A NEW APPROACH TO PREFERENTIAL STRUCTURES

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Abstract

This paper deals with some fundamental concepts and questions of preferential structures. Traditionally, a model for preferential reasoning is a strict partial order on the set of classical models of the language; in this article it will be a total order on the classical models. Instead of representing non-monotonic inference relations by individual partial orders, we represent them by sets of total orders. We thus stay close to the way completeness proofs are done in classical logic. Our new approach will also justify multiple copies (or labelling functions) present in most work on preferential structures. A representation result for the finite case is proven; for the infinite case it remains an open question.

1 INTRODUCTION

1.1 Main concepts and results

In this article we address some fundamental questions of preferential structures. Our analysis will be guided by classical propositional (or first order) logic, more precisely by the concept of their models, and by the way their completeness proofs are done.

First, we reconsider the concept of a model for preferential reasoning. Traditionally, such a model is a strict partial order (i.e. an irreflexive, transitive binary relation) on the set of classical models of the underlying language. Instead, we will work here with strict *total* orders (i.e. strict and complete binary relations) on the set of classical models of the underlying language. Such structures have maximal preferential information, just as classical propositional models have maximal propositional information.

Second, we will work in completeness proofs with sets of such total orders and thus again closely follow the approach in classical logic, whereas the traditional approach works with one canonical structure. More precisely, in classical logic, one shows $T \vdash \phi$ iff $T \models \phi$, by proving soundness and that for every T, ϕ such that $T \not\vdash \phi$ there is a T-model $m_{T,\phi}$, where ϕ fails. In traditional preferential logic, one constructs a canonical structure $\mathcal{M}$, which satisfies exactly the consequences of T, i.e. $T \models_{\mathcal{M}} \phi$ iff $T \mid\sim \phi$, simultaneously for all T and ϕ (where $T \models_{\mathcal{M}} \phi$ iff $\mu(T) \subseteq M(\phi)$, i.e. iff in all minimal models of T in $\mathcal{M}$ ϕ holds).

Third, our approach will also shed new light on the somewhat obscure question of multiple copies (equivalent to non-injective labelling functions) present in most constructions (see e.g. the work of the authors, or [KLM90], [LM92]). In our approach, it is natural to consider disjoint unions of sets of total orders over the classical models. They have (almost) the same properties as these sets have. As disjoint unions are structures with multiple copies, we have created multiple copies of models and non-injective labelling functions in a natural way.

To keep the article short, we will not give much background or motivation, and refer the interested reader to the now classical papers in the field, [KLM90] and [LM92], or to the authors' own publications like [Sch97-t1].

1.2 Motivation and overview

Our work started as an analysis of the different ways completeness proofs are made in classical logic and traditional preferential structures - the first is folklore, for the second see e.g. [KLM90]. The idea to consider total orders as models can essentially also be found in [ALS98-1], where we revised nonmonotonic databases. The justification of multiple copies arose naturally with the definition of a disjoint union of total orders.

The concept of a disjoint union of preferential structures raises the question whether a property which holds in all individual structures will also hold in the disjoint union of the structures. This is (trivially) true for entailment relations, but not necessarily for inference rules. A counterexample for the latter using definability problems is given in Section 3.

In the rest of this Section, we will first (Section 1.3) argue in more detail that total orders should be considered the models of preferential reasoning. We then introduce some notation and basic facts. In Section 1.5, we will outline our representation result and the strategy of proof.

In Section 2, we examine different kinds of property of preferential structures and their logics, and their interpretation in our new approach.

In Section 3, we introduce disjoint unions of preferential structures, and present our results on preservation of properties from sets of structures to their disjoint union. We also show that not all preferential structures are equivalent to a disjoint union of total orders, and justify in more detail the existence of multiple copies of models and non-injective labelling functions in traditional preferential structures.

In Section 4, we formulate and prove our representation result for the finite case, taking our usual algebraic detour (which has proved useful in many cases). We first characterize the choice functions of sets of total orders (or, equivalently, their disjoint unions), and translate this characterization into logic by a standard argument.

1.3 Strict total orders are the models of preferential reasoning

A classical propositional or first order model has maximal propositional or first order information: every formula is decided, either the formula or its negation holds. A set of models (in the finite case corresponding to an incomplete formula, i.e. to a formula ϕ such that there is a formula ψ with neither $\phi \vdash \psi$, nor $\phi \vdash \neg\psi$) has less information. Preferential reasoning reasons about preferences between the classical models of a given language $\mathcal{L}$. Maximal preferential information is given by a strict total order between these classical models. A strict partial order can also be considered as the set of total orders which extend it (as a set of pairs). (This can be seen as follows: Suppose a and b are incomparable in a strict partial order P. Then adding $a \prec b$ to the relation cannot introduce cycles, nor destroy irreflexivity. Thus, the transitive closure of the new relation will again be a strict partial order. But adding $b \prec a$ is also a possibility. In the limit step, we take unions; as cycles are finite, we do not introduce any cycles. We end the construction when all elements are comparable.) Thus, strict total orders on the set of classical models may be seen as the basic semantic units for preferential reasoning, just as classical propositional models are the basic semantic items of propositional reasoning.

1.4 Basic definitions and facts

We make ample and tacit use of the Axiom of Choice.

Definition 1.1

We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \to \bigcup\{X_i : i \in I\},$ $\forall i \in I. g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets. Given a set of pairs $\mathcal{X}$, and a set X, we define $\mathcal{X}\lceil X := \{<x,i>\in \mathcal{X} : x \in X\}$.

$\prec^*$ will denote the transitive closure of the relation $\prec$.

By a child (or successor) of an element x in a tree t we mean a direct child in t. A child of a child etc. will be called an indirect child. Trees will be visualized as growing downwards, so the root is the top element.

Definition 1.2

We work in propositional logic, the language will be denoted $\mathcal{L}$, the set of its propositional variables will be $v(\mathcal{L})$. Lower case Greek letters like ϕ etc. will denote formulas of a given language $\mathcal{L}$, and T etc. will denote a theory - an arbitrary set of formulas.

For two theories T and T', let $T \vee T' := \{\phi \vee \psi : \phi \in T, \psi \in T'\}$. $\perp$ stands for falsity.

$\vdash$ will denote the classical consequence relation, and $\overline{T} \subseteq \mathcal{L}$ the closure of T under this relation, i.e. $\overline{T} := \{\phi : T \vdash \phi\}$. Given some other consequence relation $\mid\sim$, $\overline{\overline{T}}$ will denote the set of consequences of T under that relation, i.e. $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$. Such a relation $\mid\sim$ will sometimes also be called a logic. Note that the double bar notation does not really conflict with the single bar notation: closing twice under classical logic makes no sense.

$M_{\mathcal{L}}$ will be the set of classical models of the propositional language $\mathcal{L}$, $M(\phi)$ will be the set of classical models of ϕ, likewise $M(T)$. To avoid too many parentheses, we will sometimes write M_ϕ and M_T.

$\boldsymbol{D}_{\mathcal{L}}$ will be the set of *definable* sets of models for $\mathcal{L}$, i.e. the set of those sets X of models such that there is a theory T with $X = M(T)$. In the finite case, $\boldsymbol{D}_{\mathcal{L}} = \mathcal{P}(M_{\mathcal{L}})$.

For a set X of classical models, $X \models \phi$ iff for all $m \in X$ $m \models \phi$, and $Th(X) := \{\phi : X \models \phi\}$, likewise, $Th(m) := \{\phi : m \models \phi\}$ for a classical model m.

For $X \subseteq \mathcal{P}(M_{\mathcal{L}})$, a function $\mu : X \to \mathcal{P}(M_{\mathcal{L}})$ will be called *definability preserving*, iff $\mu(Y) \in \boldsymbol{D}_{\mathcal{L}}$ for all $Y \in \boldsymbol{D}_{\mathcal{L}} \cap X$.

Definition 1.3

$\mathcal{Z} =< \mathcal{X}, \prec>$ will be called a preferential structure iff $\mathcal{X}$ is a set of pairs and $\prec$ is a binary relation on $\mathcal{X}$. We say that $\mathcal{Z}$ is transitive, irreflexive etc., iff $\prec$ is.

$<y,i>$ is called a minimal element of $\mathcal{X}\lceil Y$ in $\mathcal{Z}$ iff:

1. $<y,i>\in \mathcal{X}\lceil Y$ and

2. there is no $<y',i'>\in \mathcal{X}\lceil Y$ such that $<y',i'>\prec<y,i>$.

Thus, $\mathcal{Z}$ defines a function $\mu_{\mathcal{Z}} : V \to V$ (V the set-theoretic universe) by $\mu_{\mathcal{Z}}(Y) := \{y :$ there is i such that $<y,i>$ is a minimal element of $\mathcal{X}\lceil Y\}$. We call this function the $\mu-$ or choice-function of $\mathcal{Z}$. If for some function μ, $\mu = \mu_{\mathcal{Z}}$, we say that $\mathcal{Z}$ represents μ.

$\mathcal{Z} =< \mathcal{X}, \prec>$ will be called $\mathcal{Y}$-smooth (terminology of Lehmann and his co-authors) or $\mathcal{Y}$-stoppered (terminology of Makinson) iff for all $X \in \mathcal{Y}$ and $< y, i >\in \mathcal{X}\lceil X$, either $< y, i >$ is minimal in $\mathcal{X}\lceil X$, or there is $< y', i' >\prec< y, i >$, $< y', i' >$ minimal in $\mathcal{X}\lceil X$.

Definition 1.4

A preferential structure $\mathcal{M} =< \mathcal{X}, \prec>$ will be called a classical preferential model for $\mathcal{L}$, iff for all $< x, i >\in \mathcal{X}$, $x \in M_\mathcal{L}$.

We set $\mu_\mathcal{M}(\phi) := \mu_\mathcal{M}(M(\phi))$, likewise $\mu_\mathcal{M}(T)$, and define $\phi \models_\mathcal{M} \psi$ iff $\mu_\mathcal{M}(\phi) \subseteq M(\psi)$, likewise $T \models_\mathcal{M} \psi$ iff $\mu_\mathcal{M}(T) \subseteq M(\psi)$. By abuse of language, we will also write $\overline{\overline{T}} := \{\phi : T \models_\mathcal{M} \phi\}$ - context will disambiguate.

A classical preferential mdel $\mathcal{M}$ will be called *definability preserving* iff for all $X \in \boldsymbol{D}_\mathcal{L}$ $\mu_\mathcal{M}(X) \in \boldsymbol{D}_\mathcal{L}$. Note that all structures over finite languages are definability preserving.

A consequence relation $\mid\sim$ for $\mathcal{L}$ is said to be *representable* by a classical preferential model, iff there is a classical preferential model $\mathcal{M}$ for $\mathcal{L}$, such that for all T and ϕ, $T \mid\sim \phi$ iff $T \models_\mathcal{M} \phi$.

For $< m, i >\in \mathcal{X}$, we shall abuse notation and say $< m, i >\models \phi$ iff $m \models \phi$.

$\mathcal{M}$ will be called smooth iff it is $\boldsymbol{D}_\mathcal{L}$-smooth.

Definition 1.5

For a given language $\mathcal{L}$, TO etc. will stand for a strict total order on $M_\mathcal{L}$. Considering TO as a preferential model, we will slightly abuse notation here: as there will be only one copy per model, we will omit the indices i.

$\mathcal{O}$ etc. will stand for sets of such strict total orders.

If $\mathcal{O}$ is such a set, we set $\mu_\mathcal{O}(X) := \bigcup\{\mu_\mathcal{M}(X) : \mathcal{M} \in \mathcal{O}\}$, and define $T \models_\mathcal{O} \phi$ iff $T \models_\mathcal{M} \phi$ for all $\mathcal{M} \in \mathcal{O}$.

Note that for all T and all strictly totally ordered structures TO, $\mu_{TO}(T)$ is either a singleton or empty, so TO is definability preserving.

Definition 1.6

Let $\mathcal{Z} :=< \mathcal{X}, \prec>$ be a preferential structure. For $< x, i >\in \mathcal{X}$, let $< x, i >^-_\mathcal{Z}:= \{< y, j >\in \mathcal{X} :< y, j >\prec< x, i >\}$, and $< x, i >^*_\mathcal{Z}:= \{y : \exists < y, j >\in \mathcal{X}. < y, j >\prec< x, i >\}$. When the context is clear, we omit the index $\mathcal{Z}$.

Fact 1.1

Let $\mathcal{Z} :=< \mathcal{X}, \prec>$, $\mathcal{Z}' :=< \mathcal{X}', \prec'>$ be two preferential structures.

(1) Let $x \in X$. Then $x \in \mu_\mathcal{Z}(X)$ iff $\exists < x, i >\in \mathcal{X}. X\cap < x, i >^*_\mathcal{Z}= \emptyset$.

(2) If $\forall <x,i> \in \mathcal{X} \exists <x,i'> \in \mathcal{X}'. <x,i'>^*_{\mathcal{Z}'} \subseteq <x,i>^*_{\mathcal{Z}}$ and $\forall <x,i'> \in \mathcal{X}' \exists <x,i> \in \mathcal{X}. <x,i>^*_{\mathcal{Z}} \subseteq <x,i'>^*_{\mathcal{Z}'}$, then $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$.

Proof:

(1) $x \in \mu_{\mathcal{Z}}(X) \Leftrightarrow \exists <x,i> \in \mathcal{X}. \neg \exists <y,j> \in \mathcal{X}. <y,j> \prec <x,i> \wedge y \in X \Leftrightarrow \exists <x,i> \in \mathcal{X}. <x,i>^*_{\mathcal{Z}} \cap X = \emptyset$.

(2) Let $x \in \mu_{\mathcal{Z}}(X)$, then by (1) $\exists <x,i> \in \mathcal{X}. X \cap <x,i>^*_{\mathcal{Z}} = \emptyset$. By prerequisite, $\exists <x,i'> \in \mathcal{X}'. <x,i'>^*_{\mathcal{Z}'} \subseteq <x,i>^*_{\mathcal{Z}}$, so $x \in \mu_{\mathcal{Z}'}(X)$ by (1). The other direction is symmetrical. □

Fact 1.2

If $\mathcal{O}$ is a set of preferential structures, then $T \models_{\mathcal{O}} \phi$ iff $\mu_{\mathcal{O}}(M_T) \models \phi$.

Proof:

(This is also a consequence of Fact 3.2, we give here a direct proof.) $\mu_{\mathcal{O}}(M_T) \models \phi \Leftrightarrow \mu_{\mathcal{O}}(M_T) := \bigcup \{\mu_{\mathcal{Z}}(M_T) : \mathcal{Z} \in \mathcal{O}\} \subseteq M(\phi) \Leftrightarrow \forall \mathcal{Z} \in \mathcal{O}. \mu_{\mathcal{Z}}(M_T) \subseteq M(\phi) \Leftrightarrow \forall \mathcal{Z} \in \mathcal{O}. T \models_{\mathcal{Z}} \phi \Leftrightarrow T \models_{\mathcal{O}} \phi$. □

1.5 Outline of our representation result and technique

We describe here the kind of representation result we will show in Section 4.

We have characterized in [Sch92], [Sch97-t1] etc. usual smooth preferential structures first algebraically by conditions on their choice functions, and only then logically by corresponding conditions. More precisely, given a function μ satisfying certain conditions, we have shown that there is a preferential structure $\mathcal{Z}$, whose choice function $\mu_{\mathcal{Z}}$ is exactly μ. The choice functions correspond to the logics by the equation $\mu(M(T)) = M(\overline{\overline{T}})$.

We will take a similar approach here, but will first analyze the form a representation theorem will have in our context.

Our starting point was that classical completeness proofs have the form: For each ϕ such that $T \not\vdash \phi$, we can find $m_{T,\phi}$ such that $m_{T,\phi} \models T, \neg\phi$. Equivalently, we can find a set of models $\boldsymbol{M}_T$ such that for each such ϕ there is a $m_{T,\phi}$ in $\boldsymbol{M}_T$ with $m_{T,\phi} \models T, \neg\phi$. Then, using also soundness, $Th(\boldsymbol{M}_T) = \overline{T}$. Our construction will have a similar form.

First, given any strict total order TO (or any set $\mathcal{O}$ of strict total orders) over $M_{\mathcal{L}}$, the logic defined by $T \mid\sim \phi :\Leftrightarrow T \models_{TO} \phi$ (or $:\Leftrightarrow T \models_{\mathcal{O}} \phi$) satisfies our conditions $(\mid\sim 1)-(\mid\sim 5)$ (see Proposition 4.1). Second, given a logic $\mid\sim$ satisfying $(\mid\sim 1)-(\mid\sim 5)$, there is a set $\mathcal{O}$ of strict total orders over $M_{\mathcal{L}}$ such that $T \mid\sim \phi \Leftrightarrow T \models_{\mathcal{O}} \phi$. Thus, the *set* $\mathcal{O}$ represents exactly $\mid\sim$, contrary to usual preferential structures, where a single structure represents exactly $\mid\sim$.

We work again first via an algebraic characterization, and show the following: Given any strict total order TO (or any set $\mathcal{O}$ of strict total orders), the choice function μ_{TO} (the choice function $\mu_{\mathcal{O}}$) satisfies our algebraic conditions $(\mu 1)-(\mu 3)$ (see Proposition 4.2). Conversely, given a choice function μ satisfying $(\mu 1)-(\mu 3)$, there is a set $\mathcal{O}$ of strict total orders such that $\mu = \mu_{\mathcal{O}}$.

The logical part will then follow easily via a standard argument (see Proposition 4.3).

2 VALIDITY IN TRADITIONAL AND IN OUR PREFERENTIAL STRUCTURES

We distinguish here validity of type 1 and type 2, where type 1 validity is validity of entailments like $T \mid\sim \phi$, and type 2 validity is validity of rules like $\phi \mid\sim \psi \wedge \sigma \Rightarrow \phi' \mid\sim \psi'$.

(The set $\mathcal{O}$ used in this Section is motivated by Example 3.1, where we do not consider all totally ordered sets, but only those satisfying a certain property.)

2.1 Validity of type 1

This is validity of expressions like $\phi \mid\sim \psi$ (or $T \mid\sim \psi$), and is defined for a given preferential structure $\mathcal{M}$ in the usual sense by $\phi \models_{\mathcal{M}} \psi$ (or $T \models_{\mathcal{M}} \psi$). In our new interpretation we read this as: $\phi \models_{\mathcal{O}} \psi$ (or $T \models_{\mathcal{O}} \psi$).

2.2 Validity of type 2

This is validity of rules of e.g. the type

(2.1) $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow \phi \mid\sim \psi \wedge \psi'$,

(2.2) $\phi \mid\sim \psi \Rightarrow (\phi \mid\sim \neg\phi'$ or $\phi \wedge \phi' \mid\sim \psi)$,

(2.3) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$.

As strict total orders are definability preserving, we can argue semantically when dealing with them. More precisely, there is a 1-1 correspondence between theories (and formulas) and sets of models: If $\mathcal{M}$ is a definability preserving preferential model, and T a theory, then $M(\{\phi : T \models_{\mathcal{M}} \phi\}) = \mu_{\mathcal{M}}(M(T))$, so setting $\overline{\overline{T}} := \{\phi : T \models_{\mathcal{M}} \phi\}$, we have for instance $T' \vdash \overline{\overline{T}}$ iff $M(T') \subseteq \mu_{\mathcal{M}}(M(T))$.

Discussion of (2.1):

In usual preferential structures, we read (2.1) as: If in a fixed structure $\mathcal{M}$ both $\phi \models_{\mathcal{M}} \psi$ and $\phi \models_{\mathcal{M}} \psi'$ hold, then so will $\phi \models_{\mathcal{M}} \psi \wedge \psi'$.

In our new approach, we read (2.1) now as: If $\phi \models_{\mathcal{O}} \psi$ and $\phi \models_{\mathcal{O}} \psi'$ hold, then $\phi \models_{\mathcal{O}} \psi \wedge \psi'$ will also hold. In semantical terms: If $\mu_{\mathcal{O}}(\phi) \subseteq M(\psi)$ and $\mu_{\mathcal{O}}(\phi) \subseteq M(\psi')$, then $\mu_{\mathcal{O}}(\phi) \subseteq M(\psi) \cap M(\psi')$.

This is the exact analogue of the classical definition: $\alpha \models \beta$ iff in all classical models where α (and perhaps some other property, too) holds, β will also hold. Our α is here of the form $\phi \mid\sim \psi$ (or $\phi \models_{TO} \psi$) etc.

Discussion of (2.2):

The usual approach is similar to the one for rule (2.1).

For the new approach, we have to be careful with distributivity. A comparison with classical logic helps. In all classical models it is true that if $\alpha \vee \beta$ holds, then α holds, or β holds (by definition of validity of "or"). But we do not say that $\alpha \vee \beta \models \alpha$ or $\alpha \vee \beta \models \beta$ holds, as this would imply either that in all models where $\alpha \vee \beta$ holds, α holds, or that in all models where $\alpha \vee \beta$ holds, β holds, which is usually false.

So a rule of type (2.2) holds iff $\phi \models_{\mathcal{O}} \psi$ implies $\phi \models_{\mathcal{O}} \neg\phi'$ or $\phi \wedge \phi' \models_{\mathcal{O}} \psi$. In semantical terms: A rule of type (2.2) holds iff $\mu_{\mathcal{O}}(\phi) \subseteq M(\psi)$ implies $\mu_{\mathcal{O}}(\phi) \subseteq M(\neg\phi')$ or $\mu_{\mathcal{O}}(\phi \wedge \phi') \subseteq M(\psi)$.

Note that (2.2) holds in all strict total orders on $M_{\mathcal{L}}$, as such structures are ranked. But in a set of such structures, it usually fails, as it is not usually true that either in all these structures $\phi \mid\sim \neg\phi'$ holds, or that in all these structures $\phi \wedge \phi' \mid\sim \psi$ holds.

Discussion of (2.3):

(2.3) stands for: If $T \cup T' \mid\sim \phi$, then there are $\phi_1, \ldots, \phi_n$ and $\phi'_1, \ldots, \phi'_m$ such that $T \mid\sim \phi_i$ and $\phi'_i \in T'$, and $\{\phi_1, \ldots, \phi_n, \phi'_1, \ldots, \phi'_m\} \vdash \phi$.

So, in usual preferential structures, (2.3) holds in structure $\mathcal{M}$, iff: If $T \cup T' \models_{\mathcal{M}} \phi$, then there are $\phi_1, \ldots, \phi_n$ and $\phi'_1, \ldots, \phi'_m$ such that $T \models_{\mathcal{M}} \phi_i$ and $\phi'_i \in T'$, and $\{\phi_1, \ldots, \phi_n, \phi'_1, \ldots, \phi'_m\} \vdash \phi$.

In our new approach, (2.3) holds iff in all strict total orders $TO \in \mathcal{O}$ $T \cup T' \models_{TO} \phi$, there are $\phi_1, \ldots, \phi_n$ and $\phi'_1, \ldots, \phi'_m$ such that $T \models_{TO} \phi_i$ and $\phi'_i \in T'$, and

$\{\phi_1, \ldots, \phi_n, \phi'_1, \ldots, \phi'_m\} \vdash \phi$.

The discussion in semantical terms clarifies the role of the existential quantifiers (which are "ors" - see discussion of (2.2)): Condition (2.3) reads now: in all $TO \in \mathcal{O}$ $\mu_{TO}(T) \cap M(T') \subseteq \mu_{TO}(T \cup T')$ holds (and thus also $\mu_{\mathcal{O}}(T) \cap M(T') \subseteq \mu_{\mathcal{O}}(T \cup T')$).

3 THE DISJOINT UNION OF MODELS AND THE PROBLEM OF MULTIPLE COPIES

3.1 Disjoint unions and preservation of validity

We introduce the disjoint union of preferential structures and examine the question whether a property Φ which holds in all $\mathcal{M}_i$, $i \in I$, will also hold in their disjoint union $\biguplus\{\mathcal{M}_i : i \in I\}$. This is true for type 1 validity, but not for type 2 validity in the general infinite case.

3.1.1 DISJOINT UNIONS AND PRESERVATION OF TYPE 1 VALIDITY

Definition 3.1

Let $\mathcal{M}_i := < M_i, \prec_i >$ be a family of preferential structures. Let then $\biguplus\{\mathcal{M}_i : i \in I\} :=< M, \prec >$, where $M := \{< x, < k, i >>: i \in I,\ < x, k > \in M_i\}$, and $< x, < k, i >> \prec < x', < k', i' >>$ iff $i = i'$ and $< x, k > \prec_i < x', k' >$. Thus, $\biguplus\{\mathcal{M}_i : i \in I\}$ is the disjoint union of the sets and the relations, and we will call it so.

Fact 3.1

Let μ_i be the choice functions of the $\mathcal{M}_i$. Then $\mu_{\biguplus\{\mathcal{M}_i : i \in I\}}(X) = \bigcup\{\mu_i(X) : i \in I\}$, so $\mu_{\biguplus\{\mathcal{M}_i : i \in I\}} = \mu_{\{\mathcal{M}_i : i \in I\}}$.

Proof:

(Trivial.) Let $\mu := \mu_{\biguplus\{\mathcal{M}_i : i \in I\}}$. Let $< x, < k, i >> \in \mu(X)$, then $x \in X$, and there is no $x' \in X$ with $< x', < k', i >> \prec < x, < k, i >>$ for some $< x', k' > \in M_i$, so $x \in \mu_i(X)$. The converse holds by a similar argument. $\square$

Fact 3.2

$T \models_{\biguplus\{\mathcal{M}_i : i \in I\}} \phi$ iff for all $i \in I$ $T \models_{\mathcal{M}_i} \phi$. Thus $T \models_{\biguplus\{\mathcal{M}_i : i \in I\}} \phi$ iff $T \models_{\{\mathcal{M}_i : i \in I\}} \phi$, and disjoint unions preserve type 1 validity.

Proof:

(Trivial.) Let again $\mu := \mu_{\biguplus\{\mathcal{M}_i : i \in I\}}$. $T \models_{\biguplus\{\mathcal{M}_i : i \in I\}} \phi$ iff in all $m \in \mu(T)$ ϕ holds. If for all $i \in I$ in all $m \in \mu_i(T)$ ϕ holds, then ϕ holds in all $m \in \mu(T)$ by Fact 3.1. But if there is some $i \in I$ and $m \in \mu_i(T)$ such that ϕ fails in m, then ϕ will fail in some $m \in \mu(T)$, too, again by Fact 3.1. $\Box$

3.1.2 PRESERVATION OF TYPE 2 VALIDITY

Rules of type (2.1) are preserved: This is a direct consequence of Fact 3.2, the argument is similar to the following one for type (2.2) rules.

Rules of type (2.2) are preserved: We show that if in all strict total orders TO where $\phi \mid\sim \psi$ (and perhaps some other property) holds, $\phi \mid\sim \neg\phi'$ holds, then $\phi \mid\sim \neg\phi'$ holds in the disjoint union $\mathcal{M}$ of these structures, and, if in all strict total orders TO where $\phi \mid\sim \psi$ (and perhaps some other property) holds, $\phi \wedge \phi' \mid\sim \psi$ holds, then $\phi \wedge \phi' \mid\sim \psi$ holds in the disjoint union $\mathcal{M}$ of these structures. But, it is a direct consequence of Fact 3.2 that in the first case $\phi \models_{\mathcal{M}} \neg\phi'$, and in the second case $\phi \wedge \phi' \models_{\mathcal{M}} \psi$.

Rules of type (2.3) are not necessarily preserved - at least not in the general infinite case:

Example 3.1

(This is the - slightly adapted - Example 1.9.1 from [Sch92], which shows failure of infinite conditionalization in a case where definability preservation fails.)

Consider the language $\mathcal{L}$ defined by the propositional variables p_i, $i \in \omega$. Let $T_0^+ := \{p_0\} \cup \{p_i : 0 < i < \omega\}$, $T_0^- := \{\neg p_0\} \cup \{p_i : 0 < i < \omega\}$, set $T' := T_0^+ \vee T_0^-$, and $T := \emptyset$. Let the classical model m_0^+ (m_0^-) be the unique model satisfying T_0^+ (T_0^-), so $M(T') = \{m_0^+, m_0^-\}$. Consider the set $\mathcal{O}$ of all strict total orders TO on $M_{\mathcal{L}}$ satisfying $T' \models_{TO} T_0^-$. Obviously $T' \models_{TO} T_0^-$ iff $m_0^- \prec_{TO} m_0^+$. If $TO \in \mathcal{O}$ has no (global) minimum, then $T \models_{TO} \bot$, so $\neg p_0 \in \overline{\overline{\overline{\overline{T}}} \cup T'}$ - where $\overline{\overline{\overline{T}}} := \{\phi : T \models_{TO} \phi\}$. If TO has a minimum, which is neither m_0^+ nor m_0^-, then $\overline{\overline{\overline{T}}} \cup T'$ is inconsistent, and again $\neg p_0 \in \overline{\overline{\overline{\overline{T}}} \cup T'}$. The minimum cannot be m_0^+, so in all cases $\neg p_0 \in \overline{\overline{\overline{\overline{T}}} \cup T'}$. But now every model except m_0^+ can be minimal, so in the disjoint union $\mathcal{M} := \biguplus \mathcal{O}$ of these structures, $\mu_{\mathcal{M}}(T) = M_{\mathcal{L}} - \{m_0^+\}$. Thus $\overline{\overline{\overline{T}}} = \overline{T}$ (in $\mathcal{M}$), and $\overline{\overline{\overline{\overline{T}}} \cup T'} = \overline{T'}$, but $\neg p_0 \notin \overline{T'}$. In particular, the example shows that rule (2.3) of Section 1.2 might hold in all components of a disjoint union, but fail in the union: As any total order TO is definability preserving, (2.3) holds in TO (see [Sch92]). On the other hand, $\neg p_0 \in \overline{\overline{T \cup T'}}$ (in $\mathcal{M}$), so (2.3) fails in $\mathcal{M}$. $\Box$

Remarks:

(1) Failure of definability preservation in $\mathcal{M}$ is crucial for our example. More generally, definability preserving disjoint unions preserve rule (2.3). We know this already from [Sch92], but give a direct argument to illustrate which kinds of rules of type 2 will be preserved in definability preserving disjoint unions. Let $\mathcal{X}$ be some set of strict total orders and $\mathcal{M} = \biguplus \mathcal{X}$. We have to show $M(\overline{\overline{T}} \cup T') \subseteq M(\overline{\overline{T \cup T'}})$ (in $\mathcal{M}$). If $\mathcal{M}$ is definability preserving, then $M(\overline{\overline{T}}) = \mu_{\mathcal{M}}(T)$, so $M(\overline{\overline{T}} \cup T') = M(\overline{\overline{T}}) \cap M(T') = \mu_{\mathcal{M}}(T) \cap M(T') = \bigcup\{\mu_{TO}(T) : TO \in \mathcal{X}\} \cap M(T') = \bigcup\{\mu_{TO}(T) \cap M(T') : TO \in \mathcal{X}\} \subseteq \bigcup\{\mu_{TO}(T \cup T') : TO \in \mathcal{X}\} = \mu_{\mathcal{M}}(T \cup T') = M(\overline{\overline{T \cup T'}})$. (In the inclusion, we have used property $(\mu 2')$ of the proof of Proposition 4.3, which holds in all preferential structures.) Thus $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{\overline{T}} \cup T'}}$, and as $\neg p_0 \in \overline{\overline{\overline{\overline{T}} \cup T'}}$ in our Example 3.1, the Example would not work.

(2) The general argument showing preservation of a rule in a definability preserving structure proceeds semantically as above, that it is preserved under union: $\Phi(\mu_i(X), \mu_i(Y), \ldots)$ implies $\Phi(\bigcup \mu_i(X), \bigcup \mu_i(Y), \ldots)$. The semantical argument is possible by $M(\overline{\overline{T}}) = \mu_{\mathcal{M}}(T)$.

3.1.3 EQUIVALENCE OF GENERAL PREFERENTIAL STRUCTURES WITH SETS OF TOTAL ORDERS

Ideally, one would like every preferential structure to be (or at least, to be equivalent for type 1 validity to) a disjoint union of strictly totally ordered structures. This is not the case.

Example 3.2

Consider the language defined by one variable, p. Let $m \models p$, $m' \models \neg p$, and consider the structure $< m, 0 > \succ < m', 0 > \succ < m', 1 > \succ < m', 2 > \succ \ldots$. Then $\mu(true) = \emptyset$, but $\mu(p) = \{m\}$. There are only two possible total orders: $m \prec m'$, $m' \prec m$. $m \prec m'$ gives $\mu(\emptyset) = \{m\}$, $m' \prec m$ gives $\mu(\emptyset) = \{m'\}$, $(m \prec m') \uplus (m' \prec m)$ gives $\mu(\emptyset) = \{m, m'\}$. (Omitting some models totally won't help, either.)

Thus, traditional preferential structures are more expressive than strict total orders (or their disjoint union).

In Section 4, we will construct a structure that is equivalent in the finite smooth case.

3.2 Multiple copies:

The usual constructions with multiple copies (the author's notation) or non-injective labelling functions (notation e.g. of Kraus, Lehmann, Magidor) have always been intriguing for their intuitive justification, which seemed somewhat weak. (By different languages of description and reasoning, see e.g. [Imi87]. The language of reasoning is a sub-language of the language of description, so every model in the language of reasoning is a set of models in the language of description.) We give here a purely formal one. (In [Sch96-1], we have investigated the expressive strength of structures with multiple copies in more detail.)

Fact 3.2 shows that we can construct a usual structure with multiple copies out of a set of strictly totally ordered sets of classical models (without multiple copies), preserving validity of type 1. Example 3.1 shows that validity of type 2 is usually not preserved. For its failure, we needed a structure that does not preserve definability, which exists only for infinite languages. We thus conjecture that validity of type 2 is also preserved in the case of finite languages - this is difficult to prove, as we do not know any framework which describes all usual or reasonable type 2 rules.

Thus, considering sets of strict total orders of models leads us naturally to consider their disjoint unions - at least largely equivalent structures - which are constructions with multiple copies.

4 REPRESENTATION IN THE FINITE CASE

In this Section we show our main result, Proposition 4.1, a representation theorem for the finite cumulative case. The infinite case remains open.

As done before (see e.g. [Sch97-t1]), we first show an algebraic representation result, Proposition 4.2, whose proof is the main work, and translate this result by routine methods - in Proposition 4.7 - to the logical representation problem.

It is easily seen that the consequence relations of the structures examined will be cumulative: First, it is well known (see e.g. [KLM90], or [Sch97-t1]) that smooth structures define cumulative consequence relations. Second, transitive relations over finite sets are smooth, and, third, we will see that our structures will be finite (see the modifications in the proof of Proposition 4.2).

Condition 4.1

We consider the following conditions for $\mid\sim$:

($\mid\sim$ 1) $\overline{T} = \overline{T'}$ entails $\overline{\overline{T}} = \overline{\overline{T'}}$,

($\mid\sim$ 2) $\overline{\overline{T}}$ is classically closed,

($\mid\sim$ 3) $T \subseteq \overline{\overline{T}}$,

($\mid\sim$ 4) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$

($\mid\sim$ 5) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}}$ entails $\overline{\overline{T}} = \overline{\overline{T'}}$

for all $T, T' \subseteq \mathcal{L}$.

Proposition 4.1

Let $\mathcal{L}$ be a propositional language defined by a finite set of variables.

(A) (Soundness) Let $\mathcal{O}$ be a set of strict total orders over $M_{\mathcal{L}}$, defining a logic $\mid\sim$ by $T \mid\sim \phi :\Leftrightarrow T \models_{\mathcal{O}} \phi$. Then $\mid\sim$ satisfies $(\mid\sim 1) - (\mid\sim 5)$.

(B) (Completeness) If a logic $\mid\sim$ for $\mathcal{L}$ satisfies $(\mid\sim 1) - (\mid\sim 5)$, then there is a set $\mathcal{O}$ of strict total orders over $M_{\mathcal{L}}$ such that $T \mid\sim \phi \Leftrightarrow T \models_{\mathcal{O}} \phi$.

Let us first explain why Proposition 4.1 is precisely the result to be expected. Classical logic defines exactly one consequence relation, $\vdash$. The conditions for preferential structures (system P of [KLM90], or our Conditions 4.1 below) do not describe *one* consequence relation, but a whole class, which have to obey certain principles. The representation theorem of classical logic states $T \vdash \phi$ iff in *all* models, if T holds, then so will ϕ. This unrestricted universal quantifier fixes *one* consequence relation, $\vdash$. This cannot be expected in our case. In our case, each preferential consequence relation $\mid\sim$, i.e. each relation $\mid\sim$ satisfying our conditions, will have to correspond to one particular set $\mathcal{O}_{\mid\sim}$ of total orders, in the sense that $T \mid\sim \phi$ iff in all $TO \in \mathcal{O}_{\mid\sim}$, $T \models_{TO} \phi$. The quantifier is restricted to $\mathcal{O}_{\mid\sim}$. This is the completeness part of Proposition 4.1. The soundness part shows that *any* set $\mathcal{O}$ of total orders satisfies the conditions, thus a fortiori any total order will do so. Looking back at traditional preferential structures, and e.g. the classical paper [KLM90], we see an exact correspondence with our result. There, it was shown in the soundness part that *every* preferential structure satisfies the system P. The completeness part there shows that if $\mid\sim$ satisfies P, then there is *one* preferential structure $\mathcal{M}$ such that $T \mid\sim \phi$ iff $T \models_{\mathcal{M}} \phi$. As preferential structures in the usual sense correspond to sets of total orders, we see that our result is the exact analogue of e.g. the KLM-result. To summarize, we show the exact analogue to usual preferential structures, and the closest analogue possible to classical logic.

Note that validity of type (2.3) rules is not necessarily preserved in the general infinite case, so, before we can modify a given completeness proof, we have to be sure that the construction in the proof is finite. (This is not necessarily trivial if the language is finite, as we may work with arbitrarily many copies.) Thus, our result is *not* a trivial corollary of the classical result, we have to take a look at the classical proof. For completeness' sake, we repeat here one such proof.

For the algebraic representation result, we will consider some $\mathcal{Y} \subseteq \mathcal{P}(Z)$, closed under finite unions and finite intersections, and a function $\mu : \mathcal{Y} \to \mathcal{Y}$. $\mathcal{Y}$ is intended to be $\boldsymbol{D}_{\mathcal{L}}$ for some propositional language $\mathcal{L}$. The conditions for representation are:

Condition 4.2

For a function $\mu : \mathcal{Y} \to \mathcal{Y}$, we consider the conditions:

($\mu 1$) $\mu(X) \subseteq X$,

($\mu 2$) $X \subseteq Y$ entails $\mu(Y) \cap X \subseteq \mu(X)$,

($\mu 3$) $\mu(X) \subseteq Y \subseteq X$ entails $\mu(X) = \mu(Y)$

(for all $X, Y \in \mathcal{Y}$).

Proposition 4.2

Let Z be a finite set, let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be closed under finite unions and finite intersections, and $\mu : \mathcal{Y} \to \mathcal{Y}$.

(A) (Soundness) If $\mathcal{O}$ is a set of strict total orders over Z, then $\mu_{\mathcal{O}}$ satisfies $(\mu 1) - (\mu 3)$.

(B) (Completeness) If μ satisfies $(\mu 1) - (\mu 3)$, then there is a set $\mathcal{O}$ of strict total orders over Z such that $\mu = \mu_{\mathcal{O}}$.

The proof of Proposition 4.2 will be a modification of a proof for traditional preferential structures which has appeared in [Sch97-t1]. For the reader's convenience, we will repeat the proof, and then indicate the necessary modifications. The subsequent material, until the proof of Fact 4.6 included, is taken from [Sch97-t1], the modifications are done in the proof of Proposition 4.2 itself.

Definition 4.1

Define $H(U) := \bigcup\{X : \mu(X) \subseteq U\}$.

Fact 4.3

Let A, U, U', Y and all A_i be in $\mathcal{Y}$.

($\mu 1$) and ($\mu 2$) entail:

(1) If $A = \bigcup\{A_i : i \in I\}$, then $\mu(A) \subseteq \bigcup\{\mu(A_i) : i \in I\}$,

(2) $U \subseteq H(U)$, and if $U \subseteq U'$, then $H(U) \subseteq H(U')$,

(3) $\mu(U \cup Y) - H(U) \subseteq \mu(Y)$.

($\mu 1$) - ($\mu 3$) entail:

(4) if $U \subseteq A$ and $\mu(A) \subseteq H(U)$, then $\mu(A) \subseteq U$,

(5) if $\mu(Y) \subseteq H(U)$, then $Y \subseteq H(U)$ and $\mu(U \cup Y) = \mu(U)$,

(6) if $x \in \mu(U)$ and $x \in Y - \mu(Y)$, then $Y \not\subseteq H(U)$,

(7) if $Y \not\subseteq H(U)$, then $\mu(U \cup Y) \not\subseteq H(U)$.

Proof:

(1) $\mu(A) \cap A_j \subseteq \mu(A_j) \subseteq \bigcup \mu(A_i)$, so by $\mu(A) \subseteq A = \bigcup A_i$ $\mu(A) \subseteq \bigcup \mu(A_i)$. (2) trivial. (3) $\mu(U \cup Y) - H(U) \subseteq_{(2)} \mu(U \cup Y) - U \subseteq_{(\mu 1)} \mu(U \cup Y) \cap Y \subseteq_{(\mu 2)} \mu(Y)$. (4) $\mu(A) = \bigcup\{\mu(A) \cap X : \mu(X) \subseteq U\} \subseteq_{(\mu 2')} \bigcup\{\mu(A \cap X) : \mu(X) \subseteq U\}$. But if $\mu(X) \subseteq U \subseteq A$, then by $\mu(X) \subseteq X$, $\mu(X) \subseteq A \cap X \subseteq X$ entails by $(\mu 3)$ $\mu(A \cap X) = \mu(X) \subseteq U$, so $\mu(A) \subseteq U$. (5) Let $\mu(Y) \subseteq H(U)$, then by $\mu(U) \subseteq H(U)$ and (1) $\mu(U \cup Y) \subseteq \mu(U) \cup \mu(Y) \subseteq H(U)$, so by (4) $\mu(U \cup Y) \subseteq U$ and $U \cup Y \subseteq H(U)$. Moreover, $\mu(U \cup Y) \subseteq U \subseteq U \cup Y$ entails by $(\mu 3)$ $\mu(U \cup Y) = \mu(U)$. (6) If not, $Y \subseteq H(U)$, so $\mu(Y) \subseteq H(U)$, so $\mu(U \cup Y) = \mu(U)$ by (5), but $x \in Y - \mu(Y)$ entails by $(\mu 2)$ $x \notin \mu(U \cup Y) = \mu(U)$, *contradiction*. (7) $\mu(U \cup Y) \subseteq H(U)$ entails by (5) $U \cup Y \subseteq H(U)$. □

Assume now $(\mu 1) - (\mu 3)$ to hold.

Definition 4.2

For $x \in Z$, let $\mathcal{W}_x := \{\mu(Y)\colon Y \in \mathcal{Y} \wedge x \in Y - \mu(Y)\}$, $\Gamma_x := \Pi \mathcal{W}_x$, and $K := \{x \in Z\colon \exists X \in \mathcal{Y}.x \in \mu(X)\}$.

Note: Not all $z \in Z$ have to occur in our structure, therefore it is quite possible that $X \in \mathcal{Y}$, $X \neq \emptyset$, but $\mu_{\mathcal{Z}}(X) = \emptyset$. This is why we have introduced the set K in Definition 4.2, and such X will be subsets of Z-K.

Remark 4.4

(1) $x \in K$ entails $\Gamma_x \neq \emptyset$,

(2) $g \in \Gamma_x$ entails $ran(g) \subseteq K$.

Proof:

(1) We have to show that $Y \in \mathcal{Y}$, $x \in Y - \mu(Y)$ entails $\mu(Y) \neq \emptyset$. By $x \in K$, there is $X \in \mathcal{Y}$ such that $x \in \mu(X)$. Suppose $x \in Y$, $\mu(Y) = \emptyset$. Then $x \in X \cap Y$, so by $x \in \mu(X)$ and $(\mu 2)$ $x \in \mu(X \cap Y)$. But $\mu(Y) = \emptyset \subseteq X \cap Y \subseteq Y$, so by $(\mu 3)$ $\mu(X \cap Y) = \emptyset$, *contradiction*.

(2) By definition, $\mu(Y) \subseteq K$ for all $Y \in \mathcal{Y}$. □

Proposition 4.5

Let $\mathcal{Y}$ be closed under finite unions and finite intersections, and $\mu : \mathcal{Y} \to \mathcal{Y}$. Then there is a $\mathcal{Y}$-smooth transitive preferential structure $\mathcal{Z}$, such that for all $X \in \mathcal{Y}$ $\mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu 1) - (\mu 3)$.

Proof:

(Completeness)

We suppose for simplicity that $Z = K$. The general case in easy to obtain by a technique similar to that in Section 2.3 of [Sch97-t1], but complicates the picture.

For each $x \in Z$, we construct trees t_x, which will be used to index different copies of x, and control the relation $\prec$.

These trees t_x will have the following form:

(a) the root of t is $< \emptyset, x >$ or $< U, x >$ with $U \in \mathcal{Y}$ and $x \in \mu(U)$,

(b) all other nodes are pairs $< Y, y >$, $Y \in \mathcal{Y}$, $y \in \mu(Y)$,

(c) $ht(t) \leq \omega$,

(d) if $< Y, y >$ is an element in t_x, then there is some $\mathcal{Y}(y) \subseteq \{W \in \mathcal{Y} : y \in W\}$, and $f \in \Pi\{\mu(W) : W \in \mathcal{Y}(y)\}$ such that the set of children of $< Y, y >$ is $\{< Y \cup W, f(W) >: W \in \mathcal{Y}(y)\}$.

The first coordinate is used for book-keeping when constructing children, in particular for condition (d).

The relation $\prec$ will essentially be determined by the subtree relation.

We first construct the trees t_x for those sets U where $x \in \mu(U)$, and then take care of the others. In the construction for the minimal elements, at each level $n > 0$, we may have several ways to choose a selection function f_n, and each such choice leads to the construction of a different tree - we construct all these trees.

Definition 4.3

If t is a tree with root $< a, b >$, then t/c will be the same tree, only with the root $< c, b >$.

Construction 4.1

(A) The set T_x of trees t for fixed x:

(1) Construction of the set $T\mu_x$ of trees for those sets $U \in \mathcal{Y}$, where $x \in \mu(U)$:

Let $U \in \mathcal{Y}$, $x \in \mu(U)$. The trees $t_{U,x} \in T\mu_x$ are constructed inductively, observing simultaneously:

If $< U_{n+1}, x_{n+1} >$ is a child of $< U_n, x_n >$, then (a) $x_{n+1} \in \mu(U_{n+1}) - H(U_n)$, and (b) $U_n \subseteq U_{n+1}$.

Set $U_0 := U$, $x_0 := x$.

Level 0: $< U_0, x_0 >$.

Level $n \rightarrow n+1$: Let $< U_n, x_n >$ be in level n. Suppose $Y_{n+1} \in \mathcal{Y}$, $x_n \in Y_{n+1}$, and $Y_{n+1} \not\subseteq H(U_n)$. Note that $\mu(U_n \cup Y_{n+1}) - H(U_n) \neq \emptyset$ by Fact 4.3, (7), and $\mu(U_n \cup Y_{n+1}) - H(U_n) \subseteq \mu(Y_{n+1})$ by Fact 4.3, (3). Choose $f_{n+1} \in \Pi\{\mu(U_n \cup Y_{n+1}) - H(U_n) : Y_{n+1} \in \mathcal{Y}$, $x_n \in Y_{n+1} \not\subseteq H(U_n)\}$ (for the construction of this tree, at this element), and let the set of children of $< U_n, x_n >$ be $\{< U_n \cup Y_{n+1}, f_{n+1}(Y_{n+1}) >: Y_{n+1} \in \mathcal{Y},\ x_n \in Y_{n+1} \not\subseteq H(U_n)\}$. (If there is no such Y_{n+1}, $< U_n, x_n >$ has no children.) Obviously, (a) and (b) hold.

We call such trees U, x-trees.

(2) Construction of the set T'_x of trees for the non-minimal elements. Let $x \in Z$. Construct the tree t_x as follows (here, one tree per x suffices for all U):

Level 0: $< \emptyset, x >$

Level 1: Choose arbitrary $f \in \Pi\{\mu(U) : x \in U \in \mathcal{Y}\}$. Note that $U \neq \emptyset$ entails $\mu(U) \neq \emptyset$ by $Z = K$ (by Remark 4.4, (1)). Let $\{< U, f(U) >: x \in U \in \mathcal{Y}\}$ be the set of children of $< \emptyset, x >$. This assures that the element will be non-minimal.

Level > 1: Let $< U, f(U) >$ be an element of level 1, as $f(U) \in \mu(U)$, there is a $t_{U,f(U)} \in T\mu_{f(U)}$. Graft one of these trees $t_{U,f(U)} \in T\mu_{f(U)}$ at $< U, f(U) >$ on the level 1. This assures that a minimal element will be below it to guarantee smoothness.

Finally, let $T_x := T\mu_x \cup T'_x$.

(B) The relation $\lhd$ between trees: For $x, y \in Z$, $t \in T_x$, $t' \in T_y$, set $t \rhd t'$ iff for some Y $< Y, y >$ is a child of the root $< X, x >$ in t, and t' is the subtree of t beginning at this $< Y, y >$.

(C) The structure $\mathcal{Z}$: Let $\mathcal{Z} := < \{< x, t_x >: x \in Z,\ t_x \in T_x\}, < x, t_x > \succ < y, t_y >$ iff $t_x \rhd^* t_y >$.

The rest of the proof is made up of simple observations.

Fact 4.6

(1) If $t_{U,x}$ is an U, x-tree, $< U_n, x_n >$ an element of $t_{U,x}$, $< U_m, x_m >$ a direct or indirect child of $< U_n, x_n >$, then $x_m \notin H(U_n)$.

(2) Let $< Y_n, y_n >$ be an element in $t_{U,x} \in T\mu_x$, t' the subtree starting at $< Y_n, y_n >$, then t' is a Y_n, y_n-tree.

(3) $\prec$ is free from cycles.

(4) If $t_{U,x}$ is an U, x-tree, then $< x, t_{U,x} >$ is $\prec$-minimal in $\mathcal{Z} \lceil U$.

(5) No $< x, t_x >$, $t_x \in T'_x$ is minimal in any $\mathcal{Z} \lceil U$, $U \in \mathcal{Y}$.

(6) Smoothness is respected for the elements of the form $< x, t_{U,x} >$.

(7) Smoothness is respected for the elements of the form $< x, t_x >$ with $t_x \in T'_x$.

(8) $\mu = \mu_{\mathcal{Z}}$.

Proof:

(1) Trivial by (a) and (b).

(2) Trivial by (a).

(3) Note that no $< x, t_x >$ $t_x \in T'_x$ can be smaller than any other element (smaller elements require $U \neq \emptyset$ at the root). So no cycle involves any such $< x, t_x >$. Consider now $< x, t_{U,x} >$, $t_{U,x} \in T\mu_x$. For any $< y, t_{V,y} > \prec < x, t_{U,x} >$, $y \notin H(U)$ by (1), but $x \in \mu(U) \subseteq H(U)$, so $x \neq y$.

(4) This is trivial by (1).

(5) Let $x \in U \in \mathcal{Y}$, then f as used in the construction of level 1 of t_x chooses $y \in \mu(U) \neq \emptyset$, and some $< y, t_{U,y} >$ is in $\mathcal{Z}\lceil U$ and below $< x, t_x >$.

(6) Let $x \in A \in \mathcal{Y}$, we have to show that either $< x, t_{U,x} >$ is minimal in $\mathcal{Z}\lceil A$, or that there is $< y, t_y > \prec < x, t_{U,x} >$ minimal in $\mathcal{Z}\lceil A$. Case 1, $A \subseteq H(U)$. Then $< x, t_{U,x} >$ is minimal in $\mathcal{Z}\lceil A$, again by (1). Case 2, $A \not\subseteq H(U)$. Then A is one of the Y_1 considered for level 1. So there is $< U \cup A, f_1(A) >$ in level 1 with $f_1(A) \in \mu(A) \subseteq A$ by Fact 4.3, (3). But note that by (1) all elements below $< U \cup A, f_1(A) >$ avoid $H(U \cup A)$. Let t be the subtree of $t_{U,x}$ beginning at $< U \cup A, f_1(A) >$, then by (2) t is one of the $U \cup A, f_1(A)$-trees, and $< f_1(A), t >$ is minimal in $\mathcal{Z}\lceil U \cup A$ by (4), so in $\mathcal{Z}\lceil A$, and $< f_1(A), t > \prec < x, t_{U,x} >$.

(7) Let $x \in A \in \mathcal{Y}$, $< x, t_x >$, $t_x \in T'_x$, and consider the subtree t beginning at $< A, f(A) >$, then t is one of the $A, f(A)$-trees, and $< f(A), t >$ is minimal in $\mathcal{Z}\lceil A$ by (4).

(8) Let $x \in \mu(U)$. Then any $< x, t_{U,x} >$ is $\prec$-minimal in $\mathcal{Z}\lceil U$ by (4), so $x \in \mu_{\mathcal{Z}}(U)$. Conversely, let $x \in U - \mu(U)$. By (5), no $< x, t_x >$ is minimal in U. Consider now some $< x, t_{V,x} > \in \mathcal{Z}$, so $x \in \mu(V)$. As $x \in U - \mu(U)$, $U \not\subseteq H(V)$ by Fact 4.3, (6). Thus U was considered in the construction of level 1 of $t_{V,x}$. Let t be the subtree of $t_{V,x}$ beginning at $< V \cup U, f_1(U) >$, by $\mu(V \cup U) - H(V) \subseteq \mu(U)$ (Fact 4.3, (3)), $f_1(U) \in \mu(U) \subseteq U$, and $< f_1(U), t > \prec < x, t_{V,x} >$. □ (Fact 4.6 and Proposition 4.5)

We turn now to the modifications.

Proof of Proposition 4.2:

By Fact 3.1, $\mu_{\mathcal{O}} = \mu_{\biguplus \mathcal{O}}$, so we can work with the set or its disjoint union.

(A) Soundness:

Conditions ($\mu 1$) and ($\mu 2$) hold for arbitrary preferential structures, and ($\mu 3$) holds for smooth preferential structures (see e.g. [Sch97-t1]). Strict total orders over finite sets are smooth, so are their disjoint unions.

(B) Completeness:

We will modify Construction 4.1 in the proof of Proposition 4.5. We have constructed there for a function μ satisfying $(\mu 1) - (\mu 3)$ a transitive smooth preferential structure $\mathcal{Z} =< \mathcal{X}, \prec>$ representing μ. We first show in (a) that the construction is finite for finite languages. We then eliminate in (b) unnecessary copies, and construct in (c) for each remaining $< x, i >$ a total order $TO_{<x,i>}$ such that the set of all these $TO_{<x,i>}$ represents μ.

(a) Finiteness of the construction:

First, if the language $\mathcal{L}$ is finite, the constructed structure is finite, too: As $v(\mathcal{L})$ is finite, $Z = M_{\mathcal{L}}$ is finite. For each non-minimal element $x \in Z$, there is one tree in T'_x, so this is easy. Now, for the set T_x. T_x consists of trees $t_{U,x}$ where the elements of $t_{U,x}$ are pairs $< U', x' >$ with $U' \in \mathcal{Y} \subseteq \mathcal{P}(Z)$ and $x' \in Z$, so there are finitely many such pairs. Each element in the tree has at most $card(\mathcal{P}(Z))$ successors, and by Fact 4.6, (1), if $< U_m, x_m >$ is a direct or indirect successor in the tree of $< U_n, x_n >$, then $x_m \notin H(U_n)$, but $x_n \in U_n \subseteq H(U_n)$, so $x_n \neq x_m$, so branches have length at most $card(Z)$. So there is a uniform upper bound on the size of the trees, so there are only finitely many such trees.

(b) Elimination of unnecessary copies:

Next, if, for each $x \in Z$ there is a finite number of copies, then there are "best" copies $< x, i >$ in the sense that there is no $< x, i' >\prec< x, i >$ in $\mathcal{Z}$, so we can eliminate the "not so good" copies $< x, i >$ for which there is $< x, i' >\prec< x, i >$, without changing representation. (Note that, instead of arguing with finiteness, we can argue here with smoothness, too, as singletons are definable.)

Representation is not changed, as the following easy argument shows: Let $\mathcal{Z}' =< \mathcal{X}', \prec>$ be the new structure, we have to show that $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$. Suppose $X \in \mathcal{Y}$, and $x \in \mu_{\mathcal{Z}}(X)$. Then there is $< x, i >$ minimal in $\mathcal{Z}\lceil X$. But then $< x, i >\in \mathcal{X}'$ too, and, as we have not introduced new smaller elements, $x \in \mu_{\mathcal{Z}'}(X)$. Suppose now $x \in \mu_{\mathcal{Z}'}$, then there is some $< x, i >$ minimal in $\mathcal{Z}'\lceil X$. If there were $< y, j >$ smaller than $< x, i >$ in $\mathcal{Z}$, $y \in X$, then $< y, j >$ would have been eliminated, as there is minimal $< y, k >$ below $< y, j >$, but then, by transitivity, $< y, k >$ is smaller than $< x, i >$, too, but $< y, k >$ is kept in $\mathcal{Z}'$, so $< x, i >$ would not be minimal in $\mathcal{Z}'$, either. Thus, $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$.

(c) Construction of the total orders:

We take now the modified construction $\mathcal{Z}'$ to construct a set of total orders. $< x, i >^-$ etc. will now be relative to $\mathcal{Z}'$.

We construct for each $x \in Z$ a set $\mathcal{O}_x = \{TO_{<x,i>} :< x, i > \in \mathcal{X}'\}$ of total orders. $\biguplus \mathcal{O} := \biguplus \{TO : TO \in \mathcal{O}_x, x \in Z\}$ will be the final structure, equivalent to $\mathcal{Z}$. $TO_{<x,i>}$ is constructed as follows: We first put all elements $y \in < x, i >^*$ below x, and all $y \neq x$, $y \notin < x, i >^*$ above x. We then order $< x, i >^*$ totally, staying sufficiently close to the order of $\mathcal{Z}'$, and finally do the same with the remaining elements.

Fix now $< x, i >$, and let $< := <_{TO_{<x,i>}}$ be the strict total order on Z to be constructed.

First, set $y < x$ iff $y \in < x, i >^*$, and set $x < y$ iff $y \neq x$ and $y \notin < x, i >^*$.

We construct in (α) the part of the total order below x, and then in (β) the part above x.

(α) Work now inside $< x, i >^*$, and construct a total order $<$ on $< x, i >^*$ in three steps. (1) Extend the partial order $\prec$ on $< x, i >^-$ to a total order $\lhd$. (2) If $< y, j > \lhd < y, j' >$, eliminate $< y, j' >$. By finiteness, one copy of y survives. (3) For $y, z \in < x, i >^*$, let $y < z$ iff there are $< y, j >$, $< z, k >$ with $< y, j > \lhd < z, k >$ left in step (2).

By step (2), $<$ in $< x, i >^*$ is free from cycles, and by elimination of unnecessary elements in the construction of $\mathcal{Z}'$ x does not occur in $< x, i >^*$, so the entire relation constructed so far is free from cycles.

Note that for $y \in < x, i >^*$, there is some $< y, j >$ such that $< y, j >^* \subseteq \{z : z < y\}$: Let $< y, j >$ be the $\lhd$-least copy of y, i.e. the one which survives step (2). Then by (1), all $< z, k > \in < y, j >^-$ are $\lhd$-below $< y, j >$. But if some such $< z, k >$ is eliminated in (2), there is an even smaller $< z, k' > \lhd < y, j >$ which survives, so $z < y$ in step (3).

(β) Work now on $\mathcal{X}' - (\{< x, i >\} \cup < x, i >^-)$. (1) Extend the order $\prec$ on $\mathcal{X}' - (\{< x, i >\} \cup < x, i >^-)$ to a total order $\lhd$. (2) Eliminate again $< y, j' >$, if $< y, j > \lhd < y, j' >$, but eliminate also all $< y, j' >$ such that $y = x$ or $y \in < x, i >^*$. (3) Let $y < z$ iff there are $< y, j >$, $< z, k >$ with $< y, j > \lhd < z, k >$ left in step (2).

By the same argument as above, we see that for any y $<$-above x, there is some $< y, j >$ such that $< y, j >^* \subseteq \{z : z < y\}$.

Let finally $\mathcal{O} = \{TO_{<x,i>} : x \in Z, < x, i > \in \mathcal{X}'\}$ and consider $\biguplus \mathcal{O}$. Let $< x, i > \in \mathcal{X}'$. By construction, $< x, TO_{<x,i>} >^*_{\mathcal{O}} = x^*_{TO_{<x,i>}} = < x, i >^*_{\mathcal{Z}'}$. Consider now arbitrary $TO_{<x,i>}$, and $y \in Z$. It was shown in the construction that there is $< y, j > \in \mathcal{X}'$ such that $< y, j >^*_{\mathcal{Z}'} \subseteq y^*_{TO_{<x,i>}} = < y, TO_{<x,i>} >^*_{\mathcal{O}}$. So by Fact 1.1, (2) $\mu_{\mathcal{Z}'} = \mu_{\biguplus \mathcal{O}} = \mu_{\mathcal{O}}$.

□ (Proposition 4.2)

Proposition 4.7 completes the proof of Proposition 4.1, it is taken from e.g. [Sch97-t1], its proof is repeated here for the sake of completeness.

Proposition 4.7

Consider for a logic $\mid\sim$ on the finite language $\mathcal{L}$ the properties of Conditions 4.1, and for a function $\mu : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ the properties of Conditions 4.2.

We then have:

(a) If μ satisfies $(\mu 1) - (\mu 3)$, then $\mid\sim$ defined by $T \mid\sim \phi :\Leftrightarrow \mu(M_T) \models \phi$ satisfies $(\mid\sim 1) - (\mid\sim 5)$.

(b) If $\mid\sim$ satisfies $(\mid\sim 1) - (\mid\sim 5)$, then $\mu : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ defined by $\mu(M_T) := M_{\overline{\overline{T}}}$ satisfies $(\mu 1) - (\mu 3)$ and $T \mid\sim \phi \Leftrightarrow \mu(M_T) \models \phi$.

Proof of Proposition 4.7:

Note that, as $\boldsymbol{D}_{\mathcal{L}}$ is closed under finite intersections, in the presence of $(\mu 1)$, $(\mu 2)$ is equivalent to $(\mu 2')$ $\mu(X) \cap Y \subseteq \mu(X \cap Y)$, we may work with $(\mu 2')$ in the proof. Note further that, by finiteness of $\mathcal{L}$, $M(Th(X)) = X$ for any set of models, thus $\mu(M_T) = M_{\overline{\overline{T}}}$ will hold by our prerequisite $T \mid\sim \phi \Leftrightarrow \mu(M_T) \models \phi$.

(a) Suppose $T \mid\sim \phi :\Leftrightarrow \mu(M_T) \models \phi$ for some such μ, and all T and ϕ. $(\mid\sim 1)$: If $\overline{T} = \overline{T'}$, then $M_T = M_{T'}$, so $\mu(M_T) = \mu(M_{T'})$, and $\overline{\overline{T}} = \overline{\overline{T'}}$. $(\mid\sim 2)$ and $(\mid\sim 3)$ are trivial. We show $(\mid\sim 4)$: Let now $\phi \in \overline{\overline{T \cup T'}}$, so ϕ holds in all $m \in \mu(M_{T \cup T'}) = \mu(M_T \cap M_{T'})$, so by $(\mu 2')$, ϕ holds in all $m \in \mu(M_T) \cap M_{T'} = M_{\overline{\overline{T}}} \cap M_{T'} = M_{\overline{\overline{T}} \cup T'}$, so $\overline{\overline{T}} \cup T' \models \phi$, and $\phi \in \overline{\overline{\overline{T}} \cup T'}$. We turn to $(\mid\sim 5)$: Assume $T \subseteq \overline{T'} \subseteq \overline{\overline{T}}$, so $M_{\overline{\overline{T}}} = \mu(M_T) \subseteq M_{T'} \subseteq M_T$. If $\phi \in \overline{\overline{T'}} = \overline{\overline{T \cup T'}}$, then by $(\mid\sim 4)$ $\phi \in \overline{\overline{\overline{T}} \cup T'} = \overline{(\overline{\overline{T}})} = \overline{\overline{T}}$. If $\phi \in \overline{\overline{T}}$, then ϕ holds in all $m \in \mu(M_T) = \mu(M_{T'}) = M_{\overline{\overline{T'}}}$ by $(\mu 3)$. Thus $\overline{\overline{T'}} \vdash \phi$, but then by $(\mid\sim 2)$, $\phi \in \overline{\overline{T'}}$.

(b) Let $\mid\sim$ satisfy $(\mid\sim 1) - (\mid\sim 5)$ for all T. We define μ, show that μ satisfies $(\mu 1) - (\mu 3)$ and that $T \mid\sim \phi \Leftrightarrow \mu(M_T) \models \phi$. If $X = M_T$ for some $T \subseteq \mathcal{L}$, set $\mu(X) := M_{\overline{\overline{T}}}$. If $X = M_T = M_{T'}$, then $\overline{T} = \overline{T'}$, thus $\overline{\overline{T}} = \overline{\overline{T'}}$ by $(\mid\sim 1)$, so $M_{\overline{\overline{T}}} = M_{\overline{\overline{T'}}}$, and μ is well-defined. Moreover, μ satisfies by $(\mid\sim 3)$, $\mu(X) \subseteq X$. We show $T \mid\sim \phi \Leftrightarrow \mu(M_T) \models \phi$: Let now $T \subseteq \mathcal{L}$ be given. Then $\forall m \in \mu(M_T).m \models \phi \leftrightarrow \forall m \in M_{\overline{\overline{T}}}.m \models \phi \leftrightarrow \overline{\overline{T}} \vdash \phi \leftrightarrow \phi \in \overline{\overline{T}}$ (as $\overline{\overline{T}}$ is classically closed). Next, we show that the μ defined above satisfies $(\mu 2')$. Suppose $X := M_T$, $Y := M_{T'}$. Let $m \in \mu(X) \cap Y = M_{\overline{\overline{T}}} \cap M_{T'}$, so $m \models \overline{\overline{T}} \cup T'$, and $m \models \overline{\overline{\overline{T}} \cup T'}$, so by $(\mid\sim 4)$ $m \models \overline{\overline{T \cup T'}}$. As $X \cap Y = M_T \cap M_{T'} = M_{T \cup T'}$, $\mu(X \cap Y) = M_{\overline{\overline{T \cup T'}}}$, so $m \in \mu(X \cap Y)$. It remains to show $(\mu 3)$. So let $X = M_T$, $Y = M_{T'}$, and $\mu(M_T) := M_{\overline{\overline{T}}} \subseteq M_{T'} \subseteq M_T$ entails $\overline{T} \subseteq \overline{T'} \subseteq \overline{\overline{T}} = \overline{\overline{(\overline{T})}}$, so $\overline{\overline{T}} = \overline{\overline{(\overline{T})}} = \overline{\overline{(\overline{T'})}} = \overline{\overline{T'}}$ and $\mu(M_T) = M_{\overline{\overline{T}}} = M_{\overline{\overline{T'}}} = \mu(M_{T'})$, thus $\mu(X) = \mu(Y)$. $\square$ (Proposition 4.7)

Proof of Proposition 4.1:

Let $\mathcal{O}$ be any set of strict total orders over $M_{\mathcal{L}}$. Then $\mu_{\mathcal{O}} = \mu_{\biguplus \mathcal{O}}$ satisfies $(\mu 1) - (\mu 3)$ by Proposition 4.2, so the logic defined by $T \mid\sim \phi :\Leftrightarrow \mu_{\biguplus \mathcal{O}}(M_T) \models \phi \Leftrightarrow T \models_{\biguplus \mathcal{O}} \phi$ $(\Leftrightarrow T \models_{\mathcal{O}} \phi)$ satisfies $(\mid\sim 1) - (\mid\sim 5)$ by Proposition 4.7. Conversely, given a logic $\mid\sim$ which satisfies $(\mid\sim 1) - (\mid\sim 5)$, the model choice function μ defined by $\mu(M_T) := M_{\overline{\overline{T}}}$ satisfies $(\mu 1) - (\mu 3)$ and $T \mid\sim \phi \Leftrightarrow \mu(M_T) \models \phi$ by Proposition 4.7, so by Proposition 4.2, there is a set $\mathcal{O}$ of strict total orders over $M_{\mathcal{L}}$ such that $\mu = \mu_{\mathcal{O}} = \mu_{\biguplus \mathcal{O}}$, so $T \mid\sim \phi$ iff $\mu(M_T) = \mu_{\mathcal{O}}(M_T) = \mu_{\biguplus \mathcal{O}}(M_T) \models \phi$ iff $T \models_{\mathcal{O}} \phi$, the latter by Fact 1.2. □ (Proposition 4.1)

5 CONCLUSION AND OPEN PROBLEMS

We have presented a new analysis of preferential reasoning. We have modified the notion of a model, and, consequently, the way completeness results are proved. Both modifications keep us close to the concepts and techniques of classical propositional and first order logic. We have also given a new justification of preferential structures with multiple copies (or labelling functions) by introducing disjoint unions of structures. We have shown that some, but not all, properties are preserved by going from sets of structures to their disjoint unions. Finally, we have given a representation result for the finite case.

The main open problem seems to be a characterization of the infinite case, or at least the infinite smooth case.

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