Theory Revision and Probability

Karl Schlechta
IBM Germany, IWBS W&S3
POBox 80 08 80
D-7000 Stuttgart 80
West Germany

7 June 1989, revised 16 October 1989

Abstract

The problem of Theory Revision is to "add" a formula to a theory, while preserving consistency and making only minimal changes to the original theory. A natural way to uniquely determine the process is by imposing an order of "epistemic entrenchment" on the formulae, as done by Gärdenfors and Makinson. We improve their results as follows : We define orders which generate unique revision processes too, but in addition, 1) have nice logical properties, 2) are independent of the theory considered, and thus well suited for iterated revision and computational purposes, 3) have a natural probabilistic construction. Finally, we show that the completeness problems of Theory Revision carry over to a certain extent to an approach based on revising axiom systems.

1 Introduction

Recent years have seen an increasing interest in theory revision, which has partly centered around the work of Gärdenfors and his co-authors ([AGM], [AM], [G], [GM], [M]). In [G] and [GM], the problem of choice in theory revision (more precisely : maxichoice contraction/revision) finds a natural solution in the concept of epistemic entrenchment : An order on the formulae tells us which to choose. The orders of [G] and [GM] have, however, theoretical as well as computational drawbacks : 1) They are tailored to fit the requirements of theory revision and will not respect other natural demands like $\phi \leq \psi$ iff $\neg\psi \leq \neg\phi$ (if I tend to believe more in ψ than in ϕ, then I might tend to believe more in $\neg\phi$ than in $\neg\psi$) . 2) For each revision of a new theory (in a fixed language) we have to find a new order, so iterated revision means iterated effort in ordering. Note that "theory" is used here in a technical sense: a deductively closed set of formulae (the deductive closure of a database). Thus two theories might be closely related, and are not necessarily as different, as say a physical and a medical theory. So iterated theory revision in this sense is a very common phenomenon for cognitive systems. The main aim of this paper is to show how to overcome these drawbacks : 1) To prove that one order of a very simple kind will do for all theories of a given language and thus for all revisions (Propositions II.2 and II.4). 2) To show how to construct such an order with particularly nice and natural properties for countable propositional languages (Proposition II.8 and II.11). For a different treatment of iterated revision, see Spohn's work, e.g. [Sp]. The last chapter is independent of the first part. In Section I and II, we consider what Gärdenfors and his co-authors call maxichoice contraction - choosing a maximal subset K' of K, s.th. $K' \not\vdash A$. As pointed out in [AGM], [G] and [M], maxichoice contraction suffers from a completeness problem. Our point in Section III is, that Theory Revision with underlying axiom sets is plagued by essentially the same problems (and some more). We consider systems $< K, A >$, where K is a deductively closed set of formulae, and A is a set of axioms for K. Theory Revision for $< K, A >$ will essentially amount to the choice of a suitable subset of A. Proposition III.1 shows that there is a continuum between too coarse axiom sets (and too coarse revision) and too fine-grained axiom sets (resulting in full completeness at revision). For the convenience of the reader, we now repeat the - for our purposes - main definitions and results of Gärdenfors and Makinson. But before that, let's give an example to point out the basic problem of underdeterminacy.

A Problem of Theory Revision : Let T be a theory, i.e. a deductively closed set of formulae. Suppose $\{A, B\} \subseteq T$, thus $A \wedge B \epsilon T$, and we would like to revise T to a maximal theory $T' \subseteq T$ such that $A \wedge B \not\in T'$. So $\{A, B\} \subseteq T$ is impossible, and we have to withdraw A, B, or both. (Leaving aside extreme cases like $\vdash A \longleftrightarrow B$) "both" is unsatisfactory, as T' should be maximal. So we can and have to chose which of A or B, but logic won't tell us which. If we have an order $A < B$ telling us that we like A less than B, we are finished. This is the idea of

Gärdenfors's and Makinson's Solution : In the following, we adopt Gärdenfors' and Makinson's terminology to make this article more readable for those familiar with their work. They denote by "theory contraction" the process of removing a formula from a theory, and by "theory revision" adding a formula A to a theory T so that the resulting theory T' is consistent (if A is) and $A \epsilon T'$. This is made precise in the following

Definition 1.1 *Given a language $\mathcal{L}$, an inference rule $\vdash$ (we will not be more specific here, and the interested reader is referred e.g. to [G]), and a "knowledge set" K, i.e. a set of formulae of $\mathcal{L}$ closed under $\vdash$, then a function K- : Formulae of $\mathcal{L} \rightarrow$ Sets of Formulae of $\mathcal{L}$ is called a contraction function for K, iff it satisfies the postulates (K-1) to (K-8) below, and a function K* : Formulae of $\mathcal{L} \rightarrow$ Sets of Formulae of $\mathcal{L}$ is called a revision function for K, iff it satisfies the postulates (K*1) to (K*8) below.*

Proposition 1.1 *Both notions are interdefinable by the following equations : K*A := $(K - \neg A) + A$ (where L+B is the deductive closure of $L \cup \{B\}$) i.e., if K- is a contraction function, then K* so defined is a revision function, and K-A := $K \cap (K * \neg A)$ i.e. if K* is a revision function, then K- so defined will be a contraction function.*

The proofs are straightforward. □

We now state the axioms for K- and K* , some very short comments are given in parentheses, the reader will find more motivation e.g. in [G] :

Definition 1.2 *(K-1) K-A is a knowledge set (i.e. deductively closed under* $\vdash$*), (K-2) K-A* $\subseteq K$*, (K-3) If* $A \notin K$*, then K-A=K (the desired result already applies to K), (K-4) If* $\not\vdash A$*, then* $A \notin K - A$ *(success, if possible), (K-5)* $K \subseteq (K - A) + A$ *(where L+B is the deductive closure of* $L \cup \{B\}$*, the "postulate of recovery"), (K-6) If* $\vdash A \longleftrightarrow B$*, then K-A=K-B, (K-7)* $(K - A) \cap (K - B) \subseteq K - (A \wedge B)$ *(a condition of minimality), (K-8) If* $A \notin K - (A \wedge B)$*, then* $K - (A \wedge B) \subseteq K - A$ *(In general, the more specific a formula is, the less the change necessary for revision. If* $A \notin K - (A \wedge B)$*, however, then contraction by* $A \wedge B$ *will do already.)*

and

*(K*1) K*A is a knowledge set, (K*2) A* ϵ *K*A (success), (K*3) K*A* $\subseteq$ *K+A (the purpose of K*A is to "add" A to K, if consistently possible), (K*4) If* $\neg A \notin K$*, then K+A* $\subseteq$ *K*A (see K*3), (K*5)* $K * A = K_\perp$ *(*$K_\perp$ *the inconsistent theory) only if* $\vdash \neg A$ *(preserve consistency, if possible), (K*6) If* $\vdash A \longleftrightarrow B$*, then K*A=K*B, (K*7)* $K * (A \wedge B) \subseteq$ *(K*A)+B (consider K*2 and minimality for motivation), (K*8) If* $\neg B \notin K * A$*, then (K*A)+B* $\subseteq K * (A \wedge B)$ *(see K*4 !)*

By the above interdefinability result, it suffices for our purposes to consider contraction only in the following. As already pointed out, a suitable order on the formulae of $\mathcal{L}$ will give rise to a unique contraction function for maxichoice contraction. Gärdenfors and Makinson consider relations of "epistemic entrenchment", where $A \leq B$ means that B is more deeply entrenched, and we are more willing to give up A than to give up B, if need be, and provided we have a choice. This is made precise in

Definition 1.3 *Let* $\leq \; = \; \leq_K$ *be a relation relative to a knowledge set K on the formulae of* $\mathcal{L}$ *such that (EE1) If* $A \leq B$ *and* $B \leq C$*, then* $A \leq C$

(transitivity), (EE2) If $A \vdash B$, then $A \leq B$ (If $A \vdash B$, then we believe at least as much in B as in A.), (EE3) For any A and B, $A \leq A \wedge B$ or $B \leq A \wedge B$ (Essentially this property makes $\leq$ a total order, and gives the necessary decision for contraction.), (EE4) When $K \neq K_{\perp}$ (the set of all formulae of $\mathcal{L}$), then $A \notin K$ iff $A \leq B$ for all B (It is here that K matters !), (EE5) If $B \leq A$ for all B, then $\vdash A$. (Only Truth is maximally entrenched). We then call $\leq$ a relation of epistemic entrenchment for K.

We may read $\leq$ as strength of belief, where everything outside K is not believed at all.

Again, we have an interdefinability result :

Proposition 1.2 *The function K- and the ordering $\leq_K$ are interdefinable in the following sense : Define K-A by $B \epsilon K - A :\longleftrightarrow B \epsilon K$ and ($A < A \vee B$ or $\vdash A$) ($A < B$ means : $A \leq B$, and not $B \leq A$). If $\leq$ satisfies (EE1)-(EE5), then K- so defined will satisfy (K-1)-(K-8). Define $A \leq B$ (on the formulae of $\mathcal{L}$) by $A \leq B :\longleftrightarrow A \notin K - (A \wedge B)$ or $\vdash A \wedge B$. If K- satisfies (K-1)-(K-8), then $\leq$ so defined will satisfy (EE1)-(EE5).*

Outline of the results presented here : As emphasized, any $\leq$ satisfying (EE1)-(EE5) will depend essentially on K. Thus, for iterated revision, as K changes, we need a new order $\leq_K$ for every step. Our Proposition II.2 will show that, given a preference relation $\leq$ for $\mathcal{L}$, i.e. a total ordering of the formulae of $\mathcal{L}$ which satisfies some very natural requirements (and which correspond well with the order of the Lindenbaum-Tarski algebra), we can construct from that $\leq$ for all K an order $\leq_K$ satisfying (EE1)-(EE5). Proposition II.3 will show the inverse : Given an epistemic entrenchment relation, we have a preference relation too, such that the construction of Definition II.1 will recover the epistemic entrenchment relation again. A direct definition of a contraction function from a preference relation with an additional property to obtain (K-7), is given in Proposition II.4. Proposition II.8 and II.11 will show how to naturally define an order satisfying the prerequisites of Proposition II.2.

2 Preference Relations

Definition 2.1 *a. Call a relation $\leq$ on the formulae of $\mathcal{L}$ a preference relation for $\mathcal{L}$ iff $\leq$ is a binary relation such that 1. $A \vdash B \rightarrow A \leq B$ 2. $\leq$ is transitive 3. $\leq$ is total 4. $\forall B.B \leq A \rightarrow \vdash A$*
b. Let K be closed under $\vdash$, and a preference relation $\leq$ for $\mathcal{L}$ be fixed. Define $A \trianglelefteq B$ iff D1. $A \vdash B$ or D2. $A \notin K$ or D3. $B = A \wedge C$ and $A \leq C$ and $A \wedge C = B \,\epsilon\, K$. Furthermore, define $\preceq$ as the transitive closure of $\trianglelefteq$.

Fact 2.1 *There is a standard way of establishing $A \preceq B$: Let $A, C \epsilon K$, then there is B such that $A \trianglelefteq A \wedge B$ by D3, and $A \wedge B \trianglelefteq C$ by D1 iff $A \leq A \rightarrow C$.*

Proof : ” $\leftarrow$ ” : As $A, C\epsilon K$, $A \wedge (A \rightarrow C) \epsilon K$, so $A \trianglelefteq_{D3} A \wedge (A \rightarrow C) \trianglelefteq_{D1} C$. ” $\rightarrow$ ” : As $\vdash A \wedge B \rightarrow C$, $\vdash B \rightarrow (A \rightarrow C)$, and by Conditions 1 and 2 above $A \leq B \leq A \rightarrow C$. □

Proposition 2.2 *If $\leq$ is a preference relation for $\mathcal{L}$, and $\preceq$ defined as in Definition II.1, then $\preceq$ satisfies (EE1)-(EE5).*

Thus, given one global preference relation for $\mathcal{L}$, we can easily obtain epistemic entrenchment relations for all knowledge sets K of $\mathcal{L}$.

Proof : We first show two claims, the proof will then be trivial.
Claim 1 : For no $A\epsilon K$, $B \notin K$ we have $A \preceq B$. Proof : Suppose the contrary. Let $A = A_1 \trianglelefteq A_2 \trianglelefteq \ldots \trianglelefteq A_n = B$. We have to ”cross the border between K and $\mathcal{L} - K$” somewhere : There is $A_i \trianglelefteq A_{i+1}$ s.th. $A_i \epsilon K$, $A_{i+1} \notin K$. Examine the cases of $\trianglelefteq$. D1 can't be, as $A_i \epsilon K$, $A_i \vdash A_{i+1}$ implies $A_{i+1} \epsilon K$. D2 can't be, as $A_i \epsilon K$. D3 can't be, as $A_{i+1} \notin K$. Contradiction. □ (Claim 1)
Claim 2 : $\forall B.B \preceq A \rightarrow \vdash A$. Proof : (Induction on the length of the $\trianglelefteq - chain$.) By Condition 1., it suffices to consider B:=True. True $\trianglelefteq_{D1}$ A $\rightarrow \vdash A$. True $\trianglelefteq_{D2}$ A can't be, as True ϵ K. True $\trianglelefteq_{D3}$ $A = True \wedge C \rightarrow True \leq C \rightarrow \vdash C$ by maximality of True and 4. of $\leq$. □ (Claim 2)
We prove the proposition : (EE1) is trivial by definition. (EE2) by D1. (EE3) We have to prove $A \preceq A \wedge B$ or $B \preceq A \wedge B$. By 3. of $\leq$, $A \leq B$ or

$B \leq A$. Case 1 : $A \wedge B \,\epsilon$ K. If $A \leq B$, then $A \trianglelefteq A \wedge B$ by D3. $B \leq A$ analogously. Case 2 : $A \wedge B \notin$ K. Then $A \notin K$ or $B \notin K$, continue with D2. (EE4) " $\rightarrow$ " : $A \notin K \rightarrow A \trianglelefteq B$ by D2 for all B. " $\leftarrow$ " : Let $K \neq K_{\perp}$, $A \preceq B$ for all B. Suppose $A \epsilon K$. By $K \neq K_{\perp}$, there is $B \notin K$, and by prerequisite $A \preceq B$, contradicting Claim 1. (EE5) Claim 2. $\Box$ (Proposition II.2)

Gärdenfors (in personal communication) has raised the question whether we can, given a relation of epistemic entrenchment, define from this relation a preference relation and recover the epistemic entrenchment relation again as in Definition II.1. The answer is "yes" (in a simplified version due to David Makinson) :

Proposition 2.3 *Let $\leq_K$ be an epistemic entrenchment relation for a knowledge set K. Then, by definition, $\leq_K$ is a preference relation and $\preceq$ defined for this $\leq_K$ and K as in Definition II.1.b is equal to $\leq_K$.*

Proof : " $\preceq \subseteq \leq_K$ " : It suffices to prove $\trianglelefteq \subseteq \leq_K$. D1 and D2 are trivial. D3 : Let $A \trianglelefteq A \wedge B$ by $A \leq_K B$. If $B \leq_K A \wedge B$, then $A \leq_K B \leq_K A \wedge B$, and we are finished by (EE3). " $\leq_K \subseteq \preceq$ " : Let $A \leq_K B$. If $A \notin K$, then $A \trianglelefteq B$ by D2. If $A \epsilon K$, then $B \epsilon K$ by (EE4), so $A \wedge B \epsilon K$, thus $A \trianglelefteq_{A \leq_K B} A \wedge B \trianglelefteq_{\vdash} B$. $\Box$

Next, we show how to define a contraction operation directly and in a simple way from a preference relation, without taking a detour via an epistemic entrenchment relation. To show the property (K-7), we now need an additional assumption, which will, however, be satisfied in our final construction.

Proposition 2.4 *a) Let $\leq$ be a preference relation for a language $\mathcal{L}$. Define a function - : Knowledge sets in $\mathcal{L}$ x Formulae of $\mathcal{L}$ $\rightarrow$ Sets of Formulae of $\mathcal{L}$ by*

$$K - A := \begin{cases} K & if \vdash A or A \notin K \\ \{B \epsilon K : A < A \vee B\} & otherwise \end{cases}$$

The restriction K- to K will satisfy (K-1)-(K-6), (K-8) of Definition I.2. b) K- satisfies (K-7) for all K too iff the following condition holds : (*) For all formulae A, B, C, $(A \wedge B) \vee C \leq A \wedge B$ implies $A \vee C \leq A$ or

$B \vee C \leq B$. c) Condition (*) need not necessarily be satisfied for preference relations. It holds for epistemic entrenchment relations and the construction of Propositions II.8 and II.11.

Proof : a) The cases $\vdash A$ or $A \not\in K$ are trivial, so assume $\not\vdash A$, $A\epsilon K$ in the sequel. $A \leq A \vee B$ is always true for preference relations, so it suffices to show $A \vee B \not\leq A$ to prove $B\epsilon K - A$ (for $B\epsilon K$). (K-2), (K-3), (K-4), (K-6) are trivial. (K-1) : By compactness of $\vdash$, it suffices to consider $\vdash B \rightarrow B'$. Let $B\epsilon K - A$, $\vdash B \rightarrow B'$, assume $B' \not\in K - A$, i.e. $A \vee B' \leq A$. But then $\vdash A \vee B \rightarrow A \vee B'$, and by condition 1. of Definition II.1, $A \vee B \leq A \vee B' \leq A$, contradiction. (K-5) : Let $B\epsilon K$, so $\neg A \vee B\epsilon K$. Assume $A \vee (\neg A \vee B) \leq A$, then $True \leq A$, and by condition 4, $\vdash A$, contradiction. So $\neg A \vee B = A \rightarrow B\epsilon K - A$, and $B\epsilon(K - A) + A$. (K-8) : We have to show that $(A \wedge B) \vee A \leq A \wedge B$ implies $\forall C$ ($A \vee C \leq A \rightarrow (A \wedge B) \vee C \leq A \wedge B$). But then $(A \wedge B) \vee C \leq_{\vdash} A \vee C$ $\leq$ A $\leq_{\vdash}$ $(A \wedge B) \vee A \leq A \wedge B$.
b) " $\leftarrow$ " : If $\vdash A$, then $A \wedge B \longleftrightarrow B$, so by (K-6) $K - B = K - (A \wedge B)$, and (K-7) is satisfied. If $A \not\in K$, then $A \wedge B \not\in K$, and (K-7) holds, too. Likewise for B. Suppose now $C\epsilon K$, $C \not\in K - (A \wedge B)$. Thus, by condition 1 for preference relations, $(A \wedge B) \vee C \leq A \wedge B$. By (*), $C \not\in K - A$, or $C \not\in K - B$. " $\rightarrow$ " : Let A, B, C be such that $(A \wedge B) \vee C \leq A \wedge B$, but neither $A \vee C \leq A$ nor $B \vee C \leq B$. Consider K:=Th(A,B,C), the deductive closure of A, B, C. Then A, B, $A \wedge B$ ϵ K, and $\not\vdash A$, $\not\vdash B$, $\not\vdash A \wedge B$ (otherwise, $A \vee C \leq A$ etc. by maximality of True). Consequently, C ϵ $(K - A) \cap (K - B)$, but not C ϵ $K - (A \wedge B)$.
c) See Example II.1 below for a counterexample. For epistemic entrenchment relations $(A \vee C) \wedge (B \vee C) = (A \wedge B) \vee C \leq A \wedge B \leq$ A, B, and (EE3) gives the result. For the construction of Propositions II.8 and II.11, by Lemma II.7.7), $(A \wedge B) \vee C \leq A \wedge B$ implies $\vdash C \rightarrow A \wedge B$, thus $\vdash C \rightarrow A$ and $\vdash C \rightarrow B$, and $A \vee C \leq A$, $B \vee C \leq B$. □

We now turn to the task of defining such a total order on the formulae of $\mathcal{L}$ in a natural way.

Let, in the following, $\mathcal{D}$ be the Lindenbaum-Tarski algebra for the language $\mathcal{L}$ and the empty theory. (Thus, elements of $\mathcal{D}$ have the form $[\phi]$, where ϕ is a formula of $\mathcal{L}$, and $[\phi] = [\psi]$ iff $\vdash \phi \longleftrightarrow \psi$. Moreover, $[\phi] \wedge [\psi] := [\phi \wedge \psi]$, $-[\phi] := [\neg\phi]$, and $[\phi] \leq [\psi] :\longleftrightarrow [\phi] \wedge [\psi] = [\phi]$.)

We have a first constructive result :

Lemma 2.5 *Extending the natural ordering on the formulae of $\mathcal{L}$ given by $\mathcal{D}$ to a total order, preserving [True] as the only maximal element, will give a preference relation for $\mathcal{L}$, and thus, by Proposition II.2, epistemic entrenchment relations $\leq_K$ for all knowledge sets K of $\mathcal{L}$.* $\Box$

Next, we assign probability values to formulae of $\mathcal{L}$, i.e. each $\phi \epsilon \mathcal{L}$ will have a real value $\nu(\phi)$, and the natural order of the real numbers will order the formulae too. Of course, logically equivalent formulae should be given the same probability. We proceed indirectly, assigning first probabilities to models, and defining the probability of a formula as the sum of the probabilities of its models. The above equivalence condition will then be trivially true. It is easily seen (Proposition II.8), that our construction will give a preference relation $\leq$ for $\mathcal{L}$ as needed to define the epistemic entrenchment relations $\leq_K$. We can improve our result and the equivalence condition to obtain ($\phi \leq \psi$ and $\psi \leq \phi$) iff $\vdash \phi \longleftrightarrow \psi$ (Proposition II.11). For this end, we use algebraic closure properties of the reals (Fact II.10). We can thus construct in a natural way a total (and natural) extension of the natural order of the Lindenbaum-Tarski algebra $\mathcal{D}$, such that ($[\phi] \leq [\psi]$ and $[\psi] \leq [\phi]$) is equivalent to $[\phi] = [\psi]$. In conclusion, we remark that the whole process can be easily relativized to a fixed theory, by considering only models of that theory (see Definition II.5).

But first, we need some constructions :

Let $\mathcal{A}$ be the $\sigma - Algebra$ (i.e. the $\aleph_1 - complete$ Boolean algebra) of Lebesgue-measurable sets restricted to subsets of the unit interval [0,1). Let μ be the usual Lebesgue measure. (The reader unfamiliar with these notions will find definitions and properties in any book on measure and integration theory.)

Definition 2.2 *Let $< x_i : i \epsilon \omega >$ be a sequence of reals in the open interval (0,1). Define by induction : $a_0 := [0, x_0)$, $b_0 := \{0, x_0, 1\}$. Let a_n, b_n be defined ($n \epsilon \omega$). b_n will be a set of $2^{n+1}+1$ elements, a_n a disjoint union of 2^n non-empty intervals. Let $b_n = \{y_j : j < 2^{n+1}+2\}$, the y_j in increasing order. Define $a_{n+1} := \bigcup \{ \ [\ y_j \ , \ y_j + (y_{j+1} - y_j) * x_{n+1} \ ) : j < 2^{n+1}+1 \ \}$ and $b_{n+1} := b_n \cup \{y_j + (y_{j+1} - y_j) * x_{n+1} : j < 2^{n+1}+1\}$ Finally, set $\overline{a_n} :=$ [0,1) - a_n. (See Diagram II.1 below.)*

Let $\mathcal{B}$ be the $\aleph_1-$complete subalgebra of $\mathcal{A}$ generated by $\{a_i : i\epsilon\omega\}$

Fact 2.6 *For the a_i thus defined we have : 1) $\mu(a_n) = x_n$, 2) $\mu(\overline{a_n}) = 1 - \mu(a_n)$ (trivial), 3) $\mu(\bigcap\{c_n : n\epsilon X\}) = \Pi\{\mu(c_n) : n\epsilon X\}$ where c_n is either a_n or $\overline{a_n}$ for $X \subseteq \omega$ finite , by the "independence" of the construction. This property is essential to all that follows.* $\square$

Let, in the rest of the paper, $\mathcal{L} = \{p_i : i\epsilon\omega\}$ be a countable language of propositional calculus.

Definition 2.3 *a) Define $f : \mathcal{L} \rightarrow \{a_i : i\epsilon\omega\}$ by $f(p_i) := a_i$, i.e. $\mu(f(p_i)) = x_i$. b) Let $\mathcal{M}$ be the set of assignments of truth values to finite subsets of $\mathcal{L}$, $t\epsilon\mathcal{M}$, t defined on $\mathcal{L}' \subseteq \mathcal{L}$. (It suffices to consider finite subsets, as standard propositional calculus admits only finite formulae.) Define $g(t) := \bigcap\{a_i : p_i\epsilon\mathcal{L}', t(p_i) = true\} \cap \bigcap\{\overline{a_n} : p_i\epsilon\mathcal{L}', t(p_i) = false\}$.*

Thus, $\mu(g(t)) = \mu(\ \bigcap\{a_i\ :\ p_i\epsilon\mathcal{L}',\ t(p_i) = true\} \cap \bigcap\{\overline{a_n}\ :\ p_i\epsilon\mathcal{L}',\ t(p_i) = false\}\ ) = \Pi\{x_i\ :\ p_i\epsilon\mathcal{L}',\ t(p_i) = true\} * \Pi\{1 - x_i\ :\ p_i\epsilon\mathcal{L}',\ t(p_i) = false\}$, and we have defined for every assignment $t\epsilon\mathcal{M}$ a real value $\mu(g(t))$. There is a natural way to extend this function to formulae :

Definition 2.4 *Let ϕ be a formula with propositional variables $p_i\epsilon\mathcal{L}_\phi \subseteq \mathcal{L}$ finite. a) Let $Val(\phi) := \{t\epsilon\mathcal{M} : dom(t) = \mathcal{L}_\phi,\ t(\phi) = true$, i.e. ϕ is true under $t\}$. b) So we can define $\nu(\phi) := \Sigma\ \{\mu(g(t)) : t\epsilon Val(\phi)\}$. (See Diagram II.1.)*

Diagram II.1 : Let $\mathcal{L} = \{p, q\}$, t(p)=true, t(q)=false, t'(p)=false, t'(q)=true, $\phi = p \longleftrightarrow \neg q$

```
+-------------+-------------+----------------+----------------+
0           x *x            x           x +(1-x )*x           1
             0  1            0           0     0   1
+---------------------------+
```

```
                a
                 0
+-------------+             +-----------------+
      a                               a
       1                               1
              +-------------+
                    g(t)
                            +-----------------+
                                g(t')
```

Thus, $\mu(a_0) = x_0$, $\mu(a_1) = x_1$, $\nu(\phi) = \mu(g(t)) + \mu(g(t')) = x_0 * (1 - x_1) + (1 - x_0) * x_1$.

Our construction has the following properties :

Lemma 2.7 *1) $\nu(\phi)$ is independent of dom(t) in the following sense : Let $\mathcal{L}_\phi \subseteq \mathcal{L}' \subseteq \mathcal{L}$ finite. Then $\nu(\phi) := \Sigma \ \{\mu(g(t)) : t\epsilon Val(\phi)\} = \Sigma \ \{\mu(g(t)) : t\epsilon\mathcal{M}$, $dom(t) = \mathcal{L}'$, $t(\phi) = true\}$. 2) By definition of Val and ν, logically equivalent formulae will have the same real value $\nu(\phi)$. 3) $\vdash \phi \rightarrow \psi$ implies $\nu(\phi) \leq \nu(\psi)$. (To see this, consider $\mathcal{L}' = \mathcal{L}_\phi \cup \mathcal{L}_\psi$, use 1) and the fact, that every assignment which makes ϕ true, will make ψ true too.) 4) $\nu(\neg\phi) = 1 - \nu(\phi)$. (Use $\nu(true) = \Sigma \ \{\mu(g(t)) : t\epsilon\mathcal{M}$, $dom(t) = \mathcal{L}'$ finite$\} = 1$, $t(\phi) = true \longleftrightarrow t(\neg\phi) = false$, and for $t, t'\epsilon\mathcal{M}$ with the same domain $t \neq t' \rightarrow g(t) \cap g(t') = \emptyset$.) 5) Exactly the valid formulae will have real value $\nu(\phi) = 1$. ($g : \mathcal{D} \rightarrow \mathcal{B}$ (extended suitably to formulae) is an injective homomorphism of Boolean algebras, and use the above arguments.) 6) $\nu(\phi) \leq \nu(\psi) \longleftrightarrow \nu(\neg\psi) \leq \nu(\neg\phi)$ (by 4). 7) $\nu(\phi \vee \psi) \leq \nu(\phi) \longleftrightarrow \vdash \psi \rightarrow \phi$. " $\leftarrow$ " by 3). " $\rightarrow$ " : Suppose $\not\vdash \psi \rightarrow \phi$. Thus $M' := \{t : t \models \phi\} \subset M := \{t : t \models \phi \vee \psi\}$, let $t\epsilon M - M'$. As $x_i\epsilon(0, 1)$, $\mu g(t) \neq 0$, thus $\nu(\phi) := \Sigma\{\mu g(t) : t\epsilon M'\} < \Sigma\{\mu g(t) : t\epsilon M\} =: \nu(\phi \vee \psi)$. 8) We can't expect $\nu(\phi \wedge \psi) = \nu(\phi) * \nu(\psi)$ or $\nu(\phi \vee \psi) = \nu(\phi) + \nu(\psi)$, just think of $\phi = \psi$. These equations can only be* *valid if ϕ and ψ are independent. For this reason, we gave first a value to models, which are independent, and then to formulae.* $\Box$

We have thus proved our main constructive result :

Proposition 2.8 *Let $p_i : i\epsilon\omega$ be given a probability $x_i\epsilon(0,1)$, then this gives rise naturally to probabilities $\nu(\phi)$ for any formula in $\mathcal{L}$, such that 1) - 6) of Lemma II.7 are valid, and thus to a preference relation $\leq$ for $\mathcal{L}$, i.e. satisfying 1. - 4. of $\leq$ in Definition II.1, and thus the prerequisites of Propositions II.2 and II.4.* $\Box$

Fact 2.9 *Let $0 \leq a \leq b < 1$. Augment the natural order of the reals by setting $x \leq^+ y$ for all $a \leq x, y \leq b$, i.e. "identify" all elements of the interval [a,b]. Let ν be defined as in the construction leading to Proposition II.8 and set $\phi \leq \psi$ iff $\nu(\phi) \leq \nu(\psi)$ or $\nu(\psi) \leq^+ \nu(\phi)$. Then $\leq$ is still a preference relation on $\mathcal{L}$.*

Proof : In Definition II.1, 1. and 3. are trivial, 4. holds by $b < 1$. But 2 is simple too : consider e.g. $x \leq y \leq^+ z$. If $x > z$, then $a \leq z \leq x \leq y \leq b$, and $x \leq^+ z$. $\Box$

Example 2.1 *Consider now $\mathcal{L} := \{A, B, C\}$, and set $\mu f(A) := 1/2$, $\mu f(B) := 1/3$, $\mu f(C) := 1/5$, a:=5/30, b:=10/30, and identify in the interval [a,b] as described in the above Fact. Then $\nu(A) = 15/30$, $\nu(B) = 10/30$, $\nu(A \wedge B) = 5/30$, $\nu(A \vee C) = 18/30$, $\nu(B \vee C) = 14/30$, $\nu((A \wedge B) \vee C) = 10/30$. By identification, $(A \wedge B) \vee C \leq A \wedge B$, but neither $A \vee C \leq A$ nor $B \vee C \leq B$. Thus, this order is a counterexample as promised in Proposition II.4.c.* $\Box$

So far, it is quite possible that $\nu(\phi) = \nu(\psi)$, but $\not\vdash \phi \longleftrightarrow \psi$. We now make ν injective (modulo $\longleftrightarrow$). Thus, we improve our result such that ($\phi \leq \psi$ and $\psi \leq \phi$) iff $\vdash \phi \longleftrightarrow \psi$. Choosing the x_i of Definition II.2 above according to the following fact on the reals will do the trick :

Fact 2.10 *Let* $X := \{x_i : i\epsilon\omega\} \subset I \subseteq \mathcal{R}$ *(the reals), I uncountable be given. Then there is* $x'\epsilon I$ *s.th. x' is not equal to any real that can be obtained by finite addition, subtraction, multiplication, division from elements of* $\mathcal{Q} \cup X$ *(*$\mathcal{Q}$ *the rationals) .* $(Card(I) > card(\mathcal{Q} \cup X) = \aleph_0$ *suffices for the proof.)* □

We choose the x_i for the above construction of the a_i in Definition II.2 according to this fact. Suppose that ϕ, ψ are not equivalent, but $\nu(\phi) = \nu(\psi)$. Thus, there is an assignment t s.th. $t(\phi) \neq t(\psi)$. So $\bigcup\{g(t) : t\epsilon Val(\phi)\} \neq \bigcup\{g(t) : t\epsilon Val(\psi)\}$ (w.l.o.g. all t with the same domain $p_0 \ldots p_n$, and n chosen least s.th. the assumption is valid), but $\nu(\phi) = \nu(\psi)$. Thus, $\nu(\phi) = \sum_{i=0,m} \prod_{j=0,n} y_{i,j}$, $\nu(\psi) = \sum_{i=0,m'} \prod_{j=0,n} y'_{i,j}$, where the $y_{i,j}$, $y'_{i,j}$ are either x_j or $1 - x_j$. After multiplication, the equation looks like this : $s_1 + \ldots + s_k = t_1 + \ldots + t_l$, the s_u and t_u are of the form : 1 or $\pm x_{r_1} * \ldots * x_{r_h}$, and each x_j occurs at most once in each summand. After cancelling summands of the same form that occurr on both sides of the equation, x_n will still occurr in at least one of the summands, as n was chosen least. So, we can solve the equation (linear in x_n) for x_n and have $x_n = f(x_0 \ldots x_{n-1})$, where f is composed of addition, subtraction, multiplication, division - contradicting Fact II.10. As the x_i can be chosen within any distance > 0 from a desired value, choosing x_i according to this fact is no real restriction. We have thus obtained our injectivity result and shown

Proposition 2.11 *Let* $p_i : i\epsilon\omega$ *be given a probability* $x_i\epsilon(0,1)$*, chosen according to Fact II.10, then this gives rise naturally to probabilities* $\nu(\phi)$ *for any formula in* $\mathcal{L}$*, such that 1) - 6) of Remark 4 are valid, and (*$\phi \leq \psi$ *and* $\psi \leq \phi$*) iff* $\vdash \phi \longleftrightarrow \psi$ *. In other words, this defines a total (and natural) extension of the natural order of the Lindenbaum-Tarski algebra* $\mathcal{D}$*, and, in addition,* $([\phi] \leq [\psi]$ *and* $[\psi] \leq [\phi])$ *iff* $[\phi] = [\psi]$. □

Remark 2.12 *So far, we have worked over the empty theory and its Lindenbaum Tarski algebra. It is easy to extend our results to non-empty theories, by considering only models of that theory in our Definition II.4 .*

Thus, we can define e.g.

Definition 2.5 *Let T be a theory in $\mathcal{L}_T \subseteq \mathcal{L}$ finite, and $\mathcal{L}' := \mathcal{L}_T \cup \mathcal{L}_\phi$. Set* $\nu_T(\phi) := \frac{\sum\{\mu(g(t)):dom(t)=\mathcal{L}',t(\phi)=true,t(\psi)=true for all \psi \epsilon T\}}{\sum\{\mu(g(t)):dom(t)=\mathcal{L}',t(\psi)=true for all \psi \epsilon T\}}$

So $\nu_T(\phi)$ and $\nu_T(\phi')$ will be equal, iff the models that make T true treat ϕ and ϕ' in the same way.

Remark 2.13 *We can work backwards in the following sense too : Suppose we are given a set of formulae $\{\phi_i : i \epsilon I\}$ and preferences (probabilities) $\pi(\phi_i)$ for all $i \epsilon I$. Can we find a sequence $x_i : i \epsilon \omega$ s.th., constructing as above, $\pi(\phi_i) = \nu(\phi_i) := \sum\{\mu(g(t)) : dom(t) = \mathcal{L}_{\phi_i}, t \models \phi_i\}$? The answer is trivial and canonical. We have a number of equations $\pi(\phi_i) = \Sigma$ { $\Pi\{x_j : t(p_j) = true\}$ * $\Pi\{(1 - x_j) : t(p_j) = false\}$: $t \epsilon Val(\phi_i)$ } and any solution $\{x_j : j \epsilon \omega, x_j \epsilon (0, 1)\}$ (if there is one) of this system of equations, and $\{a_j : j \epsilon \omega\}$ chosen as above will do what we need.*

3 Measuring Theories, and an Outlook for a Different Treatment of Theory Revision

In this section, we discuss three somewhat different approaches to Theory Contraction. In the first two, we extend our measure from formulae to theories, and use it to do contraction. The first attempt is very naive, and mentioned only for illustration. The second approach is again (i.e. as in Section II) "maxichoice contraction" in the sense of [AGM], [G], [M], and as such plagued by the well-known completeness result : $K - A \cup \{\neg A\}$ is a complete theory (see below for a proof).

In the third case, we take a totally different approach and consider pairs $< K, X >$, where K is a theory, and X an axiom set for K. This approach suffers from another defect : it is highly dependent on the syntactic structure of the axiom set : The "coarser" the axiom set is (in the one extreme the conjunction of K), the more drastic and coarse a contraction will be; the more fine-grained it is (in the other extreme K itself), the more we approach the above completeness result. This is made precise in Proposition III.1 , which, basically, shows that splitting an axiom ψ into $\{\psi \vee \phi, \psi \vee \neg\phi\}$ will decide ϕ, i.e. give completeness wrt. ϕ.

Consider now theories T, T' .... in some finite $\mathcal{L}' \subseteq \mathcal{L}$. It is natural to define $\nu(T) := \Sigma\ \{\mu(g(t)) : dom(t) = \mathcal{L}',\ t \models T\}$. ($t \models T$ means, of course, $t(\phi) = true$ for all $\phi \epsilon T$, see Remark II.12.) In other words, $\nu(T)$ is the sum of the probabilities of all $\mathcal{L}' - models$ t that make all $\phi\epsilon T$ true. The more specific a theory is, the less likely it is, too : $T \subseteq T' \rightarrow \nu(T') \leq \nu(T)$, and the empty theory has probability 1. On the other hand, we are interested in "good choices", i.e. we prefer ϕ to $\neg\phi$ if $\nu(\phi) > \nu(\neg\phi)$. So ν will be a good measure only for theories of the same level of specificity. In other words, K-A (here, K-A means some contraction of K w.r.t. A) cannot sensibly be the $\nu - maximal$ $K' \subseteq K$ s.th. $K' \not\vdash A$, as this is always the empty theory.

A better choice might be a $\nu - maximal$ one (if it exists) among $K_A := \{K' \subseteq K$ maximal : $K' \not\vdash A$, K' is $\vdash -closed\}$, this is again "maxichoice contraction".

But there is a problem to maxichoice contraction, pointed out in [AM] : For any $K'\epsilon K_A$, $A\epsilon K$, $Th(K' \cup \{\neg A\})$ will be a complete theory. (The proof is very simple : Let B be given. As $A\epsilon K$, $A \vee B$ and $A \vee \neg B$ are in K. Suppose $A \vee B \notin K'$, $A \vee \neg B \notin K'$. As $A \vee B \notin K'$, by maximality there is $C_0 \epsilon K'$ such that $C_0 \wedge (A \vee B) \vdash A$, and as $A \vee \neg B \notin K'$, there is $C_1 \epsilon K'$ with $C_1 \wedge (A \vee \neg B) \vdash A$. Thus, for $C := C_0 \wedge C_1 \epsilon K'$, $C \wedge (A \vee B) \vdash A$, $C \wedge (A \vee \neg B) \vdash A$, consequently $C \wedge B \vdash A$, $C \wedge \neg B \vdash A$, and $C \vdash A$, contradicting $C\epsilon K' \epsilon K_A$. Thus, $A \vee B \epsilon K'$ or $A \vee \neg B \epsilon K'$, and $K' \cup \{\neg A\} \vdash B$, or $K' \cup \{\neg A\} \vdash \neg B$.)

We now show that this problem essentially carries over to Theory Revision based on axiom sets too.

So far, we have examined theories without any specified axiom system generating the theory. In the following, we consider pairs $< K, X >$, where X is an axiom set for K. Define $\overline{K}_{A,X} := \{< K', X' > : X' \subseteq X$ maximal, $X' \not\vdash A$, K'=Th(X')$\}$ and choose K-A as a $\nu - maximal$ $< K', X' >$ from $\overline{K}_{A,X}$ (if possible). Consider now $< K, X_1 >$, $< K, X_2 >$, where $X_1 := \{\phi, \phi \rightarrow \psi, \psi\}$, and $X_2 := \{\phi, \phi \rightarrow \psi\}$. In both cases, we can infer ψ, and the resulting theories are the same. Suppose we now retract $\phi \rightarrow \psi$. In case 1, it is very sensible to uphold ψ, whereas in case 2, it will not be a good choice. (This example can be found analogously in [GM] and [FUV].) So we are highly dependant on the syntactic form of the axioms, and this is certainly not very desirable. As another example, consider revising a theory which is given by the axiom sets $\{a_1, a_2\}$ or $\{a_1 \wedge a_2\}$. So revision may give different results ($\{a_1\}$ or $\{a_2\}$ vs. the empty theory), which is a doubtful outcome. To avoid this influence of the syntactic form, we might split the axioms as far as possible to obtain optimal results. This procedure, however, approaches completeness, as the following proposition will show :

Let $Y := \{y_1 \ldots y_m\}$ be minimal with $Y \vdash A$. Let $Y' := \{y_2 \ldots y_m\}$. Split y_1 into $\{y_1 \vee \phi, y_1 \vee \neg\phi\}$. Both $Y' \cup \{y_1 \vee \phi\} \vdash A$ and $Y' \cup \{y_1 \vee \neg\phi\} \vdash A$ can't be, since otherwise $Y' \vdash A$, contradicting minimality of Y. So $Y_0 := Y' \cup \{y_1 \vee \phi\}$ or $Y_1 := Y' \cup \{y_1 \vee \neg\phi\}$ is a good candidate for Y-A, i.e. for contracting Y w.r.t. A. So, let Y-A be Y_0 if $Y_0 \not\vdash A$, and Y_1 otherwise.

Proposition 3.1 *If Y-A is as just defined, then* $\overline{Y} := Y - A \cup \{\neg A\}$ *decides* ϕ.

Proof : We have $y_1 \wedge \ldots \wedge y_m \vdash A$, thus $\neg A \wedge y_2 \wedge \ldots \wedge y_m \vdash \neg y_1$. Consequently, $\overline{Y} \vdash \neg y_1$. If $Y - A$ is Y_0, then (by $y_1 \vee \phi \epsilon \overline{Y}$) $\overline{Y} \vdash \phi$, if Y-A is Y_1, then $\overline{Y} \vdash \neg\phi$ $\Box$

This is further illustrated by the following point of view : We may consider "maxichoice contraction" as theory revision with axioms - taking the full theory as axiom set, and choosing a maximal subset from which A does not follow - resulting in full completeness.

4 Conclusion

We first have shown how to construct, from a simple order on all formulae of a given language, orderings suitable for theory revision in the style of Gärdenfors/Makinson. Next, we have shown how to define naturally such a simple order with some additional very nice properties. In the end, we have reconsidered theory revision more generally. Roughly speaking, it turns out that, for theory revision, every axiom set is in a sense a compromise between completeness (too fine-grained) and over-reaction to revision (too coarse). In other words, we should consider to make the choice of the axiom set part of the revision process itself.

Acknowledgements : I would like to thank S. Koppelberg for correcting a proof, and S. Pribbenow for discussions, P. Gärdenfors for his suggestions, and D. Makinson for suggestions and discussions and the simplified version of Proposition II.3.

References

[AGM] C. Alchourron, P. Gärdenfors, D. Makinson : On the Logic of Theory Change, Journal of Symbolic Logic 50, 1985, p. 510-530

[AM] C. Alchourron, D. Makinson : On the Logic of Theory Change : Contraction functions and their associated revision functions ; Theoria 48 (1982) p.14-37

[FUV] R. Fagin, J.D. Ullmann, M.Y. Vardi : On the Semantics of Updates in Databases, Proc. Second ACM SIGACT-SIGMOD, 1983, p. 352-365

[G] P. Gärdenfors : Knowledge in Flux, MIT Press 1988

[GM] P. Gärdenfors, D. Makinson : Revisions of Knowledge Systems Using Epistemic Entrenchment; in : Theoretical Aspects of Reasoning about Knowledge, M.Y.Vardi ed., Morgan Kaufmann, Los Altos, Cal., 1988, p.83-95

[M] D. Makinson : How to give it up : A Survey of Some Formal Aspects of the Logic of Theory Change; Synthese 62 (1985), 347-363

[Sp] W. Spohn : "Ordinal conditional functions : A dynamic theory of epistemic states" in Causation in Decision, Belief Change, and Statistics, W.L.Harper and B.Skyrms, eds. Dordrecht 1988, Reidel, vol. 2, 105-134