Results on infinite extensions

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Abstract

M.Freund, D.Lehmann, and D.Makinson have introduced and examined a natural extension of finitary (non-monotonic) inference relations to the infinite case, which may be roughly described as an approximation from below. We answer two questions left open there, discuss related problems, and present examples and techniques for constructing non-monotonic inference operations.

Contents

1 Introduction

(+++ Orig.: 0. Introduction +++)

LABEL: Section 0. Introduction

In a joint paper M.Freund, D.Lehmann and D.Makinson [FLM] have examined the problem of extending non-monotonic inference relations that are defined for the finite case (i.e. for finite sets of premises) to the infinite case (i.e. for infinite sets of premises).

We present here some results related to this problem. The first shows that the extension does not preserve cautious monotony. This was formulated as a question in the original version of [FLM] as presented in a workshop at the Gesellschaft fuer Mathematik und Datenverarbeitung, Bonn, West Germany. The version as published cites our result, though without proof. The second result shows that two versions of distributivity are equivalent. This is of interest to the FLM problem, for as reported in [FLM], distributivity plus cautious monotony is strong enough to carry cautious monotony through to the extension. The third result compares two ways of applying the FLM construction to the uncountably infinite case. The fourth result cautions against one kind of weakening of the basic construction. Roughly, the weakened approach corresponds to convergent partial sequences, the original one to totally converging sequences. It is not surprising that the former can give funny logics. The fifth presents another technique for constructing still quite well-behaved non-monotonic logics.

In a way, this paper is rather more a collection of methods than of results, and might, hopefully, as such be useful to workers on abstract inference relations.

All uses of set theory are standard, and can be found in any such book.

Throughout, we work in propositional languages, $\vdash$ will denote the classical consequence relation, $\mid\!\sim$ any other, $\overline{A}$, $\overline{\overline{A}}$, $\overbrace{A}$, $C(A)$ the closures of some set A of formulas under an inference relation - where $\overline{A}$ is reserved for classical closure. Thus, $\overline{A} = \{\phi \in \mathcal{L} : A \vdash \phi\}$. (This notation has the convenience of conciseness, and is in line with other areas of mathematics.) We shall sometimes abbreviate $A \cup \{x\}$ by $A + x$.

We now formulate the basic problem and approach of [FLM]. Suppose we are given an infinite language and an inference relation $=$ on finite subsets of $\mathcal{L}$, is there a natural way to extend $=$ to some $C(.)$ which is defined on all subsets of $\mathcal{L}$? For monotonic logics, this is trivial : $\phi \in C(A)$ iff there is $B \subseteq A$ finite s.th. $\phi \in \overline{\overline{B}}$. For non-monotonic logics, this is clearly not sufficient, as B might overlook some negative (blocking) information contained in A. Let now $\mathcal{P}'(A)$ be the set of finite subsets of A. In the author's opinion, a very natural candidate for extension is that of [FLM]: $\phi \in Cl(A)$ iff there is $B \in \mathcal{P}'(A)$ s.th. for all $B' \in \mathcal{P}'(\overline{A})$ $B \subseteq B'$ $\rightarrow \phi \in \overline{\overline{B'}}$. (Why $\overline{A}$ instead of A is a minor problem and shortly discussed in [FLM].) In a sense, ϕ is then a limit of $\mathcal{P}'(A)$. Our argument against considering just one $B \in \mathcal{P}'(A)$ collapses, because we look at all the information, though only in small chunks.

As in [FLM], we shall suppose that the finitary logic $\underset{\cdot}{=}$ satisfies supraclassicality (i.e. $\overline{A} \subseteq \overline{\overline{A}}$) and (finitary) cumulativity (i.e. $x \in \overline{\overline{A}}$ $\rightarrow \overline{\overline{A + x}} = \overline{\overline{A}}$). So, our first task in constructing examples will always be to establish these two properties.

A short remark concerning the general techniques used in sections 1, 4, and 5: We strengthen classical logic by some new inferences $p \mid\!\sim q$. To get well-behaved systems, we do a mixed iteration of $\mid\!\sim$ and classical closure. As we admit only finitely many prerequisites in all examples, it suffices to iterate to ω. (If we were to use, say countably many prerequisites, we would do a mixed iteration to ω_1 , and get analogous systems, using regularity of ω_1.) So far, we are still monotonic. Next, we select a suitable subset of the thus constructed A_ω, and go non-monotonic.

2 Cautious monotony does not extend

(+++ Orig.: 1. Cautious monotony does not extend +++)

LABEL: Section 1. Cautious monotony does not extend

Idea:

(+++ Orig.: Idea: +++)

LABEL: Section Idea:

The idea is, to have positive and negative information, and to glue them strongly together, s.th. classical inference cannot separate them, only the basic entailment relation $\mid\!\sim$ can: p_i will be positive information for s, r_i negative in the sense that r_i prevents all $p_j : j < i$ to be usable for inferring s.

Construction of $\mid\!\sim$ $(or \underset{\cdot}{=})$: Let $\mathcal{L}$ be a (propositional) language, consisting of the propositional variables $p_i : i \in \omega$, $r_i : i \in \omega$, s.

We define the closure under the rules $p_i \mid\!\sim r_i$, $p_i \mid\!\sim s$:

$\overbrace{A} := A \cup \{r_i : p_i \in A\} \cup \{s : p_i \in A\}$, and strengthen classical logic by closing under $\mid\!\sim$ and $\vdash$. We thus define inductively

$A_0 := A$,

$A_{2i+1} := \overline{A_{2i}}$,

$A_{2i+2} := \overbrace{A_{2i+1}}$,

$A_\omega := \bigcup\{A_i : i < \omega\}$.

So far, we have a monotonic system. We now select (non-monotonically) a suitable subset of A_ω:

Set $I^A := \{i : r_i \in A_\omega\}$, $J^A := \{i : p_i \in A_\omega\}$.

Define $I^A \leq J^A$ iff $\exists j \in J^A \forall i \in I^A . i \leq j$.

Set $\overline{\overline{A}} := A_\omega$ iff $I^A \leq J^A$, and $:= \overline{A}$ otherwise.

We have to show that $\overline{\overline{.}}$ is supraclassical and finitary cumulative, i.e.:

1) $\overline{A} \subseteq \overline{\overline{A}}$,

2) $x \in \overline{\overline{A}} \rightarrow \overline{\overline{A+x}} = \overline{\overline{A}}$, where $A + x := A \cup \{x\}$.

1) is trivial.

2) First two simple facts:

1. $x \in A_\omega \rightarrow (A+x)_\omega = A_\omega$. Proof: $A_\omega \subseteq (A+x)_\omega$ by monotonicity of $\overline{.}$, $\overbrace{\;.\;}$. Let $\phi \in (A+x)_i$, $x \in A_j$, then $\phi \in A_{i+j} \subseteq A_\omega$.

2. $x \in A_\omega \rightarrow I^A = I^{A+x}$, $J^A = J^{A+x}$ (trivial by 1.).

Assume now $x \in \overline{\overline{A}} \subseteq A_\omega$. If $I^A \leq J^A$, then $I^{A+x} \leq J^{A+x}$ and $\overline{\overline{A}} = A_\omega = (A+x)_\omega = \overline{\overline{A+x}}$.

Otherwise, $I^A \not\leq J^A$ and $I^{A+x} \not\leq J^{A+x}$. Thus $\overline{\overline{A}} = \overline{A}$ and $x \in \overline{A}$, so $\overline{A} = \overline{A+x} = \overline{\overline{A+x}}$, we are done, and $\overline{\overline{.}}$ is as desired. $\square$

The Extension $Cl(.)$ (as in [FLM]):

(+++ Orig.: The Extension Cl(.) (as in [FLM]): +++)

LABEL: Section The Extension Cl(.) (as in [FLM]):

Define now $x \in Cl(A)$ iff ex. finite $A_x \subseteq A$ s.th. for all finite A', $A_x \subseteq A' \subseteq \overline{A}$ $x \in \overline{\overline{A'}}$ holds.

To give a counterexample as desired, we show that there is x, A, B s.th. $x \in Cl(A)$, $A \subseteq B \subseteq Cl(A)$, $x \notin Cl(B)$.

Set A $:= \{p_i : i \in \omega\}$.

We show simultaneously $s \in Cl(A)$ and $r_i \in Cl(A)$ for all i. Let $A_{r_i} := \{p_i\}$.

Assume now $A_{r_i} \subseteq A' \subseteq \overline{A}$ finite. As $p_i \in A'$, r_i and $s \in A'_2 \subseteq A'_\omega$, moreover $J^{A'} \neq \emptyset$. As A' is finite, the set $K^{A'} := \{i \in \omega : p_i$ or r_i occurrs in some $\phi \in A'\}$ is finite, too. Obviously, $I^{A'}$, $J^{A'} \subseteq K^{A'}$, so $I^{A'}$ is finite. But, as $A' \subseteq \overline{A}$, $r_i \in A'_\omega$ can only be derived from $p_i \in A'_\omega$, so $I^{A'} \leq J^{A'}$. Thus, $\overline{\overline{A'}} = A'_\omega$ and s, $r_i \in \overline{\overline{A'}}$.

Next, we show that there is B, $A \subseteq B \subseteq Cl(A)$, $s \notin Cl(B)$.

Set $B := \{p_i : i \in \omega\} \cup \{r_i : i \in \omega\}$. Thus $A \subseteq B \subseteq Cl(A)$. Assume there is $B_s \subseteq B$ finite s.th. for all finite B' $B_s \subseteq B' \subseteq \overline{B}$, $s \in \overline{\overline{B'}}$. Let K^{B_s} be the finite set of all p_i occurring in B_s, and let $k := max(K^{B_s})$. Consider $B' := B_s \cup \{r_{k+1}\}$. Then $p_j \notin B'_\omega$ for $j \geq k+1$, but $r_{k+1} \in B'_\omega$, so $I^{B'} \not\leq J^{B'}$ and $\overline{\overline{B'}} = \overline{B'}$, and $s \notin \overline{\overline{B'}}$. $\square$

3 Weak distributivity entails partial distributivity

(+++ Orig.: 2. Weak distributivity entails partial distributivity +++)

LABEL: Section 2. Weak distributivity entails partial distributivity

Assume in the sequel supraclassicality (S) and cumulativity (C) for $\overline{\overline{.}}$ ($\mid\!\sim$ resp.).

Moreover, assume weak distributivity (W), i.e. $x \mid\!\sim y \rightarrow x \vee y \mid\!\sim y$, we show partial distributivity, i.e. $x \mid\!\sim y$ and $x' \vdash y \rightarrow x \vee x' \mid\!\sim y$. As the latter is known to entail full distributivity, i.e. $x \mid\!\sim y$, $x' \mid\!\sim y \rightarrow x \vee x' \mid\!\sim y$, we can close the circle, and all are equivalent.

Fact 1: $\phi \vdash \psi$ and $\psi \vdash \phi \Rightarrow \overline{\overline{\phi}} = \overline{\overline{\psi}}$.

Proof: $\{\phi, \psi\} \subseteq \overline{\overline{\phi}} \cap \overline{\overline{\psi}}$ by S, so $\overline{\overline{\phi}} = \overline{\overline{\{\phi, \psi\}}} = \overline{\overline{\psi}}$ by C.

Fact 2: $\mid\!\sim$ concatenates with $\vdash$ on the right.

Proof: Assume $\phi \mid\!\sim \psi \vdash \sigma$. By C, $\overline{\overline{\phi}} = \overline{\overline{\{\phi, \psi\}}}$, by S, $\sigma \in \overline{\overline{\{\phi, \psi\}}}$, so $\phi \mid\!\sim \sigma$.

Fact 3: $\phi \mid\!\sim \psi$, $\phi \vdash \psi' \Rightarrow \phi \mid\!\sim \psi \wedge \psi'$.

Proof: $\overline{\overline{\phi}} = \overline{\overline{\{\phi, \psi, \psi'\}}}$, $\psi \wedge \psi' \in \overline{\overline{\{\phi, \psi, \psi'\}}}$ by $C + S$.

Assume now $a \mid\!\sim b$, $c \vdash b$.

By $a \vdash a \vee c$ and Fact 3, $a \mid\!\sim b \wedge (a \vee c)$, so by W $a \vee (b \wedge (a \vee c)) \mid\!\sim b \wedge (a \vee c)$. But, $a \vee (b \wedge (a \vee c)) \leftrightarrow (a \vee b) \wedge (a \vee c) \leftrightarrow a \vee c$. By Fact 1, $a \vee c \mid\!\sim b \wedge (a \vee c)$, and by Fact 2, $a \vee c \mid\!\sim b$. $\square$

4 On different infinite extensions of n

(+++ Orig.: 3. On different infinite extensions of n +++)

LABEL: Section 3. On different infinite extensions of n

Suppose $\mathcal{L}$ is uncountable. Instead of one big step, approximating by finite sets, one might consider a smoother procedure: inductive approximation through the cardinals. Are both approaches the same? This section gives a partial answer: We prove that if both approaches coincide for all $\alpha < \kappa$, and κ is a regular cardinal, then they coincide for κ too. The author does not know whether the induction carries through singular cardinals.

Assume the axiom of choice throughout. Let $\mid . \mid$ denote the cardinality function, $\bullet$ ordinal multiplication, and $f\lceil x$ the restriction of the function f to a subset x of its domain. κ is any infinite cardinal.

Definition of $\mid\!\sim_\kappa$ (by induction on cardinals): Let $\mid\!\sim$ be a finitary entailment relation for some language $\mathcal{L}$. Assume $\mid\!\sim_\lambda$ to be defined for all infinite cardinals $\lambda < \kappa$. Then define

$A \mid\!\sim_\kappa \phi$ iff there is $A_0 \subseteq A$, $\mid A_0 \mid < \kappa$, and for all B s.th. $\mid B \mid = \lambda < \kappa$ and $A_0 \subseteq B \subseteq \overline{A}$, $B \mid\!\sim \phi$ iff $\lambda < \omega$ and $B \mid\!\sim_\lambda \phi$ iff $\lambda \geq \omega$.

(Thus, $\mid\!\sim_\omega$ is the canonical extension of [FLM].)

Lemma 4.1

(+++ Orig. No.: Lemma 1: +++)

LABEL: Lemma 1:

$A \mid\!\sim_\omega \phi \rightarrow A \mid\!\sim_\kappa \phi$

Proof:

(+++ Orig.: Proof: +++)

LABEL: Section Proof:

By induction on infinite cardinals.

For $\kappa = \omega$, this is the definition. Assume it holds for $\lambda < \kappa$. Let $A_0 \subseteq A$ be given by the def. of $\mid\!\sim_\omega$, and $A_0 \subseteq B \subseteq \overline{A}$, $\mid B \mid = \lambda < \kappa$. If $\lambda < \omega$, $B \mid\!\sim \phi$ by $A \mid\!\sim_\omega \phi$. Otherwise, obviously $B \mid\!\sim_\omega \phi$ (A_0 works again), and by induction hypothesis, $B \mid\!\sim_\lambda \phi$. $\Box$

Lemma 4.2

(+++ Orig. No.: Lemma 2: +++)

LABEL: Lemma 2:

Assume $\mid A \mid = \kappa$, κ regular, and for all $\mu < \kappa$ and $A' \subseteq \overline{A}$ $A' \mid\!\sim_\mu \phi \rightarrow A' \mid\!\sim_\omega \phi$.

Then $A \mid\!\sim_\kappa \phi \rightarrow A \mid\!\sim_\omega \phi$.

Proof:

(+++ Orig.: Proof: +++)

LABEL: Section Proof:

To simplify notation and save us another bijection, assume wlog. $\overline{A} = \kappa$.

By assumption, there is $A_0 \subseteq A$, $\mid A_0 \mid < \kappa$, and for all B, $\mid B \mid = \lambda < \kappa$, $A_0 \subseteq B \subseteq \overline{A}$ $B \mid\!\sim_\lambda \phi$ holds. By induction hypothesis, $B \mid\!\sim_\omega \phi$. By regularity of κ, there is $\alpha < \kappa$ s.th. $A_0 \subseteq \alpha$. If κ is a limit cardinal, set $K := \{\kappa' < \kappa$: κ' is a *cardinal*$\}$, if $\kappa = \kappa'^+$, set $K := \{\kappa' \bullet \beta$: $\beta < \kappa\}$. In both cases, $K \subseteq \kappa$ is closed unbounded. Moreover, there is a bijection $g : \kappa \leftrightarrow \kappa^{<\omega}$ (where $X^{<\omega} := \{y \subseteq X$: $\mid y \mid < \omega\}$), s.th. for all $\kappa' \in K$, $g\lceil\kappa' : \kappa' \leftrightarrow \kappa'^{<\omega}$ is a bijection. (To see this, proceed inductively:

If κ' is a limit point of K, let $g\lceil\kappa' := \bigcup\{g\lceil\kappa'' : \kappa'' < \kappa'\}$.

If κ' is the successor of κ'' in K, use $\mid \kappa' - \kappa'' \mid = \mid \kappa'^{<\omega} \mid$.)

Let now $\Lambda := K - \alpha$, so $\Lambda \subseteq \kappa$ is closed unbounded, and for $\lambda \in \Lambda$, $\lambda \mid\!\sim_\omega \phi$.

Thus, for each $\lambda \in \Lambda$, there is $a_\lambda \subseteq \lambda$ finite and for all $a_\lambda \subseteq b \subseteq \overline{\lambda}$ finite, $b \mid\!\sim \phi$. Define $h(\lambda) :=$ such an a_λ (using AC). Thus, for $\lambda \in \Lambda$, $f(\lambda) := g^{-1}(h(\lambda)) \in \lambda$, so f is regressiv, and defined on a stationary subset of κ. Using Fodor's (Pressing Down) Theorem by regularity of κ, we see that there is $X \subseteq \Lambda$, $X \subseteq \kappa$ stationary, s.th. $f\lceil X$ is constant. Let $f(\lambda) = \alpha$ for $\lambda \in X$. So, as $h = g \circ f$, there is $X \subseteq \kappa$ unbounded, s.th. $h\lceil X$ is constant, say $= a$. But now, we are finished: Let $a \subseteq b \subseteq \kappa$ finite. As X is unbounded, there is $\lambda \in X$, $b \subseteq \lambda$, and $b \mid\!\sim \phi$, by definition of $a_\lambda = a$. $\Box$

5 Extension by unbounded subsets

(+++ Orig.: 4. Extension by unbounded subsets +++)

LABEL: Section 4. Extension by unbounded subsets

One might consider the definition of [FLM] as too cautious: We may think of arguments and say: "I will be prepared to accept ϕ iff *I* can win any argument against ϕ.", formally, $\phi \in Cl(A)$ iff there is some $U_\phi \subseteq \mathcal{P}'(\overline{A})$ unbounded s.th. for all $X \in U_\phi$ $\phi \in \overline{\overline{X}}$, i.e. for each $B \in \mathcal{P}'(\overline{A})$ there is $C \in \mathcal{P}'(\overline{A})$ s.th. $B \subseteq C$ and $\phi \in \overline{\overline{C}}$. The problem here is that such unbounded U, U' might have empty or bounded intersection, giving some very strange results. Comparing some $Cl(A)$ and $Cl(B)$ needs even stronger properties for reasonable results.

The following example shows that care has to be taken so the logic will not behave wildly: We show that without such precaution it is possible that $\phi \in Cl(A)$, $\psi \in Cl(A)$, but $\phi \wedge \psi \notin Cl(A)$. This is not really surprising, but the reader might want to see a proof, to get familiar with techniques.

Let $\mathcal{L} := \{p_i : i \in \omega\} \cup \{q_i : i \in \omega\} \cup \{r, s\}$.

Define $\overbrace{B} := B \cup \{r : p_i \in B\} \cup \{s : q_i \in B\}$ for $B \subseteq \mathcal{L}$, and

$B_0 := B$,

$B_{2i+1} := \overline{B_{2i}}$,

$B_{2i+2} := \overbrace{B_{2i+1}}$,

$B_\omega := \bigcup\{B_i : i < \omega\}$.

Note that all operations so far are monotonic.

Let $I^B := \{i : p_i \in B_\omega\}$, $J^B := \{i : q_i \in B_\omega\}$.

For $I, I' \subseteq \omega$ define $I < I'$ iff $\exists j \in I' \forall i \in I. i < j$, likewise $I \leq I'$ iff $\exists j \in I' \forall i \in I. i \leq j$.

Note, that if B is finite, so is I^B and J^B, and either $I^B < J^B$, or $J^B \leq I^B$, or $I^B = J^B = \emptyset$.

Thus, for finite B, the following definition is exhaustive:

$\overline{\overline{B}} := \overline{B+r}\ iff\ J^B \leq I^B,\ \overline{B+s}\ iff\ I^B < J^B,\ and\ \overline{B}\ iff\ I^B = J^B = \emptyset.$

We have to show for finite B:

1) $\overline{B} \subseteq \overline{\overline{B}}$ (this is trivial)

2) $x \in \overline{\overline{B}} \rightarrow \overline{\overline{B+x}} = \overline{\overline{B}}$

2): We first prove three facts (similar to section 1):

1. $x \in B_\omega \rightarrow (B+x)_\omega = B_\omega$. This is trivial, $\supseteq$ by monotony, for $\subseteq$, assume $\phi \in (B+x)_i$, $x \in B_j$, $i, j < \omega$, so $\phi \in B_{i+j} \subseteq B_\omega$.

2. $x \in B_\omega \rightarrow I^B = I^{B+x}$, $J^B = J^{B+x}$ (by 1.)

3. $\overline{\overline{B}} \subseteq B_\omega$, by cases:

If $I^B = J^B = \emptyset$, then $\overline{\overline{B}} = \overline{B} \subseteq B_\omega$ trivially. If $J^B \leq I^B$, then $I^B \neq \emptyset$, so there is some $p_i \in B_\omega$, so $r \in B_\omega$, but B_ω is closed under $\overline{}$. Likewise, if $I^B < J^B$.

Assume now $x \in \overline{\overline{B}}$. If $I^B = J^B = \emptyset$, then by 2. $I^{B+x} = J^{B+x} = \emptyset$, so $x \in \overline{\overline{B}} = \overline{B} = \overline{B+x} = \overline{\overline{B+x}}$. Likewise, if $J^B \leq I^B$, $x \in \overline{\overline{B}} = \overline{B+r} = \overline{B+r+x} = \overline{\overline{B+x}}$, and if $I^B < J^B$, $\overline{\overline{B}} = \overline{B+s} = \overline{B+s+x} = \overline{\overline{B+x}}$. Thus, $=$ defined on finite sets is as desired.

Consider now A $:= \{p_i : i \in \omega\} \cup \{q_i : i \in \omega\}$. $U := \{B \in \mathcal{P}'(\overline{A}) : J^B \leq I^B\}$ and $U' := \{B \in \mathcal{P}'(\overline{A}) : I^B < J^B\}$ are both unbounded (but $U \cap U' = \emptyset$), and so are $V := \{B \in \mathcal{P}'(\overline{A}) : r \in \overline{\overline{B}}\}$ and $V' := \{B \in \mathcal{P}'(\overline{A}) : s \in \overline{\overline{B}}\}$. So, by our modified definition, $r, s \in Cl(A)$. On the other hand, for no finite $B \subseteq \overline{A}$ $r \wedge s \in \overline{\overline{B}}$. □

If, however, $U \cap U'$ (U, U' as in above construction) were unbounded in $\mathcal{P}'(\overline{A})$, we would have concluded $r \wedge s \in \overline{\overline{B}}$ unboundedly often (using supraclassicality). So, any weakening of the original extension definition should have the finite intersection property, i.e. the systems considered should be closed under finite intersections. Yet, to achieve more results, e.g. to show (under additional assumptions) something like $A \subseteq B \subseteq \overline{\overline{A}} \rightarrow \overline{\overline{A}} = \overline{\overline{B}}$, one needs a global system for $\mathcal{L}$ (i.e. for all $A, B \subseteq \mathcal{L}$), which, very likely, will finally have properties similar to those of the original definition of extension of [FLM].

6 A final example

(+++ Orig.: 5. A final example +++)

LABEL: Section 5. A final example

In sections 1 and 4, we chose a relatively brutal approach: E.g. in section 1, we measured A_ω globally comparing the p_i's and $r_i's$ contained in it. This served to decide on <u>all</u> formulas of A_ω. We now give a more subtle technique, looking at the individual "proofs" of the $\phi \in A_\omega$, and show how to define this way a supraclassical cumulative inference operation.

Let $\mathcal{L} := \{p_i, q_i, r_i, s_i\colon i < \omega\}$, and consider first closure under $p_i \mid\!\sim q_i$, $p_i \mid\!\sim r_i$, $p_i \mid\!\sim s_i$, $q_i \mid\!\sim s_i$:

$\overbrace{A} := A \cup \{q_i : p_i \in A\} \cup \{r_i : p_i \in A\} \cup \{s_i : p_i \in A\ or\ q_i \in A\}$

and set as above

$A_0 :=$ A,

$A_{2i+1} := \overline{A_{2i}}$,

$A_{2i+2} := \overbrace{A_{2i+1}}$,

$A_\omega := \bigcup\{A_i : i < \omega\}$.

We encode now in an ATMS-like fashion (the for us important part of) the proofs of $\phi \in A_\omega$ from A by simultaneous induction on the length of proof:

For $\phi \in \mathcal{L} - A_\omega$, set $B^A_\phi := \emptyset$. Let now $\phi \in A_\omega$, and π be a proof of ϕ from $A = A_0$.

Case 1: $\phi \in A_0$ (π the empty proof): Set $b^\pi_\phi := \emptyset$

Case 2: The last step of π is $\phi_0 \wedge \ldots \wedge \phi_n \vdash \phi$ classically. By induction, $b^{\pi_i}_{\phi_i}$ is defined, where π_i is the branch of π proving ϕ_i. Set $b^\pi_\phi := \bigcup \{ b^{\pi_i}_{\phi_i} : i \leq n \}$

Case 3.1: $\phi = q_i$, and the last step of π is $p_i \mid\sim q_i$. Set $b^\pi_\phi := b^{\pi'}_{p_i}$, where π' is the subproof of π, showing p_i.

Case 3.2: $\phi = r_i$, and the last step of π is $p_i \mid\sim r_i$. Set $b^\pi_\phi := b^{\pi'}_{p_i}$ (π' as above)

Case 4.1: $\phi = s_i$, and the last step of π is $p_i \mid\sim s_i$. Set $b^\pi_\phi := b^{\pi'}_{p_i} \cup \{i\}$

Case 4.2: $\phi = s_i$, and the last step of π is $q_i \mid\sim s_i$. Set $b^\pi_\phi := b^{\pi'}_{q_i} \cup \{i\}$

We thus encode the uses of $p_i \mid\sim s_i$, $q_i \mid\sim s_i$. For technical convenience, we close upwards and define

$B^A_\phi := \{ a \subseteq \omega$: there is a proof π of ϕ from A_0, and $b^\pi_\phi \subseteq a \}$ for $\phi \in A_\omega$.

Obviously, for $A \subseteq A'$, $B^A_\phi \subseteq B^{A'}_\phi$.

Set $I^A := \{ i : r_i \in A_\omega \}$ and $b \geq I^A$ iff $\forall i \in b \forall j \in I^A . i \geq j$.

(Thus, in particular, $\emptyset \geq I^A$. Note that $b, b' \geq I^A \to b \cup b' \geq I^A$.)

Set finally $\overline{\overline{A}} := \{ \phi \in A_\omega$: $\exists b^\pi_\phi \in B^A_\phi . b^\pi_\phi \geq I^A \}$.

We now show as usual:

1) $\overline{A} \subseteq \overline{\overline{A}}$,

2) $x \in \overline{\overline{A}} \to \overline{\overline{A+x}} = \overline{\overline{A}}$.

Proof:

(+++ Orig.: Proof: +++)

LABEL: Section Proof:

1) is trivial by $A_1 = \overline{A}$, $\emptyset \in B^A_\phi$ for $\phi \in A_1$.

2) We first show:

1. For $x \in A_\omega$ $(A+x)_\omega = A_\omega$.
2. For $x \in A_\omega$ $I^A = I^{A+x}$.
3. Let $x \in A_\omega$, $\phi \in A_\omega$, $b^{\pi'}_\phi \in B^{A+x}_\phi$, $b^\pi_x \in B^A_x$, then $b^{\pi'}_\phi \cup b^\pi_x \in B^A_\phi$.

1.: As in above sections.

2.: trivial by 1.

3.: trivial.

Assume now $x \in \overline{\overline{A}}$. Let $\phi \in \overline{\overline{A}} \subseteq A_\omega = (A+x)_\omega$. By definition, there is $b^\pi_\phi \in B^A_\phi \subseteq B^{A+x}_\phi$ s.th. $b^\pi_\phi \geq I^A = I^{A+x}$, so $\phi \in \overline{\overline{A+x}}$. Let $\phi \in \overline{\overline{A+x}} \subseteq (A+x)_\omega = A_\omega$. Thus, there is $b^\pi_\phi \in B^{A+x}_\phi$, $b^\pi_\phi \geq I^{A+x} = I^A$. As $x \in \overline{\overline{A}}$, there is $b^{\pi'}_x \in B^A_x$, $b^{\pi'}_x \geq I^A$. By 3, $b^\pi_\phi \cup b^{\pi'}_x \in B^A_\phi$ and we are done by $b^\pi_\phi \cup b^{\pi'}_x \geq I^A$.

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LABEL: Section Acknowledgements:

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References

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