A TOPOLOGICAL CONSTRUCTION OF A NON-SMOOTH MODEL OF CUMULATIVITY

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Abstract

To solve a problem posed by Bezzazi, Makinson, Pérez, we construct an injective, non-smooth preferential structure validating Cumulativity and Weak Determinacy, in which Negation Rationality fails. We make essential use of infinite sequences of models approaching sets of models. To our knowledge, this is the first time that such topological constructions are used in the context of preferential structures.

1 INTRODUCTION

These notes are remarks to Bezzazi, Makinson, Pérez: "Beyond Rational Monotony: Some Strong Non-Horn Rules for Nonmonotonic Inference Relations", [BMP97]. Bezzazi et al. discuss closure rules of logical systems, additional to those of the well-known system P (see e.g. [KLM90]), and their relation to preferential structures. They pose a number of open questions; we solve one of them.

Among others, [BMP97] discuss the following rules:
NR (Negation Rationality): $\alpha \mid\sim \beta \Rightarrow \alpha \wedge \gamma \mid\sim \beta$ or $\alpha \wedge \neg\gamma \mid\sim \beta$ (for any γ),
WD (Weak Determinacy): $true \mid\sim \neg\alpha \Rightarrow \alpha \mid\sim \beta$ or $\alpha \mid\sim \neg\beta$ (for any β) (we say that such α decide),
DR (Disjunctive Rationality): $\alpha \vee \beta \mid\sim \gamma \Rightarrow \alpha \mid\sim \gamma$ or $\beta \mid\sim \gamma$.
We recall the rule (which is part of the system P):
CM (Cautious Monotony): $\alpha \mid\sim \beta$ and $\alpha \mid\sim \gamma \Rightarrow \alpha \wedge \gamma \mid\sim \beta$.
We use $\neg NR$ as shorthand for the existence of α, β, γ such that $\alpha \mid\sim \beta$, but neither $\alpha \wedge \gamma \mid\sim \beta$, nor $\alpha \wedge \neg\gamma \mid\sim \beta$.

In a terminology which goes - to my knowledge - back to M.Freund, [BMP97] call a preferential structure injective, iff the labelling function l is injective, i.e. iff all classical models in the structure occur at most once (≤ 1 copies in the author's terminology). As we consider here only injective preferential structures, we can identify states and models.
Remark: A preferential structure is for the authors of the above article a *smooth* preferential structure - in contrast to the terminology of the author of this paper. Moreover, the preferential relations considered will be transitive.
Section 2 of the present article addresses a question concerning consistency of WD and $\neg NR$ in injective structures. The variant posed in [BMP97] - is there an injective, smooth structure validating P, WD and $\neg NR$? - has a (quite easy) negative answer. There is a much more challenging variant - is there a non-smooth injective preferential structure validating P, WD, and $\neg NR$? - with a positive answer, which teaches us something about non-smooth structures of a cumulative inference relation: The construction and analysis show first how to achieve CM by other means than by Smoothness, i.e. through infinite descending chains of models. It also shows how to preserve desired logical properties by constructing the descending chains in a suitable manner, essentially by logically approximating a set of models. The author is optimistic that this technique can be used to prove other representation results, as it seems the "right technique, doing what has to be done".
Remark: Our results do not contradict the KLM theorem (see [KLM90]), and similar representation results e.g. by the author (see [Sch97-t1]) that all logics satisfying P have a representation by a smooth preferential structure, as the structures in the cited articles are not necessarily injective.

The main idea:
If CM is violated, there are ϕ, ψ, τ such that $\phi \mid\sim \psi$, $\phi \mid\sim \tau$, but $\phi \wedge \psi \not\mid\sim \tau$. As all minimal models of ϕ are then minimal models of $\phi \wedge \psi$, there must be a new minimal model of $\phi \wedge \psi$, which is not a minimal model of ϕ, weakening the set of consequences of $\phi \wedge \psi$, compared to the set of consequences of ϕ. Smoothness assures that this cannot happen, as any model of $\phi \wedge \psi$ must be above some minimal model of ϕ. If smoothness cannot hold, we must prevent the existence of "dangerous" (i.e. consequence changing) new minimal $\phi \wedge \psi$-models by other means, which (by transitivity) can only be infinite descending chains of $\phi \wedge \psi$-models.
But smoothness cannot hold in an injective structure showing joint consistency of the system P, WD, and $\neg NR$, and we have such "dangerous" $\phi \wedge \psi$-models, as we will see now. By failure of NR, there are α, β, γ such that $\alpha \mid\sim \beta$, $\alpha \wedge \gamma \not\mid\sim \beta$, $\alpha \wedge \neg\gamma \not\mid\sim \beta$. Thus we have a minimal model m_1 of $\alpha \wedge \gamma \wedge \neg\beta$, and a minimal model m_2 of $\alpha \wedge \neg\gamma \wedge \neg\beta$. By WD, there will be at most one $\alpha \wedge \neg\beta$-model, so they cannot both be minimal models of $\alpha \wedge \neg\beta$. Suppose m_1 is not, the other case is analogous. A simple analysis (in Fact 2.2 below) shows that there cannot be a minimal model of $\alpha \wedge \neg\beta$ below m_1, so Smoothness is indeed violated, and we must have an infinite descending chain X of $\alpha \wedge \neg\beta$-models below m_1. Let now $\phi := \alpha \wedge \neg\beta$, and m be the unique (if it exists - if not, a similar argument

applies) minimal $\alpha \wedge \neg\beta$-model, and suppose $m \models \psi$, so $\phi \mid\sim \psi$. If there were now a minimal model m' of $\phi \wedge \psi$ in X, Cumulativity would be violated: By injectivity of the structure, m' is logically different from m, and the theory determined by $\{m\}$ is stronger than the one determined by $\{m, m'\}$ (finiteness of $\{m\}$ is crucial here). Thus, in X either ψ will be infinitely often true, or not at all. We will make it infinitely often true, so "X approximates m logically".
To my knowledge, this is the first time that essentially topological properties of the space of models are used - beyond simple questions of definability, see e.g. [Sch92] - in the context of preferential structures, and in particular to define non-smooth structures validating Cumulativity.

Notation 1.1
m etc. will denote classical models, M a set of classical models, ϕ etc. formulas, r, s, t, p etc. propositional variables. $M(\phi)$ will be the set of models of ϕ in a given structure, $\mu(\phi)$ the set of minimal models. $M \models \phi :\Leftrightarrow \forall m \in M.m \models \phi$. f etc. will denote sequences of models. $\overline{\overline{\phi}} := \{\psi : \phi \mid\sim \psi\}$. $Th(M)$ will denote the set of formulas true in all $m \in M$.

2 THERE IS NO SMOOTH INJECTIVE PRFERENTIAL MODEL OF WD AND $\neg NR$

Fact 2.1
ϕ decides all formulas, i.e. $\phi \mid\sim \psi$ or $\phi \mid\sim \neg\psi$ for all ψ, in a preferential structure, iff $\mu(\phi)$ has at most one element - up to logical equivalence.

Proof:
Trivial. $\square$

Fact 2.2
There is no smooth injective preferential structure validating WD and $\neg NR$.

Proof:
Suppose NR is false, so there are α, β, γ with $\alpha \mid\sim \beta$, $\alpha \wedge \gamma \not\mid\sim \beta$, $\alpha \wedge \neg\gamma \not\mid\sim \beta$. Let $\alpha \mid\sim \beta$, so $true \mid\sim \alpha \rightarrow \beta$. (If m is a minimal model of $true$, and if $m \models \alpha$, then m is a minimal model of α, so $m \models \beta$.) So $true \mid\sim \neg(\alpha \wedge \neg\beta)$. If $\vdash \phi \rightarrow \alpha \wedge \neg\beta$, then $true \mid\sim \neg\phi$. Thus, by WD, if $\vdash \phi \rightarrow \alpha \wedge \neg\beta$, ϕ decides, thus, by injectivity, ϕ has at most one minimal model in the structure.

Let now $\alpha \wedge \gamma \not\mid\!\sim \beta$, $\alpha \wedge \neg\gamma \not\mid\!\sim \beta$, thus there is a minimal model m_1 of $\alpha \wedge \gamma$, where $\neg\beta$ holds, and a minimal model m_2 of $\alpha \wedge \neg\gamma$, where $\neg\beta$ holds. Thus, m_1 is a minimal model of $\alpha \wedge \gamma \wedge \neg\beta$, m_2 a minimal model of $\alpha \wedge \neg\gamma \wedge \neg\beta$.
(a) Suppose m_1 is not a minimal model of $\alpha \wedge \neg\beta$, then by smoothness, there is $m < m_1$, m a minimal model of $\alpha \wedge \neg\beta$. $\neg\gamma$ has to hold in m, so m is a minimal model of $\alpha \wedge \neg\beta \wedge \neg\gamma$. By uniqueness, $m = m_2$, so $m_2 < m_1$, and m_2 is a minimal model of $\alpha \wedge \neg\beta$.
(b) If m_2 is not a minimal model of $\alpha \wedge \neg\beta$, then, analogously, m_1 is, and $m_1 < m_2$. (c) m_1 and m_2 are minimal models of $\alpha \wedge \neg\beta$: Impossible, as $\alpha \wedge \neg\beta$ decides.
Suppose now e.g. m_2 is the minimal model of $\alpha \wedge \neg\beta$, and $m_2 < m_1$. As $\alpha \mid\!\sim \beta$, m_2 cannot be a minimal model of α, so there must be $m' \models \alpha$ below m_2. $m' \models \alpha \wedge \neg\beta$ is impossible (by minimality of m_2), so $m' \models \alpha \wedge \beta$. (Note that we did not need smoothness for this argument.) But $m' \models \gamma$, or $m' \models \neg\gamma$, contradicting minimality of m_1 or of m_2. The other case is analogous. □

For later use (Remark 3.2) we analyse the proof of Fact 2.2. By $\alpha \mid\!\sim \beta$, $\alpha \wedge \gamma \not\mid\!\sim \beta$, $\alpha \wedge \neg\gamma \not\mid\!\sim \beta$, we have found m_1, a minimal model of $\alpha \wedge \gamma \wedge \neg\beta$, and m_2, a minimal model of $\alpha \wedge \neg\gamma \wedge \neg\beta$. m_1 and m_2 cannot both be minimal models of $\alpha \wedge \neg\beta$, so one of them is not. Suppose m_1 is not a minimal model of $\alpha \wedge \neg\beta$. If there is a minimal model m of $\alpha \wedge \neg\beta$ below m_1, then $m = m_2$, and we run into a contradiction. Suppose m_2 is not a minimal model of $\alpha \wedge \neg\beta$. If there is a minimal model m' of $\alpha \wedge \neg\beta$ below m_2, then $m' = m_1$, and we run again into a contradiction. Smoothness guarantees that we find such minimal models m or m' in both cases.
So, if we want to construct a structure validating WD and $\neg NR$, there either must not be a minimal model of $\alpha \wedge \neg\beta$ below m_1, or there must not be a minimal model of $\alpha \wedge \neg\beta$ below m_2. Thus, if the structure is transitive, we have to construct an infinite descending chain of $\alpha \wedge \neg\beta$-models below m_1 or below m_2.
We now turn to the non-smooth structure validating Cumulativity.

3 A NON-SMOOTH INJECTIVE STRUCTURE VALIDATING P, WD, $\neg NR$

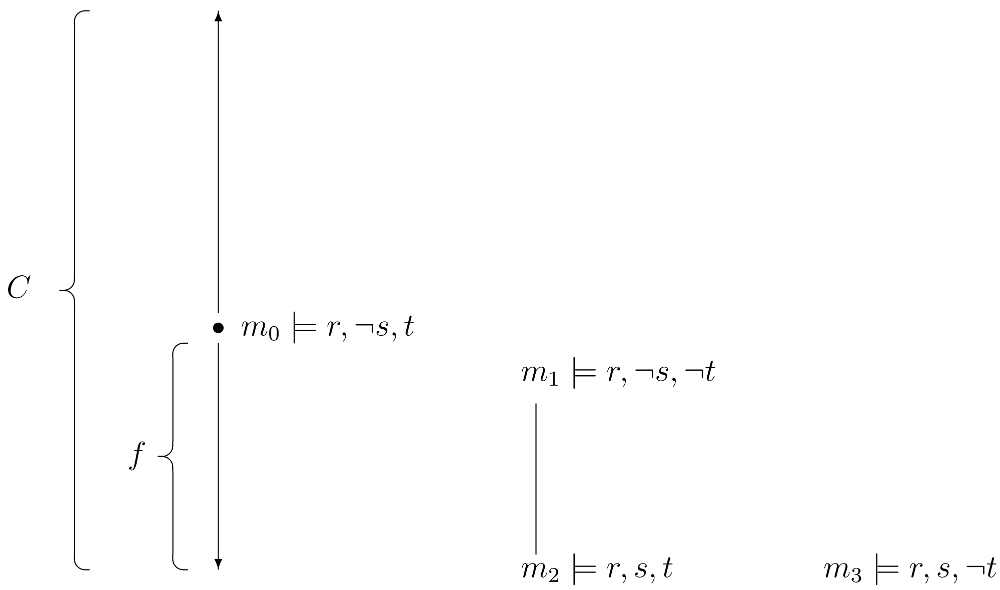
Definition 3.1
A sequence f of models converges to a set of models M, $f \to M$, iff $\forall\phi(M \models \phi \to \exists i \forall j \geq i. f_j \models \phi)$. If $M = \{m\}$, we will also write $f \to m$.

Fact 3.1
Let f be a sequence composed of n subsequences $f^1, \ldots, f^n$, e.g. $f_{n*j+0} = f^1_j$ etc., and $f^i \to M_i$. Let ϕ be a formula unboundedly often true in f. Then there is $1 \leq i \leq n$ and $m \in M_i$ s.t. $m \models \phi$.

Proof:
If for all i and all $m \in M_i$ $m \not\models \phi$, then $M_i \models \neg\phi$, so there are j_i s.t. for all $j \geq j_i$ $f^i_j \models \neg\phi$, so there is k s.t. for all $j \geq k$ $f_j \models \neg\phi$. □

Example 3.1
(A non-smooth injective structure validating P, WD, $\neg NR$.)
As any transitive acyclic relation over a finite structure is necessarily smooth, and an injective structure over a finite language is finite, Fact 2.2 shows that we need an infinite language.
Take the language defined by the propositional variables $r, s, t, p_i : i < \omega$.
Take 4 models m_i, $i = 1, \ldots, 4$, where for all i, j $m_i \models p_j$ (to be definite), and let $m_0 \models r, \neg s, t$, $m_1 \models r, \neg s, \neg t$, $m_2 \models r, s, t$, $m_3 \models r, s, \neg t$. It is important to make m_2 and m_3 identical except for t, the other values for the p_j are unimportant.
Let $m_2 < m_1$. (The other m_i are incomparable.)
Define two sequences of models $f^1 \to m_1$, $f^3 \to m_3$ s.t. for all i, j $f^i_j \models r, \neg t$. This is possible, as $m_1 \models r, \neg t$, $m_3 \models r, \neg t$.
All models in these sequences can be chosen different, and different from the m_i - this is no problem, as we have for all consistent ϕ uncountably many models where ϕ holds.
Let f be the mixture of f^i, e.g. $f_{2n+0} := f^1_n$, etc.
Put m_0 above f, with f in *descending* order. Arrange the rest of the 2^ω models above m_0 ordered as the ordinals - i.e. every subset has a minimum. Thus, there is one long chain C (i.e. C is totally ordered) of models, at its lower end a descending countable chain f, directly above f m_0, above m_0 all other models except $m_1 - m_3$, arranged in a well-order. The models $m_1 - m_3$ form a separate group. See Figure 3.1.

Figure 3.1



Note that m_0 is a minimal model of t.
Obviously, NR is false, as $r \mid\!\sim s$, but neither $r \wedge t \mid\!\sim s$, nor $r \wedge \neg t \mid\!\sim s$.
The usual rules of P hold, as this is a preferential structure, except perhaps for CM, which holds in smooth structures, and our construction is not smooth. (This is the real problem.)
Note that CM says $\phi \mid\!\sim \psi \rightarrow \overline{\overline{\phi}} = \overline{\overline{\phi \wedge \psi}}$, so it suffices to show for all ϕ $\phi \mid\!\sim \psi \rightarrow \mu(\phi) = \mu(\phi \wedge \psi)$. This is the point of the construction. The infinite descending chains converge to some minimal model, so if α holds in this minimal model, then α holds infinitely often in the chain, too. Thus there are no new minimal models of α, which might weaken the consequences.
For WD, we have to show by Fact 2.2 that, if $M(\phi) \cap \mu(true) = \emptyset$, then $\mu(\phi)$ contains at most one model (where $\mu(true) = \{m_2, m_3\}$).
We examine the possible cases of $\mu(\phi)$ ($\emptyset$, $\{m_1\}$, $\{m_2\}$, $\{m_3\}$, $\{m_1, m_3\}$, $\{m_2, m_3\}$, and $\mu(\phi) \cap C \neq \emptyset$).

For CM:
Case 1: $\mu(\phi) = \{m_2, m_3\}$: Then $\phi \mid\!\sim \psi$ iff $\{m_2, m_3\} \models \psi$. So if $\phi \mid\!\sim \psi$, then $\phi \wedge \psi$ holds in m_3, so by f^3, $\phi \wedge \psi$ is (downward) unboundedly often true in f, so $\mu(\phi \wedge \psi) = \{m_2, m_3\}$.
Case 2: $\mu(\phi) = \{m_1\}$ and Case 3: $\mu(\phi) = \{m_3\}$: as above, by f^1 and f^3.
Case 4: $\mu(\phi) = \{m_2\}$: As $m_2 \models \phi$, and $m_3 \not\models \phi$, ϕ is of the form $\phi' \wedge t$, so none of the f_i is a model of ϕ, so ϕ has a minimal model in the chain C, so this is impossible.

Case 5: $\mu(\phi) = \{m_1, m_3\}$: Then $\phi \mid\!\sim \psi$ iff $\{m_1, m_3\} \models \psi$. So as in Case 1, if $\phi \mid\!\sim \psi$, $\phi \wedge \psi$ is unboundedly often true in f^1 (and in f^3), and $\mu(\phi \wedge \psi) = \{m_1, m_3\}$.
Case 6: $\mu(\phi) = \emptyset$: This is impossible by Fact 3.1: If ϕ is unboundedly often true in C, then it must be true in one of m_1, m_3.
Case 7: $\mu(\phi) \cap C \neq \emptyset$: Then below each $m \models \phi$, there is $m' \in \mu(\phi)$. Thus, the usual argument which shows Cumulativity in smooth structures applies.

For WD:
We only have to consider the cases where $m_2, m_3 \notin M(\phi)$, so the only possible cases are: Case 2, Case 7.
In Case 2, there is nothing to show, $\mu(\phi)$ is a singleton.
In Case 7, WD is trivial, we have a unique minimum: $m_2, m_3 \notin M(\phi)$ by prerequisite. But if $m_1 \models \phi$, then ϕ would be true unboundedly often in f, so it would not have a minimal model in C. Thus, $\mu(\phi)$ is a singleton. □

Remark 3.2
In a presentation of this result in a local seminar at Marseille, C.Schwind asked whether it might be possible to prove the same result as in Example 3.1 without the technique of infinite sequences of models approaching or avoiding a set of models. This is not the case. We give an informal argument instead of a formal proof, which would only introduce new terminology without adding substance.
The analysis of the proof of Fact 2.2 (see above, after the proof of Fact 2.2) shows that we need an infinite descending chain D of $\alpha \wedge \neg\beta$-models to make WD and $\neg NR$ true in a transitive structure. But, as $\alpha \wedge \neg\beta$ decides, and by injectivity, there is at most one minimal $\alpha \wedge \neg\beta$-model, so $\mu(\alpha \wedge \neg\beta)$ is finite. But now D has to contain for each ϕ s.t. $\mu(\alpha \wedge \neg\beta) \models \phi$ an infinite number of models where ϕ holds - or no model of ϕ at all. Suppose not, so there is ϕ s.t. $\mu(\alpha \wedge \neg\beta) \models \phi$, but ϕ is not infinitely often true in D. Let $m \in D$ be minimal s.t. ϕ holds, so $m \notin \mu(\alpha \wedge \neg\beta)$, but $m \in \mu(\alpha \wedge \neg\beta \wedge \phi)$, thus $\mu(\alpha \wedge \neg\beta) \neq \mu(\alpha \wedge \neg\beta \wedge \phi)$, but, as $\mu(\alpha \wedge \neg\beta)$ is finite, and by injectivity, $Th(\mu(\alpha \wedge \neg\beta)) \neq Th(\mu(\alpha \wedge \neg\beta \wedge \phi))$. Thus $\phi \in \overline{\overline{\alpha \wedge \neg\beta}} \neq \overline{\overline{\alpha \wedge \neg\beta \wedge \phi}}$, contradicting Cumulativity. (Note that we cannot argue with WD in the last step, as $\mu(\alpha \wedge \neg\beta)$ might be empty, preserving $card(\mu(\alpha \wedge \neg\beta \wedge \phi)) \leq 1$).
If D is (inversely) well-ordered and $\mathcal{L}$ countable and contains a model of ϕ, we can choose a subsequence of D converging to $\mu(\alpha \wedge \neg\beta)$: Let ϕ_i, $i < \omega$ be an enumeration of $Th(\mu(\alpha \wedge \neg\beta))$, let $\psi_i := \bigwedge\{\phi_j : j \leq i\}$. Let $f_k :=$ the first $m \in D$ s.t. $m < f_i$ for $i < k$ and $m \models \psi_k$. By definition of the ψ_k, f_k is at most the k-th m s.t. $m \models \psi_k$, so this is well-defined. Obviously, $f \rightarrow \mu(\alpha \wedge \neg\beta)$.
Thus, at least for each $\phi \in Th(\mu(\alpha \wedge \neg\beta))$ separately, we have an infinite sequence in D where ϕ holds (or ϕ never holds), and "with some luck", we have an infinite sequence in

D converging to $\mu(\alpha \wedge \neg\beta)$.

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References

[BMP97] Bezzazi, Makinson, Pérez: "Beyond Rational Monotony: Some Strong Non-Horn Rules for Nonmonotonic Inference Relations", JLC Vol. 7, No.5, p.605, 1997.

[KLM90] S.Kraus, D.Lehmann, M.Magidor: "Nonmonotonic Reasoning, Preferential Models and Cumulative Logics", Artificial Intelligence 44 (1990), p.167-207

[Sch92] K.Schlechta: "Some Results on Classical Preferential Models", Journal of Logic and Computation, Oxford, Vol.2, No.6 (1992), *p.* 675-686

[Sch97-t1] K.Schlechta: "New techniques and completeness results for preferential structures", to appear in Journal of Symbolic Logic