COMPLETENESS AND INCOMPLETENESS FOR PLAUSIBILITY LOGIC

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Abstract

Plausibility Logic was introduced by Daniel Lehmann. We show - among some other results - completeness of a subset of Plausibility Logic for Preferential Models, and incompleteness of full Plausibility Logic for smooth Preferential Models.

1 Introduction

Plausibility Logic was introduced by D.Lehmann [Leh92a], [Leh92b] as a sequent calculus in a propositional language without connectives. Thus, a Plausibilty Logic language $\mathcal{L}$ is just a set, whose elements correspond to propositional variables, and a sequent has the form $X \mid\sim Y$, where X, Y are *finite* subsets of $\mathcal{L}$. (I use $\mid\sim$ instead of the $\vdash$ used in [Leh92a], [Leh92b] and continue to reserve $\vdash$ for classical logic.) The intended interpretation of a sequence is that the conjunction of the left hand side entails the disjunction of the right hand side, so it should be read as $\bigwedge X \mid\sim \bigvee Y$. As usual in such cases, we abuse notation, and write $X \mid\sim a$ for $X \vdash \{a\}$, $X, a \mid\sim Y$ for $X \cup \{a\} \mid\sim Y$, $ab \mid\sim Y$ for $\{a, b\} \mid\sim Y$ etc. When discussing Plausibility Logic, X, Y etc. will denote finite subsets of $\mathcal{L}$, a, b etc. elements of $\mathcal{L}$.

Definition 1.1 The base axiom and rules of Plausibilty Logic are:
I (Inclusion): $X \mid\sim a$ for all $a \in X$,
RM (Right Monotony): $X \mid\sim Y \Rightarrow X \mid\sim a, Y$,
CLM (Cautious Left Monotony): $X \mid\sim a$, $X \mid\sim Y \Rightarrow X, a \mid\sim Y$,
CC (Cautious Cut): $X, a_1 \ldots a_n \mid\sim Y$, and for all $1 \leq i \leq n$ $X \mid\sim a_i, Y \Rightarrow X \mid\sim Y$,

and as a special case of CC:
UCC (Unit Cautious Cut): $X, a \mid\sim Y$, $X \mid\sim a, Y \Rightarrow X \mid\sim Y$.
and we denote by PL, for Plausibility Logic, the full system, i.e. I+RM+CM+CC. □

For motivation, the reader is referred to [Leh92a], [Leh92b].
We show in this article that:
(1) RM is "almost" negligeable, more precisely, that RM can be treated essentially as an axiom, which considerably simplifies the search for possible proofs. (2) UCC is strictly weaker than CC, even in the presence of the other axioms and rules. (3) The system I+RM+CC is sound and complete for Minimal Preferential Models (as adapted to Plausibility Logic). The proof we give here is a simplification of our own original proof, and is due to D. Lehmann. (4) The system I+RM+CC+CLM is *not* complete for smooth Minimal Preferential Models. This is somewhat surprising, and in contrast to standard propositional non-monotonic logics, where comparable axiom systems are complete for smooth Minimal Preferential Models. Incompleteness is essentially due to the absence of an "or" on the left hand side of $\mid\sim$, so the sets of models of Plausibility formulas are not closed under finite union, violating one of the prerequisites of Theorem 3.8.
An introduction and general discussion of Preferential Structures will be found in Section 3.

Organization of the paper: In Section 2, we show the basic results on Plausibility Logic, i.e. that RM is almost superfluous, and that UCC is strictly weaker than CC. The proofs are elementary and straightforward.
Section 3 contains an introduction to Preferential Structures, and their basic definitions. The reader familiar with these structures can skip this material with the exception of the very end of Section 3.2 (from Fact 3.10 onward). We first give, in Section 3.1, a general and detailed introduction to Preferential Models. This is followed in Section 3.2 by some basic definitions and results for Preferential Structures. Most of this material will be used in the later development, or is presented, as e.g. Theorem 3.8 in contrast, to point out the differences, which are somewhat subtle. Specific results needed for the later development are to be found at the end of Section 3.2. The central definition is Definition 3.11, the main Fact is Fact 3.10, it is used to show incompleteness later on.
Section 4 contains the main results of the paper.
Section 4.1 contains the representation result for I+RM+CC and Preferential Models. The proof given is a simplification of our original proof, and is due to D. Lehmann.
Section 4.2 shows incompleteness of the full system for smooth preferential models. We construct a counterexample, a finite set of formulas which satisfies the axioms and rules of Plausibility Logic, but violates Fact 3.10, and is thus not representable by a smooth preferential model. The result came as a certain surprise, as everything but the seemingly innocuous condition of closure under finite unions of the domain (see Theorem 3.8) is satisfied. In fact, this condition thus turns out to be crucial. The result shows that the

closure conditions of Theorem 3.8 cannot be omitted. On the other hand, as we do not know whether they are necessary in their full strength, this part of Theorem 3.8 is not yet fully understood.

2 Discussion of Plausibility Logic

Fact 2.1 We note that (partially by finiteness of all sets involved)
1. In the presence of RM, I is equivalent to I': $X \mid\sim Y$ for $X \cap Y \neq \emptyset$.
2. RM is equivalent to RM': $X \mid\sim Y \rightarrow X \mid\sim Z, Y$ for all Z.
3. CC is equivalent to CC': $X, Z \mid\sim Y$ and $X \mid\sim z, Y$ for all $z \in Z \Rightarrow X \mid\sim Y$ for all X, Y, Z such that $(X \cup Y) \cap Z = \emptyset$. □

Let PL' denote CLM+CC'.
PL and PL' are equivalent in the following sense, i.e. RM can essentially be omitted as rule:

Fact 2.2 Let $\mathcal{A}$ be a set of $\mathcal{L}-$ sequences, $\mathcal{A}'$ be the closure of $\mathcal{A}$ under PL, and $\mathcal{A}$" be the closure of $\{X \mid\sim Y\colon X \cap Y \neq \emptyset\} \cup \{X \mid\sim Y\colon X \mid\sim Y' \in \mathcal{A}$ for some $Y' \subseteq Y\}$ under PL'. Then $\mathcal{A}' = \mathcal{A}$".

Proof: " $\supseteq$ " is trivial. " $\subseteq$ ": Any application of RM can be pulled back through applications of CLM and CC. More formally, let e.g. Π be a proof of $X', a' \mid\sim a, Y'$ terminating with CLM, followed by RM: $X' \mid\sim a'$, $X' \mid\sim Y' \Rightarrow_{CLM} X', a' \mid\sim Y' \Rightarrow_{RM} X', a' \mid\sim a, Y'$. Then $X' \mid\sim Y' \Rightarrow_{RM} X' \mid\sim a, Y'$ and $X' \mid\sim a'$, $X' \mid\sim a, Y' \Rightarrow_{CLM} X', a' \mid\sim a, Y'$ also is a proof of $X', a' \mid\sim a, Y'$. (The case CC is analogous.)
Thus, any proof can be assumed to start with instances of I or elements from $\mathcal{A}$, followed by some applications of RM, and only then applications of CLM or CC. □
The value of this remark lies in the fact that it considerably reduces the number of possible proofs of a sequence.

Fact 2.3 If $\mathcal{A}$ does not contain $\emptyset \mid\sim Y$ for any Y, then neither does the closure of $\mathcal{A}$ under PL.

Proof: All rules introducing $\emptyset \mid\sim Y$ contain some $\emptyset \mid\sim Y'$ on the left hand side. Moreover, I does not introduce any $\emptyset \mid\sim Y$. □

Fact 2.4 Assume a set S of sequences does not contain any sequence of the form 1. $\emptyset \mid\sim Y$, 2. $Y \mid\sim \emptyset$, 3. $e \mid\sim f$, 4. $e, f \mid\sim Y$ with $e, f \not\in Y$, where $e \neq f$ are formulas, Y is any set of formulas. Then the closure of S under RM, CLM, UCC does not either.

Proof: We examine the possible cases of introducing such sequences, and show that they all contradict the assumed properties of S:
RM: 1., i.e. there is $\emptyset \mid\sim Y$ introduced by RM, but this contradicts 1. of S, 2. is impossible, 3. contradicts 2. or 3.: $X = \{e\}$ and $Y = \emptyset$ or $Y = \{f\}$, 4. contradicts 4.
CLM: 1. is impossible, 2. contradicts 2., 3. contradicts 1. or 3., 4. contradicts 3. or 4.
UCC: 1. contradicts 1., 2. contradicts 2., 3. contradicts 3. or 4., 4. contradicts 4. $\Box$

Corollary 2.5 CC+I+RM+CLM is stronger than UCC+I+RM+CLM.

Proof: Let now $\mathcal{L} := \{a, b, c, d\}$, set $S_0 := \{cab \mid\sim d,\ c \mid\sim da,\ c \mid\sim db\} \cup \{X \mid\sim x : x \in X\}$ S_0 satisfies 1.-4. We iterate closure under RM, CLM, UCC ω many times to S_ω. Then S_ω satisfies I, RM, CLM, UCC. But S_ω does not contain $c \mid\sim d$, as CC demands. $\Box$

3 Preferential Structures

3.1 Introduction to Preferential Structures

The Intuitive Background: The basic idea is to interpret a primitive notion of "importance" or "value", introduced into a given language and logic, by a function which chooses the subset of "important" models of a theory or formula of that language.
In other words, we work on a set of "possible worlds", i.e. models of the underlying base logic, but do not accord the same importance or value to all such models. Given then a theory T of the base language and logic, we determine the semantical consequences of T in a structure $\mathcal{M}$ by considering only the subset of "important" models of T: $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all important models of T in our structure. More formally, such a structure $\mathcal{M}$ will then consist of a set M of models or possible worlds for the base logic, and a choice function f on $\mathcal{P}(M)$ - the power set of M. If $M(T)$ is the set of all base models of a theory T in $\mathcal{M}$, f singles out the set $f(M(T)) \subseteq M(T)$ of important models of T in the structure $\mathcal{M}$. We thus define $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$. (We simplify here for the moment. Later, we shall consider a more general case, where logically identical models may occur several times.)
A refinement of the idea is to work not with one subset of "maximally important" models, but with many subsets of important models, perhaps of increasing importance. We then have several choice functions f_i in the structure $\mathcal{M}$, and we define $T \models_{\mathcal{M}} \phi$ iff there is some f_i such that ϕ holds in all $m \in f_i(M(T))$. This structure captures the intuition that we may not dispose of ideal models, but can approximate them in the limit: Suppose for simplicity $f_i(M(T)) \supseteq f_{i+1}(M(T))$ for all $i \in \omega$, with $f_i(M(T))$ the set of T-models of importance i. Then, even if each $f_i(M(T))$ is non-empty, $\bigcap\{f_i(M(T)) : i \in \omega\}$ may be empty. So, there are no maximally important models, but we have non-empty sets of ever better ones, which approximate the ideal case. I shall call the latter approach the limit case, and the first variant, for historical reasons, the minimal case.

Already this very abstract description makes it plausible that representation theorems for the limit case are harder to obtain than for the minimal case. In the limit case, we have to handle a possibly infinite set of choice functions, and there need not be a global f such that for all ϕ $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$. In other words, we do not always have a set of "joint witnesses" for all consequences of a theory.

Logical Consequences: It is evident that such consequence relations will be well-behaved with respect to the base logic - provided the latter is sound and complete for the models we have chosen as possible worlds. So, if T and T' are equivalent with respect to the base logic, they will have the same set of semantic consequences, and, if $T \models_{\mathcal{M}} \phi$, and ϕ implies ψ in the base logic, then also $T \models_{\mathcal{M}} \psi$. Moreover, if ϕ is a consequence of T in the base logic, then $T \models_{\mathcal{M}} \phi$, as the choice functions will choose a subset of $M(T)$. These facts hold in both the limit and the minimal version.

Preferential Structures: Preferential Structures are a special case of the above, the choice is made *locally* by a binary relation $\prec$ on the set M of base models, m is considered to be more important than m' iff $m \prec m'$. ($m \prec m'$ instead of $m' \prec m$ for historical reasons). They are thus very similar to Kripke structures, but use the relation $\prec$ differently.
In the minimal case, we define f by means of $\prec$ by $f(A) := \{a \in A : \neg\exists b \in A.b \prec a\}$.
In the limit case, the natural definition is to consider initial segments of A: $\delta_A \subseteq A$ is called an initial segment of A iff ($\delta 1$) we find some $b \in \delta_A$ below each $a \in A$: $\forall a \in A \exists b \in \delta_A (b = a \vee b \prec a)$ ($\delta 2$) δ_A is downward closed: $\forall a \in A \forall b \in \delta_A (a \prec b \rightarrow a \in \delta_A)$. Each f_i corresponds then to the choice of one such δ_A for each $A \subseteq M$.
We thus have in the minimal case $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in \mu(T)$ - the set of $\prec -$ minimal models of T in $\mathcal{M}$. If, for instance, $M(T)$ consists of infinite descending chains, then $\mu(T) = \emptyset$, and $T \models_{\mathcal{M}} \phi$ for any ϕ, $\perp$ (=false) included. On the other hand, any $m \in \mu(T)$ will be a "witness" of *all* $\models_{\mathcal{M}} -$ consequences of T, all ϕ with $T \models_{\mathcal{M}} \phi$ will hold in such m.
In the limit case, we have $T \models_{\mathcal{M}} \phi$ iff there is some $\delta_{T,\phi} \subseteq M(T)$ which satisfies ($\delta 1$) and ($\delta 2$) with respect to $M(T)$ and such that ϕ holds in all $m \in \delta_{T,\phi}$. Thus, in the limit case, $\mu(T)$ may be empty, but if $M(T) \neq \emptyset$, we will still not have $T \models_{\mathcal{M}} \perp$, as all $\delta_{T,\phi}$ are then non-empty. It is easily seen, that if $T \models_{\mathcal{M}} \phi$ and $T \models_{\mathcal{M}} \phi'$, and $\prec$ is transitive, then also $T \models_{\mathcal{M}} \phi \wedge \phi'$: if $\delta_{T,\phi}$ and $\delta_{T,\phi'}$ are suitable, then $\delta_{T,\phi} \cap \delta_{T,\phi'}$ will be a suitable $\delta_{T,\phi\wedge\phi'}$. Moreover, if $T \models_{\mathcal{M}} \phi$, and $M(T \cup \{\phi\}) \subseteq M(T') \subseteq M(T)$, then also $T' \models_{\mathcal{M}} \phi$.
An immediate consequence of the locality of the definition of f is a kind of upward absoluteness in the minimal case. An element, which is not minimal in A, can't be minimal in any B with $A \subseteq B$:
(1) $A \subseteq B \rightarrow f(B) \cap A \subseteq f(A)$.
In contrast, in the general case of arbitrary f, the choice may depend on the "context", there need not be any interdependence between $f(A)$ and $f(B)$, even if $A \subseteq B$.

As a matter of fact, (1) is *the* crucial property for Minimal Preferential Structures. Any choice function which obeys (1) and the trivial property
(0) $f(A) \subseteq A$
can be represented by a Preferential Structure, i.e. by such a binary relation of preference (see below, Theorem 3.6). This is a very general "algebraic" characterization, the underlying set M need not consist of models, it may be just any arbitrary set.
A similar result for Limit Preferential Structures seems to be missing up to now, see [Bou90a], [Bou90b], [Bou92] and [Sch94-t4] for restricted cases. Boutilier's results are restricted in the sense that they treat finitely axiomatisable theories only, but such theories correspond exactly to clopen (=closed and open) sets in the standard topology (see below in this Section). Yet clopen sets can neither be entered nor left by approximation: If x_i is a sequence of elements in a clopen set, then its limit (if it has any) is in X, as X is closed. If x_i is a sequence of elements not in X, then its limit is outside X, as the complement of X is closed. Thus, this restriction seems to go somewhat against the spirit of the limit approach. (See below for details.) In the end, one might also criticize that Boutilier lets the modal operators $\Diamond$ and $\Box$ do all the "nasty" work, which turns out so unpleasant in an attempt of a direct construction: From the semantical point of view, $\Diamond$ and $\Box$ are quantifiers over possible worlds which cooperate with the relation $\prec$, and it is precisely the interplay of both quantifiers which presents difficulties. But, it is always easy to criticize in hindsight ....

Interpretation: We have so far deliberately left open the base logic and its models in M, as well as the intuition behind the "importance" of models of the base logic.
Non-monotonic Logic:
This "importance" may be read as "normality" in the case of non-monotonic logics: We are primarily interested in reasoning about the normal cases, and the preferred models are the most normal ones - where birds can fly, houses have doors etc.
As a matter of fact, Preferential Structures in their various forms provide an important and relatively well-studied group of semantics for non-monotonic logics. They have proved a powerful tool for investigation, providing - via additional properties of the relation $\prec$ - a technique of constructing semantics of logical systems of different strengths. Limit Preferential Structures for non-monotonic logics were introduced by G.Bossu and P.Siegel in [BS85]. The minimal case was first examined by Y.Shoham ([Sho87]) as a generalization of the Minimal Model Semantics for Circumscription. More or less general cases of Preferential Structures are characterized by soundness and completeness theorems in [KLM90], [LM92], [Sch92], [Sch96-1] for the minimal case, in [Bou90a], [Bou90b], [Bou92], and [Sch94-t4] for the limit case. For an overview, see also [Mak94].
Deontic Logic:
Deontic logic reasons about the morally acceptable situations, and about what ought to be done (by humans, robots etc.). Reasoning about morally acceptable actions can be split into two subquestions: Reasoning about the morally acceptable states, and reasoning

about the problem of acting in a way that those states are reached. The latter question can be considered separately, at least in first approximation.
In this framework, the preferred or more important models are those which are morally more acceptable. Thus, Preferential Structures also provide a natural semantics for deontic logic, and, in fact, were examined as such before the advent of non-monotonic logics [Han69]. This was pointed out by D.Makinson in [Mak93].
In hindsight, it is no surprise that a local preference by a binary relation tends to emerge, when we examine choice functions which single out some states as more important or interesting than others. Such local preferences seem to correspond well to intuitions, and simplify the situation by making the choice context-independent.
In [Mak93], still other natural applications of Preferential Structures are discussed.

An Example: Before we proceed, we give a simple example which shows that the relation $\models_{\mathcal{M}}$ defined by a Preferential Structure may indeed be a non-monotonic consequence relation.
Let $\mathcal{L}$ be the propositional language with two varibles p, q, let M consist of two (classical) models, $m \models p \wedge q$, $m' \models \neg p \wedge \neg q$, and let $m' \prec m$. Then $\emptyset \models_{\mathcal{M}} \neg q$, but $p \models_{\mathcal{M}} q$ in both the minimal and the limit definition.
As is the case already in our example, not all classical models for a given language $\mathcal{L}$ need occur in the base set M of a Preferential Structure $\mathcal{M}$ (e.g., in our example, some m" $\models p \wedge \neg q$ is missing). Moreover, some classical models might occur several times, even infinitely often. Take for example $\mathcal{L}$ with one propositional variable p and consider the structure $\mathcal{M} :=< \{< m, i >: i < \omega\}, \prec>$, m a classical model with $m \models p$, and $< m, i >\prec< m, j >$ iff $j < i$. By abuse of language, we shall also write $< m, i >\models p$, etc. Then $\mu(M) = \emptyset$, so $true \models_{\mathcal{M}} \perp$ in the minimal reading, but $true \models_{\mathcal{M}} \phi$ iff ϕ is a classical consequence of p, in the limit reading. More details and examples of logics which require several copies of classical models to be representable by preferential models can be found in [Sch96-1].

Strengthenings of the Conditions for the Relation $\prec$: Various additional conditions for the relation $\prec$ have been introduced and examined for Minimal Preferential Structures.
The most natural one is perhaps transitivity.
An important condition, which results in nice properties of the semantic consequence relation $\models_{\mathcal{M}}$ is smoothness (terminology of D.Lehmann and his co-authors) or stopperedness (terminology of D.Makinson): Given a theory T, and a non-minimal model m of T, there is $m' \prec m$, which is a minimal model of T. Consequently, if $M(T) \neq \emptyset$, then $\mu(T) \neq \emptyset$.
"Smoothness" or "stopperedness" can be violated essentially in two situations. First, suppose X consists of an infinite descending chain of elements x_i. Then, no $x_i \in X$ is minimal in X, and no x_i has a minimal $y \in X$ below it. Second, suppose that $X := \{x, y, z\}$, with

$x \prec y \prec z$, but not $x \prec z$. Then z is not minimal in X, and there is no $a \in X$ below z which is minimal in X.
The counterpart for the consequence relation $\models_{\mathcal{M}}$ is Cumulativity (see [KLM90] and [Gab85]) which says that two theories T, T' with $T \subseteq T' \subseteq \{\phi : T \models_{\mathcal{M}} \phi\}$ have the same consequences: $T \models_{\mathcal{M}} \phi$ iff $T' \models_{\mathcal{M}} \phi$. We may read this as "normal use of Lemmas": If we have already deduced the "Lemma" ϕ from T, we neither loose nor win in terms of possible deductions by starting from $T \cup \{\phi\}$.
As a matter of fact, a very general algebraic representation result can again be obtained: A choice function f can be represented by a smooth Minimal Preferential Structure iff it satisfies the conditions (0), (1) and
(2) $f(A) \subseteq B \subseteq A \rightarrow$ f(A)=f(B)
and if its domain satisfies closure under finite intersections and unions (see below, Theorem 3.8).
Another strengthening of $\prec$ is rankedness, which may be seen as the existence of a "rotating scale with fixed origin": $\prec$ is called ranked (on M), iff there is an order-preserving function $f : (M, \prec) \rightarrow (X, \prec \bullet)$, where $\prec \bullet$ is a total order on X. Then two $\prec -$ incomparable elements $m, m' \in M$ behave exactly the same way with respect to $\prec$: $n \prec m$ iff $n \prec m'$, and $m \prec n$ iff $m' \prec n$. The corresponding property of $\models_{\mathcal{M}}$ is Rational Monotony: If $\alpha \models_{\mathcal{M}} \gamma$, then $\alpha \wedge \beta \models_{\mathcal{M}} \gamma$ or $\alpha \models_{\mathcal{M}} \neg\beta$ (see [LM92]). General representation results are again to be found in [Sch96-1].

Generalizations: We can consider choice functions on arbitrary sets, which need not be sets of models.
We have already seen above two characterizations of such functions defined by Minimal Preferential Structures.
The strength of these algebraic representation results lies in their generality. We can more or less easily obtain soundness and completeness results as Corollaries for non-monotonic logics with classical propositional logic as background, see [Sch92], [Sch96-1], and [Sch94-t4]. But, these results can also be read with classical predicate logic in the background, giving representation theorems for Preferential Structures over models for predicate logic. Still more generally, they can be used for dynamic situations, see [Sch95-3] and for the present case, Plausibility Logic. As a matter of fact, the algebraic characterization of Minimal Preferential Structures (Theorem 3.6 below) gave the first proof of a representation of I+RM+CC by Preferential Models - later considerably simplified by D.Lehmann, and the seemingly unmotivated closure properties required in the algebraic characterization of smooth Minimal Preferential Structures (Theorem 3.8 below) gave an indication where to look for a counterexample when attempts to represent Plausibility Logic by smooth Preferential Models failed: It is precisely the lack of an "or" - corresponding to finite union - on the left hand side of the $\mid\sim$ in Plausibility Logic, which causes representation to fail.

3.2 Basic Definitions and Results

Definition 3.1 We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g: g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets. Given a class of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X}\lceil X := \{<x, i> \in \mathcal{X} : x \in X\}$, so if $\mathcal{X}$ is a function f, $f\lceil X$ is the usual notation for the restriction of f to a subset of its domain.
Let $\mathcal{L}$ be a propositional language, we denote by $v(\mathcal{L})$ the set of its variables, by $M_{\mathcal{L}}$ the set of its classical models, ϕ etc. shall denote formulas, T etc. theories in $\mathcal{L}$ (i.e. $T \subseteq \mathcal{L}$), and $M_T \subseteq M_{\mathcal{L}}$ the models of T.
For any classical model m, let $Th(m)$ be the set of formulas valid in m, likewise $Th(M) := \{\phi : m \models \phi$ for all $m \in M\}$, if M is a set of classical models. For two theories T and T', let $T \vee T' := \{\phi \vee \psi : \phi \in T, \psi \in T'\}$. Note that $M_T \cup M_{T'} = M_{T \vee T'}$. $\overline{T} \subseteq \mathcal{L}$ will denote the closure of T under classical logic, and $\vdash$ the classical consequence relation. Given some other logic, $\overline{\overline{T}}$ will denote the set of consequences of T under that logic, i.e. if the more conventional notation for the logic is $\mid\sim$, then $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$. $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{P}(M_{\mathcal{L}})$ shall be the set of definable subsets of $M_{\mathcal{L}}$, i.e. $A \in \boldsymbol{D}_{\mathcal{L}}$ iff there is some $T \subseteq \mathcal{L}$ such that $A = M_T$. If the context is clear, we omit the subscript $\mathcal{L}$ from $\boldsymbol{D}_{\mathcal{L}}$.
For $X \subseteq \mathcal{P}(M_{\mathcal{L}})$, a function $f : X \to \mathcal{P}(M_{\mathcal{L}})$ will be called definability preserving (dp), iff for all $Y \in \boldsymbol{D}_{\mathcal{L}} \cap X$ $f(Y) \in \boldsymbol{D}_{\mathcal{L}}$. If $\boldsymbol{D}_{\mathcal{L}} \subseteq X$, then $f : X \to \mathcal{P}(M_{\mathcal{L}})$ defines a logic $T \mapsto T^f$ on $\mathcal{L}$ by $T^f := \{\phi: \forall m \in f(M_T).m \models \phi\}$. So, if f=id, then $T^f = \overline{T}$. Note that $f(M_T) \subseteq M_{T^f}$ always holds, but not necessarily $f(M_T) = M_{T^f}$, the latter only iff f is dp.
□

Since closure conditions for the set of definable sets of models reveal themselves as crucial below, we note en passant the following (proved in [Sch92]):

Fact 3.2 1. If $v(\mathcal{L})$ is infinite, then $\boldsymbol{D}_{\mathcal{L}} \neq \mathcal{P}(M_{\mathcal{L}})$,
2. $\emptyset$, $M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$,
3. $\boldsymbol{D}_{\mathcal{L}}$ contains all singletons,
4. $\boldsymbol{D}_{\mathcal{L}}$ is closed under arbitrary intersections,
5. $\boldsymbol{D}_{\mathcal{L}}$ is closed under finite unions. □

Definition 3.3 $\mathcal{Z} =< \mathcal{X}, \prec>$ will be called a Preferential Structure iff $\mathcal{X}$ is a set of pairs and $\prec$ is a binary relation on $\mathcal{X}$. We say that $\mathcal{Z}$ is transitive, irreflexive etc., iff $\prec$ is. $< y, i >$ is called a minimal element of $\mathcal{X}\lceil Y$ in $\mathcal{Z}$ iff: 1. $< y, i > \in \mathcal{X}\lceil Y$ and 2. there is no $< y', i' > \in \mathcal{X}\lceil Y$ such that $< y', i' > \prec < y, i >$. Thus, $\mathcal{Z}$ defines a function $\mu_{\mathcal{Z}} : V \to V$ (V the set-theoretic universe) by $\mu_{\mathcal{Z}}(Y) := \{y : there$ is i such that $< y, i >$ is a minimal element of $\mathcal{X}\lceil Y\}$. Given a set Z, $\mu_{\mathcal{Z},Z}$ shall denote $\mu_{\mathcal{Z}}\lceil\mathcal{P}(Z)$. □

Definition 3.4 $\mathcal{Z} =< \mathcal{X}, \prec>$ will be called $\mathcal{Y}-$ smooth (terminology of [KLM90]) or $\mathcal{Y}-$ stoppered (terminology of [Mak94]) iff for all $X \in \mathcal{Y}$ and $< y, i > \in \mathcal{X}\lceil X$, either $< y, i >$ is minimal in $\mathcal{X}\lceil X$, or there is $< y', i' > \prec < y, i >$, $< y', i' >$ minimal in $\mathcal{X}\lceil X$. □

The following definition contains some very important notions. "Definability preserving" says that the preferential relation cooperates with logic. I.e., if we take the set of models of a theory, the subset of minimal models of that theory will again be exactly the set of models of a (in general) new theory.

Definition 3.5 A Preferential Structure $\mathcal{M} =< \mathcal{X}, \prec>$ will be called a classical preferential model (cpm) for $\mathcal{L}$, iff for all $< x, i > \in \mathcal{X}$, $x \in M_{\mathcal{L}}$. $\mathcal{M}$ will be called definability preserving (dp) iff $\mu := \mu_{\mathcal{M},M_{\mathcal{L}}} : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ is definability preserving. By the above, $\mathcal{M}$ defines a logic on $\mathcal{L}$ by $T^{\mathcal{M}} := T^{\mu}$, i.e. $T^{\mathcal{M}}:= \{\phi \in \mathcal{L}$: ϕ holds in all $m \in \mu(M_T)\}$.
A logic $^{=}$ for $\mathcal{L}$ is said to be representable by a cpm, iff there is a cpm $\mathcal{M}$ for $\mathcal{L}$, such that for all $T \subseteq \mathcal{L}$ $T^{\mathcal{M}} = \overline{\overline{T}}$. For $< m, i > \in \mathcal{X}$, we shall abuse notation and say $< m, i > \models \phi$ iff $m \models \phi$, for $\phi \in \mathcal{L}$. $\mathcal{M}$ will be called smooth or stoppered iff it is $\boldsymbol{D}_{\mathcal{L}}-$ smooth. □

The following Theorems are the basis of, or parallel or contrast the development of Section IV. Theorem 3.6, the algebraic characterization of Preferential Structures in general, was the basis for the author's older proof of the completeness result of Section 4.1. Theorem 3.7 is the propositional analogue of this completeness result. Theorem 3.8 is the algebraic characterization of smooth Preferential Structures. It seems to be applicable to the situation of Plausibility Logic, but for the condition of closure under finite unions. So an analogue of Theorem 3.9 for Plausibility Logic was to be hoped for. The counterexample in Section 4.2 demonstrates the importance of this closure condition. Theorems 3.6-3.7 are taken from [Sch92], Theorems 3.8-3.9 from [Sch96-1].

Theorem 3.6 Let Z be any set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $f : \mathcal{Y} \rightarrow \mathcal{P}(Z)$. Then there is a Preferential Structure $\mathcal{Z} =< \mathcal{X}, \prec>$ such that for all $X \in \mathcal{Y}$ $f(X) = \mu_{\mathcal{Z}}(X)$
iff
(f1) $f(X) \subseteq X$ and
(f2) $X \subseteq Y \rightarrow f(Y) \cap X \subseteq f(X)$
for all $X, Y \in \mathcal{Y}$. (Moreover, given such f, $\mathcal{Z}$ can be chosen transitive and irreflexive.) □

and as a consequence

Theorem 3.7 Let $^{=}$ be a logic for $\mathcal{L}$. Then there is a definability preserving classical preferential model $\mathcal{M}$ such that $\overline{\overline{T}} = T^{\mathcal{M}}$
iff
($^{=}$1) $\overline{T} = \overline{T'} \rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$,
($^{=}$2) $\overline{\overline{T}}$ is classically closed,
($^{=}$3) $T \subseteq \overline{\overline{T}}$,

($^=4$) $\overline{\overline{\overline{T \cup T'}}} \subseteq \overline{\overline{\overline{\overline{T}} \cup T'}}$
for all $T, T' \subseteq \mathcal{L}$.
Condition 4 is called infinite conditionalization. (Again, given $^=$, $\mathcal{M}$ can be chosen transitive and irreflexive.) □

We have further shown in [Sch96-1] that smooth Preferential Models correspond also in the infinite version to Cumulativity and quote the results. It is precisely the failure of the closure condition in Theorem 3.8 which causes incompleteness of Plausibilty Logic for smooth Preferential Models. This in turn is caused by the lack of "or" on the left hand side of $\mid\sim$.

Theorem 3.8 Let Z be any set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be closed under finite unions and finite intersections, $f : \mathcal{Y} \to \mathcal{Y}$ Then there is a $\mathcal{Y}-$ smooth Preferential Structure $\mathcal{Z}$, such that for all $X \in \mathcal{Y}$ $f(X) = \mu_{\mathcal{Z}}(X)$
iff
(f1) $f(X) \subseteq X$,
(f2) $X \subseteq Y \to f(Y) \cap X \subseteq f(X)$,
(f3) $f(X) \subseteq Y \subseteq X \to f(X) = f(Y)$
for all $X, Y \in \mathcal{Y}$. □

We might call (f3) algebraic cumulativity.
We have again as consequence:

Theorem 3.9 Let $^=$ be a logic for $\mathcal{L}$. Then there is a smooth definability preserving classical preferential model $\mathcal{M}$ such that $\overline{\overline{T}} = T^{\mathcal{M}}$ iff
($^=1$) $\overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$
($^=2$) $\overline{\overline{T}}$ is classically closed
($^=3$) $T \subseteq \overline{\overline{T}}$
($^=4$) $\overline{\overline{\overline{T \cup T'}}} \subseteq \overline{\overline{\overline{\overline{T}} \cup T'}}$
($^=5$) $T \subseteq T' \subseteq \overline{\overline{T}} \to \overline{\overline{T}} = \overline{\overline{T'}}$
for all $T, T' \subseteq \mathcal{L}$. □

Condition ($^=5$) is called general (logical) cumulativity.
We note as further consequence of smoothness:

Fact 3.10 Let R, S, T be any sets, $\mathcal{M}$ be $\{S, T\}-$ smooth and $\mu_{\mathcal{M}}(T) \subseteq R \cup S$, $\mu_{\mathcal{M}}(S) \subseteq R$, then $S \cap T \cap \mu_{\mathcal{M}}(R) \subseteq \mu_{\mathcal{M}}(T)$.

Proof: We show $m \in S \cap T$, $m \notin \mu_{\mathcal{M}}(T) \rightarrow m \notin \mu_{\mathcal{M}}(R)$. As $m \in T - \mu_{\mathcal{M}}(T)$, by smoothness, for each copy $< m, i >$ there is $< m', i' > \prec < m, i >$, $m' \in \mu_{\mathcal{M}}(T) \subseteq R \cup S$. If $m' \in S$, then, by $m \in S$, $< m, i >$ is not minimal in S, so by smoothness again, there is $< m", i" > \prec < m, i >$ with $m" \in \mu_{\mathcal{M}}(S) \subseteq R$. So, for each copy $< m, i >$ there is some $< n, k > \prec < m, i >$, $n \in R$, so $m \notin \mu_{\mathcal{M}}(R)$. □
Fact 3.10 will be used to show that Plausibility Logic is incomplete for smooth Preferential Models.
We now adapt the Definition of a Preferential Model to Plausibility Logic. This is the central definition on the semantic side.

Definition 3.11 Fix a Plausibility Logic language $\mathcal{L}$. A model for $\mathcal{L}$ is then just an arbitrary subset of $\mathcal{L}$.
If $\mathcal{M} := < M, \prec >$ is a Preferential Model such that M is a set of (indexed) $\mathcal{L} - models$, then for a finite set $X \subseteq \mathcal{L}$ (to be imagined on the left hand side of $\mid\sim$!), we define (a) $m \models X$ iff $X \subseteq m$, (b) $M(X) := \{m :< m, i > \in M$ for some i and $m \models X\}$, (c) $\mu(X) := \{m \in M(X)$: $\exists < m, i > \in M . \neg \exists < m', i' > \in M$ $(m' \in M(X) \wedge < m', i' > \prec < m, i >)\}$, (d) $X \models_{\mathcal{M}} Y$ iff $\forall m \in \mu(X) . m \cap Y \neq \emptyset$. □

(a) reflects the intuitive reading of X as $\bigwedge X$, and (d) that of Y as $\bigvee Y$ in $X \mid\sim Y$. Note that X is a set of "formulas", and $\mu(X) = \mu_{\mathcal{M}}(M(X))$, where $\mu_{\mathcal{M}}$ is as in Definition 3.3.
We note as trivial consequences of the definition (and/or of Theorem 3.6):

Fact 3.12 (a) $a \models_{\mathcal{M}} b$ iff for all $m \in \mu(a) . b \in m$,
(b) $X \models_{\mathcal{M}} Y$ iff $\mu(X) \subseteq \bigcup \{M(b) : b \in Y\}$,
(c) $m \in \mu(X) \wedge X \subseteq X' \wedge m \in M(X') \rightarrow m \in \mu(X')$. □

4 Completeness and Incompleteness Results for Plausibility Logic

4.1 I+RM+CC is complete (and sound) for Preferential Models:

The following proof is a simplification of our own original proof, and is due to D. Lehmann.
Let $\mathcal{L}$ be a Plausibility Logic language, and $\mathcal{N} \subseteq \mathcal{L}$ be an arbitrary subset.

Definition 4.1 $\mathcal{N}$ is normal for X iff $X \mid\sim X' \rightarrow \mathcal{N} \cap X' \neq \emptyset$.

Remark 4.2 If $\mathcal{N}$ is normal for X, then $X \subseteq \mathcal{N}$. (By I.) □

Definition 4.3 $\mathcal{N}$ is minimal normal for X iff $\mathcal{N}$ is normal for X, and no proper subset of $\mathcal{N}$ is normal for X.

Remark 4.4 If $\mathcal{N}$ is normal for X, then there is $\mathcal{N}' \subseteq \mathcal{N}$ minimal normal for X.

Proof: It suffices to show that for any system Σ of sets normal for X, and totally ordered by " $\subseteq$ ", $\bigcap \Sigma$ is normal too. Take then X' with $X \mid\sim X'$. By prerequisite, each $\mathcal{N} \in \Sigma$ has non-empty intersection with X'. By finiteness of X', there is $x' \in X' \cap \mathcal{N}$ for all $\mathcal{N} \in \Sigma$. $\square$

Remark 4.5 If $\mathcal{N}$ is normal for X, then $\mathcal{N}$ is minimal normal for X iff $\forall y \in \mathcal{N} \exists X_{\mathcal{N}} (X \mid\sim X_{\mathcal{N}} \wedge \mathcal{N} \cap X_{\mathcal{N}} = \{y\})$.

Proof: " $\leftarrow$ " : If $\mathcal{N}$ is not minimal normal, then we may omit one element. " $\rightarrow$ " : Suppose there is $y \in \mathcal{N}$ and for all X' with $X \mid\sim X'$ $y \in \mathcal{N} \cap X'$, and $card(\mathcal{N} \cap X') > 1$, then we may omit y, so $\mathcal{N}$ is not minimal. $\square$

Lemma 4.6 (Fundamental Lemma) If $X \subseteq W \subseteq \mathcal{N}$, and $\mathcal{N}$ is minimal normal for X, then $\mathcal{N}$ is normal for W.

Proof: Suppose $W \mid\sim W'$, we have to show $W' \cap \mathcal{N} \neq \emptyset$. Let then $W = X \cup \{w_0 \ldots w_n\}$, so $X, w_0 \ldots w_n \mid\sim W'$. By above Remarks and $W \subseteq \mathcal{N}$, for w_i there is X'_i with $X \mid\sim X'_i$ and $\mathcal{N} \cap X'_i = \{w_i\}$. Let $Z_i := X'_i - \{w_i\}$. By RM, $X, w_0 \ldots w_n \mid\sim W', Z_0 \ldots Z_n$, and by $X \mid\sim X'_i = w_i, Z_i$ and RM again, $X \mid\sim w_i, W', Z_0 \ldots Z_n$. So by CC $X \mid\sim W', Z_0 \ldots Z_n$. As $\mathcal{N}$ is normal for X, $\mathcal{N} \cap (W' \cup Z_0 \cup \ldots \cup Z_n) \neq \emptyset$, so $\mathcal{N} \cap W' \neq \emptyset$. $\square$

The construction: Set $\mathcal{M} :=< M, \prec >$ with $M := \{< X, i >: X \subseteq \mathcal{L}$ finite, $i \in \omega\} \cup \{< \mathcal{N}, X >: \mathcal{N}$ minimal normal for $X\}$ and $< X, i > \prec < X, j >$ iff $i > j$, and $< Z, 0 > \prec < \mathcal{N}, X >$ iff $X \not\subseteq Z$. So the only possibly minimal elements are of the second type. Note further that $< \mathcal{N}, X >$ is minimal in $M \lceil X := \{< m, r > \in M : m \models X\}$.
Suppose now $X \mid\sim Y$ and $m \in \mu(X)$ (in $\mathcal{M}$). Then $X \subseteq m$ and there is W such that m is minimal normal for W, and $< m, W >$ is minimal in $M \lceil X$. If $W \not\subseteq X$, then $< X, 0 > \prec < m, W >$, contradicting minimality, so $W \subseteq X \subseteq m$. By the Fundamental Lemma 4.6, m is normal for X, so by definition of normality, $m \cap Y \neq \emptyset$. So $X \models_{\mathcal{M}} Y$. Suppose on the other hand that it is not the case that $X \mid\sim Y$. Then $\mathcal{L} - Y$ is normal for X by RM. Let, by Remark 4.4, $\mathcal{N} \subseteq \mathcal{L} - Y$ be minimal normal for X, so $< \mathcal{N}, X >$ is minimal in $M \lceil X$. But $\mathcal{N} \cap Y = \emptyset$, so $X \not\models_{\mathcal{M}} Y$. $\square$

4.2 Incompleteness of the full system of Plausibilty Logic for Smooth Preferential Models

We work in PL and construct a counterexample, a set of formulas which satisfies the axiom and rules of Plausibility Logic, but violates above Fact 3.10, and thus can't be represented by a smooth Preferential Model.

Fact 4.7 Let $\mathcal{L} := \{a, b, c, d, e, f\}$, and $\mathcal{X} := \{a \mid\sim b,\ b \mid\sim a,\ a \mid\sim c,\ a \mid\sim fd,\ dc \mid\sim ba,\ dc \mid\sim e,\ fcba \mid\sim e\}$. Then $\mathcal{X}$ does not entail $a \mid\sim e$.

Proof: Let $\mathcal{A} := \{a\mid\sim b,\ a\mid\sim c,\ a\mid\sim ed,\ a\mid\sim fd,\ b\mid\sim a,\ b\mid\sim c,\ b\mid\sim ed,\ b\mid\sim fd,\ ba\mid\sim c,\ ba\mid\sim ed,\ ba\mid\sim fd,\ ca\mid\sim b,\ ca\mid\sim ed,\ ca\mid\sim fd,\ cb\mid\sim a,\ cb\mid\sim ed,\ cb\mid\sim fd,\ cba\mid\sim ed,\ cba\mid\sim fd,\ dc\mid\sim ba,\ dc\mid\sim e,\ edc\mid\sim ba,\ fcba\mid\sim e\}$
Set $\mathcal{A}_0 := \{X\mid\sim Y\text{: } X\cap Y\neq\emptyset\}$, $\mathcal{A}_1 := \{X\mid\sim Y\text{: there is } Y'\subseteq Y \text{ such that } X\mid\sim Y'\in\mathcal{A}\}$, and $\mathcal{A}" := \mathcal{A}_0\cup\mathcal{A}_1$.
As $\mathcal{A}"$ contains $\mathcal{X}$, but not $a\mid\sim e$, it suffices to show that $\mathcal{A}"$ is a Plausibility Logic, i.e. is closed under PL. By Fact 2.2, this is equivalent to showing that $\mathcal{A}"$ is closed under CLM + CC'. We note

Remark 4.8 (a) For $X\in\{a,b,ba,ca,cb,cba\}$ and $Y\in\{a,b,c,ed,fd\}$ $X\mid\sim Y\in\mathcal{A}"$, (b) for $X\in\{dc,edc,fcba,fecba\}$, $Y\in\{e,ba\}$ $X\mid\sim Y\in\mathcal{A}"$ $\Box$ (Remark)

Note also that all cases of $\mathcal{A}$ occur as cases of (a) or (b).
We first show closure of $\mathcal{A}"$ under CLM: $X'\mid\sim a',\ X'\mid\sim Y'\rightarrow X',a'\mid\sim Y'$ $(a'\notin X')$:
Thus,
$X'\mid\sim a'\in\mathcal{A}$, and
$X'=a$ and $a'=b$ or $a'=c$,
$X'=b$ and $a'=a$ or $a'=c$,
$X'=ba$ and $a'=c$,
$X'=ca$ and $a'=b$,
$X'=cb$ and $a'=a$,
$X'=dc$ and $a'=e$,
$X'=fcba$ and $a'=e$.
The case $X'\mid\sim Y'\in\mathcal{A}_0$ is trivial. Suppose $X'\mid\sim Y'\in\mathcal{A}_1$, so there is $Y\subseteq Y'$ and $X'\mid\sim Y\in\mathcal{A}$. It suffices to show that then $X',a'\mid\sim Y\in\mathcal{A}"$, as $\mathcal{A}"$ is obviously closed under RM. But all cases are handled by Remark 4.8 (a) or (b): If $X'\mid\sim a'\in\mathcal{A}$ and $X'\mid\sim Y\in\mathcal{A}$, then X' is one of the X and Y is one of the Y in (a) or (b). But then X',a' is also one of the X in (a) or (b).
We turn to closure under CC'. We have to show for all X',Y',Z' with $Z'\cap(X'\cup Y')=\emptyset$, $Z'\neq\emptyset$: $X',Z'\mid\sim Y'$, $X'\mid\sim z',Y'$ for all $z'\in Z'\rightarrow X'\mid\sim Y'$. As for no Y $\emptyset\mid\sim Y\in\mathcal{A}"$, $X'\neq\emptyset$.
The case $X',Z'\mid\sim Y'\in\mathcal{A}_0$ is again trivial, as $Z'\cap Y'=\emptyset$, likewise the case $X'\mid\sim z',Y'\in\mathcal{A}_0$ for some $z'\in Z'$, as $Z'\cap X'=\emptyset$. So assume without loss of generality $X'\neq\emptyset$, $Z'\neq\emptyset$, $X'\cap Z'=\emptyset$, $Y'\cap Z'=\emptyset$, $X',Z'\mid\sim Y'\in\mathcal{A}_1$, and for all $z'\in Z'$ $X'\mid\sim z',Y'\in\mathcal{A}_1$. We have to show $X'\mid\sim Y'\in\mathcal{A}"$. Note that by definition of $\mathcal{A}_1$, and $X'\mid\sim z',Y'\in\mathcal{A}_1$, X' has to occur on the left hand side in $\mathcal{A}$, so $X'\in\{a,b,ba,ca,cb,cba,dc,edc,fcba\}$. As $X',Z'\mid\sim Y'\in\mathcal{A}_1$, there is some $Y\subseteq Y'$ with $X',Z'\mid\sim Y\in\mathcal{A}$. Moreover, $X'\cup Z'$ has at least 2 elements. Case 1: $X'\cup Z'\in\{ba,ca,cb,cba\}$. X' is a proper, non-empty subset of $X'\cup Z'$. As $X'\neq c$, Remark 4.8 (a) shows that $X'\mid\sim Y$ too, and thus $X'\mid\sim Y'$. Case 2: $X'\cup Z'\in\{dc,edc,fcba\}$. The possible cases are: (1) $X'\cup Z'=edc$, $X'=dc$ and $Z'=e$, (2) $X'\cup Z'=fcba$, $X'\in\{a,b,ba,ca,cb,cba\}$, $Z'=fcba-X'$, so $f\in Z'$. In (1), we are

done by Remark 4.8 (b). (2): As $X' \cup Z' \mid\sim\ Y' \in \mathcal{A}_1$, Y' has to contain e. Moreover, by $f \in Z'$, $X' \mid\sim\ f, Y' \in \mathcal{A}_1$, so there must be some $Y" \subseteq f, Y'$ with $X' \mid\sim\ Y" \in \mathcal{A}$. If $Y" \subseteq Y'$, we are done, as then $X' \mid\sim\ Y' \in \mathcal{A}_1 \subseteq \mathcal{A}"$. But if $f \in Y"$, then $Y" = fd$, so $d \in Y'$. Thus, $d, e \in Y'$. But by Remark 4.8 (a), $X' \mid\sim\ ed \in \mathcal{A}"$, so $X' \mid\sim\ Y' \in \mathcal{A}"$. $\Box$ ($\mathcal{X}$ does not entail $a \mid\sim\ e$)
(Remark: The author first checked the correctness of this example by a small Pascal program, which computes the deductive closure of $\mathcal{X}$, and runs in less than 2 minutes on a 10 MHz 8088 laptop. Simplifying the output finally led to the hand-made proof given above.)
Suppose now that there is a smooth Preferential Model $\mathcal{M} =< M, \prec>$ for Plausibility Logic which represents $\mid\sim$, i.e. for all X, Y finite subsets of $\mathcal{L}$ $X \mid\sim\ Y$ iff $X \models_{\mathcal{M}} Y$. (See Definition 3.11 and Fact 3.12.) $a \mid\sim\ a$, $a \mid\sim\ b$, $a \mid\sim\ c$ implies for $m \in \mu(a)$ $a, b, c \in m$. Moreover, as $a \mid\sim\ df$, then also $d \in m$ or $f \in m$. As it is not the case that $a \mid\sim\ e$, there must be $m \in \mu(a)$ such that $e \notin m$. Suppose now $m \in \mu(a)$ with $f \in m$. So $a, b, c, f \in m$, thus by $m \in \mu(a)$ and Fact 3.12, $m \in \mu(a, b, c, f)$. But $fcba \mid\sim\ e$, so $e \in m$. We thus have shown that $m \in \mu(a)$ and $f \in m$ imply $e \in m$. Consequently, there must be $m \in \mu(a)$ such that $d \in m$, $e \notin m$. Thus, in particular, as $cd \mid\sim\ e$, there is $m \in \mu(a)$, $a, b, c, d \in m$, $m \notin \mu(cd)$. But by $cd \mid\sim\ ab$, and $b \mid\sim\ a$, $\mu(cd) \subseteq M(a) \cup M(b)$ and $\mu(b) \subseteq M(a)$ by Fact 3.12. Let now $T := M(cd)$, $R := M(a)$, $S := M(b)$, and $\mu_{\mathcal{M}}$ be the choice function of the minimal elements in the structure $\mathcal{M}$, we then have by $\mu(S) = \mu_{\mathcal{M}}(M(S))$: 1. $\mu_{\mathcal{M}}(T) \subseteq R \cup S$, 2. $\mu_{\mathcal{M}}(S) \subseteq R$, 3. there is $m \in S \cap T \cap \mu_{\mathcal{M}}(R)$, but $m \notin \mu_{\mathcal{M}}(T)$, but this contradicts above Fact 3.10. $\Box$ (Counterexample)

4.3 Acknowledgement

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