Defaults, Preorder Semantics and Circumscription

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Abstract

We examine questions related to translating defaults into circumscription. Imielinski has examined the concept of preorder semantics as an abstraction from specific systems of circumscription. We give precise definitions, characterize preorder semantics syntactically and examine the translatability of one default into preorder semantics. Finally, we give a rather bleak outlook on the translation of defaults into circumscription.

1 Introduction: Outline and Definitions

Outline: For the convenience of the reader, we repeat in the introduction some definitions of [I], and add some of our own.

Chapter 2 and 3 deal almost exclusively with finite languages of propositional calculus. A noteworthy example is lemma 2.4, which we formulate stronger than needed to help the reader finding better results than our own.

In chapter 2, we give a syntactical characterisation of preorder semantics for finite languages of propositional calculus.

In chapter 3 we examine in detail the translation of one default into preorder semantics and weak preorder semantics. The central result there is proposition 3.3.

A look at the many different situations of proposition 3.3 makes the search for translations of more than one default into preorder semantics seem quite hopeless. This is somewhat in contrast with the rather simple description in proposition 2.5. We should note, however, that our results in chapter 3 are much more detailed than those in chapter 2, and that the preorders of the positive cases there are much more natural, so we have to invest into more specialized proofs, whereas proposition 2.5 only gives the general picture. So, after all, the problem of translating several defaults into preorder semantics might not be hopeless, proposition 2.5 might give a hint where to look.

The situation changes again, when we look at translating defaults to circumscription in predicate calculus. What we feel to be a modular translation in the spirit of [I] is made precise in definition 1.9. The preorder of minimal models is a rather special one. In particular, when minimizing, we can't change the domain, which stays fixed. That gives us immediately the sad result of lemma 4.2: A normal closed default without prerequisites that can't be translated modularly into circumscription. Similar techniques give still more negative results in the subsequent lemma and corollary.

So we feel that circumscription and defaults are rather orthogonal ways of non-monotonic reasoning.

Our Lemmas 3.19 and 3.21 seem to contradict Theorem 4.4/4.5 of [I]. What is the solution? First of all, one can't really tell, because [I] does not always present precise definitions. Second, a look at the proof of Theorem 4.4 in [I] will show that the E(V(m))'s there may be formulae in the extended language. So Thm. 4.4 and 4.5 of [I] should read for clarity " .... seminormal defaults without prerequisites in the *extended* language". Third, this still leaves Lemma 3.6 c) in contradiction. As you will see, we use the "Ab" there. A look back at the proof of Thm. 4.4 in [I] will show that the "Ab" admitted in Definition 4.2 of [I] has disappeared. So "order semantics" in Thm. 4.4 in [I] should be replaced by "order semantics without any "Ab"".

Apart from the definitions in the introduction and Lemma 2.1 - 2.2, the chapters can be read independently, there is very little crossreference. The reader might find chapter 3 or 4 the easiest to begin with. We apologize for the technical character of the paper and hope that reader will easily get the spirit of constructing funny preorders.

Definition 1.1 *Let $\mathcal{L}$, $\mathcal{L}^*$, etc. be languages, most of the time of finite propositional calculus.*

Definition 1.2 *A preorder is a reflexive and transitive binary relation. In the rest of the paper, $\leq$ will always denote a preorder. x will be called minimal with respect to $\leq$, iff there is no $x' \neq x$, $x' \leq x$.*

Definition 1.3 *Let $M_{\mathcal{L}}$ be the set of models of $\mathcal{L}$. For $m \in M_{\mathcal{L}}$, $A \subseteq formulae$ of $\mathcal{L}$, $m \models A$ means that A is valid in m. $M(A) := \{m \in M_{\mathcal{L}}: m \models A\}$ In the first order case, let dom(m) be the universe, or domain of m, i.e. the underlying set of the model.*

Definition 1.4 *Let $\nu : \mathcal{P}(M_{\mathcal{L}}) \to \mathcal{P}(M_{\mathcal{L}})$, A, $\phi \in \mathcal{L}$. Then $A \models_\nu \phi: \leftrightarrow \forall m \in \nu(M(A)).m \models \phi$ $\mid\!\sim : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is called an inference rule on $\mathcal{L}$ (and written $A \mid\!\sim \phi$ for $\phi \in \mid\!\sim(A)$ most of the time). Let $\mid\!\sim$ be an inference rule on $\mathcal{L}$. Then $\nu : \mathcal{P}(M_{\mathcal{L}}) \to \mathcal{P}(M_{\mathcal{L}})$ is called a semantic representation of $\mid\!\sim$ iff for all $A, \phi \in \mathcal{L}$ $A \mid\!\sim \phi \leftrightarrow A \models_\nu \phi$.*

Definition 1.5 *Let $\leq$ be a preorder on $M_{\mathcal{L}}$. $\mu(A) := \{m \in M(A):$ m is minimal in $M(A)$, i.e. there is no $m' \neq m$, $m' \leq m$, $m' \in M(A)\}$ ($A \in \mathcal{L}$)*

If we work in different languages, we will sometimes write μ^* for μ in $M_{\mathcal{L}^*}$ etc.

$\mu(B)$ for B $\subseteq M_{\mathcal{L}}$ is defined analogously.

Let $A \subseteq formulae$ of $\mathcal{L}$, $\sigma \in$ formulae of $\mathcal{L}$ (in these cases, we will frequently abuse notation and just say A, $\sigma \in \mathcal{L}$ etc.). We define $A \models_\leq \sigma$ $:\leftrightarrow \forall m \in \mu(A).m \models \sigma$. The empty order is defined by $m \leq m' :\leftrightarrow$ m=m', and $\phi \mapsto \psi$ will denote the following (important) preorder on $M_{\mathcal{L}} : m \leq m'$ $:\leftrightarrow$ ($m \models \psi$ and $m' \models \phi$) or m=m'.

For formulae and sets of formulae A, B, σ, τ we will freely use A+B, $A \wedge B$, $A + \sigma$ etc. to denote $A \cup B$, $A \cup \{\sigma\}$ etc. The meaning will always be clear from the context.

Definition 1.6 *Let $\mathcal{L} \subseteq \mathcal{L}^*$. For $m \in M_{\mathcal{L}}$ (or $M_{\mathcal{L}^*}$) let $E(m) := \{\psi : m \models \psi$, ψ atomic or negation of atomic$\}$. For $m \in M_{\mathcal{L}^*}$ let $E^-(m) := E(m) \cap \mathcal{L}$ Let pr : $M_{\mathcal{L}^*} \rightarrow M_{\mathcal{L}}$ with pr(m') defined as that m with $E^-(m') = E(m)$, the latter formed in $\mathcal{L}$.*

Definition 1.7 *Let $W \in \mathcal{L}$, $D := \frac{A:B}{C}$, A,B,C $\in \mathcal{L}$ (we say $D \in \mathcal{L}$). Let Ext(W,D) be the (unique, if it exists) extension of the default theory (W,D) (see [R], Def.1). Furthermore, let $(W, D) \vdash_D \sigma$ iff $\sigma \in$ Ext(W,D). Let Con(A,B) mean: $A \wedge B$ is consistent. Let $\perp$ stand for anything false like $\phi \wedge \neg\phi$.*

Definition 1.8 *Let D be as above. We say*
1) $D \in MO^*_{Ab}$ iff there is an extension $\mathcal{L}^*$ of $\mathcal{L}$, and $< \mathcal{L}^*$, Ab, $\leq>$ is a modular translation of D into preorder semantics (POS), i.e. a) $Ab \subseteq \mathcal{L}^*$ b) $\leq$ a preorder on $M_{\mathcal{L}^*}$ c) $\forall$ $W, \sigma \in \mathcal{L}$ ($(W, D) \vdash_D \sigma \longleftrightarrow W \cup Ab \models_{\leq} \sigma$)
2) MO^* means MO^*_{Ab}, Ab not necessary MO_{Ab} means MO^*_{Ab}, $\mathcal{L}^* = \mathcal{L}$ suffices MO means MO_{Ab}, Ab not necessary

If we have to be very clear, we say MO^*, MO^+ for MO in $\mathcal{L}^*$, in $\mathcal{L}^+$ etc.

We will use MO etc. freely as sets, or predicates, whatever seems more natural.

Definition 1.9 *Let $\mathcal{L}$ be a first order language, for simplicity with predicates only, $\Pi \subseteq \mathcal{L}$, $P \in \mathcal{L}$, $\mathcal{L} \subseteq \mathcal{L}^*$. (In the sense on circumscription, Π will be the set of "floating" predicates which we vary, P will be minimized.)*
Then $\leq := \leq_{\mathcal{L}^*,\Pi,P}$, a preorder on $M_{\mathcal{L}^*}$ will be defined as follows: $m \leq m'$ iff

1) dom(m) = dom(m') 2) $[Q]_m = [Q]_{m'}$ *for* $Q \notin \Pi$, $Q \neq P$ *3)* $[P]_m \subseteq [P]_{m'}$. *Let* $S \subseteq M_{\mathcal{L}^*}$, *call* $m \in S$ *minimal in* S ($m \in \mu(S)$) *with respect to* $\leq := \leq_{\mathcal{L}^*,\Pi,P}$ *iff 1)* $m \in S$ *2) there is no* $m' \in S$, $m' \leq m$, *but not* $m \leq m'$, *i.e. there is no* $m' \in S$ *s.t. i) dom(m) = dom(m') ii)* $[Q]_m = [Q]_{m'}$ *for* $Q \notin \Pi$, $Q \neq P$ *iii)* $[P]_{m'} \subset [P]_m$. *See [L] for motivation.*

Definition 1.10 *Let* $\mathcal{L}$ *be a first order language, D a default in* $\mathcal{L}$. *We say that D has a modular translation into circumscriptive preorder semantics (D* $\in MCO^*_{Ab}$*) iff there is* $\mathcal{L}^* := \mathcal{L}^*_D \supseteq \mathcal{L}$, *a formula* $Ab := Ab_D \in \mathcal{L}^*$, *a set of predicates* $\Pi := \Pi_D \subseteq \mathcal{L}$, *a predicate* $P := P_D \in \mathcal{L}$, *such that for all* $W, \phi \in \mathcal{L}$ $W + Ab \models_{\leq_{\mathcal{L}^*,\Pi,P}} \phi$ *iff* $(W, D) \vdash_D \phi$.

2 Preorder Semantics

Unless otherwise stated, we will work in this chapter with finite languages of propositional calculus.

The following Lemmas 2.1 and 2.2 are very simple, but central for the further development.

Lemma 2.1 $W \subseteq W' \rightarrow \mu(W) \cap M(W') \subseteq \mu(W')$ *for* W, $W' \subseteq \mathcal{L}$.

Proof Let m $\in \mu(W) \cap M(W')$. Suppose there is $m' < m$, $m' \in M(W') \subseteq M(W)$, by $W \subseteq W'$, thus $m \notin \mu(W)$. *Contrad.* □

Lemma 2.2 $A + \phi \models_{\leq} \psi \rightarrow A \models_{\leq} \phi \rightarrow \psi$ *for all formulae* A, ϕ, ψ.

Proof Suppose there is $m \in \mu(A)$, $m \models \phi \wedge \neg\psi$. Thus $m \in M(A \wedge \phi)$, by Lemma 2.1 $m \in \mu(A \wedge \phi)$, but $m \models \neg\psi$ *Contrad.* □

Lemma 2.3 *Let* $\mathcal{L}$ *be a finite language of propositional calculus, and* $\mid\!\sim$ *an inference rule on* $\mathcal{L}$. *Then* $\mid\!\sim$ *satisfies for all* $A, \phi, \psi \in \mathcal{L}$ *a)* $A \vdash \phi \rightarrow A \mid\!\sim \phi$ *b)* $A + \phi \mid\!\sim \psi \rightarrow A \mid\!\sim \phi \rightarrow \psi$ *c)* $Th(\mid\!\sim(A)) = \mid\!\sim(A)$ *d)* $\mid\!\sim(A) = \mid\!\sim(Th(A))$

iff

there is a semantic representation $\nu : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ *of* $\mid\!\sim$ *such that a')* $\nu(W) \subseteq W$ *b')* $W \subseteq W' \rightarrow \nu(W') \cap W \subseteq \nu(W)$ *for all* $W, W' \subseteq M_{\mathcal{L}}$ *.*

Remark: The important properties are b) and b') .

Proof: " $\leftarrow$ " Let ν be a representation of $\mid\!\sim$. a) Let $A \vdash \phi$, we have to show $A \mid\!\sim \phi$. By a'), $\nu(M(A)) \subseteq M(A)$. As $A \vdash \phi$, $m \models \phi$ for all $m \in \nu(M(A))$, so $A \models_\nu \phi$. By representation, $A \mid\!\sim \phi$. The argument works for inconsistent A, too. b) By representation, it suffices to assume $A + \phi \models_\nu \psi$, and show $A \models_\nu \phi \rightarrow \psi$. Let $W := M(A \wedge \phi)$, W':=M(A) and $m \in \nu(W')$. Suppose $m \not\models \phi \rightarrow \psi$, so $m \models \phi \wedge \neg\psi$. By $m \in \nu(W') \subseteq W'$ and $m \models \phi$, $m \in W$. By b') $m \in \nu(W)$, so by $A + \phi \models_\nu \psi$, $m \models \psi$ *Contrad.* c) Let B $\subseteq$ $\mid\!\sim(A)$, $B \vdash \psi$, we have to show $A \mid\!\sim \psi$. By representation again, $A \models_\nu B$. As $B \vdash \psi$, $A \models_\nu \psi$, so $A \mid\!\sim \psi$. d) This is immediate from M(A) = M(Th(A)).
" $\rightarrow$ " Define ν as follows: Let $X \subseteq M_{\mathcal{L}}$, $A_X := \{\phi\colon \forall m \in X.m \models \phi\}$ $B_X := \{\psi\colon A_X \mid\!\sim \psi\} = \mid\!\sim(A_X)$ $\nu(X) := M(B_X)$. We have to show that ν is a representation of $\mid\!\sim$, and a') and b'). First, $A \mid\!\sim \psi \leftrightarrow A \models_\nu \psi$: We use $A_{M(A)} = Th(A)$ and d) to obtain $\nu(M(A)) := M(B_{M(A)}) := M(\mid\!\sim(A_{M(A)}))$ $= M(\mid\!\sim(Th(A))) = M(\mid\!\sim(A))$ So $A \models_\nu \psi :\leftrightarrow \forall m \in \nu(M(A)).m \models \psi \leftrightarrow$ $\forall m \in M(\mid\!\sim(A)).m \models \psi \leftrightarrow (\mid\!\sim(A)) \vdash \psi \leftrightarrow A \mid\!\sim \psi$, the last equivalence by c). a') $\nu(X) \subseteq X$: Suppose $m \in \nu(X)$, so $m \models B_X$. By a) $A_X \subseteq B_X$ and $m \models A_X$. So $m \in X$. Remark: We use here that $X \vdash\!\rightarrow A_X$ is injective for finite propositional calculus. b') $X \subseteq X' \rightarrow \nu(X') \cap X \subseteq \nu(X)$: Let $m \in \nu(X') \cap X$, we have to show $m \in \nu(X)$, or $m \models B_X$. Let $\psi \in B_X$, so $A_X \mid\!\sim \psi$. As $X \subseteq X'$, $A_{X'} \subseteq A_X$, let $A_X = A_{X'} + G$. As $m \in X$, $m \models A_X = A_{X'} + G$, so $m \models G$. As $A_X = A_{X'} + G \mid\!\sim \psi$, by b), $A_{X'} \mid\!\sim G \rightarrow \psi$ (We use here, that G can be made one single formula). Finally, as $m \in \nu(X')$, by definition of ν, $m \models G \rightarrow \psi$, and by $m \models G$, we have $m \models \psi$. $\Box$

Lemma 2.4 *Let* $Y \subseteq \mathcal{P}(A)$, $f : Y \rightarrow Y$ *such that for all* $W, W' \in Y$ *a) f(W)* $\subseteq W$ *b)* $W \subseteq W' \rightarrow f(W') \cap W \subseteq f(W)$*. Then there is I and a preorder* $\leq$ *on AxI such that f(W) =* $pr[\mu(pr^{-1}[W])]$ *(see definition 1.4 for* μ *and* $pr : AxI \rightarrow A$ *with* $pr(< x, i >) := x$ *). In other words, f can be represented* *by a suitable preorder on an extension of A. Remark: We do not need A to be finite here !*

Proof: We give two proofs, since both are constructive and might be interesting to the reader. Both are shortenings of the author's original proof, a construction of trees. The proofs use "extreme" preorders, but help to illustrate the general frame of preorder semantics.

Proof 1: Fix a well-ordering of A. Fix $m \in A$ for the moment. Let $< X_\beta : \beta < \alpha_m >$ be an enumeration of $\{X \in Y : m \in X - f(X)\}$ and consider $F_m := \Pi\{X_\beta : \beta < \alpha_m\} = \{$g: $g : \alpha_m \to \bigcup\{X_\beta : \beta < \alpha_m\}$ and $\forall \beta < \alpha_m . g(\beta) \in X_\beta\}$

(If $\alpha_m = \emptyset$, take $F_m := \{\emptyset\}$. The following claim 1 will then be trivial, and we can proceed as in the other cases.)

Claim 1: Let $B \in Y, m \in f(B)$. Then there is $g_B \in F_m$ with $ran(g_B) \cap B = \emptyset$. Proof: Consider $\beta < \alpha_m$. As $m \in f(B)$ and $m \in X_\beta - f(X_\beta)$, $X_\beta - B \neq \emptyset$ by b). Using the well-ordering of A, let $g_B(\beta)$ be the least element $x \in X_\beta - B$.

Let $I := \{< m, g, i >: m \in A, g \in F_m, i = 0, 1\}$ and define $\leq_{AxI}$ as follows: $x_{m,g,i} \leq x'_{m',g',i'}$ iff

m=m' $\wedge$ g=g' and

a) x=x' $\wedge$ $x \neq m$ or b) x'=m $\wedge$ $x \in ran(g)$ or c) x=x' $\wedge$ i=i'

Remarks: 1) Condition m=m' $\wedge$ g=g' makes the construction "local", we only have to consider two "layers" of A at a time. 2) Condition a) makes every $x \neq m$ non-minimal in $A_{m,g,0} \cup A_{m,g,1}$. 3) Condition b) makes m the "center of a star" in $A_{m,g,i}$: any element of ran(g) prevents m to be minimal.

Claim 2: f(B) = $pr[\mu(pr^{-1}[B])]$ for $B \in Y$. Proof: " $\subseteq$ " Let $m \in f(B)$. By claim 1, there is $g_B \in F_m$ s.th. $ran(g_B) \cap B = \emptyset$. By construction, $m_{m,g_B,i}$ is minimal in $pr^{-1}[B]$: Suppose $x'_{m',g',i'} \leq m_{m,g_B,i}$ and $x' \in B$, then m=m' and $g' = g_B$. x'=m and $x' \neq m' = m$ can't be, so a) is impossible. $x' \in ran(g') = ran(g_B)$ and $x' \in B$ can't be either, as $ran(g_B) \cap B = \emptyset$, so b) is impossible too. " $\supseteq$ " Let $m \notin f(B)$. By a) $f(B) \subseteq B$ and by definition $pr[\mu(pr^{-1}[B])] \subseteq B$. So, if $m \notin B$, $m \notin pr[\mu(pr^{-1}[B])]$. Suppose now $m \in B - f(B)$, so $B = X_\beta$ for some $\beta < \alpha_m$. By definition of F_m, $g \in F_m \to g(\beta) \in B = X_\beta$, so $ran(g) \cap B \neq \emptyset$ for all $g \in F_m$. Consider $m_{m',g,i}$. If $m \neq m'$, $m_{m',g,1-i} < m_{m',g,i}$. If m=m', take x $\in ran(g) \cap B$, then

$x_{m',g,1-i} < m_{m',g,i}$. So, in both cases, $m_{m',g,i}$ is not minimal in $pr^{-1}[B]$, and $m \notin pr[\mu(pr^{-1}[B])]$.

Proof 2: (Due to R. Keller) Let $S := \mathcal{P}(A)$, I:=Sx{0,1}. Define $< m, X, l > \leq < m', X', l' > :\leftrightarrow$ X=X' and
a) $m \notin X$ or b) $m' \in X - f(X)$ or c) m=m',l=l' for $m, m' \in A$, $X, X' \subseteq A$, l,l'=0,1.
This is transitive, thus a preorder. We show $m \in f(W) \leftrightarrow m \in W \wedge \exists i \in I. < m, i >$ minimal in WxI. " $\rightarrow$ " Suppose $m \in f(W) \subseteq W$, let $i :=< W, 0 >$. Then $< m', W', l > \leq < m, W, 0 >$ implies W=W' and ($m' \notin W$ or m=m',l=l') by definition of $\leq$. So $< m, W, 0 >$ is minimal in WxI. " $\leftarrow$ " Suppose $m \in W$ and $< m, W', l >$ is minimal in WxI, i.e. $\forall m' \in W \ \forall l' = 0, 1 \ (\neg < m', W', l' > \leq < m, W', l >$ or $m = m' \wedge l = l')$. Thus, for $l \neq l' \ \forall m' \in W \ \neg < m', W', l' > \leq < m, W', l >$, and, by definition of $\leq$, $m \in W \wedge \forall m' \in W (m' \in W') \wedge m \notin (W' - f(W'))$, hence $m \in W \wedge W \subseteq W' \wedge (m \in f(W') \vee m \notin W')$, but $m \notin W'$ can't be, as $m \in W \subseteq W'$. By b) then $m \in f(W)$. □

Proposition 2.5 *Let $\mathcal{L}$ be a finite language of propositional calculus, and $\mid\!\sim$ an inference rule on $\mathcal{L}$. Then $\mid\!\sim \in MO^*$ (in an obvious generalization of Definition 1.7) iff $\mid\!\sim$ satisfies for all $A, \phi, \psi \in \mathcal{L}$ a) $A \vdash \phi \rightarrow A \mid\!\sim \phi$ b) $A + \phi \mid\!\sim \psi \rightarrow A \mid\!\sim \phi \rightarrow \psi$ c) $Th(\mid\!\sim(A)) = \mid\!\sim(A)$ d) $\mid\!\sim(A) = \mid\!\sim(Th(A))$*

Proof " $\rightarrow$ " Use the ideas of Lemma 2.1 and 2.2. " $\leftarrow$ " Let $\nu : \mathcal{P}(M_\mathcal{L}) \rightarrow \mathcal{P}(M_\mathcal{L})$ be a representation of $\mid\!\sim$ by lemma 2.3. Extend $\mathcal{L}$ to $\mathcal{L}^*$ so we can define $\leq$ on $M_{\mathcal{L}^*}$ by lemma 2.4 with $\nu(W) = pr[\mu(pr^{-1}[W])]$ (duplicate layers if necessary). We then have $A \mid\!\sim \psi \leftrightarrow A \models_\nu \psi \longleftrightarrow \forall m(m \in \nu(M(A)) \rightarrow m \models \psi) \leftrightarrow \forall m(m \in \mu^*(A) \rightarrow m \models \psi)$, the latter as $\mu^*(A) = \mu(pr^{-1}[M(A)])$, and pr(m),m coincide on $\mathcal{L}$. □

3 Defaults and Preorder Semantics

We first summarize results on extensions for one default. They will be used in the sequel without further mentioning. The proofs are all direct from the definition in [R] and trivial.

Lemma 3.1 *Consider* $(W, D = \frac{A:B}{C})$.

The extensions of (W,D) can be described as follows:
Case I: W $\vdash$ C: Th(W) is (the only) extension
Case II.1: W $\not\vdash$ C, W $\vdash$ A: The following little table gives the answer:

	$W + C \vdash \neg B$	$W + C \not\vdash \neg B$
$W \vdash \neg B$	Th(W)	impossible
Con(W,B)	No Extension	Th(W+C)

Case II.2: W $\not\vdash$ C, W $\not\vdash$ A: Th(W) is (the only) extension □
Remark: If $\vdash A$, i.e. D = $\frac{:B}{C}$, we will always be in Case I or II.1 .

Proposition 3.2 $MO \subset MO_{Ab} \subset MO^* = MO^*_{Ab}$

Proof: The inequalities will be shown implicitly, by solving the different cases of proposition 3.3. The inclusions are either trivial or follow from Lemma 3.23. □

Proposition 3.3 *The modular translatability of one default of a finite propositional language into preorder semantics is completely described by the following diagram of cases:*

General Default, $D := \frac{A:B}{C}$.

$\neg Con(A, B)$*:*
MO (3.6a)
empty order
Con(A,B):

$Con(\neg A)$*:*

$Con(A, \neg C)$*:*
$\neg MO^*$ *(3.9a)*
$\vdash A \rightarrow C$*:*
MO (3.6b)

empty order

$\vdash A$:

$\vdash B \rightarrow C$:

MO (3.13)

$\neg C \mapsto B$

$\not\vdash B \rightarrow C$:

¬MO (3.8)

$\vdash C \rightarrow B$:

MO (3.19)*

$\neg C \mapsto B$

$\vdash B$:

MO_{Ab} (3.6c)

$\not\vdash B$:

$\neg MO_{Ab}$ *(3.18)*

$\not\vdash C \rightarrow B$:

Con(B,C):

¬MO (3.9b)*

$\neg Con(B, C)$:

MO (3.24)*

$\neg MO_{Ab}$ *(3.18)*

For most positive cases, we have indicated the translating preorder. The numbers show the proving lemmas.

Remark: On our way, we will meet a default translatable into preorder semantics and weak preorder semantics, which is *not* equivalent to any set of seminormal defaults in the same language (Lemma 3.19 and 3.21).

We turn to the proofs:

Lemma 3.4 *If* $\forall W\ (W, D) \vdash_D \sigma \leftrightarrow W \vdash \sigma$, *then* D *is translated by the empty order into POS.*

Proof: $W \models_{\leq} \sigma \leftrightarrow W \models \sigma \leftrightarrow W \vdash \sigma$, since $\mu(W) = M(W)$. □

Lemma 3.5 *If* $\forall W \; (W, D) \vdash_D \sigma \leftrightarrow W + C \vdash \sigma$, *then D is translated by the empty order and Ab:=C into POS.*

Proof $W + C \models_{\leq} \sigma \leftrightarrow W + C \models \sigma \leftrightarrow W + C \vdash \sigma$, as $\mu(W + C) = M(W + C)$. □

We can settle now 3 cases:

Lemma 3.6 *a)* $\neg Con(A, B)$ *implies* $D \in$ *MO, b) Con(A,B),* $Con(\neg A)$, $\vdash A \rightarrow C$ *implies* $D \in$ *MO, c) Con(A,B),* $\vdash A$, $Con(B, \neg C)$, $\vdash B$ *implies* $D \in MO_{Ab}$

Proof a) The default can never fire, use Lemma 3.4. b) The extension will always be trivial, Lemma 3.4 again. c) The extension will always be Th(W+C), use Lemma 3.5. □

Lemma 3.7 *If* $D \in$ *MO, then (W,D) has an extension for all W.*

Proof Suppose (W,D) has no extension, so W is consistent, and Con(W,B). Consider $m \models W \wedge B$, $m \in M_{\mathcal{L}}$, W':=E(m). M(W')={m}, so $\mu(W') = \{m\} \neq \emptyset$, thus $W' \not\models_{\leq} \bot$, but Con(W'), Con(W',B) by $m \models W' \wedge B$, $W' \vdash A$, $W' + C \vdash \neg B$, so (W',D) has no extension and (W',D) $\vdash_D \bot$ *Contrad.* □

Lemma 3.8 *If* $\vdash A$, $\not\vdash B \rightarrow C$, *then* $D \notin$ *MO.*

Proof $Con(B, \neg C)$, so let $W := B \wedge \neg C$, then (W,D) has no extension. Finish by Lemma 3.7. □

Lemma 3.9 *a) If Con(A,B),* $Con(\neg A)$, $Con(A, \neg C)$ *then* $D := \frac{A:B}{C} \notin MO^*_{Ab}$ *(Imielinski). b) If* $\vdash A$, $Con(B, \neg C)$, $Con(C, \neg B)$, *Con(B,C) then* $D := \frac{A:B}{C} \notin MO^*_{Ab}$

Proof (I owe this simplification of my original proof to R. Keller.) Let $D := \frac{A:B}{C}$. Suppose we have a translation $(W,D) \vdash_D \sigma \leftrightarrow$ W+Ab $\models_{\leq} \sigma$. By Lemma 2.2, $W + \phi \models_{\leq} \psi \rightarrow W \models_{\leq} \phi \rightarrow \psi$. So $(W+\phi,D) \vdash_D \psi \rightarrow (W,D) \vdash_D \phi \rightarrow \psi$. 1) Assume $Con(\neg A)$, Con(A,B,C), $Con(A,\neg C)$. Let $W := \emptyset$, $\phi := A$, $\psi := C$. So $(A,D) \vdash_D \sigma \leftrightarrow A + C \vdash \sigma$ by Con(A,B,C), so $(A,D) \vdash_D C$. $(\emptyset,D) \vdash_D \sigma \leftrightarrow \vdash \sigma$ by $Con(\neg A)$. But $\not\vdash A \rightarrow C$, as $Con(A,\neg C)$. 2) Assume $Con(\neg A)$, $Con(A,B,\neg C)$. Let $W := \emptyset$, $\phi := A \wedge (\neg C \vee \neg B)$, $\psi := \bot$. As $\phi \wedge B \leftrightarrow A \wedge B \wedge \neg C$, $Con(\phi \wedge B)$. But $\neg Con(\phi \wedge B \wedge C)$, so (ϕ,D) has no extension and $(\phi,D) \vdash_D \psi$. $(\emptyset,D) \vdash_D \sigma \leftrightarrow \vdash \sigma$, as $Con(\neg A)$. Suppose $\vdash \phi \rightarrow \psi$. Then $\vdash \neg\phi$. But $Con(\phi)$ by $Con(A,\neg C)$. 3) Assume $\vdash A$, $Con(B,\neg C)$, $Con(C,\neg B)$, Con(B,C). Let W:=A, $\phi := \neg A \vee \neg B \vee \neg C$, $\psi := \bot$. Again, $(W+\phi,D)$ has no extension, $(W+\phi,D) \vdash_D \psi$. $(A,D) \vdash_D \sigma \leftrightarrow (\emptyset,D) \vdash_D \sigma \longleftrightarrow \vdash \sigma$ by $Con(\neg B)$. Again, $Con(\phi)$. b) follows now immediately from 3) a) Suppose Con(A,B,C), then 1) solves this case If $\neg Con(A,B,C)$, then $A \wedge B \vdash \neg C$, so, as Con(A,B), $Con(A,B,\neg C)$. □

The remaining cases require more work, they all have a translation into preorder semantics.

We start by examining more closely the preorder $\neg C \mapsto B$, introduced by Imielinski. So this is the assumed preorder until further notice.

Lemma 3.10 *a)* $\neg Con(W,B) \rightarrow \mu(W) = M(W)$*, b)* $\neg Con(W,B)$ *implies* $W \models_{\leq} \sigma \leftrightarrow (W, \frac{:B}{C}) \vdash_D \sigma$*, c) M(W+C)* $\subseteq \mu(W)$.

Proof a) is trivial, b) $W \models_{\leq} \sigma \leftrightarrow \mu(W) \models \sigma \longleftrightarrow M(W) \models \sigma \leftrightarrow W \vdash \sigma$ by a) But $\neg Con(W,B) \rightarrow Ext(W, \frac{:B}{C}) = Th(W)$. So $W \models_{\leq} \sigma \leftrightarrow W \vdash \sigma \leftrightarrow (W, \frac{:B}{C}) \vdash_D \sigma$, c) is trivial again. □

Lemma 3.11 *Assume Con(W,B). Then*

a) $\exists m' \neq m" \models W \wedge B$ *or* $\neg Con(W,B,\neg C) \longleftrightarrow \mu(W) = M(W+C) \rightarrow (\mu(W) = \emptyset \leftrightarrow W \vdash \neg C)$

b) $\exists_1 m' \models W \wedge B$ *and* $Con(W,B,\neg C) \rightarrow \mu(W) = M(W+C) \cup \{m' : m' \models W \wedge B\} \rightarrow (\mu(W) = \{m' : m' \models W \wedge B\} \leftrightarrow W \vdash \neg B)$

Proof In both cases, the last implication is trivial.

a) " $\to$ ": " $\supseteq$ " by lemma 3.10 c) " $\subseteq$ " Let $m \in \mu(W)$, $m \models \neg C$. Assume $m' \neq m$" $\models W \wedge B$. By definition of $\neg C \mapsto B$, m', m" $\leq$ m. Since $m' \neq m$", $m \neq m'$ or $m \neq m$", so $m' < m$ or m" $< m$ *Contrad.* Assume $\neg Con(W, B, \neg C)$. By Con(W,B), let $m' \models W \wedge B$, so $m' \leq m$. By $\neg Con(W, B, \neg C)$, $m \neq m'$ *Contrad.*
" $\leftarrow$ ": Assume $\exists_1 m' \models W \wedge B$, $Con(W, B, \neg C)$. Thus $m' \models W \wedge B \wedge \neg C$. By assumption, there is no other $m \in M(W \wedge B)$, so m' must be minimal in M(W), but $m' \models \neg C$.
b) " $\supseteq$ " By lemma 3.10 c) and uniqueness of $m' \models W \wedge B$. " $\subseteq$ " Let $m \in \mu(W)$, assume $m \models \neg B$. By minimality of m, we must have M(W+B)={m}, as m' is the only such, m=m'. □

Remark:

By augmenting $\mathcal{L}$ to $\mathcal{L}^*$, we can always force the existence of $m' \neq m$" $\models W \wedge B$. This technique will be often used in the sequel.

We assume now $\vdash A$ and, furthermore, $\exists m' \neq m$" $\models W \wedge B$ or $\neg Con(W, B, \neg C)$, piece together our above results and compare with lemma 3.1.
Let $D := \frac{A:B}{C} = \frac{:B}{C}$.
Case 1: $\neg Con(W, B)$. Then $W \models_{\leq} \sigma \leftrightarrow (W, D) \vdash_D \sigma$ by lemma 3.10 b)
Case 2: Con(W,B). Then $W \models_{\leq} \sigma \leftrightarrow$ W+C $\vdash \sigma$ by lemma 3.11 a)
Case 2.1 $W \vdash C$. Then $(W, D) \vdash_D \sigma \leftrightarrow W \vdash \sigma \leftrightarrow W + C \vdash \sigma$
Case 2.2 $W \not\vdash C$, $W + C \not\vdash \neg B$. Then $(W, D) \vdash_D \sigma \leftrightarrow W + C \vdash \sigma$
Case 2.3 $W \not\vdash C$, $W + C \vdash \neg B$. No extension, $(W, D) \vdash_D \sigma \leftrightarrow \bot \vdash \sigma$. So, the only thing that can go wrong is Case 2.3, thus:

Lemma 3.12 *Assume* $\vdash A$. *Assume further that Con(W,B) implies* ($\exists m' \neq m$" $\models W \wedge B$ *or* $\neg Con(W, B, \neg C)$). *Then* $\neg C \mapsto B$ *is a translation of* $D = \frac{A:B}{C}$, *iff* $\forall\, W \in \mathcal{L}$ *(Con(W,B),* $W \not\vdash C$, $W + C \vdash \neg B \to W \vdash \neg C$) □

Lemma 3.13 $\vdash A$, $\vdash B \to C$ *implies* $D \in MO$ *by* $\neg C \mapsto B$. *(Imielinski)*

Proof $\neg Con(W, B, \neg C)$ by $\vdash B \to C$. Con(W,B) and $W + C \vdash \neg B$ together can't be, for the same reason. □

The condition in Lemma 3.12 is too nasty, let's improve it.

Lemma 3.14 $\forall\, W \in \mathcal{L}$ *(Con(W,B),* $W \not\vdash C$, $W + C \vdash \neg B \to W \vdash \neg C$) *iff* $\vdash B \to C$ *or* $\vdash C \to B$.

Proof a) Con(W,B), $W \not\vdash C$, $W + C \vdash \neg B \to W \vdash \neg C$ iff Con(W,B), $W + C \vdash \neg B \to W \vdash \neg C$ iff Con(W,B) $\to$ ($W + C \vdash \neg B \to W \vdash \neg C$) iff Con(W,B) $\to$ (Con(W,C) $\to$ $Con(W, B \wedge C)$) iff Con(W,B) $\wedge$ Con(W,C) $\to$ $Con(W, B \wedge C)$
b) $\forall$ $W \in \mathcal{L}$ (Con(W,B) $\wedge$ Con(W,C) $\to$ $Con(W, B \wedge C)$) iff $\vdash B \to C$ or $\vdash C \to B$ Proof: " $\leftarrow$ " is trivial. " $\to$ ": Let $W := \neg(B \wedge C)$ Then Con(W,B) iff $\not\vdash B \to C$, Con(W,C) iff $\not\vdash C \to B$ (this is simple arithmetic) and $\neg Con(W, B \wedge C)$. Thus $\vdash B \to C$ or $\vdash C \to B$. □

This, together with lemma 3.12 proves:

Lemma 3.15 *Assume* $\vdash A$ *and Con(W,B) implies* ($\exists m' \neq m" \models W \wedge B$ *or* $\neg Con(W, B, \neg C)$) *Then* $\neg C \mapsto B$ *is a translation of* $D = \frac{A:B}{C}$, *iff* $\vdash B \to C$ *or* $\vdash C \to B$. □

We now establish a limit on what we can do in MO, if we add Ab.

Lemma 3.16 *Let* $\vdash A$, $Con(B, \neg C)$, $D := \frac{A:B}{C} \in MO_{Ab}$. *Then* $\vdash Ab \leftrightarrow C$.

Proof Let $(W, D) \vdash_D \sigma \leftrightarrow W + Ab \models_{\leq} \sigma$. Choose $W_0 := \emptyset$. So $\emptyset + Ab \models_{\leq} \sigma \leftrightarrow (\emptyset, \frac{:B}{C}) \vdash_D \sigma \leftrightarrow C \models \sigma$, as Con(B). This is true for $\neg Con(C)$ too. Thus M(C) $= \mu(Ab) \subseteq$ M(Ab), and $\vdash C \to Ab$. Suppose $Con(Ab \wedge \neg C)$. Choose $W_1 := \neg C$. So $\neg C + Ab \models_{\leq} \sigma \leftrightarrow (\neg C, \frac{A:B}{C}) \vdash_D \sigma \leftrightarrow \bot \vdash \sigma$, as $Con(B, \neg C)$. Choose $W_2 := \neg C \wedge Ab$. So $(\neg C \wedge Ab, \frac{A:B}{C}) \vdash_D \sigma \leftrightarrow \neg C \wedge Ab \wedge Ab \models_{\leq} \sigma \leftrightarrow \neg C + Ab \models_{\leq} \sigma \leftrightarrow \bot \vdash \sigma$ by W_1. So $Con(\neg C \wedge Ab \wedge B)$, otherwise $(\neg C \wedge Ab, D)$ would have an extension. Take $m' \models \neg C \wedge Ab \wedge B$, consider $W_3 := E(m')$, so $M(E(m')) = \mu(E(m')) = \{m'\}$. As we work in $M_{\mathcal{L}}$, E(m') is a candidate under consideration and we obtain $E(m') = E(m') + Ab \models_{\leq} \sigma \leftrightarrow m' \models \sigma$. But, as $m' \models \neg C \wedge B$, $(E(m'), \frac{A:B}{C}) \vdash_D \sigma \leftrightarrow \bot \vdash \sigma$ *Contrad*. Thus $\vdash Ab \to C$, and we are finished. □

Lemma 3.17 $\vdash A$, $Con(B, \neg C)$, $Con(\neg B)$, $D := \frac{A:B}{C} \in MO_{Ab}$ *implies* $\vdash \neg B \to C$.

Proof By lemma 3.16, $\vdash Ab \leftrightarrow C$. Let $W := \neg B$. $\neg B \wedge C \models_{\leq} \sigma \leftrightarrow (\neg B, D) \vdash_D \sigma \leftrightarrow \neg B \vdash \sigma$. So $M(\neg B) = \mu(\neg B \wedge C) \subseteq M(\neg B \wedge C)$ and $\vdash \neg B \rightarrow C$. □

Lemma 3.18 *Let* $\vdash A$, $Con(B, \neg C)$, $Con(\neg B)$. *Then* $D := \frac{A:B}{C} \notin MO_{Ab}$.

Proof Suppose $D \in MO_{Ab}$. Let $W_0 := C \rightarrow \neg B$. 1) $Con(C \rightarrow \neg B, B)$, as $B \wedge (C \rightarrow \neg B) \leftrightarrow B \wedge \neg C$, but $Con(B, \neg C)$. 2) $Con(C \rightarrow \neg B, \neg C)$, as $\neg C \wedge (C \rightarrow \neg B) \longleftrightarrow \neg B$, and $Con(\neg B)$. Furthermore, $C \wedge (C \rightarrow \neg B) \vdash \neg B$, so (W_0, D) has no extension. Now $W_0 + Ab = (C \rightarrow \neg B) \wedge C \leftrightarrow C \wedge \neg B$ (by lemma 3.16, Ab=C) As $W_0 + Ab \models_{\leq} \sigma \leftrightarrow (W_0, D) \vdash_D \sigma$, $\mu(C \wedge \neg B) = \emptyset$ Let $W_1 := C \wedge \neg B$. So $C \wedge \neg B = (C \wedge \neg B) \wedge C \models_{\leq} \sigma \leftrightarrow (W_1, D) \vdash_D \sigma \leftrightarrow C \wedge \neg B \vdash \sigma$ and $M(C \wedge \neg B) = \mu(C \wedge \neg B) = \emptyset$, by the above, so $\neg Con(C \wedge \neg B)$. But by lemma 3.17, $\vdash \neg B \rightarrow C$, and $Con(\neg B)$, so $Con(\neg B \wedge C)$ *Contrad.* □

We now leave the simple world of $\mathcal{L}$ and enlarge $\mathcal{L}$ to $\mathcal{L}^*$. The important fact will be that we have several "layers" of $M_{\mathcal{L}}$ in $M_{\mathcal{L}^*}$: Let $\phi \in \mathcal{L}^* - \mathcal{L}$, we will then have for each $m \in M_{\mathcal{L}}$ $m_1, m_2 \in M_{\mathcal{L}^*}$, each coinciding with m on $\mathcal{L}$, but $m_1 \models \phi$, $m_2 \models \neg\phi$.

Lemma 3.19 *Let* $\vdash A$, $\not\vdash B \rightarrow C$, $\vdash C \rightarrow B$. *Then* $D := \frac{A:B}{C}$, *is translated by* $\neg C \mapsto B$ *into* $M_{\mathcal{L}^*}$.

Proof The extension of $\mathcal{L}$ to $\mathcal{L}^*$ ensures for Con(W,B) the existence of $m' \neq m" \models W \wedge B$, we can apply lemma 3.15. □

We next show that the default of lemma 3.19 is *not* equivalent to any set of seminormal defaults in the same language.

Lemma 3.20 *Let Th(W) be maximal consistent,* Δ *a set of seminormal defaults. Then Th(W) is an (and the only) extension of (W,D).*

Proof It is enough to show that Th(W) is a candidate in the definition of $\Gamma(Th(W))$ (see definition 1 in [R]). Condition D1 and D2 are trivial. Condition D3: Let $\frac{A_i:B_i^1,\ldots,B_i^n}{C_i} \in \Delta$, $A_i \in Th(W)$, $\neg B_i^j \notin Th(W)$, so $Con(Th(W), B_i^j)$, and $W \vdash B_i^j$ by maximality. By seminormality, $\vdash B_i^j \rightarrow C_i$. So $W \vdash C_i$ and $C_i \in Th(W)$. □

Lemma 3.21 *Let* $\vdash A$, $\not\vdash B \rightarrow C$, $\vdash C \rightarrow B$. *Then* $D := \frac{A:B}{C}$, *is not equivalent to any set of seminormal defaults in the same language.*

Proof $Con(B \wedge \neg C)$. Extend $B \wedge \neg C$ to a maximally consistent set of formulae, W. So Con(W,B), $W \not\vdash C$, $W + C \vdash \neg B$, as W+C is inconsistent. So (W,D) has no extension and $(W, D) \vdash_D \perp$. If Δ were an equivalent set of seminormal defaults, $(W, D) \vdash_D \sigma \leftrightarrow (W, \Delta) \vdash_D \sigma \leftrightarrow W \vdash \sigma$ by lemma 3.20. But W is consistent $Contrad.$ $\Box$

There is just one positive case left, and, so far, we have only used the trivial order or $\neg C \mapsto B$. The last case will be solved differently. We will stop here for a moment and prove the remaining inclusions of proposition 3.2.

Lemma 3.22 *Let* $pr : M_{\mathcal{L}^+} \rightarrow M_{\mathcal{L}^*}$ *and* $\leq^*$, $\leq^+$ *be preorders on* $M_{\mathcal{L}^*}$, $M_{\mathcal{L}^+}$. *Then* $\mu^*(A) = pr[\mu^+(B)]$ *implies* $A \models_{\leq^*} \sigma \leftrightarrow B \models_{\leq^+} \sigma$ *for* A, $\sigma \in \mathcal{L}^*$, $B \in \mathcal{L}^+$.

Proof " $\rightarrow$ ": Assume $A \models_{\leq^*} \sigma$, let $m \in \mu^+(B)$, we have to show $m \models \sigma$. $pr(m) \in \mu^*(A)$, so $pr(m) \models \sigma$, but m, pr(m) agree on $\mathcal{L}^*$. " $\leftarrow$ ": Assume $B \models_{\leq^+} \sigma$, let $m \in \mu^*(A)$. Thus, there is $m' \in \mu^+(B)$, pr(m')=m. Thus $m' \models \sigma$, and again $m = pr(m') \models \sigma$. $\Box$

Lemma 3.23 *Let* $\mathcal{L}^*$ *be any language,* $D \in MO^{\mathcal{L}^*}_{Ab}$. *Then there is an extension* $\mathcal{L}^+$ *of* $\mathcal{L}^*$ *and a preorder such that* $D \in MO^{\mathcal{L}^+}$. *Thus,* $MO_{Ab} \subseteq MO^*$, *and* $MO^*_{Ab} = MO^*$.

Proof Let $W + Ab \models_{\leq^*} \sigma \leftrightarrow (W, D) \vdash_D \sigma$, $Ab \in \mathcal{L}^*$. Extend $\mathcal{L}^*$ to $\mathcal{L}^+$ by one additional formula. Split $M_{\mathcal{L}^+}$ into two layers of $M_{\mathcal{L}^*}$, let pr: $M_{\mathcal{L}^+} \rightarrow M_{\mathcal{L}^*}$ as usual.
Define $\leq^+$ as follows: $m_1 \leq^+ m_2$ iff 1) m_1, m_2 in the same layer and $pr(m_1) \leq^* pr(m_2)$ and $m_1 \models Ab$ or 2) $pr(m_1) = pr(m_2)$ and $m_i \models \neg Ab$ or 3) m_1, m_2 are in different layers, $m_2 \models \neg Ab$, $m_1 \models Ab$, $pr(m_1) \leq^* pr(m_2)$ or 4) $m_1 = m_2$.

Explanation: Condition 1 erases all models non-minimal because of some element in $M(\neg Ab)$. Condition 2 makes all models of $\neg Ab$ non-minimal Condition 3 is technical, it ensures transitivity.

Transitivity is now easily checked by examining all possible cases and left as a (trivial) exercise. So our conditions fully describe the preorder. We have to show $W + Ab \models_{\leq^*} \sigma \leftrightarrow W \models_{\leq^+} \sigma$ for all $W, \sigma \in \mathcal{L}^*$. By lemma 3.22, it suffices to show $\mu^*(W + Ab) = pr[\mu^+(W)]$.

Proof: " $\supseteq$ " Let $m \in \mu^+(W) \rightarrow m \not\models \neg Ab$, $pr(m) \in M^*(W + Ab)$. Suppose there is $m' <^* pr(m)$, $m' \models W + Ab$. Let m" be in the same layer as m, pr(m")=m'. As $m" \models Ab$, and $m" \neq m$, $m" <^+ m$ *Contrad.* " $\subseteq$ " Let $m \in \mu^*(W + Ab)$, pr(m')=m. So $m' \models W \wedge Ab$. Suppose there is $m" \in M^+(W)$, $m" \leq^+ m'$. If $m" \leq^+ m'$ by condition 1, then $pr(m") \leq^* m$, and $pr(m") \models W + Ab$. By minimality of m, we have m=pr(m") and m'=m". Condition 2 and 3 can't apply, since $m' \models Ab$. So m"=m', and $m' \in \mu^+(W)$. □

There is one positive case left to show. Feeling comfortable now with layers, we will do it next, and will be finished.

Lemma 3.24 *Let* $\vdash A$, $\neg Con(B, C)$, $Con(B, \neg C)$, $Con(C, \neg B)$. *Then* $D := \frac{A:B}{C} \in MO^*$.

Proof Extend $\mathcal{L}$ to $\mathcal{L}^*$, so $M_{\mathcal{L}^*}$ has 4 layers of $M_{\mathcal{L}}$. Divide $M_{\mathcal{L}^*}$ into two layers with 2 sublayers each.
Define a preorder $\leq$ on $M_{\mathcal{L}^*}$ as follows: $m_1 \leq m_2$ iff 1) $m_i \in$ layer 1 and $m_1 \models B$ or 2) $m_i \in$ layer 2 and $m_2 \models \neg C$ or 3) $m_1 = m_2$.
Transitivity is trivial, so this is indeed a preorder.
If $W \vdash \neg B$, then Ext(W,D)=Th(W), so $(W, D) \vdash_D \sigma \longleftrightarrow W \vdash \sigma$. If Con(W,B), $W + C \vdash \neg B$, so (W,D) has no extension, so $(W, D) \vdash_D \sigma \leftrightarrow \bot \vdash \sigma$. Thus, we have to show: a) $W \vdash \neg B$ implies $pr[\mu^*(W)] = M_{\mathcal{L}}(W)$. b) Con(W,B) implies $\mu^*(W) = \emptyset$.

On a): " $\subseteq$ " is trivial. " $\supseteq$ ": Let $m \in M_{\mathcal{L}^*}$, $m \models W$. Let m' $\in$ layer 1, pr(m')=m. By $W \vdash \neg B$, $m' \models W \wedge \neg B$. So m' is minimal in layer 1, and globally, so $m' \in \mu^*(W)$ and m $\in pr[\mu^*(W)]$. On b): As Con(W,B) and $\vdash B \rightarrow \neg C$, $Con(W, \neg C)$. Assume there is $m \in \mu^*(W)$. If m $\in$ layer 1, by Con(W,B), there is $m' \models W \wedge B$, m' $\in$ layer 1. So $m' \leq^* m$. But, remember, layer 1 consists of 2 sublayers, so there is always 2 of m', and m is not minimal. *Contrad.* If m $\in$ layer 2, argue just the same way, using $Con(W, \neg C)$. So, $\mu^*(W) = \emptyset$. $\Box$

4 Defaults and Circumscription

In this chapter, we work in predicate calculus.

Lemma 4.1 *Let dom(m) be finite,* $m \in S \subseteq M_{\mathcal{L}}$. *Then* $\exists m' \in \mu(S).m' \leq m$ *(the order as in definition 1.8).*

Proof: If not, construct a strictly descending chain of models $m = m_0 > m_1 > \ldots$ in S. But $[P]_{m_i} \subseteq$ dom(m), so that's impossible. $\Box$

Lemma 4.2 *Let* $A := \exists x, x'.x \neq x'$, $D := \frac{:A}{A}$, *then* $D \notin MCO^*_{Ab}$.

Proof: Suppose D $\in MCO^*_{Ab}$, fix appropriate $\mathcal{L}^*$, Π, P. Let $W := \emptyset$. As Con(W,A), $(W, A) \vdash_D \sigma \leftrightarrow A \vdash \sigma$ Let $W' := \exists_1 x.x = x$. As $\neg Con(W', A)$, $(W', A) \vdash_D \sigma \leftrightarrow W' \vdash \sigma$. Now, $W + Ab = Ab \models_{\leq} \sigma \leftrightarrow A \vdash \sigma$, so $Ab \models_{\leq} A$. Thus, in all minimal models of Ab, A is valid. Claim: A is valid in all models of Ab. Proof: Suppose not, let $m \in M(Ab)$, $m \models \neg A$, so dom(m)={x}. By lemma 4.1, there is $m' \leq m$, $m' \in \mu(Ab)$, so dom(m')={x}, (remember: the domain can't change) and by $Ab \models_{\leq} A$, we have $m' \models A$ *Contrad.* So $Ab \models A$. As $W' = \exists_1 x.x = x$ and $Ab \models A$, there is no model of W'+Ab, thus $\perp \vdash \sigma \leftrightarrow W' + Ab \models_{\leq} \sigma \longleftrightarrow W' \vdash \sigma$, but Con(W') *Contrad.* $\Box$

The idea of the next lemma is, of course, the same as in lemma 4.2. We fix the extension of a predicate so it will be rigid under circumscription.

Lemma 4.3 *Let P,P' be unary predicates,* $A := \exists x, x'(x \neq x' \wedge Px \wedge Px')$, $D := \frac{:A}{A}$, $D \in MCO^*_{Ab}$ *with* $\mathcal{L}^*, \Pi, P_m$ *then* $P' \in \Pi$ *or* $P' = P_m$, *the minimized predicate. I.e., to obtain a modular translation, we have to vary every other unary predicate !*

Proof: Suppose not. Let $W := \forall x(Px \leftrightarrow P'x) \wedge \exists x, x'\forall y(y = x \vee y = x')$ $\wedge\, \exists x Px$. As Con(W,A), $(W, A) \vdash_D \sigma \leftrightarrow W{+}A \vdash \sigma$. Let $W' := \forall x(Px \leftrightarrow P'x)$ $\wedge\, \exists x, x'\forall y(y = x \vee y = x') \wedge \exists_1 x Px$. As $\neg Con(W', A)$, $(W', A) \vdash_D \sigma \leftrightarrow$ $W' \vdash \sigma$. Thus, $W + Ab \models_{\leq} A$. Claim: A is valid in all models of W+Ab. Proof: Suppose there is $m \in M(W + Ab)$, $m \models \neg A$, so $[P]_m = \{x\}$. As $m \models W$, $[P']_m = \{x\}$. Again by lemma 4.1 (and the second condition in W) , there is $m' \leq m$, $m' \in \mu(W + Ab)$. As $P' \notin \Pi$, and $P' \neq P_m$, $[P']_{m'} = \{x\}$. As $m' \models W$, $[P]_{m'} = \{x\}$ and $m' \models \neg A$ *Contrad.* So $W + Ab \models A$. As $\vdash W' \rightarrow W$ and $W + Ab \models A$, $W' + Ab \models A$, so there is no model of W'+Ab $(\neg Con(W', A))$. Thus $\perp \vdash \sigma \leftrightarrow W' + Ab \models_{\leq} \sigma \leftrightarrow W' \vdash \sigma$, but Con(W') *Contrad.* □

Corollary 4.4 *Let A and D be as in lemma 4.3. Suppose we can do pairing, e.g. we have enough set theory. Then we can forget the arity of the predicates and have to let every predicate P' vary in order to obtain a modular translation !*

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