NON-PRIORITIZED BELIEF REVISION BASED ON DISTANCES BETWEEN MODELS

Karl Schlechta
Laboratoire d'Informatique de Marseille, CNRS ESA 6077
CMI, Technopôle de Château-Gombert
F-13453 Marseille Cedex 13, France
ks@gyptis.univ-mrs.fr

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Abstract

We base Theory Revision on a notion of distance between the models of the underlying logic. Revisions constructed from such distances have nice properties: The AGM postulates are (with a minor exception) satisfied, and additional properties, e.g. for iterated revision, hold. The present article adapts this idea to non-prioritized Theory Revision. Some motivation and comparison to other, similar approaches are given, and so is a representation result.

1 INTRODUCTION

1.1 Overview

We base Theory Revision on a notion of distance between models of the underlying logic. In the case of prioritized Theory Revision, the revision of a theory T by a formula ϕ, $T * \phi$, will be the set of formulas valid in those models of ϕ, which are closest to some T-model. In the case of non-prioritized Theory Revision, the revision of a theory T by a formula ϕ, $T \otimes \phi$, will be the set of formulas valid in those models of T or ϕ, which have minimal distance among all pairs of T and $\phi - models$.
More precisely, assume a logic, a language $\mathcal{L}$, a distance d on $M_{\mathcal{L}}$ (the set of $\mathcal{L} - models$) to be given, and $A, B \subseteq M_{\mathcal{L}}$. Let, for the prioritized case, $A \mid B$ be the set of those $b \in B$, s.t. there is $a_b \in A$ and for all $a\prime \in A$, $b\prime \in B$ $d(a_b, b) \leq d(a\prime, b\prime)$, and for the non-prioritized case, $A \uparrow B$ be the set of those $a \in A$ and $b \in B$, s.t. for all $a\prime \in A$, $b\prime \in B$ $d(a, b) \leq d(a\prime, b\prime)$. Given a theory T, and a formula ϕ, with $M(T)$

and $M(\phi)$ their respective sets of models, we define $T * \phi =_{df} Th(M(T) \mid M(\phi))$ and $T \otimes \phi =_{df} Th(M(T) \uparrow M(\phi))$, where for $X \subseteq M_{\mathcal{L}}$ $Th(X)$ is the set of formulas which hold in X.
To prove a representation theorem, we first show an "algebraic" representation result for arbitrary sets (to be thought of as sets of models), and then translate this result to logic. This technique has the advantage that the result is almost totally robust with respect to the underlying logic, changing the logic changes the result only minimally. Moreover, our result gives thus immediately a representation result for revision of one theory by another, and not only for revision by one formula. This seems even more desirable in the case of non-prioritized Theory Revision.
Note that $A \uparrow B = (A \mid B) \cup (B \mid A)$, so we could base the non-prioritized case directly on the prioritized one. Nonetheless, we present a direct proof, for the following reason: Non-prioritized Theory Revision is not yet a very established theory, and intuitions might change. At this stage, it might be preferable not to mix non-prioritized and prioritized Theory Revision. We therefore prefer a direct construction, independent of intuitions about prioritized Theory Revision, even if the result is the same.
We discuss some properties and difficulties of the construction in Sections 2.3 and 2.4. The representation result is prepared by an order-theoretic result in Section 2.5.

1.2 Motivation

It might be appropriate to give some motivation for both approaches, i.e. to base Theory Revision on a notion of distance between models of the underlying logic, and to consider non-prioritized Theory Revision.
Instead of trying to give an exhaustive motivation - something the author is certainly not very good at, and he refers the reader e.g. to [Rab95] for a motivation of non-prioritized revision - we will try to give our personal motivation to base revision on a notion of distance between models, and mention a situation where we think that non-prioritized revision, perhaps better called theory merger (expression due to D.Makinson), is justified. First, we turn to a motivation of our semantical foundation, and then to that of non-prioritized Theory Revision.

1.2.1 A motivation of Theory Revision based on distance between models

It is the author's conviction that there are some, perhaps not too many, basic notions behind the semantics of many non-classical logics, and that the notions of distance and size are among them.
We use the word "distance" here in a rather large sense. First, formally, a distance between two points is not necessarily a real number. It is just a value in some totally ordered set, so we can compare distances, but not necessarily add them etc. (see Definition 2.1). Second, in the intuitive discussion, we will speak of distances in the case of ternary relations (in counterfactual conditionals) or even binary relations (in preferential models). We see

these relations as abstractions of a distance, first from one of the three points ($w\prime <_w w\prime\prime$ expressing that $w\prime$ is closer to w than $w\prime\prime$ is to w), second from a fixed ideal point, $w < w\prime$ expressing that w is closer to this point than is $w\prime$. This rough use is sufficient for our purposes.
The use of an abstract notion of distance is perhaps most obvious in the Stalnaker/Lewis semantics for counterfactual conditionals. An example of a counterfactual conditional is: "If it were to rain, I would take an umbrella." The Stalnaker/Lewis semantics consists of a set of possible worlds (as for a Kripke semantics for modal logic), with a distance relation. Let $\phi \gg \psi$ stand for the counterfactual conditional "if ϕ were true, then ψ would hold", and let m be one of the possible worlds. $\phi \gg \psi$ is defined to hold in m iff in those possible worlds, which are closest to m among those where ϕ holds, ψ holds, too. The intuition is the following: If my world changes *minimally* so that it rains, then I will take an umbrella. There may be worlds very distant from mine, where there is e.g. always a very strong wind, and where it makes no sense to take an umbrella. But these distant worlds will not count.
The basic concept of this semantics is thus one of "relative closeness", sometimes called "relative similarity" or conversely "relative distance". It is given formal representation in various equivalent ways, notably by a family of indexed two-placed relations $\prec_a$ between worlds, with $x \prec_a y$ read intuitively as "x is closer to a than is $y\prime\prime$.
Nonmonotonic logics speaks and reasons about what normally holds, deontic logic about what "morally" holds, i.e. which states of affairs are morally acceptable. A preferential model $\mathcal{M}$ for nonmonotonic logics or deontic logic consists of a set of possible worlds, with a preference relation $\prec$ defined on this set. In the case of nonmonotonic logics, this preference relation chooses the more normal worlds, in the case of deontic logic, the more moral worlds. The semantic consequence relation $\phi \models_{\mathcal{M}} \psi$ with respect to such a structure $\mathcal{M}$ is then defined to hold iff in the most normal (moral) worlds, where ϕ holds, ψ holds, too.
With a little bit of abstraction, we can consider this preference relation as an abstract distance relation too: $m \prec m\prime$ iff m is closer to an ideal point of maximal normality or morality - the paradise of dullness (or saintety).
The notion of size can be seen behind another aspect of nonmonotonic reasoning, reasoning about the "usual" case. We will then say ϕ "usually" implies ψ, iff in "most" cases, where ϕ holds, ψ holds, too. A semantical structure $\mathcal{M}$ will then be a set of possible worlds, with an (abstract) measure of size defined on it. Then $\phi \models_{\mathcal{M}} \psi$ will be defined to hold, iff the set of worlds, where $\phi \wedge \psi$ holds, is large in the set of worlds where ϕ holds.
In legal reasoning, the concept of size has importance, too. A postulate can only be considered a law, if in most cases, people obey to it. (This was pointed out to the author by Otto Pfersmann, Lyon, in personal communication.)
The author sees the work of discovering such basic notions of (intuitive and formal) semantics as one of the aims in research on nonclassical logics. This work has a philosophical aspect, but also a mathematical side, in making the connections precise, and examining

various abstractions of such notions (e.g. distance expressed by a binary relation - of different strengths -, by metrics etc.). We thus strive to reduce one notion, like nonmonotonic reasoning, to another notion, like distance.
Such a reductionist approach does not claim to give absolute explanations, it rather tries to establish connections. Moreover, it can point out similarities, by reducing different concepts to the same basic ideas. As pointed out above, the Stalnaker/Lewis semantics for counterfactual conditionals, preferential semantics for nonmonotonic and deontic logic reduce to the notion of distance. Consequently, results from one field can be carried over to other cases.
Of course, this defense of the reductionist approach is no argument against explaining some notion by still other means.

1.2.2 A (very brief) motivation of non-prioritized Theory Revision

In usual (prioritized) Theory Revision, (see e.g. [AGM85]) "new" information has precedence over old information. This has historical reasons: The AGM approach was motivated by legal reasoning, where newer laws have precedence over older laws. In our framework, this is not the case. We try to preserve as much as possible of both the old and the new information, but are also prepared to sacrifice some of both.
We think that inheritance diagrams can be considered as guidelines for prioritized and non-prioritized Theory Revision. Inheritance diagrams describe the inheritance of properties from superclasses to subclasses. If a subclass inherits information from two superclasses, this might lead to contradictions. In some cases, a preference of one information over the other can be established, in other cases not. Treatment of the conflict by prioritized Theory Revision in the former case, by unprioritized Revision in the second case seems to be the adequate solution. We refer the reader e.g. to [Sch97] for a discussion of inheritance diagrams.

1.3 Historical remarks

Non-prioritized Theory Revision seems to have been first considered by Hansson (see [Han94]) and, independently, under the label "arbitration", by Revesz (see [Rev93]). Liberatore and Schaerf ([LS95], [LS98]) continue the work of Revesz, discuss arbitration based on a distance between models (without a representation theorem), and give a Katsuno/Mendelzon style representation result for a family of partial orders. Theory Revision based on a distance between models of the underlying logic was also considered independently by Rabinowicz [Rab95] and Lehmann, Magidor, Schlechta ([LMS95], [SLM96], [Sch97-t2]).

1.4 Notation:

In our algebraic approach, we work over a fixed set Z - intuitively, $Z = M_{\mathcal{L}}$. We will often use a for $\{a\}$, ab for $\{a, b\}$ etc., context will disambiguate. $|$, $\uparrow$ and $\Uparrow$ have precedence over $\cap$ and $\cup$: $A \mid B \cap C =_{df} (A \mid B) \cap C$, etc. We use the convention that a, a_b etc. stand for elements in A, b, b_a etc. stand for elements in B.

2 TECHNICAL DEVELOPMENT

2.1 Overview

We first give an introduction to the technical development. We then demonstrate by an example a problem with constructing a distance function from a revision operation. Revision reveals itself as a relatively coarse instrument of investigation. Even in simple examples, we cannot totally determine the relation between all distances by examining the results of revision. Next, we show, again by an example, that the conditions considered by Rabinowicz in [Rab95] - the AGM postulates, "Unrestricted Back and Forth", and "Unrestricted Revision of Subsets" - are incomplete, essentially by lack of a loop condition. We then present an explicit method of constructing a symmetric distance function from two partial relations of comparison, $\prec$ and $\approx$, of distances. We need this construction because we cannot always determine the relation between distances. This result is used in the proof of the representation theorem. We finish by translating the algebraic representation result to logic, and comparing our approach to the postulates of Hansson in [Han97].

2.2 Introduction to the technical part

Our aim is to construct a distance function from the revision operation.
We recall the definition of a distance:

Definition 2.1
$d : ZxZ \to X$ is called a symmetric distance on Z iff $(d1) - (d3)$ hold:
(d1) X is totally ordered by an acyclic, transitive relation $<$, i.e. either $x = y$ or $x < y$ or $y < x$ will hold for any $x, y \in X$, and X has a smallest element 0,
(d2) $d(a, b) = 0$ iff $a = b$,
(d3) $d(a, b) = d(b, a)$
for any $a, b \in Z$.
Let $\leq$ stand for $<$ or $=$.

Remark 2.1
The author had first considered a "weakly symmetric distance" defined as follows:
$d : ZxZ \to X$ is called a weakly symmetric distance on Z iff $(d1) - (d4)$ hold:

(d1) X is totally ordered by an acyclic, transitive relation $<$, i.e. either $x = y$ or $x < y$ or $y < x$ will hold for any $x, y \in X$, and X has a smallest element 0,
(d2) $d(a,b) = 0$ iff $a = b$,
(d3) $d(a,b) = d(c,d) \rightarrow d(b,a) = d(d,c)$,
(d4) $d(a,b) < d(c,d) \rightarrow d(b,a) < d(d,c)$,
for any $a, b, c, d \in Z$.
An anonymous referee pointed out that "weak symmetry" entails "symmetry", by the following argument:
Suppose $d(a,b) \neq d(b,a)$, let e.g. $d(a,b) < d(b,a)$. So by weak symmetry $d(b,a) < d(a,b)$, contradicting absence of cycles.
This observation considerably shortened the proof, which is now a standard construction (see Section 2.5).

We take an "algebraic" approach, and consider first a "revision" on an arbitrary set, i.e. - in the prioritized case - a function $|: \mathcal{P}(Z)x\mathcal{P}(Z) \rightarrow \mathcal{P}(Z)$, where, intuitively, $A \mid B$ is the set of those $b \in B$ which are closest to some $a_b \in A$, and, in the non-prioritized case, a function $\uparrow: \mathcal{P}(Z)x\mathcal{P}(Z) \rightarrow \mathcal{P}(Z)$, where, intuitively, $A \uparrow B$ is the set of those $a \in A$ and $b \in B$ s.t. the distance between a and b is minimal among the distances between elements from A and B.
We try to define - given sufficiently strong conditions on $|$ or $\uparrow$ - a distance d between elements of Z. One might hope that the operations $|$ and $\uparrow$ are sufficiently discriminating to determine uniquely the relations between the distances - if such a distance exists. The example in Section 2.3 shows that this is not always the case. We give two situations of 4 points in the real plane, which have the same revision operations, but differ in their distance relations. Basically, the reason is that, even if we can always compare distances with common start or end, i.e. $d(a,b)$ and $d(a,c)$, or $d(b,a)$ and $d(c,a)$, this is not always possible when they have neither common end nor start. The problem is that one point (y in the example) might always interfere as the closest, and make comparisons with other points impossible. Comparison in general is possible if the sets $\{a,b\}$ and $\{x,y\}$ are sufficiently far from each other. Of course, by transitivity, we can sometimes use an intermediate pair $\{c,d\}$, sufficiently far away from both $\{a,b\}$ and $\{x,y\}$ to compare, indirectly, $d(a,b)$ and $d(x,y)$.
In our representation construction, a loop condition reveals itself as essential, e.g. in the prioritized case, an operation satisfying $a \mid bc = b$, $b \mid ac = c$, $c \mid ab = a$ cannot be represented by a symmetric distance, as $a \mid bc = b$ implies $d(a,b) < d(a,c)$, $b \mid ac = c$ implies $d(b,c) < d(b,a)$, and $c \mid ab = a$ implies $d(c,a) < d(c,b)$, so $d(a,b) < d(a,c) < d(b,c) < d(b,a) = d(a,b)$.
We use this property to show that Rabinowicz's conditions - which lack an exclusion of such loops - are not complete for representability by a symmetric distance. We work with three points (if you wish, one can add a forth point to have all models of the propositional language with variables p and q, and make the fourth point sufficiently far away from the

others, so that it does not disturb the picture) and define an operator $\mid$ s.t. $a \mid bc = b$, $b \mid ac = c$, $c \mid ab = a$. We show that $\mid$ satisfies the (set theoretic) equivalent of the AGM postulates, and Rabinowicz's conditions "Unrestricted Back and Forth" and "Unrestricted Revision of Subsets", but, as said above, cannot be represented by a symmetric distance function.
In Section 2.5, we present a construction of a distance function, respecting two partial relations, $\prec$ and $\approx$ between distances - provided certain conditions hold.
We turn to the proof of the completeness result. We first (Definition 2.2) present four conditions, most of which have a flavour close to those presented by Rabinowicz, in particular, ($\uparrow$ 3) is a back-and-forth condition. ($\uparrow$ 1) diverges from Rabinowicz's conditions, as we postulate $A \uparrow B = \emptyset$ iff $A = \emptyset$ or $B = \emptyset$. But this is a minor detail: We can either follow more closely AGM, or more the intuition of distances between sets. Given a revision operation $\uparrow$, we define (partial) relations between pairs of elements (intuitively: distances) $\prec$ and $\approx$ in Definition 2.4 and postulate a Loop condition in Proposition 2.5 ($\prec$ 1). The conditions imposed on $\prec$ and $\approx$ permit to use the construction of Section 2.5, it remains to show that the revision defined from the distances thus constructed is indeed the original one. This is done by an induction on the cardinality of the sets considered.
The converse, i.e. to show that any revision defined by a distance satisfying a limit condition ($A, B \neq \emptyset \rightarrow A \Uparrow B \cap A, A \Uparrow B \cap B \neq \emptyset$) will satisfy our properties, is straightforward.

2.3 Revision is a coarse instrument to investigate distances

We work here in the prioritized case.
Distances with common start (or end, by symmetry) can always be compared by looking at the result of revision:
$a \mid bb\prime = b$ iff $d(a,b) < d(a,b\prime)$,
$a \mid bb\prime = b\prime$ iff $d(a,b) > d(a,b\prime)$,
$a \mid bb\prime = bb\prime$ iff $d(a,b) = d(a,b\prime)$.
This is not the case with arbitrary distances $d(x,y)$ and $d(a,b)$, or $d(x,y)$ and $d(a\prime,b\prime)$, as the following example will show.
(The picture shows both situations together.)

```
                           * a'
                          .
                         .
                        * a
                       .
                      .
     *................*
```

```
     x                  y.
                          .
                           * b
                             .
                              .
                               * b'
```

We work in the real plane, with the standard distance, the angles have 120 degrees. a' is closer to y than x is to y, a is closer to b than x is to y, but a' is farther away from b' than x is from y. Similarly for b,b'. But we cannot distinguish the situation $\{a,b,x,y\}$ and the situation $\{a',b',x,y\}$ through $|$, i.e. $|$ does not always suffice to compare distances. This is the main technical problem: $|$ is a very coarse tool.

Proof:
Seen from a, the distances are in that order: y,b,x.
Seen from a', the distances are in that order: y,b',x.
Seen from b, the distances are in that order: y,a,x.
Seen from b', the distances are in that order: y,a',x.
Seen from y, the distances are in that order: a/b,x.
Seen from y, the distances are in that order: a'/b',x.
Seen from x, the distances are in that order: y,a/b.
Seen from x, the distances are in that order: y,a'/b'.
Thus, any $c \mid C$ will be the same in both situations (with a interchanged with a', b with b').
Moreover, $ab \mid xy = y$, $a'b' \mid xy = y$, $ax \mid by = y$, $a'x \mid b'y = y$, $ay \mid bx = b$, $a'y \mid b'x = b'$, $bx \mid ay = y$, $b'x \mid a'y = y$, $by \mid ax = a$, $b'y \mid a'x = a'$, $xy \mid ab = ab$, $xy \mid a'b' = a'b'$.
Thus, any $C \mid D$ will be the same in both situations, when we interchange a with a', and b with b'. $\square$

2.4 Incompleteness of Rabinowicz's system

Again, we work in the prioritized case, as does Rabinowicz in [Rab95].
Consider $Z =_{df} \{a,b,c\}$, and define $|$ by the following restrictions
(1) $A \mid B \subseteq B$,
(2) $A \mid B \neq \emptyset$ iff $A, B \neq \emptyset$,
(3) $A \mid B = A \cap B$ if $A \cap B \neq \emptyset$,
and the following cases:
$a \mid bc = b$, $b \mid ac = c$, $c \mid ab = a$.

Note that $|$ cannot be represented by a symmetric distance, as $a \mid bc = b$ implies $d(a,b) < d(a,c)$, $b \mid ac = c$ implies $d(b,c) < d(b,a)$, and $c \mid ab = a$ implies $d(c,a) < d(c,b)$, so $d(a,b) < d(a,c) < d(b,c) < d(b,a) = d(a,b)$.
Yet, $|$ satisfies all AGM postulates except some cases of empty sets (but this presents no real problem), Unrestricted Revision of Subsets, and Unrestricted Back and Forth, two conditions Rabinowicz considers.
In the following, we shall use repeatedly
$(*)$ If $card(Y) = 1$, then $X \mid Y = Y$ for any $X \neq \emptyset$.
This is an immediate consequence of (1) and (2) above.
The interesting cases of the AGM-conditions are $K*3$, $K*4$, $K*7$, $K*8$, which we write for sets of models as follows:
(a) $X \cap Y \neq \emptyset \rightarrow X \mid Y = X \cap Y$,
(b) $(X \mid Y) \cap W \neq \emptyset \rightarrow (X \mid Y) \cap W = X \mid (Y \cap W)$ for all $W \subseteq Z$.
(a) is guaranteed by one of the restrictions.
(b) Note that it suffices to consider $W \subseteq Y$: $(X \mid Y) \cap W = (X \mid Y) \cap (W \cap Y)$, as $X \mid Y \subseteq Y$, moreover $X \mid (Y \cap W) = X \mid (Y \cap (Y \cap W))$. Assume $(X \mid Y) \cap W \neq \emptyset$. Then $X, Y \neq \emptyset$. Suppose $X \cap Y \neq \emptyset$. Then $X \mid Y = X \cap Y$, so $X \cap Y \cap W \neq \emptyset$, thus $X \mid (Y \cap W) = X \cap Y \cap W$, and $(X \mid Y) \cap W = X \cap Y \cap W$, too. Assume now $X \cap Y = \emptyset$. Then $card(X \mid Y) = 1$, and $card(Y) \leq 2$. Assuming $W \subseteq Y$, there are two possible cases: Case 1: $Y = W$, then $X \mid (Y \cap W) = X \mid Y$, and $(X \mid Y) \cap W = X \mid Y$. Case 2: $X \mid Y = W$, then $X \mid (Y \cap W) = X \mid (Y \cap (X \mid Y)) = X \mid (X \mid Y) = X \mid Y$ (the last equality by $(*)$), and $(X \mid Y) \cap W = X \mid Y$.
We now show $(Y \mid X) \mid Y = X \mid Y$ (Unrestricted Back and Forth) - the same argument shows also directly $((X \mid Y) \mid X) \mid Y = X \mid Y$:
If $X \cap Y \neq \emptyset$, or $X = \emptyset$, or $Y = \emptyset$, this is trivial. So suppose $X \cap Y = \emptyset$, then $card(Y) \leq 2$. If $card(Y) = 1$, this is trivial. So suppose $card(Y) = 2$, then $card(X) = 1$, and it is trivial again.
Next, we show that $Y \subseteq X$, $(W \mid X) \cap Y \neq \emptyset$ implie $Y \mid W = \bigcup\{y \mid W : y \in (W \mid X) \cap Y\}$ (Unrestricted Revision of Subsets):
Note that then $X, Y, W \neq \emptyset$. Case 1: $X \cap W \neq \emptyset$. Then $Y \cap W = X \cap W \cap Y = (W \mid X) \cap Y \neq \emptyset$, and $Y \mid W = Y \cap W$. So if $z \in Y \mid W$, then $z \in (W \mid X) \cap Y$, and $z \in z \mid W$. If $z \in y \mid W$ for $y \in (W \mid X) \cap Y$, then $y \in W$, so $z = y$, $z \in Y \cap W$, and $z \in Y \mid W$. Case 2: $X \cap W = \emptyset$, $card(X) = 1$, then $Y = X$, and $(W \mid X) \cap Y = X = Y$, so we have on both sides $y \mid W$ for the unique $y \in Y$. Case 3: $X \cap W = \emptyset$, $card(W) = 1$, this is trivial.

2.5 A result on partial orders

Proposition 2.2
Let two (mutually exclusive) relations $\prec$, $\approx$ between pairs of elements from Z be given s.t.
(1) The role of 0:

(a) $(a,a) \prec \prime(b,c)$ for all $a,b,c \in Z$ with $b \neq c$, where $\prec \prime$ is the closure of $\prec$ under transitivity and well-behaviour with respect to $\approx$: $x \prec y \prec z \rightarrow x \prec \prime z$, $x \prec y \approx z \rightarrow x \prec \prime z$, $x \approx y \prec z \rightarrow x \prec \prime z$.
(b) $(a,a) \approx (b,b)$ for all a,b.
(2) Symmetry:
$(a,b) \approx (b,a)$ for all $a,b \in Z$.
(3) Acyclicity:
If a sequence $< a_0, b_0 >, \ldots, < a_n, b_n >$ exists s.t. for all pairs $< a_i, b_i >$, $< a_{i+1}, b_{i+1} >$ $(0 \leq i \leq n,\ n+1 =_{df} 0)$ $(a_i,b_i) \preceq (a_{i+1},b_{i+1})$ holds, then always $(a_i,b_i) \approx (a_{i+1},b_{i+1})$. (Here, $\preceq$ stands for $\prec$ or $\approx$.)
Then there is a symmetric distance on Z s.t.
(α) $(a,b) \approx (c,d) \rightarrow d(a,b) = d(c,d)$,
(β) $(a,b) \prec (c,d) \rightarrow d(a,b) < d(c,d)$.

Proof:
Intuitively, (a,b) is the distance between a and b. We know some relations between distances (expressed by $\prec$ and $\approx$), but not all, and try to construct a symmetric distance function d which respects the given relations $\prec$ and $\approx$, i.e. $(a,b) \prec (c,d) \rightarrow d(a,b) < d(c,d)$, and $(a,b) \approx (c,d) \rightarrow d(a,b) = d(c,d)$.
We close $\prec$ under transitivity and well-behaviour with respect to $\approx$: $x \prec y \prec z \rightarrow x \prec z$, $x \prec y \approx z \rightarrow x \prec z$, $x \approx y \prec z \rightarrow x \prec z$. This preserves acyclicity.
Let $[(a,b)]$ be the equivalence class of (a,b) under the transitive, reflexive closure of $\approx$.

Fact 2.3
Let $\prec$ be as constructed above, and $\mathcal{X}$ be the set of equivalence classes as defined above. Define $\prec$ on $\mathcal{X}$ by $[x] \prec [y]$ iff there is $x\prime \in [x]$, $y\prime \in [y]$ with $x\prime \prec y\prime$. Then $\prec$ on $\mathcal{X}$ is transitive and free of cycles too.

Proof:
Transitivity of $\prec$ on $\mathcal{X}$ is trivial by transitivity of $\prec$ on ZxZ and well-behaviour with respect to $\approx$, as introduced above. For acyclicity: Let $[x_0] \prec [x_1] \prec \ldots \prec [x_n] \prec [x_0]$. Then we have $x^0_0 \prec x^0_1 \approx x^1_1 \prec x^1_2 \approx x^2_2 \prec \ldots \prec x^{n-1}_n \approx x^n_n \prec x^n_0 \approx x^0_0$ with $x^k_i \in [x_i]$, contradiction. □

Note, that by condition (1)(b) above, $0 =_{df} [(a,a)]$ for any a, is well-defined. Moreover, by (1)(a) $0 \prec [(a,b)]$ for any $a \neq b$.
Extend $\prec$ on $\mathcal{X}$ to a strict total order $<$ on $\mathcal{X}$.
Define $d : ZxZ \rightarrow \mathcal{X}$ by $d(a,b) =_{df} [(a,b)]$.
d is a symmetric distance: (d1) by construction, (d2) by above remark, (d3) by condition (2). Finally (α) and (β) hold by construction. □

(Proposition 2.2)

2.6 The algebraic representation result

Definition 2.2
Consider the following conditions for an operation $\uparrow$ on $\mathcal{P}(Z)$, $\uparrow: \mathcal{P}(Z)x\mathcal{P}(Z) \rightarrow \mathcal{P}(Z)$:
($\uparrow$ 0) $A \uparrow B = \{b \in B : \exists a_b \in A.\forall a\prime \in A, \forall b\prime \in B.b \in a_b a\prime \uparrow bb\prime\} \cup \{a \in A : \exists b_a \in B.\forall a\prime \in A, \forall b\prime \in B.a \in b_a b\prime \uparrow aa\prime\}$,
($\uparrow$ 1) if $A, B \neq \emptyset$, then $(A \uparrow B) \cap A \neq \emptyset$, $(A \uparrow B) \cap B \neq \emptyset$, and if $A = \emptyset$ or $B = \emptyset$, then $A \uparrow B = \emptyset$,
($\uparrow$ 2) if $A \cap B \neq \emptyset$, then $A \uparrow B = A \cap B$,
($\uparrow$ 3) $b \in A \uparrow B$, $a_b \in b \uparrow A$, $a_b \in A\prime \subseteq A \rightarrow b \in A\prime \uparrow B \cap B \subseteq A \uparrow B \cap B$.

Fact 2.4
(a) Note that, by ($\uparrow$ 0), $A \uparrow B \subseteq A \cup B$ and $A \uparrow B = B \uparrow A$.
(b) ($\uparrow$ 0) implies the following condition ($\uparrow$ 4) $((\bigcup A_i) \uparrow B) \cap B \subseteq (\bigcup(A_i \uparrow B)) \cap B$.

Proof:
(b) Let $b \in (\bigcup A_i) \uparrow B \rightarrow \exists a_b \in \bigcup A_i.\forall a\prime \in \bigcup A_i \forall b\prime \in B.b \in a_b a\prime \uparrow bb\prime \rightarrow \exists i(a_b \in A_i \wedge \forall a\prime \in A_i \forall b\prime \in B.b \in a_b a\prime \uparrow bb\prime) \rightarrow b \in \bigcup(A_i \uparrow B)$. □

Definition 2.3
Given a distance d, set $A \Uparrow_d B =_{df} \{b \in B : \exists a_b \in A.\forall a\prime \in A, \forall b\prime \in B.d(a_b, b) \leq d(a\prime, b\prime)\}$
$\cup \{a \in A : \exists b_a \in B.\forall a\prime \in A, \forall b\prime \in B.d(b_a, a) \leq d(b\prime, a\prime)\}$
Note that then again $A \Uparrow_d B \subseteq A \cup B$ and $A \Uparrow_d B = B \Uparrow_d A$.

Definition 2.4
Given $\uparrow$, we define the relations $\prec$, $\approx$ between pairs of elements from Z (intuitively, a pair (a,b) is the distance from a to b). We will use properties ($\uparrow$ 3) and ($\uparrow$ 4) above to simplify the situation.
By Fact 2.4, (a), $A \uparrow B \cap A = B \uparrow A \cap A$, so it suffices to consider $A \uparrow B \cap B$ for all A,B. By ($\uparrow$ 0), it suffices to consider sets A,B of cardinality ≤ 2. The cases $A, B = \emptyset$, and $card(B) = 1$ are trivial by ($\uparrow$ 1) and give no information.
0. $(a, b) \approx (b, a)$ for all a,b.
1. $(a \uparrow bb\prime) \cap bb\prime = b \rightarrow (a, b) \prec (a, b\prime)$,
$(a \uparrow bb\prime) \cap bb\prime = b\prime \rightarrow (a, b\prime) \prec (a, b)$,
$(a \uparrow bb\prime) \cap bb\prime = bb\prime \rightarrow (a, b) \approx (a, b\prime)$.
2. $aa\prime \uparrow bb\prime$:
2.1 $(a \uparrow bb\prime) \cap bb\prime = (a\prime \uparrow bb\prime) \cap bb\prime$: Then $(aa\prime \uparrow bb\prime) \cap bb\prime = (a \uparrow bb\prime) \cap bb\prime = (a\prime \uparrow bb\prime) \cap bb\prime$ by ($\uparrow$ 3) and ($\uparrow$ 4)

2.2 $(a \uparrow bb\prime) \cap bb\prime = b$, $(a\prime \uparrow bb\prime) \cap bb\prime = b\prime$: 2.2.1 $(aa\prime \uparrow bb\prime) \cap bb\prime = b \rightarrow (a, b) \prec (a\prime, b\prime)$,
2.2.2 $(aa\prime \uparrow bb\prime) \cap bb\prime = b\prime \rightarrow (a\prime, b\prime) \prec (a, b)$,
2.2.3 $(aa\prime \uparrow bb\prime) \cap bb\prime = bb\prime \rightarrow (a, b) \approx (a\prime, b\prime)$.
2.3 $(a \uparrow bb\prime) \cap bb\prime = b$, $(a\prime \uparrow bb\prime) \cap bb\prime = bb\prime$: 2.3.1 $(aa\prime \uparrow bb\prime) \cap bb\prime = b \rightarrow (a, b) \prec (a\prime, b)$,
2.3.2 $(aa\prime \uparrow bb\prime) \cap bb\prime = b\prime$ impossible by $(\uparrow 3)$,
2.3.3 $(aa\prime \uparrow bb\prime) \cap bb\prime = bb\prime \rightarrow (a\prime, b) \prec (a, b)$ or $(a\prime, b) \approx (a, b)$.
The other cases of 2.3, e.g. $(a \uparrow bb\prime) \cap bb\prime = b\prime$, $(a\prime \uparrow bb\prime) \cap bb\prime = bb\prime$ are symmetrical.
The cases 2.1 and 2.3 do not give any information we cannot obtain by other means using symmetry: 2.3.1 and 2.3.3: $(a\prime, b) \prec (a, b)$ or $(a\prime, b) \approx (a, b)$ can be decided by considering $b \uparrow aa\prime$. Thus, the closures under symmetry of the relations $\approx$ and $\prec$ are fully determined by the cases 1 and 2.2.

Proposition 2.5

Consider the conditions $(\uparrow 0) - (\uparrow 3)$ for $\uparrow$, as well as the following condition for the relations $\prec$ and $\approx$, as defined from $\uparrow$ in Definition 2.4:
$(\prec 1)$ Acyclicity:
Let a sequence $< a_0, b_0 >, \ldots, < a_n, b_n >$ be given s.t. for all pairs $< a_i, b_i >$, $< a_{i+1}, b_{i+1} >$ $(0 \leq i \leq n, n+1 =_{df} 0)$ $(a_i, b_i) \preceq (a_{i+1}, b_{i+1})$, then always $(a_i, b_i) \approx (a_{i+1}, b_{i+1})$ ($\preceq$ stands for $\prec$ or $\approx$),

and the following condition for $\Uparrow$:
$(\Uparrow 1)$ If $A, B \neq \emptyset$, then $A \Uparrow B \cap B \neq \emptyset$, $A \Uparrow B \cap A \neq \emptyset$, and if $A = \emptyset$ or $B = \emptyset$, then $A \Uparrow B = \emptyset$.
Then:
(A) Given a symmetric distance function d, satisfying $(\Uparrow 1)$, and setting $A \uparrow B =_{df} A \Uparrow_d B$, $A \uparrow B$ will satisfy $(\uparrow 0) - (\uparrow 3)$, and $\prec$, $\approx$ defined from $\Uparrow$ will satisfy $(\prec 1)$.
(B) Given an operation $\uparrow$, satisfying $(\uparrow 0) - (\uparrow 3)$, and $\prec$, $\approx$ defined from $\uparrow$ satisfying $(\prec 1)$, we can define a symmetric distance function d satisfying $(\Uparrow 1)$ s.t. $A \uparrow B = A \Uparrow_d B$.

Proof:

The proof is tedious, but rather straightforward.
(A): Set $\Uparrow =_{df} \Uparrow_d$.
$(\uparrow 0)$: It suffices to show $A \Uparrow B \cap B = X =_{df} \{b \in B : \exists a_b \in A.\forall a\prime \in A, \forall b\prime \in B.b \in a_b a\prime \Uparrow bb\prime\}$. Recollect that $A \Uparrow B \cap B =_{df} A \Uparrow_d B \cap B =_{df} \{b \in B : \exists a_b \in A.\forall a\prime \in A, \forall b\prime \in B.d(a_b, b) \leq d(a\prime, b\prime)\}$. Let $b \in A \Uparrow B$. Take a_b by definition of $A \Uparrow B$. It suffices to show $\forall a\prime \in A \forall b\prime \in B$ $b \in a_b a\prime \Uparrow bb\prime$, and, using the definition of $a_b a\prime \Uparrow bb\prime$ from d, it suffices to show $\forall a\prime\prime \in a_b a\prime \forall b\prime\prime \in bb\prime.d(a_b, b) \leq d(a\prime\prime, b\prime\prime)$. But, by $b \in A \Uparrow B$ and choice of a_b, this is trivial, as $a\prime\prime \in A$, $b\prime\prime \in B$. Let $b \in X$. Take $a^*_b \in b \Uparrow A \cap A$. We will show $\forall a\prime \in A \forall b\prime \in B.d(a^*_b, b) \leq d(a\prime, b\prime)$. By symmetry, $d(a^*_b, b) \leq d(a\prime, b)$ for all $a\prime \in A$. By prerequisite, $\exists a_b \in A \forall a\prime \in A \forall b\prime \in B.b \in a_b a\prime \Uparrow bb\prime$. Fix such a_b. But $b \in a_b a\prime \Uparrow bb\prime$ implies $\exists a_b\prime\prime \in a_b a\prime \forall a\prime\prime \in a_b a\prime \forall b\prime\prime \in bb\prime.d(a_b\prime\prime, b) \leq d(a\prime\prime, b\prime\prime)$, thus $d(a_b, b) \leq d(a\prime, b\prime)$

or $d(a\prime, b) \leq d(a\prime, b\prime)$. So for that a_b, and any $a\prime \in A$, $b\prime \in B$ $d(a_b, b) \leq d(a\prime, b\prime)$ or $d(a\prime, b) \leq d(a\prime, b\prime)$ hold. Thus $d(a^*_b, b) \leq d(a\prime, b\prime)$ for all $a\prime \in A$, $b\prime \in B$.
$(\uparrow 1)$ is immediate from $(\Uparrow 1)$, $(\uparrow 2)$ follows from (d2).
$(\uparrow 3)$: Let $b \in A \Uparrow B$, then $\exists a_b \in A.\forall a\prime \in A \forall b\prime \in B.d(a_b, b) \leq d(a\prime, b\prime)$. Suppose $a\prime\prime \in b \Uparrow A$, then $\forall a\prime \in A.d(b, a\prime\prime) \leq d(b, a\prime)$, so by symmetry $\forall a\prime \in A.d(a\prime\prime, b) \leq d(a\prime, b)$. So we have
$d(a\prime\prime, b) \leq d(a_b, b) \leq d(a\prime, b\prime)$, for any $a\prime \in A$, $b\prime \in B$, and $a\prime\prime$ is a suitable a_b. By prerequisite, there is $a\prime\prime \in (b \Uparrow A) \cap A\prime$. Then, by $A\prime \subseteq A$, and Definition of $\Uparrow$,
$b \in A\prime \Uparrow B$. Suppose $b\prime \in A\prime \Uparrow B - A \Uparrow B$. Take $a_{b\prime} \in b\prime \Uparrow A\prime$. As $b\prime \notin A \Uparrow B$, $\exists a^* \in A \exists b^* \in B.d(a^*, b^*) < d(a_{b\prime}, b\prime)$. By $b \in A \Uparrow B$ and $a\prime\prime \in (b \Uparrow A) \cap A\prime$, we have $d(a\prime\prime, b) \leq d(a^*, b^*) < d(a_{b\prime}, b\prime)$, a contradiction to $b\prime \in A\prime \Uparrow B$, $a_{b\prime} \in b\prime \Uparrow A\prime$.
$(\prec 1)$ follows from (d1).
(B):
Assume then $\uparrow$ to be given, and $\prec$, $\approx$ etc. constructed as in Definition 2.4.
They satisfy the prerequisites of Proposition 2.2:
We have to check (1) (a) and (b).
For (a):
Case 1, $a = b$: Then $(a \uparrow ac) \cap ac = a$ by $(\uparrow 2)$, so $(a, a) \prec (a, c)$ by Definition 2.4, 1 ...(The case $a = c$ is analogous.) Case 2, $a \neq b$, $a \neq c$: Then $(a \uparrow ab) \cap ab = (a \uparrow ac) \cap ac = (ab \uparrow ac) \cap ac = a$ by $(\uparrow 2)$, thus $(a, a) \prec (a, b)$. If $(b \uparrow ac) \cap ac = c$, $(a, a) \prec (b, c)$ by 2.2.1 there. If $(b \uparrow ac) \cap ac = a$, we have $(b, a) \prec (b, c)$, so $(a, a) \prec (a, b) \approx (b, a) \prec (b, c)$. If $(b \uparrow ac) \cap ac = ac$, we have $(b, a) \approx (b, c)$, so $(a, a) \prec (a, b) \approx (b, a) \approx (b, c)$. For (b): By $(a \uparrow ab) \cap ab = a$, $(b \uparrow ab) \cap ab = b$, $(ab \uparrow ab) \cap ab = ab$, this follows from 2.2.3.
We define a distance function d by Proposition 2.2, it remains to show that $\uparrow = \Uparrow_d$, where $\Uparrow = \Uparrow_d$ is as defined in Definition 2.3.
Note that $(\uparrow 3)$ and $(\uparrow 4)$ hold for $\Uparrow$, as just demonstrated. Note further that we shall use in our proof only symmetry of d, transitivity of $<$, and $(a, b) \prec (c, d) \rightarrow d(a, b) < d(c, d)$, $(a, b) \approx (c, d) \rightarrow d(a, b) = d(c, d)$.
It suffices to show $A \Uparrow B \cap B = A \uparrow B \cap B$, as then $A \Uparrow B \cap A = B \Uparrow A \cap A = B \uparrow A \cap A = A \uparrow B \cap A$.
The cases $A = \emptyset$ or $B = \emptyset$ or $card(B) = 1$ are trivial.
We show the result by induction for the cases
(α) $card(A) = 1$, $card(B) = 2$,
(β) $card(A) = 1$, $card(B) > 2$,
(γ) $card(A) = 2$, $card(B) = 2$,
(δ) $card(A) \geq 2$, $card(B) \geq 2$, $card(A) + card(B) > 4$,
in that order.
(α)
If $a \uparrow bb\prime \cap bb\prime = bb\prime$, then $(a, b) \approx (a, b\prime)$, so $d(a, b) = d(a, b\prime)$ and $a \Uparrow bb\prime \cap bb\prime = bb\prime$ by Definition 2.3. If $a \uparrow bb\prime \cap bb\prime = b$, then $(a, b) \prec (a, b\prime)$, so $d(a, b) < d(a, b\prime)$ and $a \Uparrow bb\prime \cap bb\prime = b$. If $a \uparrow bb\prime \cap bb\prime = b\prime$: similar.

(β)
$b \in a \uparrow B \leftrightarrow \forall b\prime \in B.b \in a \uparrow bb\prime$ by ($\uparrow 0$) $\leftrightarrow \forall b\prime \in B.b \in a \Uparrow bb\prime$ by (α) $\leftrightarrow \forall b\prime \in B.d(a,b) \leq d(a,b\prime)$ by Definition 2.3 $\leftrightarrow b \in a \Uparrow B$ by Definition 2.3.
(γ)
Case 2.1: $a \uparrow bb\prime \cap bb\prime = a\prime \uparrow bb\prime \cap bb\prime \rightarrow a \Uparrow bb\prime \cap bb\prime = a \uparrow bb\prime \cap bb\prime = a \uparrow bb\prime \cap bb\prime = a \Uparrow bb\prime \cap bb\prime$ and $a \uparrow bb\prime \cap bb\prime = aa\prime \uparrow bb\prime \cap bb\prime$, and $a \Uparrow bb\prime \cap bb\prime = aa\prime \Uparrow bb\prime \cap bb\prime$ (by ($\uparrow 3$) and ($\uparrow 4$) for $\uparrow$ and $\Uparrow$), so $aa\prime \uparrow bb\prime \cap bb\prime = aa\prime \Uparrow bb\prime \cap bb\prime$.
Case 2.2.1: $(a \uparrow bb\prime) \cap bb\prime = b$, $(a\prime \uparrow bb\prime) \cap bb\prime = b\prime$, $(aa\prime \uparrow bb\prime) \cap bb\prime = b$, so $(a,b) \prec (a\prime,b\prime)$, so $d(a,b) < d(a\prime,b\prime)$, but also $d(a,b) < d(a,b\prime)$ and $d(a\prime,b\prime) < d(a\prime,b)$, thus by transitivity of $<$ $aa\prime \Uparrow bb\prime \cap bb\prime = b$ by Definition 2.3.
Case 2.2.2: $(a \uparrow bb\prime) \cap bb\prime = b$, $(a\prime \uparrow bb\prime) \cap bb\prime = b\prime$, $(aa\prime \uparrow bb\prime) \cap bb\prime = b\prime$: similar to Case 2.2.1.
Case 2.2.3: $(a \uparrow bb\prime) \cap bb\prime = b$, $(a\prime \uparrow bb\prime) \cap bb\prime = b\prime$, $(aa\prime \uparrow bb\prime) \cap bb\prime = bb\prime$, so $(a,b) \approx (a\prime,b\prime)$, so $d(a,b) = d(a\prime,b\prime)$, but also $d(a,b) < d(a,b\prime)$ and $d(a\prime,b\prime) < d(a\prime,b)$, thus $aa\prime \Uparrow bb\prime \cap bb\prime = bb\prime$ by Definition 2.3.
Case 2.3.1: If $a \uparrow bb\prime \cap bb\prime = b$, $a\prime \uparrow bb\prime \cap bb\prime = bb\prime$, $aa\prime \uparrow bb\prime \cap bb\prime = b$, then $(a,b) \prec (a\prime,b)$, $(a,b) \prec (a,b\prime)$, $(a\prime,b) \approx (a\prime,b\prime)$, so $d(a,b) < d(a\prime,b)$, $d(a,b) < d(a,b\prime)$, $d(a\prime,b) = d(a\prime,b\prime)$, and $aa\prime \Uparrow bb\prime \cap bb\prime = b$.
Case 2.3.2: As ($\uparrow 3$) holds for $\uparrow$ and $\Uparrow$, this case can neither arise for $\uparrow$, nor for $\Uparrow$.
Case 2.3.3: We have $(a,b) \prec (a,b\prime)$, $(a\prime,b) \approx (a\prime,b\prime)$. If $(a\prime,b) \prec (a,b)$, then $d(a\prime,b) = d(a\prime,b\prime) < d(a,b) < d(a,b\prime)$, so $aa\prime \Uparrow bb\prime \cap bb\prime = bb\prime$. If $(a\prime,b) \approx (a,b)$, then $d(a\prime,b) = d(a\prime,b\prime) = d(a,b) < d(a,b\prime)$, so $aa\prime \Uparrow bb\prime \cap bb\prime = bb\prime$.
(δ)
We show $A \uparrow B \cap B \subseteq A \Uparrow B \cap B$, and then $A \Uparrow B \cap B \subseteq A \uparrow B \cap B$.
Let $b \in A \uparrow B$. So there is a_b s.t. $\forall a\prime \in A \forall b\prime \in B.b \in a_b a\prime \uparrow bb\prime$, thus, by ($\alpha$), if $a\prime = a_b$, or by (γ), if $a\prime \neq a_b$, $\forall a\prime \in A \forall b\prime \in B.b \in a_b a\prime \Uparrow bb\prime$, so for fixed $a\prime$, $b\prime$ by Definition 2.3, either $d(a_b,b)$ or $d(a\prime,b)$ is a minimal element in $\{d(a_b,b), d(a_b,b\prime), d(a\prime,b), d(a\prime,b\prime)\}$ Take $a_b^* \in b \Uparrow A$. Then by symmetry, $d(a_b^*,b) \leq d(a_b,b)$ and $d(a_b^*,b) \leq d(a\prime,b)$, thus $d(a_b^*,b) \leq d(a\prime,b\prime)$ for fixed arbitrary a',b', so $b \in A \Uparrow B$ by Definition 2.3.
Let $b \in A \Uparrow B$, and suppose $b \notin A \uparrow B$. By $b \in A \Uparrow B$, $\exists a_b \in A \forall a\prime \in A \forall b\prime \in B.d(a_b,b) \leq d(a\prime,b\prime)$. Fix one such a_b. So for any $b\prime$ $d(a_b,b) \leq d(a_b,b\prime)$, thus $b \in a_b \Uparrow bb\prime$ and $b \in a_b \uparrow bb\prime$ by (α). On the other hand, $b \notin A \uparrow B$, so $\forall a_b\prime \in A \exists a\prime \in A, \exists b\prime \in B.b \notin a_b\prime a\prime \uparrow bb\prime$, so for that a_b, there are a',b' with $a_b a\prime \uparrow bb\prime = b\prime$.
Case (1): $a_b \uparrow bb\prime = b$. So by $a_b a\prime \uparrow bb\prime = b\prime$ and ($\uparrow 4$) $a\prime \uparrow bb\prime = b\prime$, so $(a\prime,b\prime) \prec (a_b,b)$, but then $d(a\prime,b\prime) < d(a_b,b)$, contradicting the choice of a_b.
Case (2): $a_b \uparrow bb\prime = bb\prime$. So by $a_b a\prime \uparrow bb\prime = b\prime$ again $a\prime \uparrow bb\prime = b\prime$, so $(a\prime,b\prime) \prec (a_b,b\prime) \approx (a_b,b)$, so $d(a\prime,b\prime) < d(a_b,b)$ again, contradiction.
□

(Proposition 2.5)

2.7 The logical representation result

Translation into syntax is straightforward - we give the recipe.
Let $M(T)$ denote the set of models of a theory T, conversely, $Th(M)$ the set of formulas valid in all models $m \in M$. For a set T of formulas, let $\overline{T}$ be its closure under classical logic: $\overline{T} =_{df} \{\phi : T \vdash \phi\}$.
We assume in the sequel that any set of models M considered is the set of models of some theory T, $M = M(T)$ (this is neccessary e.g. for (4) below). To satisfy this proviso, we need the following new condition on $\Uparrow$: (definability preservation): If $M = M(T)$, $M\prime = M(T\prime)$, then $M \Uparrow M\prime = M(T\prime\prime)$ for some theory $T\prime\prime$.
We then have:

Fact 2.6
(1) $Th(M \cup M\prime) = \overline{\{\phi \vee \phi\prime : \phi \in Th(M), \phi\prime \in Th(M\prime)\}}$.
(2) $Th(M \cap M\prime) = \overline{Th(M) \cup Th(M\prime)}$.
(3) Single models correspond to consistent complete theories.
(4) $m \in M$ translates to $Th(\{m\}) \vdash Th(M)$ - provided $M = M(Th(M))$.
(5) $M \subseteq M\prime$ translates to $Th(M) \vdash Th(M\prime)$ - provided $M\prime = M(Th(M\prime))$.
(6) $M \neq \emptyset$ translates to $Con(Th(M))$ - where Con is consistency with respect to the underlying logic.

Proof:
We show (1) and (2).
(a) Note first: $Th(M(S)) = \overline{S}$. Proof: $\phi \in Th(M(S)) \Leftrightarrow \forall m \in M(S).m \models \phi \Leftrightarrow S \models \phi \Leftrightarrow S \vdash \phi \Leftrightarrow \phi \in \overline{S}$.
(1) Let $T \vee T\prime =_{df} \{\phi \vee \phi\prime : \phi \in T, \phi\prime \in T\prime\}$. First, $Th(M(T) \cup M(T\prime)) = \overline{T \vee T\prime}$: By $M(T) \cup M(T\prime) = M(T \vee T\prime)$, we have
$Th(M(T) \cup M(T\prime)) = Th(M(T \vee T\prime)) = \overline{T \vee T\prime}$. Applying this to $\overline{T}$ and $\overline{T\prime}$, and using $M(T) = M(\overline{T})$, $M(T\prime) = M(\overline{T\prime})$, we have $Th(M(T) \cup M(T\prime)) = Th(M(\overline{T}) \cup M(\overline{T\prime})) = \overline{\overline{T} \vee \overline{T\prime}}$.
(2) $Th(M(T) \cap M(T\prime)) = \overline{Th(M(T)) \cup Th(M(T\prime))}$. Proof: By $M(T) \cap M(T\prime) = M(T \cup T\prime)$, we have $Th(M(T) \cap M(T\prime)) = Th(M(T \cup T\prime)) = \overline{T \cup T\prime}$, moreover $\overline{Th(M(T)) \cup Th(M(T\prime))} = \overline{\overline{T} \cup \overline{T\prime}} = \overline{T \cup T\prime}$. □

Remark:
Note that $M(T) \cup M(T\prime) = M(T \vee T\prime)$ has over $M(T) \cup M(T\prime) = M(T \cap T\prime)$ the following advantages: First, it stays true even if T and $T\prime$ are not deductively closed, second, and more importantly, it gives a hint why the generalization to the infinite fails: $\bigcap M(T_i) = M(\bigcup T_i)$ is true in the infinite case ($i \in I$ with I infinite), but, in general,

there is no T s.t. $\bigcup M(T_i) = M(T)$. $\bigvee T_i$ makes no sense, as we have no infinite "or". Infinite "and" can be imitated by an infinite theory, but there is no analogue for infinite "or".
The condition ($\prec 1$) in Proposition 2.5 involves all possible cases in 1. and 2.2 of Definition 2.4.

2.8 Comparison to Hansson's conditions for "semi-revision"

Hansson considers in [Han97] - among other properties - the following ones for semi-revision (in our notation):
(H1) Weak success: $\neg\alpha \notin K \rightarrow \alpha \in K \otimes \alpha$
(H2) Internal exchange: $\alpha, \beta \in K \rightarrow K \otimes \alpha = K \otimes \beta$
(H3) Inclusion: $K \otimes \alpha \subseteq K + \alpha$ ($K + \alpha$ is the deductive closure of $K \cup \{\alpha\}$)
(H4) Negation-neutrality: $K \otimes \alpha = K \otimes \neg\alpha$
(H5) Vacuity: $\neg\alpha \notin K \rightarrow K \otimes \alpha = K + \alpha$
(H6) Consistency retainment: If K is consistent, then so is $K \otimes \alpha$.
It is easily seen that (H4) and (H6) do not hold in our approach, but our definitions can easily be modified to obtain (H6). (H4), however, is far away from our system. The other properties are equally easily seen to be valid for revisions based on a distance between models.

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