other initial segments of $M(T)$, so that its elements are $\prec -minimal$. We thus have $T \models_{M(T),\prec} \phi \leftrightarrow X \models \phi$. Note that, by finiteness, X is definable, and there is a formula ϕ' such that $m \in X \leftrightarrow m \models \phi'$. Suppose that $\prec''$ is any other

relation on $M(T)$ with $T \models_{M(T),\prec''} \phi \leftrightarrow T \models_{M(T),\prec} \phi$. Then there must be an $\prec''$ $-initial$ segment X' of $M(T)$ with $X' \models \phi'$, so $X' \subseteq X$. But if $X' \overset{\subset}{\neq} X$, then X' defines, by finiteness, a strictly stronger theory, so $X' = X$. On the other hand, e.g. $< P = \{a\}, Q = \emptyset > \prec_d < P = \emptyset, Q = \{a,b\} >$, so X is not closed in $M(T)$ for $\prec''$. Consequently, the relations $\prec$ and $\prec''$ do not define the same inference relation. □

Remark
Given a default theory D', we can, for instance, choose D' either itself as the D which determines the relation in Definition 2.15, or the set of all defaults that can be deduced in our extended first-order logic.
The example in section 1.4 seems to show that the latter solution is not always desirable. But the conservative solution $D := D'$ might not always be the best either. Consider the case $\sigma \rightarrow \nabla x\psi(x)$. Here, we might like to extend $\psi(x)$ in all models m, where σ holds, thus considering not only defaults directly present in T, but also defaults, which hold (in the sense of the generalized quantifier) in the model m.
These considerations suggest that the set D of defaults we really want to use in the "inference-greedy" step has to be noted explicitly. In all our examples, however, we let D to be the set of defaults given explicitly in the theory.

2.4 Some Logical Properties of the Two-Stage Approach

It is clear that our approach is robust under logically equivalent reformulation, not only for clasical logic $\vdash$, but also for our stronger logic $\vdash_\nabla$. That is, if T and T' are equivalent under $\vdash_\nabla$, they have the same $\nabla - models$. Provided the D used to define $\prec_D$ is fixed, the preferential structures will be the same, too. Thus, if we note by $\models_{\prec_D}$ the semantical entailment relation defined by our approach, $T \models_{\prec_D} \phi$ iff $T' \models_{\prec_D} \phi$. On the other hand, if ϕ and ϕ' are $\vdash_\nabla$ $-equivalent$, then $T \models_{\prec_D} \phi$ iff $T \models_{\prec_D} \phi'$.
In particular, the problem with Reiter's approach does not arise. In Reiter's approach classically equivalent theories T and T' need not have the same set of consequences - see [BH92] and [BS93] for a discussion.
Consider now again the inference rules defined in Definition 2.13. We fix a default theory T, a set of defaults D to determine $\prec_D$, and read $\alpha{\mid\!\sim}\ \beta$ for two clasical formulas α, β as $T \cup \{\alpha\} \models_{\prec_D} \beta$. Thus, e.g. the "And" rule says that $T \cup \{\alpha\} \models_{\prec_D} \beta$ and $T \cup \{\alpha\} \models_{\prec_D} \gamma$ implies $T \cup \{\alpha\} \models_{\prec_D} \beta \wedge \gamma$.
As a consequence of the above, and the fact that the preferential construction chooses subsets of the original models, Right Weakening, Reflexivity, and Left Logical Equivalence will hold trivially. This is also true for formulas of the extended language, and if we replace $\vdash$ by the stronger logic $\vdash_\nabla$.

Before we examine the other rules of Definition 2.13, we show a very general fact about limit preferential structures whose relation $\prec$ is transitive. This fact and its proof are almost identical with Lemma 3.5 below, where, however, we consider the more general case of preferential structures admitting several (logically identical) copies of models. As we do not want to burden the reader here with the additional formalism, we state the Lemma here for the simpler situation. The arguments are the same.

Lemma 2.6

(Taken from [BS93].)

Let $\mathcal{W} :=< W, \prec>$ be a preferential structure over Z, with a transitive relation $\prec$. Let $X, Y, S \subseteq Z$.

(a) If α and β are minimizing segments of S, then so is $\alpha \cap \beta$,

(b) If $X, Y \subseteq S$, α is a minimizing segment of X, and β is a minimizing segment of Y, then there is some $\gamma \subseteq \alpha \cup \beta$ such that γ is a minimizing segment of $X \cup Y$.

Proof:

(a) If $a \in \alpha \cap \beta$, $x \in S$, $x \prec a$, then $x \in \alpha \cap \beta$, as α and β are closed in S. If $x \in S$, and $x \notin \alpha$, then there is $a \in \alpha$ with $a \prec x$. If $a \notin \beta$, then there is $b \in \beta$ with $b \prec a$, so $b \in \alpha$ too. By transitivity, $b \prec x$.

(b) Set $\alpha' := \{a \in \alpha : \neg\exists b \prec a(b \in Y \wedge b \notin \beta)\}$, likewise $\beta' := \{b \in \beta : \neg\exists a \prec b(a \in X \wedge a \notin \alpha)\}$, and set $\gamma := \alpha' \cup \beta'$. Let $x \in W$, $x \in X \cup Y$, without loss of generality $x \in X$, then there is $a \in \alpha$, $a \prec x$ (or $a = x$). Suppose $a \notin \alpha'$, so there is $b \notin \beta$, $b \in Y$ with $b \prec a$, but then there is $b' \in \beta$ with $b' \prec b$, and by transitivity $b' \prec x$. It remains to show $b' \in \beta'$. But if $b' \in \beta - \beta'$, there is $a' \notin \alpha$, $a' \in X$ with $a' \prec b'$, so $a' \prec b' \prec a$, contradicting by transitivity again the closure of α in X. Let, on the other hand, $c \in \gamma$, and $x \in X \cup Y$ with $x \prec c$. Suppose without loss of generality $c \in \alpha'$. Case 1: $x \in X$, so $x \in \alpha$ by $c \in \alpha' \subseteq \alpha$. If $x \notin \alpha'$, there is $b \notin \beta$, $b \in Y$ with $b \prec x$, so $b \prec c$, contradicting $c \in \alpha'$. Case 2: $x \in Y$, then $x \in \beta$, as otherwise by $x \prec c$, $c \notin \alpha'$. But if $x \in \beta - \beta'$, then there is $a' \notin \alpha$, $a' \in X$ with $a' \prec x$, so $a' \prec c$, and $c \in \alpha$, contradicting closure of α in X. So $x \in \beta'$. $\Box$

Lemma 2.7

And, Or, Cautious Monotony, and one half of the Deduction Theorem hold. (Transitivity of $\prec$ is crucial here.)

Proof:

We write $M(T)$ for the set of T-models. "And": Suppose $T \cup \{\alpha\} \models_{\prec_D} \beta$ and $T \cup \{\alpha\} \models_{\prec_D} \gamma$. Then there is a minimizing segment S_β of $M(T \cup \{\alpha\})$, where β holds, and a minimizing

segment S_β of $M(T\cup\{\alpha\})$, where β also holds. By Lemma 2.6 (a), $S_\beta\cap S_\gamma$ is a minimizing segment S_β of $M(T\cup\{\alpha\})$, and in all models in $S_\beta\cap S_\gamma$, $\beta\wedge\gamma$ holds.
"Or": If $T\cup\{\alpha\}\models_{\prec_D}\gamma$ and $T\cup\{\beta\}\models_{\prec_D}\gamma$, and if S_α is a minimizing segment of $M(T\cup\{\alpha\})$ where γ holds, and S_β is a minimizing segment of $M(T\cup\{\beta\})$ where γ holds, then there is a minimizing segment $S\subseteq S_\alpha\cup S_\beta$ of $M(T\cup\{\alpha\})\cup M(T\cup\{\beta\})$ by Lemma 2.6 (b). But, in S, γ will hold.
"Cautious Monotony": Let S_β and S_γ be minimizing segments, which prove $\alpha\mid\sim\beta$ and $\alpha\mid\sim\gamma$. Then $S_\beta\cap S_\gamma$ is a minimizing segment in $M(T\cup\{\alpha\})$ by Lemma 2.6 (a). Now, $S_\beta\cap S_\gamma\subseteq M(T\cup\{\alpha,\beta\})$, and as $S_\beta\cap S_\gamma$ is a minimizing segment of $M(T\cup\{\alpha\})$, so a fortiori it is one of $M(T\cup\{\alpha,\beta\})$. For the Deduction Theorem, let S_α be a minimizing segment of $M(T\cup\{\alpha\})$, where β holds, and $S_{\neg\alpha}$ be an arbitrary minimizing segment of $M(T\cup\{\neg\alpha\})$, e.g. $M(T\cup\{\neg\alpha\})$ itself. By Lemma 2.6 (b), there is a minimizing segment $S\subseteq S_\alpha\cup S_{\neg\alpha}$ of $M(T)$. But is S, $\beta\vee\neg\alpha$ will hold. □

Example 2.2
(Taken essentially from [BS93].)
Rational Monotony does not hold: Set $D:=\{\nabla xP(x)\}$, $\alpha:=\exists x\neg P(x)$, $\beta:=\exists x(\neg P(x)\wedge\forall y(x\neq y\rightarrow P(y)))$, $\gamma:=\exists x\forall y\neq x.\neg P(y)$. It is shown in section 2.5, Example (4), that $\alpha\mid\sim\beta$ holds, and that $<M,\mathcal{N}>$ with the universe $\{a,b\}$, $P(a)$, $\neg P(b)$, $\{a\}\in\mathcal{N}$ is minimal, so α does not entail $\neg\gamma$. But $\alpha\wedge\gamma$ does not entail β : Take $<M,\mathcal{N}>$ with the universe $\{a,b,c\}$, and $P(a)$, $\neg P(b)$, $\neg P(c)$, $\{a\}\in\mathcal{N}$. Changing one negative instance, e.g. b, into a positive one, while preserving the positive instance a leaves intact the models of $\alpha\wedge\gamma$, so this model is minimal, and β does not hold there. □

2.5 Examples

We conclude this section by discussing the examples presented in [BS93]. Note that, with the exception of example (1), we consider theories with just one default, so $\prec$ and $\prec'$ coincide, and the results and arguments are often similar to those in [BS93]. In all cases, P and Q are unary predicates.
(1) $T:=\{\nabla xP(x),\nabla x\neg P(x)\}$ is inconsistent.
(2) $T:=\{\nabla P(x):\neg P(x)\}$ is inconsistent too.
(3) $T:=\{\nabla xP(x)\}$. Take $<M,\mathcal{N}>\models T$, so there is $A\in\mathcal{N}$ with $\forall x\in A.P(x)$. If $<M,\mathcal{N}>\models\exists x\neg P(x)$, then we can improve $<M,\mathcal{N}>$ to $<M',\mathcal{N}>$ where $P(.)$ will hold for that x too. If $\forall xP(x)$, then $<M,\mathcal{N}>$ cannot be improved, as we have no negative instances, so condition (3) in Definition 2.15 will not hold. So we can conclude $\forall xP(x)$, as this will hold in all $\prec-$ (or $\prec'-$) best models.

(4) $T := \{\exists x \neg P(x), \nabla x P(x)\}$. Take $< M, \mathcal{N} > \models T$, so there is $A \in \mathcal{N}$ with $\forall x \in A.P(x)$. As we have one negative instance, and $\nabla x P(x) \rightarrow \exists x P(x)$, the universe of M has at least two elements. If we have two different x with $\neg P(x)$, we can improve, if not, we have just one negative instance. Turning that into a positive instance would force us to introduce a new negative instance so, by condition (2) in Definition 2.14, the new model is not better than the old one. So the best models will all contain exactly one negative instance. We conclude that $\exists x(\neg P(x) \wedge \forall y(x \neq y \rightarrow P(y)))$ holds there.
(5) $T := \{\nabla x P(x)\} \cup \{\phi_{\geq n}[P] \rightarrow \phi_{n+1} : n \in \omega\}$. We can derive that all best models satisfy $\forall x P(x)$, and that we have an infinite universe.
(6) $T := \{\nabla x P(x)\} \cup \{\phi_n[\neg P] : n \in \omega\}$. There are no best models, but all sufficiently good models will contain an infinite set of positive instances.
(7) $T := \{\exists x P(x), \nabla P(x) : Q(x)\}$. $\exists x Q(x)$ will hold in all best models, which are those in which the extensions of P and Q coincide.
(8) $T := \{\neg P(b) \wedge \neg P(c), \nabla x P(x)\}$. In the best models $b = c$ will hold.

3 CHARACTERIZATION OF A RESTRICTED CLASS OF LIMIT PREFERENTIAL STRUCTURES

3.1 Combinatorial Characterization

Remark:
Our result treats a special class of limit preferential structures. Although the relation is not necessarily transitive, the structures we consider satisfy condition (5) in Definition 3.1 below, which we may call a principle of dispersedness. In the end, we show by an example that this condition (5) does not hold in all limit preferential structures. We also give a sketch of how to patch up the construction for the more general case without (5), though we have not pursued this formally.

Definition 3.1
Let Z be a set, and $\mathcal{Y} \subseteq \mathcal{P}(Z)$ ($\mathcal{P}(Z)$ the power set of Z). Let $\Sigma_Y \subseteq \mathcal{Y}$ be defined for all $Y \in \mathcal{Y}$, and set $\Sigma := \{(Y, S_Y) : Y \in \mathcal{Y}, S_Y \in \Sigma_Y\}$. We then say that Σ is over Z.
We consider the following conditions for $\mathcal{Y}$ and Σ:
(1) If $X, Y \in \mathcal{Y}$, $X - Y \neq \emptyset$, then there is $V \in \mathcal{Y}$, $\emptyset \neq V \subseteq X - Y$,
(2) $\Sigma_Y \subseteq \mathcal{P}(Y)$,
(3) $Y \in \Sigma_Y$, $\emptyset \notin \Sigma_Y$ (the latter if $Y \neq \emptyset$),
(4) $Y' \in \mathcal{Y} \wedge S_Y \in \Sigma_Y \wedge S_Y \subseteq Y' \subseteq Y \rightarrow S_Y \in \Sigma_{Y'}$,
(5) If n is finite, $S_{Y_i} \in \Sigma_{Y_i}$, $S_Y \in \Sigma_Y$, and $S_{Y_1} \cap \ldots \cap S_{Y_n} \cap (Y - S_Y) \neq \emptyset$, then $S_Y - \bigcup\{Y_i - S_{Y_i} : 1 \leq i \leq n\} \neq \emptyset$.

Remark:
We intend Z to be the set of models of a language $\mathcal{L}$, and $\mathcal{Y}$ to be the family of definable sets of models of that language. Condition (1) is satisfied in the propositional case, as singletons are definable by a theory. In the first-order case, if X and Y correspond to the sets of models of two theories T and T', and $m \in X - Y$, take $Th(m)$, the set of formulas valid in m, then the models of $Th(m)$, $M(Th(m))$ are included in X-Y, so condition (1) is satisfied here too.

Definition 3.2
Let $\mathcal{W} :=< W, \prec>$ be a preferential structure over Z, where the elements $w \in W$ are $< z, i >$ with $z \in Z$, and i an index, and $\prec$ is a binary relation on W. For $Y \subseteq Z$, we say that $\delta_Y \subseteq W$ is a minimizing segment of Y in $\mathcal{W}$, iff
1. $\forall < y, i > \in \delta_Y . y \in Y$,
2. $\forall < y, i > \in W$ with $y \in Y$ there is $< z, j > \in \delta_Y$ with $< y, i > \succ < z, j >$ or $< y, i > = < z, j >$,
3. $\forall < y, i > \in \delta_Y \ \forall < z, j > \in W$ ($z \in Y, < y, i > \succ < z, j > \to < z, j > \in \delta_Y$).
For $S_Y \subseteq Y \subseteq Z$ we say that S_Y minimizes Y in $\mathcal{W}$ iff there is δ_Y which is a minimizing segment of Y in $\mathcal{W}$ and snug in S_Y, i.e. $\forall < y, i > \in \delta_Y . y \in S_Y$ and $\forall y \in S_Y \exists i . < y, i > \in \delta_Y$.
Given $\mathcal{Y} \subseteq \mathcal{P}(Z)$, we set for $Y \in \mathcal{Y}$ $\Sigma_Y := \{S_Y \in \mathcal{Y} : S_Y$ minimizes Y in $\mathcal{W}\}$, and $\Sigma_{\mathcal{W}} := \{(Y, S_Y) : Y \in \mathcal{Y}, S_Y \in \Sigma_Y\}$.

Theorem 3.1
(a) Let $\mathcal{W}$ be a preferential structure over Z and $\mathcal{Y} \subseteq \mathcal{P}(Z)$, then $\Sigma_{\mathcal{W}}$ satisfies conditions $(2)-(4)$ in Definition 3.1.
(b) Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ satisfy (1), and Σ be over Z and satisfy the conditions $(2)-(5)$. Then there is a preferential structure $\mathcal{W}$ over Z such that
1. $(Y, S_Y) \in \Sigma \to S_Y$ minimizes Y in $\mathcal{W}$,
2. If X, S_X are such that $S_X \subseteq X$ minimizes X in $\mathcal{W}$, then there is $(Y, S_Y) \in \Sigma$ with $S_Y \subseteq S_X \subseteq X \subseteq Y$.

Proof:
(a) is trivial.
(b)

The construction
For $u \in Z$, let $\Pi_u := \{ \ f : \{(Y, S_Y) : (Y, S_Y) \in \Sigma, u \in Y\} \to Z\text{: } f(Y, S_Y) \in S_Y \ \}$.
We note that, by (5), if $\Sigma_0 \subseteq \Sigma$ is finite and $\forall (Y', S_{Y'}) \in \Sigma_0 . u \in S_{Y'}$, then $\exists f \in \Pi_u . [ran(f) \cap \bigcup\{Y' - S_{Y'} : (Y', S_{Y'}) \in \Sigma_0\} = \emptyset]$: Let $(Y, S_Y) \in \Sigma$, with $u \in Y$. If $u \in S_Y$, then let $f(Y, S_Y) := u \in \bigcap\{S_{Y'} : (Y', S_{Y'}) \in \Sigma_0\}$, if $u \notin S_Y$, choose $f(Y, S_Y)$ by (5) in $S_Y - \bigcup\{Y' - S_{Y'} : (Y', S_{Y'}) \in \Sigma_0\}$.

Consider the elements
$< u, f, \Sigma_0, i >$: $u \in Z$,
$f \in \Pi_u$,
$\Sigma_0 \subseteq \Sigma$ finite,
$(Y, S_Y) \in \Sigma_0 \rightarrow u \in S_Y$,
$ran(f) \cap \bigcup\{Y - S_Y : (Y, S_Y) \in \Sigma_0\} = \emptyset$,
$i = 0$ or $i = 1$.
Define the relation $\succ$ by
$< u, f, \Sigma_0, 0 > \succ < v, g, \Sigma_1, 0 > :\longleftrightarrow \exists (Y, S_Y).[f(Y, S_Y) = v \wedge \Sigma_1 = \{(Y, S_Y)\} \cup \{(Y', S_{Y'}) \in \Sigma_0 : v \in S_{Y'}\}]$
(Note: This condition, and thus $\succ$, are therefore independent of g.)
and
$< u, f, \Sigma_0, 1 > \succ < v, g, \Sigma_1, 1 > :\longleftrightarrow \exists (Y, S_Y).[f(Y, S_Y) = v \wedge \Sigma_1 = \{(Y, S_Y)\} \cup \{(Y', S_{Y'}) \in \Sigma_0 : v \in S_{Y'}\}]$ *or* there is $(Y, S_Y) \in \Sigma$ such that $\Sigma_0 = \{(Y, S_Y)\}$ and $v \notin Y$.
We shall distinguish the last case typographically, by writing $\bullet \succ$ instead of $\succ$, though it is the same relation.
Let W be the set of legal elements, i.e. elements of the form $< u, f, \Sigma_0, i >$ as defined above, and $\mathcal{W} :=< W, \prec >$.
Note that, by above remark, for $u \in Z$, $\Sigma_0 \subseteq \Sigma$ finite with $u \in S_Y$ for all $(Y, S_Y) \in \Sigma_0$, $i \in 2$, we find f such that $< u, f, \Sigma_0, i > \in W$.

Claim 3.2
(b) 1. holds.

Proof:
Fix $(Y, S_Y) \in \Sigma$. Set $\delta_Y := \{< v, g, \Sigma_1, i > \in W$: $v \in S_Y$, $(Y, S_Y) \in \Sigma_1\}$.
(a) δ_Y minimizes Y.
Let $< u, f, \Sigma_0, i > \in W$ with $u \in Y$, so $f(Y, S_Y) \in S_Y$ is defined. Let $v := f(Y, S_Y) \in S_Y$ and $\Sigma_1 := \{(Y, S_Y)\} \cup \{(Y', S_{Y'}) \in \Sigma_0 : v \in S_{Y'}\}$. Σ_1 is finite, for $(Y', S_{Y'}) \in \Sigma_1$ $v \in S_{Y'}$, and there is $g \in \Pi_v$ such that $ran(g) \cap \bigcup\{Y' - S_{Y'} : (Y', S_{Y'}) \in \Sigma_1\} = \emptyset$ by the above remark. Then $< v, g, \Sigma_1, i > \in W$, and $< u, f, \Sigma_0, i > \succ < v, g, \Sigma_1, i > \in \delta_Y$.
(b) δ_Y is closed in Y.
We distinguish the cases $\succ$ and $\bullet \succ$.
Let $x :=< v, g, \Sigma_1, i > \succ y :=< w, h, \Sigma_2, i > \in W$ with $x \in \delta_Y$ and $w \in Y$, we have to show $< w, h, \Sigma_2, i > \in \delta_Y$. As $(Y, S_Y) \in \Sigma_1$, $ran(g) \cap (Y - S_Y) = \emptyset$. But $w \in ran(g)$ and $w \in Y$, so $w \in S_Y$. By $(Y, S_Y) \in \Sigma_1$, $w \in S_Y$, and $x \succ y$, we have $(Y, S_Y) \in \Sigma_2$. Thus $y \in \delta_Y$.
Let $x :=< v, g, \{(Y', S_{Y'})\}, 1 > \bullet \succ y :=< w, h, \Sigma_2, 1 > \in W$ with $x \in \delta_Y$ and $w \in Y$. By the definition of δ_Y, $(Y', S_{Y'}) = (Y, S_Y)$, so by the definition of $\bullet \succ$ $w \notin Y$, so this case cannot arise. $\Box$

Claim 3.3
(b) 2. holds.

Proof:
Suppose there are $X \in \mathcal{Y}$, and $S_X \subseteq X$ minimizing X by some δ_X in $\mathcal{W}$. We show that there is a $(Y, S_Y) \in \Sigma$ with $S_Y \subseteq S_X \subseteq X \subseteq Y$. If $X = S_X$, we are done, as $(X, X) \in \Sigma$. So we assume there is $u \in X - S_X$, and fix this u.
We first show

Claim 3.4
For $i \in 2$, there is $(Y, S_Y) \in \Sigma$ such that
(1) $u \in Y$, and (2) for all $v \in S_Y$, there is g with $< v, g, \{(Y, S_Y)\}, i > \in \delta_X$.

Proof:
We fix i, and assume that the Claim is false. By (3) of Definition 3.1 $(X, X) \in \Sigma$, so there is some $(Y, S_Y) \in \Sigma$ satisfying (1). For each $(Y, S_Y) \in \Sigma$ with $u \in Y$ choose an element $v \in S_Y$ such that $\neg\exists g. < v, g, \{(Y, S_Y)\}, i > \in \delta_X$ by the choice function f. Consider the element $< u, f, \emptyset, i >$. $< u, f, \emptyset, i > \notin \delta_X$ by $u \notin S_X$. So there must be some $< v, g, \Sigma', i > \in \delta_X$ below it. This cannot be by $\bullet \succ$, as $\bullet \succ$ is defined only for $\Sigma'' \neq \emptyset$ on the left hand side. So $< v, g, \Sigma', i > = < v, g, \{(Y', S_{Y'})\}, i >$, a contradiction. $\Box$ (Claim 3.4)

Choose by Claim 3.4 for $i = 1$ $(Y, S_Y) \in \Sigma$ with $u \in Y$ such that for all $v \in S_Y$, there is g with $< v, g, \{(Y, S_Y)\}, 1 > \in \delta_X$. Then $S_Y \subseteq S_X$, by the fact that the first components of the elements of δ_X are in S_X. It remains to show $X \subseteq Y$. Assume $X \not\subseteq Y$. Then, by (1) in Definition 3.1, there is $V \in \mathcal{Y}$ such that $\emptyset \neq V \subseteq X - Y$. Let $< u, f, \emptyset, 1 > \in W$ for some suitable f, then there is $< f(Y, S_Y), g, \{(Y, S_Y)\}, 1 > \in \delta_X$ with $< u, f, \emptyset, 1 > \succ < f(Y, S_Y), g, \{(Y, S_Y)\}, 1 >$. Moreover, we have for some $v' \in V$ and $g' \in \Pi_{v'}$ $< v', g', \{(V, V)\}, 1 > \in W$, and, by definition of $\bullet \succ$, $< f(Y, S_Y), g, \{(Y, S_Y)\}, 1 >$ $\bullet \succ < v', g', \{(V, V)\}, 1 > \bullet \succ < u, f, \emptyset, 1 >$ (the latter by $u \in Y$), contradicting the downward closure of δ_X in X by $u \in X - S_X$. $\Box$ (Claim 3.3 and Theorem)

We now present a counterexample to (5), and indicate how things might be patched up.

Example 3.1
Consider the elements $\{a, b, c, d, e, f, g\}$. Let all but g occur exactly once, and g occur twice, as g_1 and g_2.
Set $Y_1 := \{a, b, c, g\}$, $Y_2 := \{d, e, f, g\}$, $Y := \{c, f, g\}$.
Set $a \succ g_1$, $c \succ b$, $d \succ g_2$, $f \succ e$, $g_1 \succ e$, $g_1 \succ f$, $g_2 \succ c$, $g_2 \succ b$.

Then we have $\delta_{Y_1} = \{g_1, b\}$, $S_{Y_1} = \{b, g\}$, $\delta_{Y_2} = \{g_2, e\}$, $S_{Y_2} = \{e, g\}$, $\delta_Y = \{c, f\}$, $S_Y = \{c, f\}$. So $g \in S_{Y_1} \cap S_{Y_2} \cap (Y - S_Y)$, but $S_Y \subseteq (Y_1 - S_{Y_1}) \cup (Y_2 - S_{Y_2})$. $\square$

How to patch things up?
(1) The condition with $n = 1$ has to hold: Suppose $u' \in S_{Y'} \cap (Y - S_Y)$ with $S_Y \subseteq Y' - S_{Y'}$. As $S_{Y'}$ minimizes Y', there is a copy $< u', i' > \in \delta_{Y'}$ below which exists $< u, i > \in \delta_Y$, but $u \in Y' - S_{Y'}$, contradiction.
(2) The problem arises, if we consider a copy $< u_i, i >$ that has to be in several δ_{Y_i}, and u_i is in some Y, and $S_Y \subseteq \bigcup (Y_i - S_{Y_i})$. This is impossible, as we have to find below $< u_i, i >$ a $< u, j >$ such that $u \in S_Y$, but this contradicts the closure of at least one of the δ_{Y_i}.
(3) What can force us to have such $< u_i, i >$ in several δ_{Y_i} ? Suppose we have $u_1 \in S_{Y_1}$, $u_1 \in Y_2$. As above in the Example, this does not yet force us to have $< u_1, i_1 >$ in $\delta_{Y_1} \cap \delta_{Y_2}$. Let $< u_1, i_1 >$ be in δ_{Y_1}, then it suffices to have a possible $< u_2, i_2 >$ in δ_{Y_2}, $< u_1, i_1 > \succ < u_2, i_2 >$ with $u_2 \notin Y_1$, and we can forget Y_1, i.e. $< u_2, i_2 >$ need not be in δ_{Y_1}. This is impossible if $S_{Y_2} \subseteq Y_1$. Then we have to be able to find $< u_2, i_2 > \in \delta_{Y_2}$ with $u_2 \in Y_1$, so $< u_2, i_2 >$ has to be in $\delta_{Y_1} \cap \delta_{Y_2}$, and $u_2 \in S_{Y_1} \cap S_{Y_2}$. Now, u_2 *can't* be in any Y_3 with $S_{Y_3} \subseteq (Y_1 - S_{Y_1}) \cup (Y_2 - S_{Y_2})$. So we have to choose u_2 outside such Y_3. If we have $u_2 \in Y_3$ with $S_{Y_3} \subseteq Y_1 \cup Y_2$, then we need $< u_3, i_3 > \in \delta_{Y_3}$ with $< u_2, i_2 > \succ < u_3, i_3 >$. $< u_3, i_3 >$ has to be in $\delta_{Y_1} \cap \delta_{Y_3}$, if $u_3 \in Y_1$, $< u_3, i_3 >$ has to be in $\delta_{Y_2} \cap \delta_{Y_3}$, if $u_3 \in Y_2$, $< u_3, i_3 >$ has to be in $\delta_{Y_1} \cap \delta_{Y_2} \cap \delta_{Y_3}$, if $u_3 \in Y_1 \cap Y_2$, etc.
Essentially, we can encode the construction process into a tree of height $\leq \omega$, where each choice corresponds to a branching in the tree. One branch has to be free for the construction, i.e. there must not be any S_{Y_i} as above, which would destroy the construction.
Alternatively, the construction can be seen as directed by finite non-empty intersections of the type $S_{Y_1} \cap S_{Y_2} - \bigcup \{Y : S_Y \subseteq (Y_1 - S_{Y_1}) \cup (Y_2 - S_{Y_2})\}$ $S_{Y_1} \cap S_{Y_2} \cap S_{Y_3} - \bigcup \{Y : S_Y \subseteq (Y_1 - S_{Y_1}) \cup (Y_2 - S_{Y_2}) \cup (Y_3 - S_{Y_3})\}$, where the sequence stops if there is e.g. no Y_3 with $S_{Y_3} \subseteq Y_1 \cup Y_2$.
Making the relation $\prec$ transitive changes the situation considerably. It also has logical repercussions, as the counterexamples at the end of this section on the one hand, the positive results discussed in section 2.4 on the other hand show. See also [Sch94-t1] for a different restricted completeness result with transitive relations.

Lemma 3.5
(Taken from [BS93].)
Let $\mathcal{W} := < W, \prec >$ be a preferential structure over Z, with a transitive relation $\prec$. Let $X, Y, S \subseteq Z$.
(a) If α and β are minimizing segments of S, then so is $\alpha \cap \beta$,
(b) If $X, Y \subseteq S$, α is a minimizing segment of X, and β is a minimizing segment of Y, then there is some $\gamma \subseteq \alpha \cup \beta$ such that γ is a minimizing segment of $X \cup Y$

Proof:
(a) If $< a,i >\in \alpha \cap \beta$, $x \in S$, $< x,j >\prec< a,i >$, then $< x,j >\in \alpha \cap \beta$, as α and β are closed in S. If $x \in S$, and $< x,j >\notin \alpha$, then there is $< a,i >\in \alpha$, $< a,i >\prec< x,j >$. If $< a,i >\notin \beta$, then there is $< b,k >\in \beta$, $< b,k >\prec< a,i >$, so $< b,k >\in \alpha$ too. By transitivity, $< b,k >\prec< x,j >$.
(b) Set $\alpha' := \{< a,i >\in \alpha : \neg\exists < b,k >\prec< a,i > (b \in Y\wedge < b,k >\notin \beta)\}$, likewise $\beta' := \{< b,k >\in \beta : \neg\exists < a,i >\prec< b,k > (a \in X\wedge < a,i >\notin \alpha)\}$, and set $\gamma := \alpha' \cup \beta'$. Let $< x,j >\in W$, $x \in X \cup Y$, without loss of generality $x \in X$, then there is $< a,i >\in \alpha$, $< a,i >\prec< x,j >$ (or $< a,i >=< x,j >$). Suppose $< a,i >\notin \alpha'$, so there is $< b,k >\notin \beta$, $b \in Y$ with $< b,k >\prec< a,i >$, but then there is $< b',k' >\in \beta$ with $< b',k' >\prec< b,k >$, and by transitivity $< b',k' >\prec< x,j >$. It remains to show $< b',k' >\in \beta'$. But if $< b',k' >\in \beta - \beta'$, there is $< a',i' >\notin \alpha$, $a' \in X$ with $< a',i' >\prec< b',k' >$, so $< a',i' >\prec< b',k' >\prec< a,i >$, contradicting, by transitivity again, closure of α in X. Let, on the other hand, $< c,m >\in \gamma$, and $x \in X \cup Y$ with $< x,j >\prec< c,m >$. Suppose without loss of generality $< c,m >\in \alpha'$. Case 1: $x \in X$, so $< x,j >\in \alpha$ by $< c,m >\in \alpha' \subseteq \alpha$. If $< x,j >\notin \alpha'$, there is $< b,k >\notin \beta$, $b \in Y$ with $< b,k >\prec< x,j >$, so $< b,k >\prec< c,m >$, contradicting $< c,m >\in \alpha'$. Case 2: $x \in Y$, then $< x,j >\in \beta$, as otherwise by $< x,j >\prec< c,m >$, $< c,m >\notin \alpha'$. But if $< x,j >\in \beta - \beta'$, then there is $< a',i' >\notin \alpha$, $a' \in X$ with $< a',i' >\prec< x,j >$, so $< a',i' >\prec< c,m >$, and $< c,m >\in \alpha$, contradicting closure of α in X. So $< x,j >\in \beta'$. $\Box$

3.2 Logical Characterization

We turn to the logical counterpart of Theorem 3.1.
Let $\vdash$ be a logic for a language $\mathcal{L}$, $M_{\mathcal{L}}$ the set of all $\mathcal{L}-models$ of a semantics for which $\vdash$ is sound and complete. Denote by $\models$ this relation of validity. Suppose further that $\boldsymbol{D}$, the family of definable sets of models, satisfies condition (1) of Definition 3.1. For a set $X \subseteq M_{\mathcal{L}}$ of models, let $Th(X)$ be the set of formulas which hold in all $m \in X$. Conversely, for a theory T, let $M(T)$ be the set of all its models. For a theory T, let $\overline{T}$ be the set of $\vdash -consequences$ of T, and for a (fixed) logic $\mid\sim$ other than $\vdash$ let $\overline{\overline{T}}$ be the set of $\mid\sim -consequences$ of T.

Definition 3.3
Let a limit preferential structure $\mathcal{W}$ be given over $M_{\mathcal{L}}$, and let $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ contain $\boldsymbol{D}$. Define $T \models_{\mathcal{W}} \phi$ iff there is some $S_\phi \in \Sigma_{M(T)}$ in $\mathcal{W}$ such that for all $m \in S_\phi$ $m \models \phi$ holds, and say that $\mathcal{W}$ represents $\mid\sim$ iff $\mid\sim = \models_{\mathcal{W}}$.

Theorem 3.6
Consider the set of conditions:

(1L) $\overline{T} = \overline{T'} \rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$,
(2L) $\overline{\overline{T}}$ is closed under weakening, i.e. if $\phi \in \overline{\overline{T}}$, and $\vdash \phi \rightarrow \psi$, then $\psi \in \overline{\overline{T}}$,
(3L) $T \subseteq \overline{\overline{T}}$,
(4L) $Con(T) \rightarrow Con(\overline{\overline{T}})$ - where Con stands for consistency under $\vdash$,
(5L) There is for each theory T a system Σ_T of theories, closed under $\vdash$, such that
(α) $\overline{T} = \overline{T'} \rightarrow \Sigma_T = \Sigma_{T'}$,
(β) $\overline{T} \in \Sigma_T$,
(γ) $S_T \in \Sigma_T \rightarrow T \subseteq S_T \subseteq \overline{\overline{T}}$,
(δ) $\bigcup \Sigma_T = \overline{\overline{T}}$,
($\in$) If $S_T \in \Sigma_T$ and $\overline{T} \subseteq \overline{T'} \subseteq S_T$, then $S_T \in \Sigma_{T'}$,
(ζ) Let $S_{T_i} \in \Sigma_{T_i}$ for $i = 1, \ldots, n$, $S_T \in \Sigma_T$, and let there be a theory T' with $\overline{T'} \neq \mathcal{L}$ *with* $T' \vdash S_{T_i}$ *for all* i, $T' \vdash T$ *and* $\neg Con(T' \cup S_T)$. *Then there is a* theory T'' with $\overline{T''} \neq \mathcal{L}$ with $T'' \vdash S_T$ and ($T'' \vdash S_{T_i}$ or $\neg Con(T'' \cup T_i)$) for all i. (T' and T'' may be assumed complete under $\vdash$.)

(a) If $\mid\!\sim$ is represented by a limit preferential structure $\mathcal{W}$ in which all models occur, then $(1L) - (4L)$, and $(\alpha) - (\in)$ of (5L) hold for $\Sigma_T := \{Th(S) : S \in \Sigma_{M(T)}\}$, where $\Sigma_{M(T)}$ is defined for $\mathcal{W}$ by Definition 3.2.
(b) If all conditions hold, then we can represent $\mid\!\sim$ by such a structure in which all models occur.

Proof:
(a):
(Occurrence of all models is needed for (4L) - but this is a minor point.) Let $\mid\!\sim$ be represented by $\mathcal{W}$. $(1L) - (3L)$ are trivial. $(\alpha) - (\delta)$ of (5L) are immediate from the definition of representation, $(5L)(\in)$ is a consequence of condition (4). (4L): By compactness of classical logic, and (5L) (δ), it suffices to show that finite intersections of elements of $\Sigma_{M(T)}$ are nonempty, if T is consistent. $Con(T)$ implies $M(T) \neq \emptyset$, so by Definition 3.1, condition (3), for each $S \in \Sigma_{M(T)}$, $S \neq \emptyset$. The following argument proves the induction step. Let $\Sigma' := \{S_i : i \in n\} \subseteq \Sigma_Y$ with $\bigcap \Sigma' \neq \emptyset$. Let δ_i be the associated minimizing segments. A straightforward inductive argument shows that $\bigcap\{\delta_i : i \in n\} \neq \emptyset$, too. Now let $S \in \Sigma_y$ be arbitrary, with minimizing segment δ. Choose $< x, i > \in \bigcap\{\delta_i : i \in n\}$. If $< x, i > \notin \delta$, then there is $< y, j > \prec < x, i >$ with $< y, j > \in \bigcap\{\delta_i : i \in n\} \cap \delta$. So $\bigcap \Sigma' \cap S \neq \emptyset$.
(b): Let $\mathcal{Y} := \boldsymbol{D}$, for $Y = M(T)$, let $\Sigma_Y := \{M(S_T) : S_T \in \Sigma_T\}$. Then $(1) - (5)$ hold. Construct $\mathcal{W}$ over $M_{\mathcal{L}}$ by Theorem 3.1 (b). Assume $T \mid\!\sim \phi$. Then there is by $(5L)(\delta)$ $S_T \in \Sigma_T$ with $\phi \in S_T$, so for $m \in M(S_T)$ $m \models \phi$, so by Definition 3.3, $T \models_{\mathcal{W}} \phi$. Assume $T \models_{\mathcal{W}} \phi$, then by the definition of representation there is some $S \in \Sigma_{M(T)}$ in $\mathcal{W}$ with $m \models \phi$ for all $m \in S$, and by 2. of Theorem 3.1 (b) there is $(Y, S_Y) \in \Sigma$ with $S_Y \subseteq S \subseteq M(T) \subseteq Y$. Let $Y = M(T')$, and $S_Y = M(S_{T'})$ for $S_{T'} \in \Sigma_{T'}$. Then $\phi \in S_{T'}$,

and by $\overline{T'} \subseteq \overline{T} \subseteq S_{T'}$ *we have* $S_{T'} \in \Sigma_T$ *by* $(5L)(\in)$, *so* $T \mid\sim \phi$ *by* $S_{T'} \subseteq \overline{\overline{T}}$. □

We fix a structure $\mathcal{W}$ and consider the semantical consequence relation $\alpha \mid\sim \beta$ iff $\alpha \models_{\mathcal{W}} \beta$.
We examine validity of the "classical" inference rules recalled in Definition 2.13.
Right Weakening, Reflexivity, and Left Logical Equivalence hold trivially.
We now present counterexamples for the others. Take the classical propositional language $\mathcal{L}$ with two variables, p, q. Take the set of models $M := \{m_0 \models p, q,\ m_1 \models p, \neg q,\ m_2 \models \neg p, q,\ m_3 \models \neg p, \neg q\}$.
For "Or" and the Deduction Theorem we order M by $m_3 \succ m_2 \succ m_1 \succ m_0$ (without transitivity!). The set of p-models in M, $M(p)$ is $\{m_0, m_1\}$, so $\{m_0\}$ is a minimizing segment of $M(p)$. The set of $\neg p-models$ in M, $M(\neg p)$ is $\{m_2, m_3\}$, so $\{m_2\}$ is a minimizing segment of $M(\neg p)$. Thus $p \mid\sim q$, $\neg p \mid\sim q$. But any minimizing segment of M has to contain m_2, and then by closure m_1. Thus, neither $true \mid\sim q$, nor $true \mid\sim p \rightarrow q$, since m_1 is a model of $p \wedge \neg q$.
For a counterexample to "And", order M by $m_3 \succ m_1 \succ m_0$, $m_3 \succ m_2 \succ m_0$ (without transitivity). $\{m_0, m_1\}$ and $\{m_0, m_2\}$ are the only minimizing segments of M, so $True \mid\sim p$ and $true \mid\sim q$, but $true \not\mid\sim p \wedge q$.
For a counterexample to "Rational Monotony", order M by $m_3 \succ m_0$, $m_2 \succ m_1$. Then $true \mid\sim p$, but neither $true \mid\sim \neg q$ nor $q \mid\sim p$.
For a counterexample to "Cautious Monotony", choose $M_0 := \{m \models p, q,\ m_{1,1} \models p, \neg q,\ m_{1,2} \models p, \neg q,\ m_2 \models \neg p, q,\ m_3 \models \neg p, \neg q\}$, ordered by $m_3 \succ m_{1,1}$, $m_3 \succ m_2$, $m_{1,2} \succ m_{1,1}$, $m_{1,2} \succ m_2$, $m_{1,1} \succ m_0$, $m_2 \succ m_0$ (again without transitivity). $\{m_{1,1}, m_0\}$ is a minimizing segment of M, so is $\{m_0, m_2\}$, thus $true \mid\sim p$ and $true \mid\sim q$. But $M(p) = \{m_0, m_{1,1}, m_{1,2}\}$ has $M(p)$ and $\{m_0, m_{1,1}\}$ as minimizing segments, so $p \not\mid\sim q$.
The situation changes when we consider transitive relations - see Lemma 3.5 above and Lemma 2.7.

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