Some Completeness Results for Stoppered and Ranked Classical Preferential Models

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Abstract

We extend the work begun in [Sch92] to stoppered (or smooth) and ranked classical preferential models, giving several soundness and completeness results for these structures. In addition, we discuss the number of copies of models needed to represent arbitrary logics defined by preferential structures.

Keywords: Nonmonotonic reasoning, defeasible reasoning, preferential models

1 Introduction

Throughout, we work in propositional logic.

1.1 Outline of this paper:

Section 1: We first give a detailed introduction to preferential structures, and then present the basic definitions as given already in [Sch92] (Definitions 1.1, 1.3-1.5).
We then discuss a question the author has been asked repeatedly: Do we need several copies of logically identical models in preferential structures, or, more precisely, how many do we need? We first present as illustration a logic defined by a preferential structure which cannot be represented by such a structure where each model occurs at most once. In the subsequent discussion of the infinite case with κ propositional variables, we show that there are logics which need κ many copies for a representation by preferential models, but also discuss a case where $\lambda < \kappa$ many suffice, when $2^{\lambda} \geq \kappa$.

Section 2: In [Sch92], we characterized definability preserving preferential models by the combinatorial properties (f1) $f(X) \subseteq X$ and (f2) $X \subseteq Y \to f(Y) \cap X \subseteq f(X)$, and the logical properties ($\mid\!\sim$ 1) $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$, ($\mid\!\sim$ 2) $\overline{\overline{T}}$ is classically closed, ($\mid\!\sim$ 3) $T \subseteq \overline{\overline{T}}$, and ($\mid\!\sim$ 4) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$. (For details, see there, or Theorems 2.1 and 2.12 of Section 2.) We extend these results to characterize definability preserving stoppered preferential models, by showing that the addition of the combinatorial property (f3) $f(X) \subseteq Y \subseteq X \to$ f(X)=f(Y) of Theorem 2.1, and ($\mid\!\sim$ 5) $T \subseteq T' \subseteq \overline{\overline{T}} \to \overline{\overline{T}} = \overline{\overline{T'}}$ of Theorem 2.12 suffices. As in [Sch92], we proceed by first characterizing such structures by their combinatorial properties (Theorem 2.1), and then by characterizing the combinatorial properties logically (Proposition 2.13). This has the advantage that the combinatorial result - which is the part more difficult to prove - can be reused in other contexts, as has been done by the author already repeatedly (see [Sch92-n9] and [Sch92-n3]). Thus, the main work of this Section is the proof of Theorem 2.1, its extension to logics (Theorem 2.12) is more or less straightforward and prepared in Proposition 2.13.

Section 3: In Section 3, we characterize ranked, and ranked and stoppered definability preserving classical preferential models, again by first presenting a combinatorial characterization (Theorems 3.10 and 3.14) and obtain from these results a logical characterization (Theorems 3.16 and 3.18). Theorems 3.10 and 3.14 are prepared in Lemma 3.5, where we prove that for ranked preferential models, we can do with either infinitely many or 1 copy of each propositional model, more precisely, minimal models need be present only once. This is a direct consequence of rankedness. We therefore divide the underlying set Z into $A \cup B$, where A contains the elements which "kill themselves", i.e. for $x \in A$ $f(\{x\}) = \emptyset$, and B the others. Elements from A will be represented by infinite descending chains of copies, elements from B by single copies. In the stoppered case, we can neglect the elements from A, eliminating them totally from the structure. Rankedness then further translates into condition 2 of Theorems 3.10 and 3.14, and a rankedness condition for an appropriately defined relation on elements of Z (Theorem 3.10) or B (Theorem 3.14). Condition 2. expresses sufficiency of one model for minimization, in contrast to "The finite case", below in the Introduction. The logical counterparts to Theorems 3.10 and 3.14 are shown as usual and presented in Theorems 3.16 and 3.18.

Related Work: Most papers presenting preferential representation results (e.g. [GM94], [KLM90], [LM92], [Mak94]) do so for single formulas on the left: $\alpha \mid\!\sim \beta$. [FL93], [FL94] have as elements of the structure *sets* of models, which radically changes the situation. There are still other representation results, e.g. [KL95], which, however, do not describe logics by a semantics, but by other means, e.g. as closures under certain operations.

1.2 Introduction to preferential structures

1.2.1 The Intuitive Background:

The basic idea is to interpret a primitive notion of "importance" or "value", introduced into a given language and logic, by a function which chooses the subset of "important" models of a theory or formula of that language.
In other words, we work on a set of "possible worlds", i.e. models of the underlying base logic, but do not accord the same importance or value to all such models. Given then a theory T of the base language and logic, we determine the semantical consequences of T in a structure $\mathcal{M}$ by considering only the subset of "important" models of T: $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all important models of T in our structure. More formally, such a structure $\mathcal{M}$ will then consist of a set M of models or possible worlds for the base logic, and a choice function f on $\mathcal{P}(M)$ - the power set of M - which, for each base theory T, singles out the set $f(M(T)) \subseteq M(T)$ of important models of T in that structure $\mathcal{M}$, where $M(T)$ is the set of all base models of T in $\mathcal{M}$. We thus define $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$.
A refinement of the idea is to work not with one subset of "maximally important" models, but with many subsets of important models, perhaps of increasing importance. This translates into the existence of several choice functions f_i in the structure $\mathcal{M}$, and we define $T \models_{\mathcal{M}} \phi$ iff there is some f_i such that ϕ holds in all $m \in f_i(M(T))$. This captures the intuition that we may not dispose of ideal models, but of ever better ones, which, in a sense, approximate the limit of the ideal case. Thus, each $f_i(M(T))$ may be non-empty, but $\bigcap\{f_i(M(T)) : i \in I\}$ may be empty.
Already this very abstract description makes it plausible that representation theorems for the latter approach - which I shall call the limit case - are harder to obtain than for the first variant - which I shall call, for historical reasons, the minimal case: In the latter we have to handle a possibly infinite set of choice functions, and there need not be a global f such that for all ϕ $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$. In other words, we do not always have a set of "joint witnesses" for all consequences of a theory.
This introduction apart, the present article deals only with the minimal variant.

1.2.2 Logical Consequences:

It is evident that such consequence relations will be well-behaved with respect to the base logic - provided the latter is sound and complete for the models we have chosen as possible worlds - i.e. if T and T' are equivalent with respect to the base logic, they will have the same set of semantic consequences, and, if $T \models_{\mathcal{M}} \phi$, and ϕ implies ψ in the base logic, then also $T \models_{\mathcal{M}} \psi$. Moreover, if ϕ is a consequence of T in the base logic, then $T \models_{\mathcal{M}} \phi$, as the coice functions will choose a subset of M(T). These facts hold in both the limit and the minimal version.

1.2.3 Preferential Structures:

Preferential Structures are a special case of the above, the choice is made *locally* by a binary relation $\prec$ on the set M of base models, m is considered to be more important than m' iff $m \prec m'$ ($m \prec m'$ instead of $m' \prec m$ for historical reasons). They are thus very similar to Kripke structures, but use the relation $\prec$ differently.
In the minimal case, we define f from $\prec$ by $f(A) := \{a \in A : \neg\exists b \in A.b \prec a\}$.
In the limit case, the natural definition is to consider initial segments of A: $\delta_A \subseteq A$ is called an initial segment of A iff
($\delta 1$) we find some $b \in \delta_A$ below each $a \in A$: $\forall a \in A \exists b \in \delta_A (b = a \vee b \prec a)$,
($\delta 2$) δ_A is downward closed: $\forall a \in A \forall b \in \delta_A (a \prec b \rightarrow a \in \delta_A)$.
Each f_i corresponds then to the choice of one such δ_A for each $A \subseteq M$.
We thus have in the minimal case $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in \mu(T)$ - the set of $\prec -$ minimal models of T in $\mathcal{M}$. If, for instance, $M(T)$ consists of infinite descending chains, then $\mu(T) = \emptyset$, and $T \models_{\mathcal{M}} \phi$ for any ϕ, $\perp$ included. On the other hand, any $m \in \mu(T)$ will be a "witness" of *all* $\models_{\mathcal{M}} -$ consequences of T, all ϕ with $T \models_{\mathcal{M}} \phi$ will hold in such m.
In the limit case, we have $T \models_{\mathcal{M}} \phi$ iff there is some $\delta_{T,\phi} \subseteq M(T)$ which satisfies ($\delta 1$) and ($\delta 2$) with respect to $M(T)$ and such that ϕ holds in all $m \in \delta_{T,\phi}$. Thus, in the limit case, $\mu(T)$ may be empty, but if $M(T) \neq \emptyset$, we will still not have $T \models_{\mathcal{M}} \perp$, as all $\delta_{T,\phi}$ are then non-empty. It is easily seen, that if $T \models_{\mathcal{M}} \phi$ and $T \models_{\mathcal{M}} \phi'$, and $\prec$ is transitive, then also $T \models_{\mathcal{M}} \phi \wedge \phi'$: if $\delta_{T,\phi}$ and $\delta_{T,\phi'}$ are suitable, then $\delta_{T,\phi} \cap \delta_{T,\phi'}$ will be a suitable $\delta_{T,\phi\wedge\phi'}$. Moreover, if $T \models_{\mathcal{M}} \phi$, and $M(T \cup \{\phi\}) \subseteq M(T') \subseteq M(T)$, then also $T' \models_{\mathcal{M}} \phi$.
An immediate consequence of the locality of the definition of f is a kind of upward absoluteness in the minimal case. An element, which is not minimal in A, can't be minimal in any B with $A \subseteq B$:
(1) $A \subseteq B \rightarrow f(B) \cap A \subseteq f(A)$.
In contrast, in the general case of arbitrary f, the choice may depend on the "context", there need not be any interdependence between f(A) and f(B), even if $A \subseteq B$.
As a matter of fact, (1) is *the* crucial property for Minimal Preferential Structures, in the sense that any choice function which obeys (1) and the trivial property
(0) $f(A) \subseteq A$
can be represented by a Preferential Structure, i.e. by such a binary relation of preference (see [Sch92], Proposition 3.3). This is a very general "algebraic" characterization, the underlying set M need not consist of models, it may be just any arbitrary set.
A similar result for Limit Preferential Structures seems to be missing up to now, see [Bou90a], [Bou90b], [Bou92] and [Sch94-t4] for restricted cases. Boutilier's results are restricted in the sense that they treat finitely axiomatisable theories only, but such theories correspond exactly to clopen sets in the standard topology. Yet clopen sets can neither be entered nor left by approximation, so this seems to go somewhat against the spirit of the limit approach. (See [Sch94-t4] for details.) In the end, one might also criticize that

Boutilier lets the modal operators $\Diamond$ and $\Box$ - from the semantical point of view quantifiers over possible worlds which cooperate with the relation $\prec$ - do all the "nasty" work, which turns out so unpleasant in an attempt of a direct construction. But, it is always easy to criticize in hindsight ....

1.2.4 Interpretation:

We have so far deliberately left open the base logic and its models in M, as well as the intuition behind the "importance" of models of the base logic.
Non-monotonic Logic:
This "importance" may be read as "normality" in the case of non-monotonic logics: We are primarily interested in reasoning about the normal cases, and the preferred models are the most normal ones - where birds can fly, houses have doors etc.
As a matter of fact, Preferential Structures in their various forms provide an important and relatively well-studied group of semantics for non-monotonic logics and have proved a powerful tool for investigation, providing - via additional properties of the relation $\prec$ - a technique of constructing semantics of logical systems of different strengths. Limit Preferential Structures for non-monotonic logics were introduced by G.Bossu and P.Siegel in [BS85], the minimal case was first examined by Y.Shoham ([Sho87]) as a generalization of the Minimal Model Semantics for Circumscription. More or less general cases of Preferential Structures are characterized by soundness and completeness theorems in [KLM90], [LM92], [Sch92], and in this paper for the minimal case, in [Bou90a], [Bou90b], [Bou92], and in [Sch94-t4] for the limit case. For an overview, see also [Mak94].
Deontic Logic:
Deontic logic reasons about the morally acceptable situations, and about what ought to be done (by humans, robots etc.). Reasoning about morally acceptable actions can be split into two subquestions: Reasoning about the morally acceptable states, and reasoning about the problem of acting in a way that those states are reached. The latter question can be considered separately, at least in first approximation.
In this framework, the preferred or more important models are those which are morally more acceptable. Thus, Preferential Structures also provide a natural semantics for deontic logic, and, in fact, were examined as such before the advent of non-monotonic logics [Han69]. This was pointed out by D.Makinson in [Mak93].
In hindsight, it is no surprise that, when examining choice functions which single out some states as more important or interesting than others, a local preference by a binary relation tends to emerge in many cases. Such local preferences seem to correspond well to intuitions, and simplify the situation by making the choice context-independent.
In [Mak93], still other natural applications of Preferential Structures are discussed.

1.2.5 An Example:

Before we proceed, we give a simple example which shows that the relation $\models_{\mathcal{M}}$ defined by a Preferential Structure may indeed be a non-monotonic consequence relation.
Let $\mathcal{L}$ be the propositional language with two varibles p, q, let M consist of two (classical) models, $m \models p \wedge q$, $m' \models \neg p \wedge \neg q$, and let $m' \prec m$. Then $\emptyset \models_{\mathcal{M}} \neg q$, but $p \models_{\mathcal{M}} q$ in both the minimal and the limit definition.
As is the case already in our example, not all classical models for a given language $\mathcal{L}$ need occur in the base set M of a Preferential Structure $\mathcal{M}$ (e.g., in our example, some $m" \models p \wedge \neg q$ is missing). Moreover, some classical models might occur several times, even infinitely often. Take for example $\mathcal{L}$ with one propositional variable p and consider the structure $\mathcal{M} :=< \{< m, i >: i < \omega\}, \prec>$ with $m \models p$, and $< m, i >\prec< m, j >$ iff $j < i$. Then $\mu(M) = \emptyset$, so *true* $\models_{\mathcal{M}} \perp$ in the minimal reading, but *true* $\models_{\mathcal{M}} \phi$ iff ϕ is a classical consequence of p, in the limit reading.

1.2.6 Strengthenings of the Conditions for the Relation $\prec$:

Various additional conditions for the relation $\prec$ have been introduced and examined for Minimal Preferential Structures.
The most natural one is perhaps transitivity.
An important condition, which results in nice properties of the semantic consequence relation $\models_{\mathcal{M}}$ is smoothness (terminology of D.Lehmann and his co-authors) or stopperedness (terminology of D.Makinson): Given a theory T, and a non-minimal model m of T, there is $m' \prec m$, which is a minimal model of T. (This condition can e.g. be violated through the existence of infinite descending chains or by non-transitive relations.) Consequently, if $M(T) \neq \emptyset$, then $\mu(T) \neq \emptyset$. The counterpart for the consequence relation $\models_{\mathcal{M}}$ is Cumulativity (see [KLM90] and [Gab85]) which says that two theories T, T' with $T \subseteq T' \subseteq \{\phi : T \models_{\mathcal{M}} \phi\}$ have the same consequences: $T \models_{\mathcal{M}} \phi$ iff $T' \models_{\mathcal{M}} \phi$. We may read this as "normal use of Lemmas": If we have already deduced the "Lemma" ϕ from T, we neither loose nor win in terms of possible deductions by starting from $T \cup \{\phi\}$.
As a matter af fact, again a very general algebraic representation result can be obtained: A choice function f can be represented by a stoppered Minimal Preferential Structure iff it satisfies the conditions (0), (1) and
(2) $f(A) \subseteq B \subseteq A \rightarrow$ f(A)=f(B)
and if its domain satisfies closure under finite intersections and unions, see below, Section 2. In fact, this is the central result of the present paper - Theorem 2.1.
Another strengthening of $\prec$ is rankedness, which may be seen as the existence of a "rotating scale with fixed origin": $\prec$ is called ranked (on M), iff there is an order-preserving function $f : (M, \prec) \rightarrow (X, \lhd)$, where $\lhd$ is a total order on X. Then two $\prec$-incomparable elements $m, m' \in M$ behave exactly the same way with respect to $\prec$: $n \prec m$ iff $n \prec m'$, and $m \prec n$ iff $m' \prec n$. The corresponding property of $\models_{\mathcal{M}}$ is Rational Monotony: If

$\alpha \models_{\mathcal{M}} \gamma$, then $\alpha \wedge \beta \models_{\mathcal{M}} \gamma$ or $\alpha \models_{\mathcal{M}} \neg\beta$ (see [LM92]). General representation results are to be found below in Section 3.

1.2.7 Generalizations:

Besides the fact that our results hold for the infinite case too - see [Sch92] for a counterexample to the Kraus/Lehmann/Magidor results in the infinite situation - our essentially algebraic approach has a certain advantage over those which immediately work with logics - it is easier to adapt it to other situations: We can consider choice functions on arbitrary sets, which need not be sets of models.

The strength of our algebraic representation results lies in their generality, and we can more or less easily obtain soundness and completeness results as corollaries for non-monotonic logics with classical propositional logic as background in [Sch92] and in [Sch94-t4] (which can also be read with classical predicate logic in the background), and for Plausibility Logic (a sequent calculus for a very poor language without connectives, introduced by D.Lehmann, see [Leh92a], [Leh92b], [Sch94-t2]).

But these representation results (or at least their ideas) can be used in still more general situations, where we do not compare single models, but whole "threads" of developments in dynamic situations. This is done in [Sch92-n9], where we consider preferences on developments.

In the first part (Theorem 2.8 there), we show that a deontic choice function of "good" developments defined in [Tho84] can be represented by a ranked, stoppered relation on all developments. Thus, we do not compare single models, but developments in a branching time structure. Again, the question of acting in a way that those preferred developments are reached (or not left), is left open, we only discuss - as R.Thomason does in [Tho84] - the "quality" of the developments, and show that again a local choice by a binary preference relation suffices, and even a very nice one.

In the second part (Theorem 3.4 there), we give a characterization of coupled logics which can be obtained from a preference relation on developments: Given the information S and T at time point s and time point t about a development, and a preference over developments, we examine the resulting preferred theories S' and T', where S' and T' are the theories determined by the end-points (i.e. models) of the preferred developments among those which pass through S- and T-models. This defines a pair of coupled logics, $< S, T > \mid\!\sim < S', T' >$.

1.3 The basic definitions

Definition 1.1 We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g: g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets. Given a class of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X}\lceil X := \{< x, i > \in \mathcal{X} : x \in X\}$,

so if $\mathcal{X}$ is a function f, $f\lceil X$ is the usual notation for the restriction of f to a subset of its domain.
Let $\mathcal{L}$ be a propositional language, we denote by $v(\mathcal{L})$ the set of its variables, by $M_{\mathcal{L}}$ the set of its classical models, ϕ etc. shall denote formulas, T etc. theories in $\mathcal{L}$ (i.e. $T \subseteq \mathcal{L}$), and $M_T \subseteq M_{\mathcal{L}}$ the models of T.
For any classical model m, let $Th(m)$ be the set of formulas valid in m, likewise $Th(M) := \{\phi : m \models \phi$ for all $m \in M\}$, if M is a set of classical models. For two theories T and T', let $T \vee T' := \{\phi \vee \psi : \phi \in T, \psi \in T'\}$.
$\overline{T} \subseteq \mathcal{L}$ will denote the closure of T under classical logic, and $\vdash$ the classical consequence relation. Given some other logic, $\overline{\overline{T}}$ will denote the set of consequences of T under that logic, i.e. if the more conventional notation for the logic is $\mid\sim$, then $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$.
$\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{P}(M_{\mathcal{L}})$ shall be the set of definable subsets of $M_{\mathcal{L}}$, i.e. $A \in \boldsymbol{D}_{\mathcal{L}}$ iff there is some $T \subseteq \mathcal{L}$ s.t. $A = M_T$. If the context is clear, we omit the subscript $\mathcal{L}$ from $\boldsymbol{D}_{\mathcal{L}}$.
For $X \subseteq \mathcal{P}(M_{\mathcal{L}})$, a function $f : X \to \mathcal{P}(M_{\mathcal{L}})$ will be called definability preserving (dp), iff for all $Y \in \boldsymbol{D}_{\mathcal{L}} \cap X$ $f(Y) \in \boldsymbol{D}_{\mathcal{L}}$.
If $\boldsymbol{D}_{\mathcal{L}} \subseteq X$, then $f : X \to \mathcal{P}(M_{\mathcal{L}})$ defines a logic $T \mapsto T^f$ on $\mathcal{L}$ by $T^f := \{\phi$: $\forall m \in f(M_T).m \models \phi\}$. So, if f=id, then $T^f = \overline{T}$.
Note that $f(M_T) \subseteq M_{T^f}$ always holds, but not necessarily $f(M_T) = M_{T^f}$, the latter only iff f is dp. □

We note en passant the following (proved in [Sch92]):

Fact 1.2 1. If $v(\mathcal{L})$ is infinite, then $\boldsymbol{D}_{\mathcal{L}} \neq \mathcal{P}(M_{\mathcal{L}})$,
2. $\emptyset$, $M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$,
3. $\boldsymbol{D}_{\mathcal{L}}$ contains all singletons,
4. $\boldsymbol{D}_{\mathcal{L}}$ is closed under arbitrary intersections,
5. $\boldsymbol{D}_{\mathcal{L}}$ is closed under finite unions. □

Definition 1.3 $\mathcal{Z} =< \mathcal{X}, \prec>$ will be called a preferential structure iff $\mathcal{X}$ is a set of pairs and $\prec$ is a binary relation on $\mathcal{X}$. We say that $\mathcal{Z}$ is transitive, irreflexive etc., iff $\prec$ is.
$< y, i >$ is called a minimal element of $\mathcal{X}\lceil Y$ in $\mathcal{Z}$ iff:
1. $< y, i > \in \mathcal{X}\lceil Y$ and,
2. there is no $< y', i' > \in \mathcal{X}\lceil Y$ s.t. $< y', i' > \prec < y, i >$.
Thus, $\mathcal{Z}$ defines a function $\mu_{\mathcal{Z}} : V \to V$ (V the set-theoretic universe) by $\mu_{\mathcal{Z}}(Y) := \{y :$ there is i s.t. $< y, i >$ is a minimal element of $\mathcal{X}\lceil Y\}$. Given a set Z, $\mu_{\mathcal{Z},Z}$ shall denote $\mu_{\mathcal{Z}}\lceil\mathcal{P}(Z)$. □

Definition 1.4 $\mathcal{Z} =< \mathcal{X}, \prec>$ will be called $\mathcal{Y}$-smooth (terminology of [KLM90]) or $\mathcal{Y}$-stoppered (terminology of [Mak94]) iff for all $X \in \mathcal{Y}$ and $< y, i > \in \mathcal{X}\lceil X$, either $< y, i >$ is minimal in $\mathcal{X}\lceil X$, or there is $< y', i' > \prec < y, i >$, $< y', i' >$ minimal in $\mathcal{X}\lceil X$. □

Definition 1.5 A preferential structure $\mathcal{M} =< \mathcal{X}, \prec>$ will be called a classical preferential model (cpm) for $\mathcal{L}$, iff for all $< x, i >\in \mathcal{X}$, $x \in M_{\mathcal{L}}$. $\mathcal{M}$ will be called definability preserving (dp) iff $\mu := \mu_{\mathcal{M},M_{\mathcal{L}}} : \mathcal{P}(M_{\mathcal{L}}) \to \mathcal{P}(M_{\mathcal{L}})$ is definability preserving.
By the above, $\mathcal{M}$ defines a logic on $\mathcal{L}$ by $T^{\mathcal{M}} := T^{\mu}$, i.e. $T^{\mathcal{M}}:= \{\phi \in \mathcal{L}$: ϕ holds in all $m \in \mu(M_T)\}$.
A logic $\overline{\overline{\;}}$ for $\mathcal{L}$ is said to be representable by a cpm, iff there is a cpm $\mathcal{M}$ for $\mathcal{L}$, s.t. for all $T \subseteq \mathcal{L}$ $T^{\mathcal{M}} = \overline{\overline{T}}$.
For $< m, i >\in \mathcal{X}$, we shall abuse notation and say $< m, i >\models \phi$ iff $m \models \phi$, for $\phi \in \mathcal{L}$. $\mathcal{M}$ will be called smooth or stoppered iff it is $\boldsymbol{D}_{\mathcal{L}}$-stoppered. □

We recollect and note:

Fact 1.6 Let $\mathcal{L}$ be a fixed propositional language, $\boldsymbol{D}_{\mathcal{L}} \subseteq X$, $f : X \to \mathcal{P}(M_{\mathcal{L}})$, for a $\mathcal{L}$-theory T $\overline{\overline{T}} := \{\phi : \forall m \in f(M_T).m \models \phi\} = Th(f(M_T))$, $W := \{S \subseteq \mathcal{L} : S$ is complete, $\vdash$-closed and $\vdash$-consistent $\}$. For $S \in W$, let m_S be its only model, finally, let T, T' be arbitrary theories, $S \in W$, then:
(a) $M_S \subseteq M_T \leftrightarrow S \vdash T$,
(b) $m_S \in M_T \leftrightarrow S \vdash T$,
(c) If $M \subseteq M_{\mathcal{L}}$ is finite, then $M = M_{Th(M)}$,
(d) $f(M_T) \subseteq M_{\overline{\overline{T}}}$,
(e) $M_T \cup M_{T'} = M_{T \vee T'}$,
(f) $f(M_T) = \emptyset \leftrightarrow \bot \in \overline{(\overline{\overline{T}})}$.
If f is dp or $f(M_T)$ is finite, then the following also hold:
(g) $f(M_T) = M_{\overline{\overline{T}}}$,
(h) $T' \vdash \overline{\overline{T}} \leftrightarrow M_{T'} \subseteq f(M_T)$,
(i) $S \vdash \overline{\overline{T}} \leftrightarrow m_S \in f(M_T)$,
(k) $f(M_T) = M_{T'} \leftrightarrow \overline{T'} = \overline{(\overline{\overline{T}})}$

Proof: The proofs are easy, we only note: (c) " $\subseteq$ " will always hold. " $\supseteq$ ": Let $M = \{m_0 \ldots m_n\}$. Suppose there is $m \notin M$, $m \models Th(M)$. Then for all $m' \in M$ there is $\phi_{m'}$ with $m' \models \phi_{m'}$, $m \not\models \phi_{m'}$. Then $\phi := \phi_{m_0} \vee \ldots \vee \phi_{m_n} \in Th(M)$, but $m \not\models \phi$, $Contrad \ldots$ □

1.4 How many logically identical copies of models do we need in preferential structures?

Several representation results use in their constructions several copies of (logically identical) models (see e.g. [Sch92], or the present paper). Thus, we may have in those constructions m and m' with the same logical properties, but with different "neighbourhoods" in the preferential structure, for example, there may be some m" with $m" \prec m$, but $m" \not\prec m'$. David Makinson and Hans Kamp have asked the author whether such repetitions of models are sometimes necessary to represent a logic, we now give a (positive) answer. For the connection of the question to ranked structures see Section 3 below, in particular Lemma 3.5.

1.4.1 The Finite Case, an Illustration

Consider the propositional language $\mathcal{L}$ of 2 propositional variables p, q, and the classical preferential model $\mathcal{M}$ defined by
$m_0 \models p \wedge q$, $m_1 \models p \wedge q$, $m_2 \models \neg p \wedge q$, $m_3 \models \neg p \wedge \neg q$, and $m_2 \prec m_0$, $m_3 \prec m_1$. Let $\models_{\mathcal{M}}$ be its consequence relation.
Obviously, $Th(m_0) \vee \{\neg p\} \models_{\mathcal{M}} \neg p$, but there is no complete theory T' s.t. $Th(m_0) \vee T' \models_{\mathcal{M}} \neg p$. (If there were one, T' would correspond to $m_0 = m_1$, m_2, m_3, or the missing $m_4 \models p \wedge \neg q$, but we need two models to kill all copies of m_0.) On the other hand, if there were just one copy of m_0, then one other model, i.e. a complete theory would suffice. More formally, if we admit at most one copy of each model in a structure $\mathcal{M}$, $m \not\models T$, and $Th(m) \vee T \models_{\mathcal{M}} \phi$ for some ϕ s.t. $m \models \neg\phi$ - i.e. m is not minimal in the models of $Th(m) \vee T$ - then there is a complete T' with $T' \vdash T$ and $Th(m) \vee T' \models_{\mathcal{M}} \phi$, i.e. there is m' with $m' \models T'$ and $m' \prec m$.

1.4.2 The Infinite Case

Let κ, λ be infinite cardinals. Let $\mathcal{L}$ have κ propositional variables, p_i, $i < \kappa$. Consider any $\vdash$-consistent $\mathcal{L}$-theory T, a model m s.t. $m \not\models T$, and the following structure $\mathcal{M}$: $\mathcal{X} := \{< m, n >: n \models T\} \cup \{< n, 0 >: n \models T\}$, with $< n, 0 > \prec < m, n >$. Let $\phi \in T$ be s.t. $m \not\models \phi$. Obviously, $T \vee Th(m) \models_{\mathcal{M}} \phi$, as all copies of m are destroyed by the full set of models of T, but no T' truly stronger than T will do, as some copy of m will not be destroyed.
In general, however, the same logic as defined by $\mathcal{M}$ can be represented by structures with considerably less copies. It suffices to find a set of models $M \subset M_T$, where exactly the formulas of the classical closure of T hold - i.e. $M \models \phi$ iff $T \vdash \phi$, we shall then call M dense in M_T - and to take as $\mathcal{M}'$ the structure $\mathcal{X}' := \{< m, n >: n \in M\} \cup \{< n, 0 >: n \in M\}$, again with $< n, 0 > \prec < m, n >$. So we can rephrase the question to: What is the minimal size of M dense in M_T?

(a) A nice case: Take for m the model that makes all p_i true, and $T := \{\neg p_0\}$, so $card(M_T) = 2^\kappa$, and the first construction of $\mathcal{M}$ as above will need 2^κ copies of m. As $\mathcal{L}$ has only κ formulas, and any subset of M_T makes all formulas of T true, we see that there is a dense subset $M \subseteq M_T$ of size κ: For any ϕ s.t. $T \not\vdash \phi$ take some $m_\phi \in M_T$ s.t. $m_\phi \not\models \phi$.
But, in our nice case, considerably less than κ models might do: Assume there is $\lambda < \kappa$ s.t. $2^\lambda \geq \kappa$, so there is an injection $h : \{p_i : 0 < i < \kappa\} \to \mathcal{P}(\lambda)$. Let now $0 < i \neq j < \kappa$. For

$\alpha < \lambda$, define the model m_α by $m_\alpha \models \neg p_0$ and $m_\alpha \models p_i :\leftrightarrow \alpha \in h(p_i)$. By $h(p_i) \neq h(p_j)$, there is $\alpha < \lambda$ s.t. $\alpha \in h(p_i)-h(p_j)$ or $\alpha \in h(p_j)-h(p_i)$, so $m_\alpha \models p_i \wedge \neg p_j$ or $m_\alpha \models \neg p_i \wedge p_j$, i.e. there is some m_α which discerns p_i, p_j. This is essentially enough:
Let M be the closure of $\{m_\alpha : \alpha < \lambda\}$ under the finite operations -,+,* defined by
$(-m) \models p_i :\leftrightarrow m \models \neg p_i$
$(m+m') \models p_i :\leftrightarrow m \models p_i$ or $m' \models p_i$
$(m * m') \models p_i :\leftrightarrow m \models p_i$ and $m' \models p_i$.
M still has cardinality λ, and $M \subseteq M_T$.
Let ϕ be s.t. $\neg p_0 \not\vdash \phi$, we have to find $m \in M$ s.t. $m \models \neg\phi$. Let $\neg\phi \equiv \phi_0 \vee \ldots \vee \phi_n$, where each $\phi_k = \pm p_{i_0} \wedge \ldots \wedge \pm p_{i_r}$ for some $i_0..i_r$. By $\neg p_0 \not\vdash \phi$, $Con(\neg p_0, \neg\phi)$ ($\vdash$-consistency), so $Con(\neg p_0, \phi_k)$ for some $0 \leq k \leq n$. Fix such $\phi_k = \pm p_{i_0} \wedge \ldots \wedge \pm p_{i_r}$, say $\phi_k = p_{j_0} \wedge \ldots \wedge p_{j_s} \wedge \neg p_{g_0} \wedge \ldots \wedge \neg p_{g_t}$. By $Con(\neg p_0, \phi_k)$, p_0 is none of the p_{j_x}. (If one of the $\neg p_{g_y}$ is $\neg p_0$, it can be neglected, it will come out true anyway.) Fix $0 \leq x \leq s$, let $0 \leq y \leq t$. Then there is m_α s.t. $m_\alpha \models p_{j_x} \wedge \neg p_{g_y}$ or $-m_\alpha \models p_{j_x} \wedge \neg p_{g_y}$. Let $m_{x,y}$ be the m_α or $-m_\alpha$, and set $m_x := m_{x,0} * \ldots * m_{x,t}$. Then $m_x \models p_{j_x} \wedge \neg p_{g_0} \wedge \ldots \wedge \neg p_{g_t}$. For $m := m_0 + \ldots + m_s$, $m \models \phi_k$, so $m \models \neg\phi$, and $m \in M$.
On the other hand, in our example, λ many models with $2^\lambda < \kappa$ will not do: Assume that for each $0 < i \neq j < \kappa$ there is $\alpha < \lambda$ and $m_\alpha \in M_T$ with $m_\alpha \models p_i \wedge \neg p_j$. Then there is a function $f : 2^\lambda \rightarrow \kappa - \{0\}$ onto: For $A \subseteq \lambda$, let $f(A) := \bigcup\{j : 0 < j < \kappa \wedge \forall\alpha \in A.m_\alpha \models p_j\}$. But, for $0 < i < \kappa$, and $A_i := \{\alpha < \lambda : m_\alpha \models p_i\}$ $f(A_i) = i$: Obviously, for $\alpha \in A_i$, $m_\alpha \models p_i$. But, if $i \neq j$, then there is $\alpha \in A_i$ with $m_\alpha \models p_i \wedge \neg p_j$.

(b) There are, however, examples where we need the full size κ: Let $\mathcal{L}$ be as above, consider $m^- \models \{\neg p_j : j < \kappa\}$, $T := \{p_i \vee p_j : i \neq j < \kappa\}$, and let $m^+ \models \{p_j : j < \kappa\}$ and $m_i^- \models \{\neg p_i\} \cup \{p_j : i \neq j < \kappa\}$ for $i < \kappa$.
Let the structure $\mathcal{M}$ be defined by $\mathcal{X} := \{< m^-, m^+ >\} \cup \{< m^-, m_i^- >: i < \kappa\} \cup \{< m^+, 0 >\} \cup \{< m_i^-, 0 >: i < \kappa\}$ and $< n, 0 > \prec < m^-, n >$ for $n = m^+$ or $n = m_i^-$, some $i < \kappa$. Then $Th(m^-) \vee T \models_{\mathcal{M}} T$. But there is no $M \subseteq M_T$ dense with $card(M) < \kappa$. Obviously, $M_T = \{m^+\} \cup \{m_i^- : i < \kappa\}$, and $\{m_i^- : i < \kappa\} \subseteq M_T$ is dense (see [Sch92]), but taking away any m_i^- will change T: p_i becomes true.

2 A Completeness Result for Stoppered Classical Preferential Models

We characterize definability preserving stoppered preferential models, extending our result of [Sch92] where we characterized general definability preserving preferential models. Again, our results hold in the arbitrary infinite case, too, in contrast to the Kraus/-Lehmann/Magidor results (see [Sch92] for a counterexample). The central combinatorial result of this Section is Theorem 2.1, it is applied to logic in Theorem 2.12, using Proposition 2.13 as intermediary step. Apart from the combinatorial core, this section is largely in parallel to [Sch92], replacing Proposition 3.3 there by Theorem 2.1 here. This section is self-contained, but may perhaps better be read with a copy of [Sch92] by the side. We make ample and tacit use of the Axiom of Choice.

2.0.3 The method and the result:

As said in the introduction, and as done already in [Sch92], we first give a very general representation result for arbitrary stoppered preferential structures (Theorem 2.1). This is the main tool to prove the logical representation result (Theorem 2.12), which, as a matter of fact, is then a more or less straightforward corollary of Theorem 2.1.
This technique has not only the advantage of giving representation results for the infinite case, but, by its generality and independence of logic, provides the essential building block for quite different situations. This was exploited in the case of Plausibility Logic (see [Leh92a], [Leh92b]), where the techniques of [Sch92] gave a positive representation result, and Theorem 2.1 below seemed to promise another representation result. In the end, however, the seemingly minor and auxiliary prerequisite of Theorem 2.1 - closure under finite unions and intersections - turned out to be crucial: Closure under finite unions does not hold in the case of Plausibility Logic (due to the lack of "or" on the left hand side of the sequents), and a counterexample emerged, first checked on a computer. See [Sch94-t2] for details.
Theorem 2.1 gives a characterization of the functions $f : \mathcal{Y} \rightarrow \mathcal{Y}$ choosing the minimal elements in stoppered structures. Comparison with [Sch92] shows that the additional property (f3), "algebraic cumulativity", corresponds exactly to to the stopperedness property. The translation into logic (Theorem 2.12) parallels that of [Sch92], we characterize definability preserving stoppered preferential models by the conditions of [Sch92], augmented with (logical) cumulativity.
The proof of Theorem 2.1: We show that a function $f : \mathcal{Y} \rightarrow \mathcal{Y}$ can be represented by a stoppered preferential structure iff f satisfies (f1)-f(3). The more difficult part is, of course, to construct a representing structure. Claim 2.2 contains a number of auxiliary results to be used in the construction. We then (up to Claim 2.7 included) define a primitive preferential structure $\mathcal{Z}$, which represents f, but is perhaps not stoppered. This structure $\mathcal{Z}$ is then refined to $\mathcal{Z}'$, which also represents f (Claim 2.9), and is stoppered (Claim 2.11). In the structure $\mathcal{Z}'$, all pairs destroying stopperedness in $\mathcal{Z}$ are successively repaired, by adding minimal elements: If $< y, j >$ is not minimal, and has no minimal $< x, i >$ below it, we just add one such $< x, i >$. As the repair process might itself generate "bad" pairs, the process may have to be repeated infinitely often. Of course, one has to take care that the representation property is preserved.
The proofs of Proposition 2.13 and Theorem 2.12 are straightforward.

Theorem 2.1 Let Z be any set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be closed under finite unions and intersections, and let $f : \mathcal{Y} \rightarrow \mathcal{Y}$. Then there is a $\mathcal{Y}$-stoppered preferential structure $\mathcal{Z}$, s.t. for all $X \in \mathcal{Y}$ $f(X) = \mu_{\mathcal{Z}}(X)$ iff

(f1) $f(X) \subseteq X$,
(f2) $X \subseteq Y \rightarrow f(Y) \cap X \subseteq f(X)$,
(f3) $f(X) \subseteq Y \subseteq X \rightarrow$ f(X)=f(Y),
for all $X, Y \in \mathcal{Y}$.

Note that, as $\mathcal{Y}$ is closed under finite intersections, in the presence of (f1), (f2) is equivalent to (f2'), where (f2') $f(X) \cap Y \subseteq f(X \cap Y)$

Proof: " $\rightarrow$ ": Soundness is easy, so we do it first. (f1) trivial.
(f2) Let $\mu := \mu_{\mathcal{Z}}$, $X \subseteq Y$, and $x \in \mu(Y) \cap X$. So $x \in X \cap Y$, and there is i s.t. $< x, i > \in \mathcal{X}$, and there is no $< x', i' > \in \mathcal{X}$, $x' \in Y$, $< x', i' > \prec < x, i >$. But then there can be no such $< x', i' >$ with $x' \in X$. Consequently, $x \in \mu(X)$.
(f3) $\mu(X) \subseteq Y \subseteq X \rightarrow \mu(X) \cap Y \subseteq \mu(Y) \rightarrow \mu(X) \subseteq \mu(Y)$ by (f2). Assume there is $x \in \mu(Y) - \mu(X)$, so $x \in Y \subseteq X$, so for each $< x, i > \in \mathcal{X}$ there is $< x', i' > \in \mathcal{X}$, $x' \in \mu(X)$ with $< x', i' > \prec < x, i >$, but by $\mu(X) \subseteq \mu(Y)$, $x \notin \mu(Y)$, *contradiction* $\Box$ (" $\rightarrow$ ")
Outline of " $\leftarrow$ ":
We first prove a number of elementary facts about such f, which will be used later on, then define a structure $\mathcal{Z}$, which represents f, but is not necessarily $\mathcal{Y}$-stoppered, refine it to $\mathcal{Z}'$ and show that $\mathcal{Z}'$ represents f too, and that $\mathcal{Z}'$ is $\mathcal{Y}$-stoppered.

Claim 2.2 Let A, A', B, U, and all $A_i \in \mathcal{A}$ be in $\mathcal{Y}$.
Properties (f1) and (f2) entail:
(1) $f(A \cup B) - A \subseteq f(B)$,
(2) $f(A \cup B) \subseteq A \rightarrow f(A \cup B) \subseteq f(A)$,
(3) $f(A) \subseteq B \rightarrow f(A \cup B) \subseteq B$,
(4) $A = \bigcup A_i \rightarrow f(A) \subseteq \bigcup f(A_i)$
Properties (f1) - (f3) entail
(5) $f(A) \subseteq A' \rightarrow (A - f(A)) \cap f(A') = \emptyset$,
(6) $f(A \cup B) \subseteq A \rightarrow f(A \cup B) = f(A)$,
(7) $f(A) \subseteq B \rightarrow f(B) \cap A \subseteq f(A)$,
(8) $f(U \cup A) \subseteq U$, $A' \subseteq A \rightarrow f(U \cup A') \subseteq U$,
(9) $f(A) \subseteq U \rightarrow f(U \cup (B \cap A)) \subseteq U$ for all $B \in \mathcal{Y}$,
(10) $\exists u(u \in f(U \cup A) - U,\ u \in A') \rightarrow f(U \cup A') \not\subseteq U$,
(11) $f(B) \subseteq \bigcup \mathcal{A}$, $f(A) \subseteq U$ for $A \in \mathcal{A}$, then $\forall x(x \in B - f(B) \rightarrow x \notin f(U))$,
(12) $\forall x(x \in B - f(B),\ x \in f(U) \rightarrow f(B) - \bigcup\{A \in \mathcal{Y} : x \in f(A) \subseteq U\} \neq \emptyset)$

Proof: (1) $f(A \cup B) \subseteq A \cup B$, thus $f(A \cup B) - A \subseteq f(A \cup B) \cap B \subseteq f(B)$.
(2) $f(A \cup B) = f(A \cup B) \cap A \subseteq f(A)$.
(3) $f(A \cup B) \cap A \subseteq f(A) \subseteq B$. $f(A \cup B) \cap B \subseteq B$, thus, by $f(A \cup B) \subseteq A \cup B$, $f(A \cup B) \subseteq B$.
(4) $f(A) \cap A_i \subseteq f(A_i) \subseteq \bigcup f(A_i)$, by $A = \bigcup A_i$ thus $f(A) \subseteq \bigcup f(A_i)$.
(5) $f(A) \subseteq A' \rightarrow f(A) \subseteq A' \cap A \subseteq A \rightarrow f(A' \cap A) = f(A)$. So by $f(A') \cap (A \cap A') \subseteq f(A \cap A')$

$f(A') \cap A \subseteq f(A \cap A') = f(A)$ and $(A - f(A)) \cap f(A') = \emptyset$.
(6) $f(A \cup B) \subseteq A \subseteq A \cup B \rightarrow f(A) = f(A \cup B)$.
(7) $f(A) \subseteq B \rightarrow$ (by (3)) $f(A \cup B) \subseteq B \rightarrow f(A \cup B) \subseteq B \subseteq A \cup B \rightarrow f(A \cup B) = f(B)$. But $f(A \cup B) \cap A \subseteq f(A)$, thus $f(B) \cap A \subseteq f(A)$.
(8) $f(U \cup A) \subseteq U \rightarrow f(U \cup A) \subseteq U \cup A' \subseteq U \cup A \rightarrow f(U \cup A') = f(U \cup A) \subseteq U$.
(9) $f(A) \subseteq U \rightarrow$ (by (3)) $f(U \cup A) \subseteq U \rightarrow$ (by (8)) $f(U \cup (B \cap A)) \subseteq U$.
(10) Suppose $f(U \cup A') \subseteq U$, thus $f(U \cup A') \subseteq U \subseteq U \cup A \rightarrow$ (by 5)) $((U \cup A') - f(U \cup A')) \cap f(U \cup A) = \emptyset$. But $u \in f(U \cup A)$, $u \in U \cup A'$, $u \notin U$, thus by supposition $u \in f(U \cup A') \subseteq U$, *Contradiction.*
(11) We show that $f(B) \subseteq \bigcup \mathcal{A}$ and $f(A) \subseteq U$ for $A \in \mathcal{A}$ implies $f(U) \cap B \subseteq f(B)$.
$f(B) \subseteq \bigcup \mathcal{A}$, $f(A) \subseteq U$ for $A \in \mathcal{A} \rightarrow f(U \cup f(B)) = f(\bigcup\{U \cup (f(B) \cap A) : A \in \mathcal{A}\}) \subseteq$ (by (4)) $\bigcup\{f(U \cup (f(B) \cap A)) : A \in \mathcal{A}\} \subseteq$ (by (9)) $U \rightarrow$ (by (6)) $f(U \cup f(B)) = f(U)$. By $f(B) \subseteq U \cup f(B)$ and (7) $f(U \cup f(B)) \cap B \subseteq f(B)$, thus by the above $f(U) \cap B \subseteq f(B)$.
(12) Suppose the contrary, then $f(B) \subseteq \bigcup\{A \in \mathcal{Y} : x \in f(A) \subseteq U\}$, for such A $f(A) \subseteq U$, $x \in B - f(B)$ and $x \in f(U)$, contradicting (11). □ (Claim 2.2)

Definition 2.3 For $x \in Z$, let $\mathcal{Y}_x := \{f(Y)\colon Y \in \mathcal{Y} \wedge x \in Y - f(Y)\}$, $F_x := \Pi \mathcal{Y}_x$, and $K := \{x \in Z\colon \exists X \in \mathcal{Y}. x \in f(X)\}$.

Remark 2.4 $g \in F_x \rightarrow ran(g) \subseteq K$ (By definition, $f(Y) \subseteq K$ for all $Y \in \mathcal{Y}$.)

Definition 2.5 Let $\mathcal{X} := \{< x, g >\colon x \in K, g \in F_x\}$, $< x', g' > \prec < x, g > :\leftrightarrow x' \in ran(g)$, $\mathcal{Z} := < \mathcal{X}, \prec >$, let $\mu := \mu_{\mathcal{Z}}$.

Claim 2.6 Let $X \in \mathcal{Y}$.
(1) If $x \in f(X)$, then ex. $g_X \in F_x$ with $ran(g_X) \cap X = \emptyset$,
(2) $x \in f(X) \leftrightarrow x \in X \wedge \exists g \in F_x. ran(g) \cap X = \emptyset$.

Proof: (2),"$\rightarrow$" follows from (1) and $f(X) \subseteq X$.
Case 1: $\mathcal{Y}_x = \emptyset$, thus $F_x = \{\emptyset\}$. (1): Choose $g_X := \emptyset$. (2),"$\leftarrow$": $x \in X \in \mathcal{Y}$, $\mathcal{Y}_x = \emptyset \rightarrow x \in f(X)$ by definition of $\mathcal{Y}_x$.
Case 2: $\mathcal{Y}_x \neq \emptyset$. (1): It suffices to show $x \in f(X)$, $Y \in \mathcal{Y}$, $x \in Y - f(Y) \rightarrow f(Y) \not\subseteq X$. Suppose $f(Y) \subseteq X$. By Claim 2.2, (7) then $f(X) \cap Y \subseteq f(Y)$, but $x \in f(X) \cap Y$, $x \notin f(Y)$.
(2),"$\leftarrow$": Assume hypotheses and suppose $x \in X - f(X)$. If $\emptyset \in \mathcal{Y}_x$, then $F_x = \emptyset$ - a contradiction. If $\emptyset \notin \mathcal{Y}_x$, then $\emptyset \neq f(X) \in \mathcal{Y}_x$, $f(X) \subseteq X$, and $\forall g \in F_x. ran(g) \cap f(X) \neq \emptyset$ - again a contradiction. □ (Claim 2.6)

Claim 2.7 $\forall U \in \mathcal{Y}. f(U) = \mu(U)$

Proof: Case 1: $x \notin K$. Then $x \notin f(U)$ and $x \notin \mu(U)$ for all $U \in \mathcal{Y}$.
Case 2: $x \in K$. By Claim 2.6, (2), it suffices to show that for all $U \in \mathcal{Y}$ $x \in \mu(U) \leftrightarrow x \in U$ $\wedge\ \exists g \in F_x.ran(g) \cap U = \emptyset$. Fix $U \in \mathcal{Y}$. " $\rightarrow$ ": $x \in \mu(U) \rightarrow$ ex. $< x, g >$ minimal in $\mathcal{X}\lceil U$, thus $x \in U$ and there is no $< x', g' > \prec < x, g >$, $x' \in U$, $x' \in K \rightarrow \forall x' \in ran(g).x' \notin U$ or $x' \notin K$. But $ran(g) \subseteq K$, so $ran(g) \cap U = \emptyset$. " $\leftarrow$ ": If $x \in U$, $g \in F_x$ s.t. $ran(g) \cap U = \emptyset$, then $< x, g >$ is minimal in $\mathcal{X}\lceil U$. □ (Claim 2.7)
We now proceed to construct the refined structure $\mathcal{Z}'$.

Definition 2.8 σ is called x-admissible sequence iff
1. σ is a sequence of length ω, $\sigma = \{\sigma_i : i \in \omega\}$,
2. $\sigma_o \in \Pi\{f(Y)$: $Y \in \mathcal{Y} \wedge x \in Y - f(Y)\}$,
3. $\sigma_{i+1} \in \Pi\{f(X)$: $X \in \mathcal{Y} \wedge x \in f(X) \wedge ran(\sigma_i) \cap X \neq \emptyset\}$.
Let Σ_x be the set of x-admissible sequences, for $\sigma \in \Sigma_x$ let $\sigma^* := \bigcup\{ran(\sigma_i) : i \in \omega\}$. Let $\mathcal{X}' := \{< x, \sigma >$: $x \in K \wedge \sigma \in \Sigma_x\}$ and $< x', \sigma' > \prec' < x, \sigma > :\leftrightarrow x' \in \sigma^*$. Finally, let $\mathcal{Z}' := < \mathcal{X}', \prec' >$, and $\mu' := \mu_{\mathcal{Z}'}$.

Claim 2.9 For all $U \in \mathcal{Y}$ $f(U) = \mu(U) = \mu'(U)$.

Proof: Obviously, for all $U \in \mathcal{Y}$, $\mu'(U) \subseteq \mu(U) = f(U)$. It remains to show $x \in f(U) \rightarrow x \in \mu'(U)$. Assume $x \in f(U)$, $U \in \mathcal{Y}$, we are finished if we can show that there is $\sigma \in \Sigma_x$ s.t. $\sigma^* \cap U = \emptyset$.

Claim 2.10 $U \in \mathcal{Y}$, $x \in f(U) \rightarrow \exists \sigma \in \Sigma_x.\sigma^* \cap U = \emptyset$.

Proof: Fix $U \in \mathcal{Y}$, $x \in f(U)$. We inductively construct σ, showing simultaneously for $i \geq 0$: $X \in \mathcal{Y} \wedge ran(\sigma_i) \cap X \neq \emptyset \rightarrow f(U \cup X) \not\subseteq U$. (Thus, in particular, $ran(\sigma_i) \cap U = \emptyset$.)
σ_0 :
For $Y \in \mathcal{Y}$ with $x \in Y - f(Y)$ let $Y' := f(Y) - \bigcup\{X \in \mathcal{Y} : x \in f(X) \subseteq U\}$. By Claim 2.2, (12), $Y' \neq \emptyset$. Note that $x \in f(U) \subseteq U$, so $Y' \cap U = \emptyset$. Let $\sigma_0 \in \Pi\{Y' : Y \in \mathcal{Y} \wedge x \in Y - f(Y)\}$, thus $ran(\sigma_0) \cap U = \emptyset$ and for all $X \in \mathcal{Y}$ s.t. $x \in f(X) \subseteq U$ $ran(\sigma_0) \cap X = \emptyset$.
Claim:
$X \in \mathcal{Y} \wedge ran(\sigma_0) \cap X \neq \emptyset \rightarrow f(U \cup X) \not\subseteq U.$
Proof:
Suppose $f(X \cup U) \subseteq U$, so $f(X \cup U) \subseteq U \subseteq X \cup U$, so $f(X \cup U) = f(U)$, so by $x \in f(U)$ $x \in f(X \cup U) \subseteq U$, thus $ran(\sigma_0) \cap (X \cup U) = \emptyset$ and $ran(\sigma_0) \cap X \neq \emptyset$, *contradiction* □ (Claim)
$\sigma_i \rightarrow \sigma_{i+1}$:
By induction hypothesis, $X \in \mathcal{Y} \wedge ran(\sigma_i) \cap X \neq \emptyset \rightarrow f(U \cup X) \not\subseteq U$. Let $\sigma_{i+1} \in \Pi\{f(U \cup X) - U$: $X \in \mathcal{Y} \wedge x \in f(X) \wedge ran(\sigma_i) \cap X \neq \emptyset\}$ Note that by Claim 2.2, (1) $f(U \cup X) - U \subseteq f(X)$, so σ_{i+1} satisfies the x-admissibility condition. Moreover, $u \in ran(\sigma_{i+1}) \rightarrow u \in f(U \cup X') - U$ for some $X' \in \mathcal{Y}$.

Claim:
$X \in \mathcal{Y} \wedge ran(\sigma_{i+1}) \cap X \neq \emptyset \to f(U \cup X) \not\subseteq U$
Proof:
Let $u \in ran(\sigma_{i+1}) \cap X \to u \in f(U \cup X') - U$ for some $X' \in \mathcal{Y} \to$ (by $u \in X$ and Claim 2.2, (10)) $f(U \cup X) \not\subseteq U$. □ (Claim)
□ (Claims 2.10 and 2.9)
It remains to show:

Claim 2.11 $\mathcal{Z}'$ is $\mathcal{Y}$-stoppered.

Proof: So let $X \in \mathcal{Y}$, $< x, \sigma > \in \mathcal{X}' \lceil X$. Let $x \in X - f(X)$, then $< x, \sigma > \in \mathcal{X}' \to x \in K \to \exists U \in \mathcal{Y}.x \in f(U)$. Thus $\emptyset \neq X' \subseteq f(X)$ (X' as defined for the construction of σ_0), and $ran(\sigma_0) \cap f(X) \neq \emptyset$. So there is $x' \in f(X) \subseteq K$ and all $< x', \sigma' > \prec < x, \sigma >$, as $ran(\sigma_0) \cap f(X) \neq \emptyset$. But, by Claim 2.9, $x' \in \mu'(X)$, so there is $< x', \sigma' >$ minimal in $\mathcal{X}' \lceil X$. Assume $x \in f(X) = \mu(X) = \mu'(X)$. If $< x, \sigma >$ is minimal in $\mathcal{X}' \lceil X$, we are done. So suppose there is $< x', \sigma' > \prec < x, \sigma >$, $x' \in X$. Thus $x' \in \sigma^*$. Let i be (for definiteness) minimal s.t. $x' \in ran(\sigma_i)$. So $x \in f(X)$ and $ran(\sigma_i) \cap X \neq \emptyset$. But $\sigma_{i+1} \in \Pi\{f(X'):$ $X' \in \mathcal{Y} \wedge x \in f(X') \wedge ran(\sigma_i) \cap X' \neq \emptyset\}$, so X is a candidate, moreover $f(X) \subseteq K$, so there is $x" \in f(X) \cap ran(\sigma_{i+1}) \cap K$, so all $< x", \sigma" > \prec < x, \sigma >$. But again by $\mu'(X) = \mu(X) = f(X)$, one $< x", \sigma" >$ has to be minimal in $\mathcal{X}' \lceil X$.
□ (Claim 2.11 and Theorem 2.1)
We obtain essentially as Corollary (though there is some more routine work to do):

Theorem 2.12 Let $\mid\!\sim$ be a logic for $\mathcal{L}$. Then there is a stoppered definability preserving classical preferential model $\mathcal{M}$ s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff
($\mid\!\sim$ 1) $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$,
($\mid\!\sim$ 2) $\overline{\overline{T}}$ is classically closed,
($\mid\!\sim$ 3) $T \subseteq \overline{\overline{T}}$,
($\mid\!\sim$ 4) $\overline{\overline{\overline{T} \cup \overline{T'}}} \subseteq \overline{\overline{\overline{T}} \cup T'}$,
($\mid\!\sim$ 5) $T \subseteq T' \subseteq \overline{\overline{T}} \to \overline{\overline{T}} = \overline{\overline{T'}}$
for all $T, T' \subseteq \mathcal{L}$.

The proof of Theorem 2.12 - given below that of Proposition 2.13 - will largely follow the proof of Theorem 3.1 in [Sch92], but replacing Proposition 3.3 there by above Theorem 2.1.

Proposition 2.13 Consider for a logic $\mid\!\sim$ on $\mathcal{L}$ the properties
($\mid\!\sim$ 1) $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$,
($\mid\!\sim$ 2) $\overline{\overline{T}}$ is classically closed,
($\mid\!\sim$ 3) $T \subseteq \overline{\overline{T}}$,

($\mid\sim$ 4) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$,
($\mid\sim$ 5) $T \subseteq T' \subseteq \overline{\overline{T}} \to \overline{\overline{T}} = \overline{\overline{T'}}$
for all $T, T' \subseteq \mathcal{L}$
and for a function $f : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ the properties
(f0) f is definability preserving,
(f1) $f(X) \subseteq X$,
(f2) $X \subseteq Y \to f(Y) \cap X \subseteq f(X)$,
(f3) $f(X) \subseteq Y \subseteq X \to$ f(X)=f(Y)
for all $X, Y \in \boldsymbol{D}_{\mathcal{L}}$.
It then holds:
(a) If f satisfies (f0)-(f3), then $\overline{\overline{\ }}$ defined by $\overline{\overline{T}} := T^f$ satisfies ($\mid\sim$ 1) $-$ ($\mid\sim$ 5).
(b) If $\overline{\overline{\ }}$ satisfies ($\mid\sim$ 1) $-$ ($\mid\sim$ 5), then there is $f : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ s.t. $\overline{\overline{T}} = T^f$ for all $T \subseteq \mathcal{L}$ and f satisfies (f0)-(f3).

Proof of Proposition 2.13: We recall that, as $\boldsymbol{D}_{\mathcal{L}}$ is closed under finite intersections, in the presence of (f1), (f2) is equivalent to (f2') $f(X) \cap Y \subseteq f(X \cap Y)$, we work with (f2') in the proof.
(a)
Suppose $\overline{\overline{T}} = T^f$ for some such f, and all T.
($\mid\sim$ 1): If $\overline{T} = \overline{T'}$, then $M_T = M_{T'}$, so $f(M_T) = f(M_{T'})$, and $T^f = T'^f$.
($\mid\sim$ 2) is trivial by definition, and ($\mid\sim$ 3) is trivial by $f(X) \subseteq X$.
We show ($\mid\sim$ 4): Let now $\phi \in \overline{\overline{T \cup T'}}$, so ϕ holds in all $m \in f(M_{T \cup T'}) = f(M_T \cap M_{T'})$, so by (f2'), ϕ holds in all $m \in f(M_T) \cap M_{T'}$. By (f0), $f(M_T) = M_{T^f} = M_{\overline{\overline{T}}}$, so ϕ holds in all $m \in M_{\overline{\overline{T}}} \cap M_{T'} = M_{\overline{\overline{T}} \cup T'}$, so $\overline{\overline{T}} \cup T' \models \phi$, and $\phi \in \overline{\overline{\overline{T}} \cup T'}$.
We turn to ($\mid\sim$ 5): Assume $T \subseteq T' \subseteq \overline{\overline{T}}$, so $M_{\overline{\overline{T}}} = f(M_T) \subseteq M_{T'} \subseteq M_T$ by (f0). If $\phi \in \overline{\overline{T'}} = \overline{\overline{T \cup T'}}$, then by ($\mid\sim$ 4) $\phi \in \overline{\overline{\overline{T}} \cup T'} = \overline{(\overline{\overline{T}})} = \overline{\overline{T}}$ $(by(\mid\sim$ 2)). Let $\phi \in \overline{\overline{T}}$, so ϕ holds in all $m \in f(M_T) = f(M_{T'}) = M_{\overline{\overline{T'}}}$ by (f3) and (f0). Thus $\overline{\overline{T'}} \vdash \phi$, but then by ($\mid\sim$ 2), $\phi \in \overline{\overline{T'}}$.
(b)
Let $\overline{\overline{\ }}$ satisfy ($\mid\sim$ 1) $-$ ($\mid\sim$ 5) for all T. We define f and show $\overline{\overline{T}} = T^f$.
If $X = M_T$ for some $T \subseteq \mathcal{L}$, set $f(X) := M_{\overline{\overline{T}}}$.
If $X = M_T = M_{T'}$, then $\overline{T} = \overline{T'}$, thus $\overline{\overline{T}} = \overline{\overline{T'}}$ by ($\mid\sim$ 1), so $M_{\overline{\overline{T}}} = M_{\overline{\overline{T'}}}$, and f is well-defined. Moreover, f satisfies (f0), and by ($\mid\sim$ 3), $f(X) \subseteq X$.
We show $\overline{\overline{T}} = T^f$: Let now $T \subseteq \mathcal{L}$ be given. Then $\phi \in T^f :\leftrightarrow \forall m \in f(M_T).m \models \phi \leftrightarrow \forall m \in M_{\overline{\overline{T}}}.m \models \phi \leftrightarrow \overline{\overline{T}} \vdash \phi \leftrightarrow \phi \in \overline{\overline{T}}$ (as $\overline{\overline{T}}$ is classically closed).
Next, we show that the above defined f satisfies (f2'). Suppose $X := M_T$, $Y := M_{T'}$. Let $m \in f(X) \cap Y = M_{\overline{\overline{T}}} \cap M_{T'}$, so $m \models \overline{\overline{T}} \cup T'$, and $m \models \overline{\overline{\overline{T}} \cup T'}$, so by ($\mid\sim$ 4) $m \models \overline{\overline{T \cup T'}}$.
As $X \cap Y = M_T \cap M_{T'} = M_{T \cup T'}$, $f(X \cap Y) = M_{\overline{\overline{T \cup T'}}}$ by (f0), so $m \in f(X \cap Y)$.

It remains to show (f3). So let $X = M_T$, $Y = M_{T'}$, and $f(M_T) := M_{\overline{\overline{T}}} \subseteq M_{T'} \subseteq M_T \to \overline{T} \subseteq \overline{T'} \subseteq \overline{\overline{T}} = \overline{\overline{(\overline{T})}} \to \overline{\overline{T}} = \overline{\overline{(\overline{T})}} = \overline{\overline{(\overline{T'})}} = \overline{\overline{T'}} \to f(M_T) = M_{\overline{\overline{T}}} = M_{\overline{\overline{T'}}} = f(M_{T'})$, thus f(X)=f(Y). □ (Proposition 2.13)

Proof of Theorem 2.12: " $\to$ ": Let $\mathcal{M}$ be a stoppered dp cpm, then $f := \mu_{\mathcal{M}} \lceil \boldsymbol{D}_{\mathcal{L}} : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ is dp, and satisfies (f1) - (f3) of Theorem 1 for all $X, Y \in \boldsymbol{D}_{\mathcal{L}}$. By Proposition 2.13, (a), the logic defined by $\overline{\overline{T}} := T^{\mathcal{M}}$ satisfies $(\mid\!\sim 1) - (\mid\!\sim 5)$.
" $\leftarrow$ ": Let $\mid\!\sim$ be a logic for $\mathcal{L}$ which satisfies $(\mid\!\sim 1) - (\mid\!\sim 5)$. By Proposition 2.13, (b), there is $f : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ s.t. f satisfies (f0)-(f3) and for all $T \subseteq \mathcal{L}$ $\overline{\overline{T}} = T^f$. By Theorem 1, for $\mathcal{Y} := \boldsymbol{D}_{\mathcal{L}}$, there is a stoppered classical preferential model $\mathcal{M}$ s.t. $f(X) = \mu_{\mathcal{M}}(X)$ for all $X \in \boldsymbol{D}_{\mathcal{L}}$. But now $\overline{\overline{T}} = T^f = T^{\mathcal{M}}$ for all $T \subseteq \mathcal{L}$, and we are done. □ (Theorem 2.12)

3 Representation Results for Ranked Preferential Structures

We show here two combinatorial representation results for ranked or modular preferential structures. They are formulated in Theorems 3.10 and 3.14. The logical counterparts are given in Theorems 3.16 and 3.18. For a more detailed discussion, see Section 1, Outline.
We follow again the strategy of first proving algebaic representation results (Theorems 3.10 and 3.14), which are then translated into logics (Theorems 3.16 and 3.18). The crucial fact is Lemma 3.5, it shows that we can do with either 1 or infinitely many copies of each model. The reason behind it is the following: Suppose we have exactly 2 copies of one model, m, m', where m and m' have the same logical properties. If, e.g., $m \prec m'$, then, as we consider only minimal elements, m' will be "invisible". If m and m' are incomparable, then, by rankedness (modularity), they will have the same elements above (and below) themselves: they have the same behaviour in the preferential structure. (In the first example in Section 1.4, non-rankedness is crucial.)
An immediate consequence is the "singleton property" of Fact 3.7: One element suffices to destroy minimality, and it suffices to look at pairs (and singletons) (Propositions 3.8, 3.9, Theorem 3.10).
The stoppered case simplifies to non-existence of the A-elements in the structure (Proposition 3.13 and Theorem 3.14).
The details are tedious, but rather straightforward.
The translation into logic paralleles that of [Sch92] and Section 2. Singletons on the semantic side correspond to complete consistent theories, and pairs to theories of the form $T \vee T'$, where T and T' are complete and consistent.

Fact 3.1 (Folklore) Let $\prec$ be an irreflexive, binary relation on X, then the following two conditions are equivalent:

(1) There is Ω and an irreflexive, total, binary relation $\prec'$ on Ω and a function $f : X \to \Omega$ s.t. $x \prec y \leftrightarrow fx \prec' fy$ for all $x, y \in X$ (we sometimes write fx for $f(x)$ etc.).
(2) Let $x, y, z \in X$ and $x \bot y$ wrt. $\prec$ (i.e. neither $x \prec y$ nor $y \prec x$), then $z \prec x \to z \prec y$ and $x \prec z \to y \prec z$.

Proof: (1) $\to$ (2): Let $x \bot y$, thus neither $fx \prec' fy$ nor $fy \prec' fx$, but then $fx = fy$. Let now $z \prec x$, so $fz \prec' fx = fy$, so $z \prec y$. $x \prec z \to y \prec z$ is similar.
(2) $\to$ (1): For $x \in X$ let $[x] := \{x' \in X : x \bot x'\}$, and $\Omega := \{[x] : x \in X\}$. For $[x], [y] \in \Omega$ let $[x] \prec' [y] :\leftrightarrow x \prec y$. This is well-defined: Let $x \bot x'$, $y \bot y'$ and $x \prec y$, then $x \prec y'$ and $x' \prec y'$. Obviously, $\prec'$ is an irreflexive, total binary relation. Define $f : X \to \Omega$ by $fx := [x]$, then $x \prec y \leftrightarrow [x] \prec' [y] \leftrightarrow fx \prec' fy$. $\Box$

Definition 3.2 Call an irreflexive, binary relation $\prec$ on X, which satisfies (1) (equivalently (2)) above, ranked. By abuse of language, we also call the structure $< X, \prec >$ ranked.

Fact 3.3 If $\prec$ on X is ranked, and free of cycles, then $\prec$ is transitive.

Proof: Let $x \prec y \prec z$. If $x \bot z$, then $y \succ z$, resulting in a cycle of length 2. If $z \prec x$, then we have a cycle of length 3. So $x \prec z$. $\Box$

Definition 3.4 Let $\mathcal{Z} =< \mathcal{X}, \prec >$ be a preferential structure. Call $\mathcal{Z}$ $1 - \infty$ over Z, iff for all $x \in Z$ there are exactly 1 or infinitely many copies of x, i.e. for all $x \in Z$ $\{u \in \mathcal{X} : u =< x, i > \text{ for some i}\}$ has cardinality 1 or $\geq \omega$.

Lemma 3.5 Let $\mathcal{Z} =< \mathcal{X}, \prec >$ be a preferential structure and $f : \mathcal{Y} \to \mathcal{P}(Z)$ with $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be represented by $\mathcal{Z}$, i.e. for $X \in \mathcal{Y}$ $f(X) = \mu_{\mathcal{Z}}(X)$, and $\mathcal{Z}$ be ranked and free of cycles. Then there is a structure $\mathcal{Z}'$, $1 - \infty$ over Z, ranked and free of cycles, which also represents f.

Proof: We construct $\mathcal{Z}' =< \mathcal{X}', \prec' >$.
Let $A := \{x \in Z$: there is some $< x, i > \in \mathcal{X}$, but for all $< x, i > \in \mathcal{X}$ there is $< x, j > \in \mathcal{X}$ with $< x, j > \prec < x, i >\}$, let $B := \{x \in Z$: there is some $< x, i > \in \mathcal{X}$, s.t. for no $< x, j > \in \mathcal{X}$ $< x, j > \prec < x, i >\}$, let $C := \{x \in Z$: there is no $< x, i > \in \mathcal{X}\}$.
Let $c_i : i < \kappa$ be an enumeration of C. We introduce for each such c_i ω many copies $< c_i, n >$: $n < \omega$ into $\mathcal{X}'$, put all $< c_i, n >$ above all elements in $\mathcal{X}$, and order the $< c_i, n >$ by $< c_i, n > \prec' < c_{i'}, n' > :\leftrightarrow (i = i'$ and $n > n')$ or $i > i'$. Thus, all $< c_i, n >$ are comparable.
If $a \in A$, then there are infinitely many copies of a in $\mathcal{X}$, as $\mathcal{X}$ was cycle-free, we put them all into $\mathcal{X}'$. If $b \in B$, we choose exactly one such minimal element $< b, m >$ (i.e. there is no $< b, n > \prec < b, m >$) into $\mathcal{X}'$, and omit all other elements. (For definiteness, assume

in all applications $m = 0$.) For all elements from A and B, we take the restriction of the order $\prec$ of $\mathcal{X}$. This is the new structure $\mathcal{Z}'$.
Obviously, adding the $< c_i, n >$ does not introduce cycles, irreflexivity and rankedness are preserved. Moreover, any substructure of a cycle-free, irreflexive, ranked structure also has these properties, so $\mathcal{Z}'$ is $1-\infty$ over Z, ranked and free of cycles.
We show that $\mathcal{Z}$ and $\mathcal{Z}'$ are equivalent. Let then $X \subseteq Z$, we have to prove $\mu(X) = \mu'(X)$ ($\mu := \mu_{\mathcal{Z}}$, $\mu' := \mu_{\mathcal{Z}'}$).
Let $z \in X - \mu(X)$. If $z \in C$ or $z \in A$, then $z \notin \mu'(X)$. If $z \in B$, let $< z, m >$ be the chosen element. As $z \notin \mu(X)$, there is $x \in X$ s.t. some $< x, j > \prec < z, m >$. x can't be in C. If $x \in A$, then also $< x, j > \prec' < z, m >$. If $x \in B$, then there is some $< x, k >$ also in $\mathcal{X}'$. $< x, j > \prec < x, k >$ is impossible. If $< x, k > \prec < x, j >$, then $< z, m > \succ < x, k >$ by transitivity. If $< x, k > \perp < x, j >$, then also $< z, m > \succ < x, k >$ by rankedness. In any case, $< z, m > \succ' < x, k >$, and thus $z \notin \mu'(X)$.
Let $z \in X - \mu'(X)$. If $z \in C$ or $z \in A$, then $z \notin \mu(X)$. Let $z \in B$, and some $< x, j > \prec' < z, m >$. x can't be in C, as they were sorted on top, so $< x, j >$ exists in $\mathcal{X}$ too and $< x, j > \prec < z, m >$. But if any other $< z, i >$ is also minimal in $\mathcal{Z}$ among the $< z, k >$, then by rankedness also $< x, j > \prec < z, i >$, as $< z, i > \perp < z, m >$, so $z \notin \mu(X)$. □
Assume in the sequel that $\mathcal{Y}$ contains all singletons and pairs, and fix $f : \mathcal{Y} \to \mathcal{P}(Z)$. We also fix the following notation: $A := \{x \in Z : f(x) = \emptyset\}$ and B:=Z-A (here and in future we sometimes write $f(x)$ for $f(\{x\})$, likewise $f(x, x') = x$ for $f(\{x, x'\}) = \{x\}$ etc., when the meaning is obvious).

Corollary 3.6 If f can be represented by a ranked $\mathcal{Z}$ free of cycles, then there is $\mathcal{Z}'$, which is also ranked and cycle-free, all $b \in B$ occur in 1 copy, all $a \in A$ ∞ often.

Fact 3.7 1) If $\mathcal{Z}'$ is as in Corollary 3.6, $b \in B$, $a \in A$, $f(a, b) = b$, then for all $< a, i > \in \mathcal{X}'$ $< a, i > \succ' < b, 0 >$.
2) If f can be represented by a cycle-free ranked $\mathcal{Z}$, then it has the "singleton property": If $x \in X$, then $x \notin f(X) \leftrightarrow \exists x' \in X . x \notin f(x, x')$.
3) If f is as in 2), $b, b' \in B$, then $f(b, b') \neq \emptyset$.

Proof: 1) For no $< a, i > < b, 0 > \succ' < a, i >$, since otherwise $f(a, b) = \emptyset$. If $< b, 0 > \perp < a, i >$, then as there is $< a, j > \prec < a, i >$, $< a, j > \prec' < b, 0 >$ by rankedness, contradiction.
2) " $\leftarrow$ " holds for all preferential structures. " $\rightarrow$ ": If $x \in A$, then $x \notin f(x, x)$. Let $x \in B$, $\mathcal{Z}$ a $1-\infty$ over Z structure representing f as above. So there is just one copy of x in $\mathcal{X}$, $< x, 0 >$, and there is some $< y, j > \prec < x, 0 >$, $y \in X$, thus $x \notin f(x, y)$.
3) In any $1-\infty$ over Z representation of f, $< b, 0 > \perp < b', 0 >$, or $< b, 0 > \prec < b', 0 >$, or $< b', 0 > \prec < b, 0 >$. $< b, 0 > \prec < b', 0 > \prec < b, 0 >$ can't be, as this is a cycle. □

Proposition 3.8 Let $\mathcal{Y}$ contain singletons and pairs, and f be represented by a cycle-free ranked structure. Define the following relation on Z:

For $b, b' \in B$ $b \prec b' :\leftrightarrow f(b, b') = b$ and $b \neq b'$,
for $a \in A$, $b \in B$ $a \prec b :\leftrightarrow f(a, b) = \emptyset$, $b \prec a :\leftrightarrow f(a, b) = b$,
for $a, a' \in A$ $a \prec a' :\leftrightarrow \exists b \in B.a \prec b \prec a'$.
Then $\prec$ is free of cycles and ranked.

Proof: Let $\mathcal{Z}' =< \mathcal{X}', \prec' >$ be a cycle-free ranked $1 - \infty$ over Z representation of f. We first show that $\prec$ is free of cycles:
Assume there is a cycle $x_0 \succ \ldots x_n \succ x_0$. If two successors $x \succ x'$ are in A, then there is by definition $b \in B$ s.t. $x \succ b \succ x'$. So we may assume that never two successors are both in A. Work now in $\mathcal{Z}'$, and let $x \succ x'$ be two successors. If both are in B, then also $< x, 0 > \succ' < x', 0 >$ in $\mathcal{Z}'$. If $x \in A$, $x' \in B$, then by Fact 3.7, 1) all copies $< x, i > \succ' < x', 0 >$. If $x \in B$, $x' \in A$, then there is some copy $< x', i > \prec' < x, 0 >$. But then we have a cycle in $\mathcal{Z}'$. (The reason is, that between A and B always all copies are minimized - the sceptical reader is invited to draw a diagram.)
$\prec$ is ranked: Irreflexivity is clear from the definition, and the fact that $\prec$ is free of cycles. We have to show $x \bot y$, $z \prec x \rightarrow z \prec y$ and $x \bot y$, $x \prec z \rightarrow y \prec z$.
Case 1: $x \in A$, $y \in B$: $x \bot y$ is impossible, as all a/b are comparable.
Case 2: $x, y \in A$: This will follow directly from the definition of $\prec$: First, by definition, for all $b \in B$ $f(x, b) = f(y, b)$. Case 2.1, $z \in B$: $z \prec x \rightarrow f(x, z) = z = f(y, z) \rightarrow z \prec y$, and $x \prec z \rightarrow f(x, z) = \emptyset = f(y, z) \rightarrow y \prec z$. Case 2.2, $z \in A$: $z \prec x \rightarrow \exists b \in B.z \prec b \prec x$ $\rightarrow \exists b \in B.z \prec b \prec y \rightarrow z \prec y$, and $x \prec z \rightarrow \exists b \in B.x \prec b \prec z \rightarrow \exists b \in B.y \prec b \prec z$ $\rightarrow y \prec z$. Case 3: $x, y \in B$: Thus neither $< x, 0 > \prec' < y, 0 >$ nor $< y, 0 > \prec' < x, 0 >$. Case 3.1, $z \in A$: $z \prec x$, so there is $< z, i > \prec' < x, 0 >$, and as $< x, 0 > \bot < y, 0 >$, $< z, i > \prec' < y, 0 >$, so $z \prec y$. $x \prec z$, so all $< z, i > \succ' < x, 0 >$, so all $< z, i > \succ' < y, 0 >$, thus, as $\mathcal{Z}'$ is free of cycles, no $< z, i > \prec' < y, 0 >$, so $y \prec z$. Case 3.2, $z \in B$: immediate by rankedness of $\mathcal{Z}'$. □

Proposition 3.9 Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ contain all singletons and pairs, and $f : \mathcal{Y} \rightarrow \mathcal{P}(Z)$. Let the following conditions hold for f:
(f1) $f(X) \subseteq X$ for $X \in \mathcal{Y}$,
(f2') if $x \in X$, then $x \notin f(X) \leftrightarrow \exists x' \in X.x \notin f(x, x')$ for $X \in \mathcal{Y}$,
(f3) for $b, b' \in B$ $f(b, b') \neq \emptyset$,
(f4) the above (in Proposition 3.8) defined order $\prec$ on Z is cycle-free and ranked (thus transitive).
Then there is a cycle-free ranked $\mathcal{Z}$, which represents f.

Proof: For $a \in A$ let $[a] := \{a' \in A : a \bot a'\}$ ($\bot$ wrt. $\prec$). For each $[a]$, let some arbitrary enumeration $\{a_i : i < \kappa\}$ of $[a]$ be given. For each $a_i \in [a]$, we take ω many copies $< a_i, j >$. Let $\mathcal{X} := (Bx1) \cup \{< a_i, j >: a_i \in [a]$, some $a \in A$, $j < \omega\}$. We order $\mathcal{X}$ by $\prec'$ as follows:
(1) for $b, b' \in B$ $< b, 0 > \prec' < b', 0 > :\leftrightarrow b \prec b'$

(2) for $a \in A$, $b \in B$: $< a, i > \prec' < b, 0 > :\leftrightarrow a \prec b$ (all $i < \omega$), $< b, 0 > \prec' < a, i > :\leftrightarrow b \prec a$ (all $i < \omega$)
(3) for $a, a' \in A$:
$< a, i > \prec' < a', j > :\leftrightarrow$
(i) $a = a'$ and $j < i$, or
(ii) $a \neq a'$, but $a \perp a'$ and $a > a'$ in the enumeration of $[a]$, or
(iii) $a \not\perp a'$, but $a \prec a'$.
Condition (i) makes infinite descending chains of the a-copies, condition (ii) creates an arbitrary total order on all $< a', i' >$ for $a' \in [a]$.
We show:
a) $\prec'$ is irreflexive and transitive
b) $\mathcal{Z} =< \mathcal{X}, \prec' >$ represents f
c) $\prec'$ is ranked
d) $\prec'$ is free of cycles

a) Irreflexivity is trivial. Transitivity: Let $< x, i > \prec' < y, j > \prec' < z, k >$. If both $\prec'$ are defined by (1), (2), or (3) (iii), transitivity is inherited from $\prec$. In the remaining cases, a definition by (3) (i) or (ii) is at least once involved. If this is the case by (3) (i), then the other definition is by (2), and transitivity is trivial. If it is the case by (3) (ii), and the other definition is by (2), transitivity holds by rankedness of $\prec$. So the now remaining 9 cases are all defined totally by condition (3). They are all easily checked case by case.
b) Let $X \in \mathcal{Y}$, $x \in X - f(X)$. We have to show $x \notin \mu(X)$. Case 1, $x \in A$: then $x \notin \mu(X)$. Case 2, $x \in B$: by (f2'), there is $x' \in X$ with $x \notin f(x, x')$. If $x' \in A$, then $x' \prec x$, and $< x', i > \prec < x, 0 >$ for all i, so $x \notin \mu(X)$. If $x' \in B$, then f(x,x')=x' by (f3), thus $x' \prec x$ and $< x', 0 > \prec' < x, 0 >$, so $x \notin \mu(X)$. Assume now $x \notin \mu(X)$. Case 1, $x \in A$: then $x \notin f(X)$ by (f2'). Case 2, $x \in B$: then $\exists x' \in X. < x', i > \prec' < x, 0 >$. If $x' \in B$, then $x' \prec x$, thus $x \notin f(x, x')$, so $x \notin f(X)$ by (f2'). If $x' \in A$, then $x' \prec x$, so $f(x, x') = \emptyset$ and $x \notin f(X)$ by (f2') again.
c) Let $< x, i > \perp < y, j >$ wrt. $\prec'$. Then $x, y \in B$, as all others are comparable, $x \perp y$ wrt. $\prec$, and $i = j = 0$. Let $< z, k > \prec' < x, 0 >$. Then $z \prec x \rightarrow$ (by rankedness of $\prec$) $z \prec y \rightarrow < z, k > \prec' < y, 0 >$. Let $< x, 0 > \prec' < z, k >$. Then $x \prec z \rightarrow y \prec z \rightarrow < y, 0 > \prec' < z, k >$.
d) Suppose there is a cycle $x_0 \succ' \ldots \succ' x_n \succ' x_0$. Not all x_i can be in B, since this would result in a cycle in $\prec$. Not all x_i can be from the same $[a]$, as this was cycle-free too. But, if $< a_i, j > \succ' < a_k, m >$ are from different $[a]$'s, then $a_i \succ a_k$ and there is some $b \in B$ between them. So between any pair from A of different $[a]$'s, we can put some such b in, possibly enlarging the cycle. By transitivity, cancelling the $a \in A$ leaves a cycle in B, contradiction. $\Box$
We obtain as corollary:

Theorem 3.10 Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ contain all singletons and pairs, let $f : \mathcal{Y} \rightarrow \mathcal{P}(Z)$, let

$A := \{x \in Z : f(x) = \emptyset\}$ and $B := Z - A$. Then there is a cycle-free ranked $\mathcal{Z}$, which represents f iff
(f1) $f(X) \subseteq X$ for $X \in \mathcal{Y}$,
(f2') if $x \in X$, then $x \notin f(X) \leftrightarrow \exists x' \in X.x \notin f(x, x')$ for $X \in \mathcal{Y}$,
(f3) for $b, b' \in B$ $f(b, b') \neq \emptyset$,
(f4) the order on Z defined by
for $b, b' \in B$, $b \prec b' :\leftrightarrow f(b, b') = b$ and $b \neq b'$,
for $a \in A$, $b \in B$, $a \prec b :\leftrightarrow f(a, b) = \emptyset$, and $b \prec a :\leftrightarrow f(a, b) = b$,
for $a, a' \in A$, $a \prec a' :\leftrightarrow \exists b \in B.a \prec b \prec a'$
is cycle-free and ranked. $\Box$

Remark 3.11 (f2') of Theorem 3.10 entails (f2) of Theorem 2.1.

Proof: Suppose $X \subseteq Y$, $x \in X - f(X)$, we have to show $x \notin f(Y)$. By (f2'), there is $x' \in X$ s.t. $x \notin f(x, x')$, by $X \subseteq Y$, $x, x' \in Y$, so again by (f2'), $x \notin f(Y)$. $\Box$
Let in the sequel $\mathcal{Y}$ contain singletons, be closed under finite unions, and let $f : \mathcal{Y} \to \mathcal{Y}$.

Remark 3.12 If f is representable by a cycle-free, ranked, $\mathcal{Y}$-stoppered $\mathcal{Z}$, then also $x \in X - f(X) \to x \notin f(f(X) \cup \{x\})$ for $X \in \mathcal{Y}$.

Proof: This holds for all f representable by a $\mathcal{Y}$-stoppered structure. $\Box$

Proposition 3.13 Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ contain all singletons and pairs, and $f : \mathcal{Y} \to \mathcal{P}(Z)$. Let the following conditions hold for f:
(f1) $f(X) \subseteq X$ for $X \in \mathcal{Y}$,
(f2') if $x \in X$, then $x \notin f(X) \leftrightarrow \exists x' \in X.x \notin f(x, x')$ for $X \in \mathcal{Y}$,
(f3') $x \in X - f(X) \to x \notin f(f(X) \cup \{x\})$ for $X \in \mathcal{Y}$,
(f4) the relation on $B = \{x \in Z : f(x) = x\}$ defined by $b \prec b' :\leftrightarrow f(b, b') = b$ and $b \neq b'$, is ranked and free of cycles.
Then there is a $\mathcal{Y}$-stoppered, ranked, cycle-free structure $\mathcal{Z} =< \mathcal{X}, \prec>$ representing f. Moreover, all $b \in B$ occur in $\mathcal{X}$ exactly once, all $a \in A$ 0 times.

Proof: (Note that condition (f3') and the definition of B in (f4) imply condition (f3) of Proposition 3.9 above.)
We take $\mathcal{Z} :=< B, \prec>$ where $\prec$ is as defined in (f4) and show that a) $\mathcal{Z}$ represents f, i.e. $f(X) = \mu(X) := \mu_{\mathcal{Z}}(X)$, and that b) $\mathcal{Z}$ is $\mathcal{Y}$-stoppered. (We omit indices, since every element in the structure occurs exactly once.)
a): Let $x \in X - f(X)$. If $x \in A$, $x \notin \mu(X)$. If $x \in B$, then by condition (f2') there is $x' \in X$ s.t. $x \notin f(x, x')$. If $x' \in B$, we are done. But $x' \in A$ can't be, as then $f(x, x') = \emptyset$, and thus $f(x) = \emptyset$ by condition (f3'), contradiction. Let $x \in X - \mu(X)$. If $x \in A$, $x \notin f(X)$. If $x \in B$, there is $x' \in B \cap X.x' \prec x$, so $x \notin f(x, x')$ and $x \notin f(X)$.

b) Let $X \in \mathcal{Y}$, $x \in X - \mu(X)$. By a) and (f3'), $x \notin \mu(\mu(X) \cup \{x\})$, so there is $x' \in \mu(X)$, $x' \prec x$. $\square$

We obtain, again as corollary:

Theorem 3.14 Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ contain all singletons and be closed under finite unions, let $f : \mathcal{Y} \to \mathcal{Y}$, let $A := \{x \in Z : f(x) = \emptyset\}$ and $B := Z - A$.
Then there is a cycle-free ranked $\mathcal{Y}$-stoppered $\mathcal{Z}$, which represents f iff
(f1) $f(X) \subseteq X$ for $X \in \mathcal{Y}$,
(f2') if $x \in X$, then $x \notin f(X) \leftrightarrow \exists x' \in X.x \notin f(x, x')$ for $X \in \mathcal{Y}$,
(f3) $f(X) \subseteq Y \subseteq X \to f(X) = f(Y)$,
(f4) the order on B defined by $b \prec b' :\leftrightarrow f(b, b') = b$ and $b \neq b'$ is cycle-free and ranked.

Proof: It suffices to show that the conditions (f1), (f2'), (f3) of Theorem 3.14 are equivalent with the conditions (f1), (f2'), (f3') of Proposition 3.13. (f3') is a special case of (f3). We show (f3) from (f1), (f2'), (f3'). Let then $f(X) \subseteq Y \subseteq X$, we have to prove $f(X) = f(Y)$.
" $\subseteq$ " : Suppose $x \notin f(Y)$. If $x \notin Y$, then by $f(X) \subseteq Y$ $x \notin f(X)$. If $x \in Y - f(Y)$, then by (f2'), there is $y \in Y$, $x \notin f(x, y)$. But then, by (f2') again, and by $Y \subseteq X$, $x \notin f(X)$.
" $\supseteq$ " : Suppose $x \in f(Y) - f(X)$. Then $x \notin f(f(X) \cup \{x\})$ by (f3') and $x \in f(Y) \subseteq Y \subseteq X$. But $f(X) \cup \{x\} \subseteq Y$, so $x \notin f(Y)$ by (f2') and its consequence (f2) of Theorem 2.1 (by Remark 3.11), *Contradiction.* $\square$

We turn to the logical counterparts of Theorems 3.10 and 3.14.
Let $\mathcal{L}$ be a fixed propositional language, $\overline{\overline{\ }}$ a fixed logic for $\mathcal{L}$ s.t. for all T $\overline{\overline{T}}$ is closed under $\vdash$. Let $W := \{S \subseteq \mathcal{L} : S$ is complete, $\vdash$-consistent, $\vdash$-closed $\}$, $U := \{S \in W : \bot \in \overline{\overline{S}}\}$, $V := W - U$.

Fact 3.15 Obviously, there is a 1-1 correspondence $g : W \to M_{\mathcal{L}}$, $S \mapsto m_S$. Let $f : X \to \mathcal{P}(M_{\mathcal{L}})$ with $\boldsymbol{D}_{\mathcal{L}} \subseteq X$, and $f(M_T) = M_{\overline{\overline{T}}}$. Let $A := \{m \in M_{\mathcal{L}} : f(\{m\}) = \emptyset\}$, $B := M_{\mathcal{L}} - A$. Then $S \in U \leftrightarrow m_S \in A$, $S \in V \leftrightarrow m_S \in B$, likewise $m \in A \leftrightarrow Th(m) \in U$, $m \in B \leftrightarrow Th(m) \in V$.

Proof: $S \in U \leftrightarrow \bot \in \overline{\overline{S}} \leftrightarrow M_{\overline{\overline{S}}} = \emptyset \leftrightarrow f(m_S) = \emptyset \leftrightarrow m_S \in A$. $\square$

Theorem 3.16 Consider the following logical properties:
($\mid\sim$ 1) $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$,
($\mid\sim$ 2) $\overline{\overline{T}}$ is classically closed,
($\mid\sim$ 3) $T \subseteq \overline{\overline{T}}$,
($\mid\sim$ 4') $S \in W$, $S \vdash T$ implies $S \not\vdash \overline{\overline{T}} \longleftrightarrow \exists S' \in W.S' \vdash T$, $S \not\vdash \overline{\overline{S \vee S'}}$,
($\mid\sim$ 5) for $T, T' \in V$: $\bot \notin \overline{\overline{T \vee T'}}$,
($\mid\sim$ 6) the order defined on W by

for $S,S' \in V$ $S \lhd S' :\leftrightarrow \overline{S} = \overline{\overline{S \vee S'}}$ and $\overline{S} \neq \overline{S'}$,
for $S \in U$, $S' \in V$ $S \lhd S' :\leftrightarrow \bot \in \overline{\overline{S \vee S'}}$, $S' \lhd S :\leftrightarrow \overline{\overline{S \vee S'}} = \overline{S'}$,
for $S,S' \in U$ $S \lhd S' :\leftrightarrow \exists S" \in V.S \lhd S" \lhd S'$ is cycle-free and ranked.
The logic $\mid\!\sim$ can be represented by a cycle free ranked dp preferential model $\mathcal{M}$, i.e. there is such $\mathcal{M}$ s.t. for all $\mathcal{L}$-theories T $\overline{\overline{T}} = T^{\mathcal{M}}$ iff $\mid\!\sim$ satisfies above properties $(\mid\!\sim\ 1) - (\mid\!\sim\ 6)$.

Proof: We make tacit use of Fact 1.2 of the Introduction, and of Theorem 3.10.
" $\rightarrow$ ":
Let $\mathcal{M}$ be cycle free, ranked, and definability preserving. Set $f := \mu_{\mathcal{M}}$. For a theory T, let again $\overline{\overline{T}} := \{\phi\colon \forall m \in f(M_T).m \models \phi\} = Th(f(M_T))$. We show properties $(\mid\!\sim\ 1) - (\mid\!\sim\ 6)$.
$(\mid\!\sim\ 1) - (\mid\!\sim\ 3)$ are obvious.
$(\mid\!\sim\ 4')$ " $\leftarrow$ ": Let S' be as by prerequisite, then by $S' \vdash T$ and $S \vdash T$, $M_{S \vee S'} \subseteq M_T$. Moreover, $S \not\vdash \overline{\overline{S \vee S'}} \rightarrow m_S \notin f(M_{S \vee S'}) \rightarrow m_S \notin f(M_T)$ (by the fundamental property of preferential models) $\rightarrow S \not\vdash \overline{\overline{T}}$.
" $\rightarrow$ ":
Let $S \not\vdash \overline{\overline{T}}$. $S \not\vdash \overline{\overline{T}} \rightarrow m_S \notin f(M_T) \rightarrow$ (by $m_S \in M_T$ and (f2') of Theorem 3.10) $\exists x' \in M_T.m_S \notin f(m_S, x') \rightarrow \exists S' \in W$ $(m_{S'} \in M_T,\ m_S \notin f(M_{S \vee S'})\ ) \rightarrow \exists S' \in W$ $(S' \vdash T,\ S \not\vdash \overline{\overline{S \vee S'}})$.
$(\mid\!\sim\ 5)$ $T, T' \in V \rightarrow m_T, m_{T'} \in B \rightarrow f(m_T, m_{T'}) \neq \emptyset \rightarrow \bot \notin \overline{\overline{T \vee T'}}$.
$(\mid\!\sim\ 6)$ Claim: For $S,S' \in W$ $S \lhd S' \leftrightarrow m_S \prec m_{S'}$, where $\prec$ is as defined in Theorem 3.10.
Proof: Case 1: $S,S' \in V$. Then $m_S, m_{S'} \in B$ by Fact 3.15. $S \lhd S' :\leftrightarrow S = \overline{\overline{S \vee S'}} \wedge S \neq S'$ $\longleftrightarrow f(m_S, m_{S'}) = m_S \wedge m_S \neq m_{S'} \leftrightarrow: m_S \prec m_{S'}$. Case 2: $S \in U$, $S' \in V$. Then $m_S \in A$, $m_{S'} \in B$ by Fact 3.15. $S \lhd S' :\leftrightarrow \bot \in \overline{\overline{S \vee S'}} \leftrightarrow f(m_S, m_{S'}) = \emptyset \leftrightarrow: m_S \prec m_{S'}$. $S' \lhd S :\leftrightarrow \overline{\overline{S \vee S'}} = S' \leftrightarrow f(m_S, m_{S'}) = m_{S'} \leftrightarrow: m_{S'} \prec m_S$. Case 3: $S,S' \in U$. Then $m_S, m_{S'} \in A$ by Fact 11. $S \lhd S' :\leftrightarrow \exists S" \in V.S \lhd S" \lhd S' \leftrightarrow$ (by Case 2) $\exists m_{S"} \in B.m_S \prec m_{S"} \prec m_{S'}$ $\longleftrightarrow: m_S \prec m_{S'}$. $\Box$ (Claim)
Consequently, $\lhd$ is cycle-free and ranked.
" $\leftarrow$ ":
Let $\mid\!\sim$ be a logic satisfying $(\mid\!\sim\ 1) - (\mid\!\sim\ 6)$ above, set $Z := M_{\mathcal{L}}$, let $\mathcal{Y} := \{M_T \subseteq M_{\mathcal{L}} : T$ is a $\mathcal{L}$-theory $\}$, $f(M_T) := M_{\overline{\overline{T}}}$. $\mathcal{Y}$ will contain all singletons and pairs, by $(\mid\!\sim\ 1)$, f is well-defined, and it is dp.
We check Properties (f1)-(f4) of Theorem 3.10. Note that $f(m) = \emptyset \leftrightarrow \bot \in \overline{\overline{Th(m)}}$, thus $A := \{m \in M_{\mathcal{L}}\colon \bot \in \overline{\overline{Th(m)}}\}$. (f1) follows from property $(\mid\!\sim\ 3)$. (f2') Let $m_S \in M_T$. " $\rightarrow$ ": $m_S \notin f(M_T) \rightarrow S \vdash T$ and $S \not\vdash \overline{\overline{T}} \rightarrow \exists S' \in W.S' \vdash T,\ S \not\vdash \overline{\overline{S \vee S'}} \rightarrow \exists m_{S'} \in M_T$, $m_{S'} \notin f(m_S, m_{S'})$. " $\leftarrow$ ": Let $m_{S'}$ be s.t. $m_{S'} \in M_T$, $m_{S'} \notin f(m_S, m_{S'}) \rightarrow S' \vdash T$, $S \not\vdash \overline{\overline{S \vee S'}} \rightarrow S \not\vdash \overline{\overline{T}} \rightarrow m_S \notin f(M_T)$. (f3) Let $m_S, m_{S'} \in B$, then $\bot \notin \overline{\overline{S}}$, $\bot \notin \overline{\overline{S'}}$, so $S,S' \in V$, so $\bot \notin \overline{\overline{S \vee S'}}$ by $(\mid\!\sim\ 5)$, so $f(m_S, m_{S'}) \neq \emptyset$. (f4) The Claim in the first half of the proof holds again: Fact 3.15, which was used in its proof, holds. Thus, $\prec$ is cycle-free and ranked. $\Box$

Remark 3.17 ($\mid\sim$ 3) and ($\mid\sim$ 4$'$) of Theorem 3.16 entail ($\mid\sim$ 4) of Theorem 2.12.

Proof: Suppose $\overline{\overline{T \cup T'}} \not\subseteq \overline{\overline{\overline{T}} \cup T'}$, so there is $\phi \in \overline{\overline{T \cup T'}}$, $\phi \notin \overline{\overline{\overline{T}} \cup T'}$, so $Con(\overline{\overline{\overline{T}} \cup T'}, \neg\phi)$. Choose $S \in W$ with $S \vdash \neg\phi$, $S \vdash \overline{\overline{\overline{T}} \cup T'}$, so $S \vdash T \cup T'$ by ($\mid\sim$ 3). By $S \vdash \neg\phi$, and consistency of S, $S \not\vdash \phi$, so $S \not\vdash \overline{\overline{T \cup T'}}$. By ($\mid\sim$ 4$'$), there is $S' \in W$ s.t. $S' \vdash T \cup T'$, $S \not\vdash \overline{\overline{S \vee S'}}$. But now $S \vdash T$, $S' \vdash T$, so by ($\mid\sim$ 4$'$) again, $S \not\vdash \overline{\overline{T}}$, $Contrad \ldots$ □

Theorem 3.18 Consider the following logical properties:
($\mid\sim$ 1) $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$,
($\mid\sim$ 2) $\overline{\overline{T}}$ is classically closed,
($\mid\sim$ 3) $T \subseteq \overline{\overline{T}}$,
($\mid\sim$ 4$'$) $S \in W$, $S \vdash T$ implies $S \not\vdash \overline{\overline{T}} \longleftrightarrow \exists S' \in W.S' \vdash T$, $S \not\vdash \overline{\overline{S \vee S'}}$,
($\mid\sim$ 5) $T \subseteq T' \subseteq \overline{\overline{T}} \to \overline{\overline{T}} = \overline{\overline{T'}}$,
($\mid\sim$ 6) the order defined on V by $S \lhd S' :\leftrightarrow \overline{S} = \overline{\overline{S \vee S'}}$ and $\overline{S} \neq \overline{S'}$ is cycle-free and ranked.
The logic $\mid\sim$ can be represented by a cycle free ranked stoppered dp preferential model $\mathcal{M}$, i.e. there is such $\mathcal{M}$ s.t. for all $\mathcal{L}$-theories T $\overline{\overline{T}} = T^{\mathcal{M}}$ iff $\mid\sim$ satisfies above properties ($\mid\sim$ 1) – ($\mid\sim$ 6).

Proof: We use Theorem 3.14.
" $\to$ ":
($\mid\sim$ 1) – ($\mid\sim$ 4$'$) and ($\mid\sim$ 6) follow from Theorem 3.14 as in the proof of Theorem 3.16. ($\mid\sim$ 5): Assume $T \subseteq T' \subseteq \overline{\overline{T}}$, so $M_{\overline{\overline{T}}} = f(M_T) \subseteq M_{T'} \subseteq M_T$ by dp. If $\phi \in \overline{\overline{T'}} = \overline{\overline{T \cup T'}}$, then by ($\mid\sim$ 4$'$) and Remark 3.17, ($\mid\sim$ 4) of Theorem 2.12 holds, so $\phi \in \overline{\overline{\overline{T}} \cup T'} = \overline{(\overline{\overline{T}})} = \overline{\overline{T}}$ (by($\mid\sim$ 2)). Let $\phi \in \overline{\overline{T}}$, so ϕ holds in all $m \in f(M_T) = f(M_{T'}) = M_{\overline{\overline{T'}}}$ by (f3) and dp. Thus $\overline{\overline{T'}} \vdash \phi$, but then by ($\mid\sim$ 2), $\phi \in \overline{\overline{T'}}$.
" $\leftarrow$ ":
Again, (f1), (f2'), (f4) follow from the conditions of Theorem 3.18 as in the proof of Theorem 3.16. It remains to show (f3). So let $X = M_T$, $Y = M_{T'}$, and $f(M_T) := M_{\overline{\overline{T}}} \subseteq M_{T'} \subseteq M_T \to \overline{T} \subseteq \overline{T'} \subseteq \overline{\overline{T}} = \overline{\overline{(\overline{T})}} \to \overline{\overline{T}} = \overline{\overline{(\overline{T})}} = \overline{\overline{(\overline{T'})}} = \overline{\overline{T'}} \to f(M_T) = M_{\overline{\overline{T}}} = M_{\overline{\overline{T'}}} = f(M_{T'})$, thus $f(X) = f(Y)$. □

Remark 3.19 We note the following connections - provided all prerequisites about $\mathcal{Y}$ are satisfied. They are shown in Remarks 3.11 and 3.17.
(f1)+(f2') of Theorem 3.10 entail (f1)+(f2) of Proposition 3.4 in [Sch92]. (f1)+(f2')+(f3) of Theorem 3.14 entail (f1)+(f2)+(f3) of Theorem 2.1. ($\mid\sim$ 1) + ($\mid\sim$ 2) + ($\mid\sim$ 3) + ($\mid\sim$ 4$'$) of Theorem 3.16 entail ($\mid\sim$ 1) + ($\mid\sim$ 2) + ($\mid\sim$ 3) + ($\mid\sim$ 4) of Theorem 3.1 in [Sch92]. ($\mid\sim$ 1) + ($\mid\sim$ 2) + ($\mid\sim$ 3) + ($\mid\sim$ 4$'$) + ($\mid\sim$ 5) of Theorem 3.18 entail ($\mid\sim$ 1) + ($\mid\sim$ 2) + ($\mid\sim$ 3) + ($\mid\sim$ 4) + ($\mid\sim$ 5) of Theorem 2.12. □

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