FILTERS AND PARTIAL ORDERS

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Abstract

We discuss several abstract semantics for nonmonotonic logics. We present their motivations, their development and some historical origins, and show that the three systems considered are essentially equivalent: (a) the coherent systems of filters of S.Ben-David and R.Ben-Eliyahu, (b) the coherent systems of filters developed by the author, (c) the partial order semantics of N.Friedman and J.Halpern.

1 INTRODUCTION

1.1 Overview

We first present a quite abstract introduction to abstract semantics (Section 1.2). This part is for the reader familiar with or uninterested in the details of their motivation by nonmonotonic logics. Sections 1.3 and 1.4 provide this motivation, and trace the origins and developments of the abstract ideas from more concrete examples of nonmonotonic reasoning. In particular, we discuss the well-examined preferential model semantics, the weak filter semantics developed by the author, and very briefly partial orders used in various completeness proofs. Section 1.5 gives the basic definitions and results of Ben-David/Ben-Eliyahu [BB94] and Friedman/Halpern [FH95], as far as relevant for our development.
We work in Sections 1.5, 2, and 3 in a fixed set U, the "universe", to be interpreted as the set of models of some fixed language. In the filter approach (Ben-David/Ben-Eliyahu and the author), a filter is associated to every $A \subseteq U$ - in the case of the $\mathcal{N}(A)\prime s$ a filter over U, in the case of the $\mathcal{N}\prime(A)\prime s$, as well as in the case of the system described by the properties (P0)-(P4), a filter over A. Coherence properties of the different filters are examined. In the partial order approach (Friedman/Halpern), an order is given on U.

Section 2 presents first a slight modification of the system of Ben-David/Ben-Eliyahu, which is perhaps a little more intuitive than the original version. Equivalence of both is shown in Proposition 2.4. We then discuss the author's system of coherent systems of subsets, and show its equivalence to that of Ben-David/Ben-Eliyahu in Proposition 2.8.
Section 3 presents equivalence results between partial orders on formulas and coherent sets of filters. Total equivalence (Proposition 3.3) can be obtained by modifying the system of Friedman/Halpern slightly. For their original system, we obtain only partial equivalence (Proposition 3.4).
In conclusion, we hope to have demonstrated that various abstract semantics have far reaching roots in history and development of nonmonotonic logics. Our technical results show that three such abstract semantics are (essentially) equivalent. We see this as indication of a common intuition behind various forms of nonmonotonic logics.
The proofs in the present paper are elementary and straightforward. But we think that this comparison - though quite obvious - had to be written down by someone. So we did it.

1.2 Introduction from an abstract point of view

There are two classical abstract ways to express that set A is bigger than set B : First, by a partial order, $B < A$, second, by giving a filter $\mathcal{F}$ on $A \cup B$, such that e.g. $A - B \in \mathcal{F}$. Re-interpreting "size" by "importance" or "normally ...", we have a natural and abstract way to speak about normal cases, i.e. do nonmonotonic reasoning. It is shown here that these two points of view, developed by Ben-David/Ben-Eliyahu [BB94], independently by the author [Sch95-t1], and by Friedman/Halpern [FH95] respectively, are essentially equivalent.
The idea of filters - or, more precisely, weak filters - was also used in the author's work on a semantics for defaults in a first order framework. As our system there was held deliberately weak, however, we did not postulate any coherence properties between individual weak filters - they were just mentioned as an extension to cover the KLM axioms (see [KLM90]).
For motivation, consider e.g. the Friedman/Halpern approach. The "soft" or defeasible rule $\alpha \mid\sim \beta$ can be read as: "$\alpha \wedge \beta$ is more plausible or more probable than $\alpha \wedge \neg\beta$" and formalized by $\alpha \wedge \beta > \alpha \wedge \neg\beta$ (in some order $>$). This order can be seen (and was created as such) as an abstraction of probability. Thus, $\alpha \wedge \beta > \alpha \wedge \neg\beta$ is to be read as expressing that the probability $P(\alpha \wedge \beta)$ is greater than the probability $P(\alpha \wedge \neg\beta)$.
We see these approaches as somewhat intermediary between proof theory and semantics. From a perhaps somewhat naive point of view, a semantics gives meaning to a language and logic, tells us what they speak about. Thus, a semantics gives us in abstract form the basic entities and objects we want to speak and reason about in the language and logic under consideration. On the other hand, above formalisms seem to be more an abstraction of the concepts of the language and logic itself.
For instance, Kripke structures seem to speak directly about objects, and their possible

developments. On the other hand, the interpretation of "typically" by some kind of "large" subset seems more an abstraction of the concept "typically" itself.
To give another example, preferential models for nonmonotonic reasoning seem a much more genuine "object driven" semantic concept. A relation of preference between individual models (or objects) gives there a local choice of "better" elements. Representation results showing that certain logics can be represented by such local choice seem much more surprising than representation results for abstract semantics. This might be a further indication that the conceptual difference between "object driven semantics" and language is bigger than the conceptual difference between "concept driven semantics" and language.
The fact that many systems of nonmonotonic logics reveal themselves as equivalent and can be unified by such abstract semantics is perhaps a further illustration of their intermediary status: These systems reflect common intuitions of people working on logical approaches to nonmonotonic reasoning. The abstract semantics formalize these intuitions about reasoning - not about the objects of reasoning.
Of course, these comments do not judge the quality of the different approaches, our objective is just to make things clear.
But these remarks are perhaps too "philosophical", and too platonist in orientation.

1.3 Introduction from a logical point of view: reasoning and size

Several forms of reasoning involve a weighing of cases, or of sets of cases. Nonmonotonic reasoning considers the normal cases, like normal birds which fly, lay eggs etc. Statistical reasoning might accept a conclusion if it holds in the large majority of cases. Deontic reasoning considers the morally acceptable cases. Reasoning with counterfactual conditionals works on those cases which are as close as possible to the actual case (at least in the Stalnaker/Lewis semantics).
Thus, abstractly, α entails β, $\alpha \mid\sim \beta$ if in an important set of α -cases β holds, or if in the most important α -cases β holds, in other words, if we have some relation like $\alpha \wedge \beta > \alpha \wedge \neg\beta$.
We have so far left open the meaning of the "cases". In a first order framework, the α-cases might be those elements of the universe where α holds, in a propositional setting with a "possible worlds" semantics, the α -cases may be those worlds where α holds, or, we might leave things in the abstract and just continue to say $\alpha \wedge \beta > \alpha \wedge \neg\beta$. In the concrete cases of elements of the universe or of worlds in a possible worlds semantics the set of important cases can be chosen in different ways: Either abstractly by a suitable system of subsets, or locally, by comparing single cases. E.g. case a might be more important, natural, normal, etc. than case b. The important sets might then be those which include all preferred elements. In both cases, we will say $\alpha \mid\sim \beta$ if there is an important subset of the α-cases where β holds. Note that this coincides with the abstract relation $\alpha \wedge \beta > \alpha \wedge \neg\beta$ mentioned above, if we assume that a superset of an "important"

set is important, too.
The local choice, i.e. the preference relation between single cases, introduces immediately coherence properties. If $f(X)$ are the preferred cases of X, and $A \subseteq B$, then $f(B) \cap A \subseteq f(A)$: if $x \in f(B) \cap A$, then there is no $x\prime$ better than x in B, so a fortiori not in A. So the preferred cases of B and A are not independent of each other: if x and $x\prime$ are both in B and in A, and $x\prime$ is better than x, then it will be so for B and for A.
The abstract choice of important subsets is a priori free from such constraints, and thus more general. Some natural constraints one might consider are given in the following

Definition 1.1 $\mathcal{N}(X) \subseteq \mathcal{P}(X)$ - $\mathcal{P}(X)$ the powerset of X - is an $\mathcal{N}$-system ($\mathcal{N}$ for "normal") over X iff
(1) $X \in \mathcal{N}(X)$,
(2) $A \in \mathcal{N}(X)$, $A \subseteq B \subseteq X \rightarrow B \in \mathcal{N}(X)$,
(3) $A, B \in \mathcal{N}(X) \rightarrow A \cap B \neq \emptyset$.

This is the base system chosen by the author in [Sch95-1]. Note that condition (3) makes this system almost a filter.
We recall the definitions of a filter and of its dual, an ideal:

Definition 1.2 Let X be any set.
$\mathcal{F} \subseteq \mathcal{P}(X)$ is a filter over X iff
(1) $X \in \mathcal{F}$,
(2) $A \in \mathcal{F}$, $A \subseteq B \subseteq X \rightarrow B \in \mathcal{F}$,
(3) $A, B \in \mathcal{F} \rightarrow A \cap B \in \mathcal{F}$.
$\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal over X iff
(1) $\emptyset \in \mathcal{I}$,
(2) $A \in \mathcal{I}$, $B \subseteq A \rightarrow B \in \mathcal{I}$,
(3) $A, B \in \mathcal{I} \rightarrow A \cup B \in \mathcal{I}$.

Intuitively, a filter consists of the large subsets of X, an ideal of the small subsets of X.
The above conditions are restrictions on the important subsets of one base set. There are natural coherence conditions between the important subsets of different base sets, too. E.g., if A is an important subset of X, and $A \subseteq Y \subseteq X$, then A is likely to be an important subset of Y, too. Such coherence conditions will be considered in detail below.
To summarize, in all situations considered, we say $\alpha \mid\sim \beta$ if there is an important subset of the α-cases, where β holds. The important subsets might be chosen abstractly, e.g. by a suitable filter or a similar structure, or by an abstract order between subsets. They might also be chosen locally, e.g. by comparison between single cases.
We have so far left open whether there is, in the abstract approach, a minimal important set of α-worlds. In the local comparison, this corresponds to the existence of maximally important cases. An alternative is to have descending chains of ever more important subsets, or of ever more important cases in the local comparison approach, without reaching

the limit. In the case of local preference the former situation is the "minimal" approach, the second the "limit" variant. They will be discussed briefly below.
It should be emphasized that this abstract point of view was not the original one, but a later step in a succession of abstractions. So our outline is given in hindsight.
We now present a more detailed introduction and some historical background to the properties and origins of the different approaches discussed above in abstract terms. More precisely, we discuss:
- preferential models, i.e. local choice of more important cases,
- systems of (weak) filters,
- abstract orders on formulas.

1.4 Origins and historical background of the different ideas

1.4.1 Preferential Structures:

Preferential structures are probably the best examined semantics for nonmonotonic (and some other) logics, so we give a detailed introduction for readers not familiar with them. As said already above, in preferential structures the choice of important subsets is made *locally* by a binary relation $\prec$ on the set M of base models, where $m \prec m\prime$ iff m is considered to be more important than $m\prime$. So, a preferential structure $\mathcal{M}$ is a pair $< M, \prec>$. They are thus very similar to Kripke structures, but the relation $\prec$ will be used differently.
In the "minimal" variant (see above), we define a choice function f of best elements of A by means of $\prec$ by $f(A) := \{a \in A : \neg \exists b \in A . b \prec a\}$.
In the "limit" variant, the natural definition is to consider initial segments of A ($A \neq \emptyset$) : $\delta_A \subseteq A$ is called an initial segment of A iff
($\delta 1$) there is some $b \in \delta_A$ below each $a \in A$: $\forall a \in A \exists b \in \delta_A (b = a$ or $b \prec a)$
($\delta 2$) δ_A is downward closed: $\forall a \in A \forall b \in \delta_A (a \prec b \rightarrow a \in \delta_A)$.
We thus have in the minimal case for a theory T and a formula ϕ the definition $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in \mu(T)$ - the set of $\prec$-minimal models of T in $\mathcal{M}$. If, for instance, $M(T)$ - the set of models of T - consists of infinite descending chains, then $\mu(T) = \emptyset$, and $T \models_{\mathcal{M}} \phi$ for any ϕ, $\bot$ (=false) included. On the other hand, any $m \in \mu(T)$ will be a "witness" of *all* $\models_{\mathcal{M}}$-consequences of T, in the sense that all ϕ with $T \models_{\mathcal{M}} \phi$ will hold in any such m.
In the limit case, we have $T \models_{\mathcal{M}} \phi$ iff there is some $\delta_{T,\phi} \subseteq M(T)$ which satisfies ($\delta 1$) and ($\delta 2$) with respect to $M(T)$ and such that ϕ holds in all $m \in \delta_{T,\phi}$. Thus, in the limit case, $\mu(T)$ may be empty, but if $M(T) \neq \emptyset$, we will still not have $T \models_{\mathcal{M}} \bot$, as all $\delta_{T,\phi}$ are then non-empty. It is easily seen, that if $T \models_{\mathcal{M}} \phi$ and $T \models_{\mathcal{M}} \phi\prime$, and $\prec$ is transitive, then also $T \models_{\mathcal{M}} \phi \wedge \phi\prime$: if $\delta_{T,\phi}$ and $\delta_{T,\phi\prime}$ are suitable, then $\delta_{T,\phi} \cap \delta_{T,\phi\prime}$ will be a suitable $\delta_{T,\phi \wedge \phi\prime}$. Moreover, if $T \models_{\mathcal{M}} \phi$, and $M(T \cup \{\phi\}) \subseteq M(T\prime) \subseteq M(T)$, then also $T\prime \models_{\mathcal{M}} \phi$.
We recall the abovementioned property

(1) $A \subseteq B \to f(B) \cap A \subseteq f(A)$.
which turns out to be the crucial one for minimal preferential structures. Any choice function which obeys (1) and the trivial property
(0) $f(A) \subseteq A$
can be represented by a preferential structure, i.e. by such a binary relation of preference (see [Sch92], Proposition 3.3). This is a very general "algebraic" characterization, the underlying set M need not consist of models, it may be just any arbitrary set.

Interpretation: We turn to the intuition behind the "importance" of models of the base logic in various contexts.
Deontic Logic:
Deontic logic reasons about the normatively acceptable situations, and about what ought to be done (by humans, robots etc.). Reasoning about normatively acceptable actions can be split into two subquestions: Reasoning about the normatively acceptable states, and reasoning about the problem of acting in a way conducive to those states. The latter question can be considered separately, at least in a first approximation.
In this framework, the preferred or more important models are those which are normatively more acceptable. Thus, preferential structures also provide a natural semantics for deontic logic, and, in fact, were examined as such before the advent of nonmonotonic logics [Han69]. This was pointed out by D.Makinson in [Mak93].
Nonmonotonic Logic:
This "importance" may be read as "normality" in the case of nonmonotonic logics: We are primarily interested in reasoning about the normal cases, and the preferred models are the most normal ones - where birds can fly, houses have doors etc.
Preferential structures in their various forms provide an important and well-studied kind of semantics for nonmonotonic logics. They have been a powerful tool for investigation, providing - via additional properties of the relation $\prec$ - a technique for constructing semantics of logical systems of different strengths.
This might be the right place to introduce a number of axioms for logical systems, closely connected by representation results to preferential models. Some connections will be hinted at briefly below, for more details the reader is referred to the quoted articles. The axioms play also a central role in the articles of Ben-David/Ben-Eliyahu and Friedman/Halpern, see Section 1.5 below.

Definition 1.3 (Axioms of P and R, see [Gab85], [KLM90], [LM92])
(Strictly speaking - i.e. in object language -, the axioms are rules, and a system X of $\alpha \mid\sim \beta\prime s$ is said to satisfy P (or R), iff it is closed under the corresponding rules.)
1. Right Weakening (RW): $\models \alpha \to \beta$, $\gamma \mid\sim \alpha \Rightarrow \gamma \mid\sim \beta$
2. Reflexivity: $\alpha \mid\sim \alpha$
3. AND: $\alpha \mid\sim \beta$, $\alpha \mid\sim \gamma \Rightarrow \alpha \mid\sim \beta \wedge \gamma$
4. OR: $\alpha \mid\sim \gamma$, $\beta \mid\sim \gamma \Rightarrow \alpha \vee \beta \mid\sim \gamma$

5. Left Logical Equivalence (LLE): $\models \alpha \leftrightarrow \beta, \beta \mid\sim \gamma \Rightarrow \alpha \mid\sim \gamma$
6. Cautious Monotony (CM): $\alpha \mid\sim \beta, \alpha \mid\sim \gamma \Rightarrow \alpha \wedge \beta \mid\sim \gamma$
7. Rational Monotony (RM): $\alpha \mid\sim \beta \Rightarrow \alpha \mid\sim \neg\gamma$ or $\alpha \wedge \gamma \mid\sim \beta$
8. CUT: $(\alpha \wedge \beta) \mid\sim \gamma, \alpha \mid\sim \beta \Rightarrow \alpha \mid\sim \gamma$
9. Cumulativity (CUM): $\alpha \mid\sim \beta \Rightarrow (\alpha \mid\sim \gamma \Leftrightarrow (\alpha \wedge \beta) \mid\sim \gamma)$,
10. Monotony: $\alpha \mid\sim \gamma \Rightarrow (\alpha \wedge \beta) \mid\sim \gamma$
The system P consists of 1-6, R of 1-7.

Monotony is, of course, not part of nonmonotonic logics and mentioned here together with the other axioms for later use (see Theorem 1.9).
"Limit" preferential structures for nonmonotonic logics were introduced by G.Bossu and P.Siegel in [BS85]. The "minimal" case was first examined by Y.Shoham ([Sho87]) as a generalization of the minimal model semantics for Circumscription. More or less general cases of preferential structures are characterized by soundness and completeness theorems in [KLM90], [LM92], [Sch92], [Sch96-1] for the minimal case, in [Bou90a], [Bou90b], [Bou92], and in [Sch96-5] for the limit case. For an overview, see also [Mak94].
In hindsight, it is no surprise that a local preference by a binary relation tends to emerge, when we examine choice functions which single out some states as more important or interesting than others. Such local preferences seem to correspond well to intuitions, and simplify the situation by making the choice context-independent.
In [Mak93], still other natural applications of preferential structures are discussed.

An Example: We give a simple example which shows that the relation $\models_{\mathcal{M}}$ defined by a preferential structure $\mathcal{M}$ may indeed be a nonmonotonic consequence relation.
Let $\mathcal{L}$ be the propositional language with two varibles p,q, let M consist of two (classical) models, $m \models p \wedge q$, $m\prime \models \neg p \wedge \neg q$, and let $m\prime \prec m$. Let $\mathcal{M} :=< M, \prec>$. Then $\emptyset \models_{\mathcal{M}} \neg q$, but $p \models_{\mathcal{M}} q$ in both the minimal and the limit definition.
As is the case already in our example, not all classical models for a given language $\mathcal{L}$ need occur in the base set M of a preferential structure $\mathcal{M}$ (e.g., in our example, an $m\prime\prime$ with $m\prime\prime \models p \wedge \neg q$ is missing). Moreover, some classical models might occur several times, even infinitely often. Take for example $\mathcal{L}$ with one propositional variable p and consider the structure $\mathcal{M} :=< \{< m, i >: i < \omega\}, \prec>$, m a classical model with $m \models p$, and $< m, i > \prec < m, j >$ iff $j < i$. By abuse of language, we shall also write $< m, i > \models p$, etc. Then $\mu(M) = \emptyset$, so $true \models_{\mathcal{M}} \perp$ in the minimal reading, but $true \models_{\mathcal{M}} \phi$ iff ϕ is a classical consequence of p, in the limit reading. More details and examples of logics which require several copies of classical models to be representable by preferential models can be found in [Sch96-1].

Strengthenings of the Conditions for the Relation $\prec$: Various additional conditions for the relation $\prec$ have been introduced and examined for minimal preferential structures.

The most natural one is perhaps transitivity.
An important condition, which results in nice properties of the semantic consequence relation $\models_{\mathcal{M}}$ is smoothness (terminology of D.Lehmann and his co-authors) or stopperedness (terminology of D.Makinson): Given a theory T, and a non-minimal model m of T, there is $m\prime \prec m$, which is a minimal model of T. Consequently, if $M(T) \neq \emptyset$, then $\mu(T) \neq \emptyset$. "Smoothness" or "stopperedness" can be violated in essentially two kinds of situation. First, suppose X consists of an infinite descending chain of elements x_i. Then, no $x_i \in X$ is minimal in X, and no x_i has a minimal $y \in X$ below it. Second, suppose that $X := \{x, y, z\}$, with $x \prec y \prec z$, but not $x \prec z$. Then z is not minimal in X, and there is no $a \in X$ below z which is minimal in X.
The counterpart for the consequence relation $\models_{\mathcal{M}}$ is Cumulativity (see [KLM90], [Gab85], and Definition 1.3) which says that two theories T, $T\prime$ with $T \subseteq T\prime \subseteq \{\phi : T \models_{\mathcal{M}} \phi\}$ have the same consequences: $T \models_{\mathcal{M}} \phi$ iff $T\prime \models_{\mathcal{M}} \phi$. We may read this as "normal use of lemmas": If we have already deduced the "lemma" ϕ from T, we neither lose nor win in terms of possible deductions by starting from $T \cup \{\phi\}$.
As a matter af fact, again a very general algebraic representation result can be obtained: A choice function f can be represented by a smooth minimal preferential structure iff it satisfies the conditions (0), (1) and
(2) $f(A) \subseteq B \subseteq A \rightarrow f(A) = f(B)$
and if its domain satisfies closure under finite intersections and unions (see [Sch96-1], Theorem 1).
Another strengthening of $\prec$ is rankedness, which may be seen as the existence of a "rotating scale with fixed origin": $\prec$ is called ranked (on M), iff there is an order-preserving (in both directions) function $f : (M, \prec) \rightarrow (X, \prec \bullet)$, where $\prec \bullet$ is a total order on X. Then two $\prec$ $-incomparable$ elements $m, m\prime \in M$ behave exactly the same way with respect to $\prec$: $n \prec m$ iff $n \prec m\prime$, and $m \prec n$ iff $m\prime \prec n$. The corresponding property of $\models_{\mathcal{M}}$ is Rational Monotony: If $\alpha \models_{\mathcal{M}} \gamma$, then $\alpha \wedge \beta \models_{\mathcal{M}} \gamma$ or $\alpha \models_{\mathcal{M}} \neg\beta$ (see [LM92] and Definition 1.3). General representation results are again to be found in [Sch96-1].

1.4.2 Nonmonotonic logics and (weak) filters

This approach, developed by the author, was first presented in a German workshop, see [Lor89], and in more detail in [Lor90], [Sch90], in particular in [Sch92-t1], [Sch95-1].
We use Reiter's notation (see [Rei80]) and write open normal defaults without prerequisites as $\frac{:\phi(x)}{\phi(x)}$, and those with prerequisites as $\frac{\psi(x):\phi(x)}{\phi(x)}$. The former means roughly the following: if we can safely assume - i.e. without running into contradiction - $\phi(x)$, we do so. The latter: if we know $\psi(x)$, and can safely assume $\phi(x)$, we do so. For example, the default $\frac{bird(x):fly(x)}{fly(x)}$ means: if we know bird(Tweety), and have no information to the contrary, we assume that Tweety flies.
We have interpreted in our approach open normal defaults as generalized quantifiers. "Normally, birds fly", $\frac{bird(x):fly(x)}{fly(x)}$, is then translated into "Most birds fly" or "The elements

of a large or important subset of the set of birds fly", and written $\nabla x bird(x) : fly(x)$ - $\nabla x \phi(x)$ for the default without prerequisite $\frac{:\phi(x)}{\phi(x)}$ - in an extension of the language of first order logic.
This is given a clear semantics in the form of a system of "important" subsets of the set of birds. We thus add additional structure to first order models to provide a semantics for the new quantifier. In such an enlarged structure, the default "Normally, birds fly" will hold, iff there is an "important" subset of the set of birds, all of whose elements fly.
We have chosen a very weak - and thus easily extendable - formalization of the concept of "large" subsets, weaker than a filter, the conditions were given in Definition 1.1 above. We do not think that any weaker system could still capture the notion of "important" or "large" subsets.
The third property in Definition 1.1 is the most interesting one. Its weakness permits e.g. a simple probabilistic interpretation by "more than half". Also by (3), we can deduce that "normally $\phi(x)$" and "normally $\neg\phi(x)$" together are impossible: there is no element which satisfies both $\phi(x)$ and $\neg\phi(x)$.
On the syntactic side, ∇x was treated as some kind of quantifier between $\exists$ and $\forall$. It seems - at least to us - unreasonable to say that normally $\phi(x)$ holds, when there is no x s.th. $\phi(x)$. On the other hand, if $\forall x \phi(x)$ holds, it seems reasonable to say that ϕ normally holds. So, we have $\forall x \phi(x) \rightarrow \nabla x \phi(x)$ and $\nabla x \phi(x) \rightarrow \exists x \phi(x)$ as axioms. As said above, it seems unreasonable to say that normally $\phi(x)$ holds, and at the same time, that normally $\neg\phi(x)$ holds. We thus have the axiom $\nabla x \phi(x) \rightarrow \neg \nabla x \neg \phi(x)$, i.e. $\{\nabla x \phi(x), \nabla x \neg \phi(x)\}$ will be inconsistent. Moreover, upward closure $\nabla x \phi(x) \wedge \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \nabla x \psi(x)$ is an axiom, as is a condition about renaming of bound variables.
The basic condition of validity of $\nabla x \phi(x)$ in a structure $< M, \mathcal{N}(U) >$, with M a classical first order structure, $\mathcal{N}(U)$ an $\mathcal{N}$-system over its universe U, is given by the condition
$< M, \mathcal{N}(U) > \models \nabla x \phi(x)$ iff $\{x \in U : < M, \mathcal{N}(U) > \models \phi(x)\} \in \mathcal{N}(U)$.
The extension of first order logic sketched above is a sound and complete axiomatisation for our semantics, and allows us not only to derive defaults - e.g., if "normally, $\phi(x)$", and $\phi(x)$ implies $\psi(x)$, then also "normally, $\psi(x)$" - but gives us a notion of consistency of default theories too.

Discussion of our system Note that we work in the full first order framework, with defaults in the object language. So not only negated defaults - in our birds example, there would then be no important subset of "birds" all of whose elements fly - have a clear meaning, but so do arbitrary boolean combinations of defaults, nested defaults, arbitrary combinations of defaults with classical quantifiers etc.
The open normal default with prerequisite $\frac{\psi(x):\phi(x)}{\phi(x)}$ is read as a bounded generalized quantifier, we understand it as saying that all x which satisfy ψ, will normally satisfy ϕ too. Again, we shall have rules like $\forall x(\psi(x) \rightarrow \phi(x)) \rightarrow \nabla x \psi(x) : \phi(x)$ and $\nabla x \psi(x) : \phi(x)$ $\rightarrow [\exists x \psi(x) \rightarrow \exists x(\psi(x) \wedge \phi(x))]$. All essential techniques and ideas are present in the case without prerequisite, the extension to defaults with prerequisites is straightforward and

works by simple relativization.
We can give predicate logic versions of the systems P and R (see Definition 1.3) an alternative semantics by translating $\phi(x) \mid\!\sim \psi(x)$ into $\nabla\phi(x) : \psi(x)$. We see that Right Weakening, Reflexivity, and Left Logical Equivalence are already built into our system. The others have to be introduced by additional axiom schemata, like $\nabla x\alpha(x) : \beta(x) \to (\nabla x\alpha(x) : \neg\gamma(x) \vee \nabla x(\alpha \wedge \gamma)(x) : \beta(x))$ for Rational Monotony. The translation is obvious, so we discuss the semantical counterparts. "AND" says that $\mathcal{N}$-systems are closed under finite intersections, i.e. are filters. "OR", Cautious and Rational Monotony concern the relation of the $\mathcal{N}$-systems of the subsets, i.e. coherence properties:
OR: $A \in \mathcal{N}(X)$, $B \in \mathcal{N}(Y) \to A \cup B \in \mathcal{N}(X \cup Y)$, Cautious Monotony: $A, B \in \mathcal{N}(X) \to A \cap B \in \mathcal{N}(A)$, Rational Monotony: $A \in \mathcal{N}(X)$, $B \subseteq X \to B \in \mathcal{N}(X)$ or $A - B \in \mathcal{N}(X - B)$.

1.4.3 Orderings on $\mathcal{L}$

Orderings between formulas of a language can be found in the context of theory revision ("epistemic entrenchment" relations, see e.g. [AGM85] and [Gărd88]), but also in various completeness proofs, see [KLM90], [LM92], [GM91]. In those proofs, several orderings between formulas are used, which arise more or less naturally from the logic $\mid\!\sim$ - in the spirit of the above abstract description, but are not elaborated in detail.
We find for instance

Definition 1.4 $\alpha \leq_{KLM} \beta :\leftrightarrow \alpha \vee \beta \mid\!\sim \alpha$ ([KLM90])
and
$\alpha <_{LM} \beta :\leftrightarrow \alpha \vee \beta \mid\!\sim \neg\beta$, $\alpha \leq_{LM} \beta :\leftrightarrow \beta \not<_{LM} \alpha$ ([LM92]).
We have modified $<_{LM}$ slightly, and defined in [Sch92-t1] and the forthcoming [Sch96-2]
$\alpha > \beta :\leftrightarrow \alpha \vee \beta \mid\!\sim \neg\beta$ and $\alpha \vee \beta \not\mid\!\sim \bot$, $\alpha \geq \beta :\leftrightarrow \alpha \not< \beta$
(if $\alpha \mid\!\sim \bot$ (=false), then any subset of α has identical size 0).

A detailed examination of the latter relation is given in [Sch92-t1] and [Sch96-2].

1.5 Basic definitions and results of Ben-David/Ben-Eliyahu and Friedman/Halpern

We use the notation: $\boldsymbol{C}(X) := U - X$ (U the universe we work in), and $\mathcal{P}(X)$ for the powerset of X.
We now present the approaches of Ben-David/Ben-Eliyahu and Friedman/Halpern, and some of their main results. (The system considered by the author - see Definition 2.7 - is very similar to the one of Ben-David/Ben-Eliyahu.) We have concentrated on those parts essential to understand our comparisons in Sections 2 and 3. For details, the reader is referred to the original papers.

1.5.1 The system of Ben-David/Ben-Eliyahu

Ben-David/Ben-Eliyahu consider a conditional language with a binary operator $\Rightarrow$, and their structures are as usual in the conditionals framework, i.e. relativized to all points in the structure. We will drop this relativization in the later development, as we are mainly interested in the nonmonotonic framework. Their language is the usual one for propositional conditionals, and admits full nestedness etc. of $\Rightarrow$. $\rightarrow$ will denote classical implication.

Definition 1.5 (Ben-David/Ben-Eliyahu)
$M =< U, l, \mathcal{N} >$ is a filter based model (FBM) for a set V of propositional variables
iff
(1) U (the universe) is a set (of worlds),
(2) $l : U \rightarrow \mathcal{P}(V)$ is a labeling function, which assigns as usual to each $w \in U$ the set of variables which hold in w,
(3) $\mathcal{N} : U \times \mathcal{P}(U) \rightarrow \mathcal{P}(\mathcal{P}(U))$ is a function s.t. $\mathcal{N}_w(A) := \mathcal{N}(w, A)$ is a filter over U with $A \in \mathcal{N}_w(A)$.

(We shall later modify equivalently so that $\mathcal{N}_w(A)$ will be a filter over A.)

Definition 1.6 (Ben-David/Ben-Eliyahu)
Given $M =< U, l, \mathcal{N} >$, define validity of an arbitrary $\Rightarrow$-formula in M at a world w and the set $\parallel \phi \parallel$ of worlds in M where a formula ϕ holds by simultaneous induction:
(1) for $\phi \in V$ $M \models_w \phi$ iff $\phi \in l(w)$,
(2) classical propositional connectives are treated as usual, i.e. $M \models_w \neg\phi$ iff $M \not\models_w \phi$ etc.,
(3) $M \models_w \phi \Rightarrow \psi$ iff $\parallel \psi \parallel \in \mathcal{N}_w(\parallel \phi \parallel)$.

The authors then turn to proof theory, first consider a basic system of axioms and rules, called F (for filter), and various extensions.
We have seen some of the conditions already before, with $\mid\sim$ instead of $\Rightarrow$. For instance, Reflexivity reads now $\alpha \Rightarrow \alpha$.

Definition 1.7 The system F consists of of the following two axioms and four rules:
- all instances of classical tautologies and Reflexivity,
- Modus ponens: $\frac{\alpha \rightarrow \beta, \alpha}{\beta}$, Left logical equivalence, Right Weakening, AND.

Theorem 1.8 (Ben-David/Ben-Eliyahu)
The system F is sound and complete for the family of filter based models.

Ben-David/Ben-Eliyahu obtain among other results the following representation theorem:

Theorem 1.9 (Ben-David/Ben-Eliyahu)
Consider the following coherence properties for $\mathcal{N}$:
UC: $B \in \mathcal{N}_w(A) \to \mathcal{N}_w(A \cap B) \subseteq \mathcal{N}_w(A)$
DC: $B \in \mathcal{N}_w(A) \to \mathcal{N}_w(A) \subseteq \mathcal{N}_w(A \cap B)$
RBC: $\mathcal{N}_w(A) \cap \mathcal{N}_w(B) \subseteq \mathcal{N}_w(A \cup B)$
SRM: (re-written) $X \in \mathcal{N}_w(A) \to X \in \mathcal{N}_w(A \cap Y) \vee \boldsymbol{C}(Y) \in \mathcal{N}_w(A)$
GTS: $\mathcal{N}_w(A \cup B) \subseteq \mathcal{N}_w(A) \cap \mathcal{N}_w(B)$.
(Remark to our version of SRM: By $F_*(\parallel \alpha \parallel, \parallel \beta \parallel) = \{w : w \models \alpha \Rightarrow \beta\}$ and $\mathcal{N}_w(\alpha) := \{\parallel \beta \parallel : w \models \alpha \Rightarrow \beta\}$, $\mathcal{N}_w(\alpha) = \{\parallel \beta \parallel : w \in F_*(\parallel \alpha \parallel, \parallel \beta \parallel)\}$. So, essentially, $\mathcal{N}_w(A) = \{B : w \in F_*(A, B)\}$. Then $F_*(A, B) \subseteq F_*(A \cap Y, B) \cup F_*(A, U - Y)$ is equivalent with $X \in \mathcal{N}_w(A) \to X \in \mathcal{N}_w(A \cap Y) \vee \boldsymbol{C}(Y) \in \mathcal{N}_w(A)$.)
Then:
$F + CUT$ is sound and complete for FBM's satisfying UC,
$F + CM$ is sound and complete for FBM's satisfying DC,
P is sound and complete for FBM's satisfying UC,DC,RBC,
$F + RM$ is sound and complete for FBM's satisfying SRM,
$F + Monotony$ is sound and complete for FBM's satisfying GTS.

1.5.2 The system of Friedman/Halpern

Definition 1.10 (Friedman/Halpern)
Let U be a set, $\leq$ a partial order on some set D (i.e. $\leq$ is reflexive, transitive, anti-symmetric). Let $\perp, \boldsymbol{T} \in D$ with $\perp \leq d \leq \boldsymbol{T}$ for all $d \in D$. Let $Pl : \mathcal{P}(U) \to D$ s.t. $Pl(U) = \boldsymbol{T}$, $Pl(\emptyset) = \perp$, and the following conditions hold:
(A1) $A \subseteq B \to Pl(A) \leq Pl(B)$,
(A2) If A,B,C are pairwise disjoint, then $Pl(C) < Pl(A \cup B)$, $Pl(B) < Pl(A \cup C) \to Pl(B \cup C) < Pl(A)$,
(A2') $Pl(A - B) < Pl(A \cap B)$, $Pl(A - B\prime) < Pl(A \cap B\prime) \to Pl((A - B) \cup (A - B\prime)) < Pl(A \cap B \cap B\prime)$,
(A3) $Pl(A) = Pl(B) = \perp \to Pl(A \cup B) = \perp$.
Then Pl is called a qualitative plausibility measure, and (U, Pl) a qualitative plausibility space.

Fact 1.11 (Friedman/Halpern)
In the presence of (A1), (A2) and (A2') are equivalent.

Definition 1.12 (Friedman/Halpern) Given a qualitative plausibility space (U, Pl), a propositional language $\mathcal{L}$ with set of variables $v(\mathcal{L})$, and a truth assignment function $\pi : U \to \mathcal{P}(v(\mathcal{L}))$, (U, Pl, π) is called a qualitative plausibility structure. For a classical formula ϕ, $[\![\phi]\!]$ is the set of $w \in U$, where ϕ holds - the latter defined as usual.
Given a flat conditional $\phi \Rightarrow \psi$, we define $(U, Pl, \pi) \models_{Pl} \phi \Rightarrow \psi$ iff $Pl([\![\phi]\!]) = \perp$ or $Pl([\![\phi \wedge \psi]\!]) > Pl([\![\phi \wedge \neg\psi]\!])$

Given a set $\boldsymbol{P}$ of qualitative plausibility structures, a set Δ of flat conditionals, and a flat conditional $\phi \Rightarrow \psi$, we define $\Delta \models_{\boldsymbol{P}} \phi \Rightarrow \psi$ iff for all $(U, Pl, \pi) \in \boldsymbol{P}$ ($\forall \delta \in \Delta((U, Pl, \pi) \models_{Pl} \delta)$ implies $(U, Pl, \pi) \models_{Pl} \phi \Rightarrow \psi$).

Definition 1.13 (Friedman/Halpern)
A set $\boldsymbol{P}$ of qualitative plausibility structures is called rich iff for all sets $\{\phi_1 \ldots \phi_n\}$ of mutually exclusive classical formulas, there is a plausibility structure $(U, Pl, \pi) \in \boldsymbol{P}$ s.t. $\bot = Pl([\![\phi_1]\!]) < Pl([\![\phi_2]\!]) < \ldots < Pl([\![\phi_n]\!])$.

Theorem 1.14 (Friedman/Halpern)
Let Δ be a set of flat conditionals, $\phi \Rightarrow \psi$ a flat conditional. Let further $\Delta \vdash_P \phi \Rightarrow \psi$ denote that $\phi \Rightarrow \psi$ follows from Δ in the system P (see Definition 1.9).
If $\boldsymbol{P}$ is a set of qualitative plausibility structures, and $\Delta \vdash_P \phi \Rightarrow \psi$, then $\Delta \vdash_{\boldsymbol{P}} \phi \Rightarrow \psi$.
Conversely, a set of qualitative plausibility structures $\boldsymbol{P}$ is rich, iff for all sets Δ of flat conditionals, and all flat conditionals $\phi \Rightarrow \psi$, $\Delta \vdash_{\boldsymbol{P}} \phi \Rightarrow \psi$ implies $\Delta \vdash_P \phi \Rightarrow \psi$.

Friedman/Halpern then show that a number of well-known systems, e.g. (essentially) that of preferential reasoning, give rise to equivalent sets of qualitative plausibility structures, which satisfy the richness condition. Consequently, they are all characterized by the system P - despite their other differences.
We have modified Definition 1.10 slightly - see Definitions 3.1, 3.2 - and obtain with this modified version a very close connection between the systems of Ben-David/Ben-Eliyahu and Friedman/Halpern (see Proposition 3.3). The connection between the former and the original version of the latter is a bit looser (see Proposition 3.4).

2 RESULTS ON COHERENT SYSTEMS OF FILTERS

2.1 The system of Ben-David/Ben-Eliyahu

We remind the reader that we shall henceforth drop the indices w of the filter systems. We collect the modified conditions of Theorem 1.9 in the following

Definition 2.1 (Ben-David/Ben-Eliyahu)
Let $\mathcal{N} := \{\mathcal{N}(A) : A \subseteq U\}$, where each $\mathcal{N}(A)$ is a filter over U. We define the conditions:
UC: $B \in \mathcal{N}(A) \rightarrow \mathcal{N}(A \cap B) \subseteq \mathcal{N}(A)$,
DC: $B \in \mathcal{N}(A) \rightarrow \mathcal{N}(A) \subseteq \mathcal{N}(A \cap B)$,
RBC: $\mathcal{N}(A) \cap \mathcal{N}(B) \subseteq \mathcal{N}(A \cup B)$,
SRM: (re-written) $X \in \mathcal{N}(A) \rightarrow X \in \mathcal{N}(A \cap Y) \vee A - Y \in \mathcal{N}(A)$,
GTS: $\mathcal{N}(A \cup B) \subseteq \mathcal{N}(A) \cap \mathcal{N}(B)$.

We modify the system of Ben-David/Ben-Eliyahu slightly and obtain conditions, which are less elegant, but perhaps more intuitive. Their equivalence with the original version is shown in Proposition 2.4.

Fact 2.2 Let $A \subseteq U$, and $\mathcal{N}(A)$ be a filter over U, with $A \in \mathcal{N}(A)$. Then
(1) $\mathcal{N}\prime(A) := \{A \cap B : B \in \mathcal{N}(A)\}$ is a filter over A,
(2) $\mathcal{N}\prime(A) = \mathcal{N}(A) \cap \mathcal{P}(A)$,
(3) $\mathcal{N}(A) = \{C \subseteq U : \exists B \in \mathcal{N}\prime(A).B \subseteq C\}$.

Proof: (1) $A \in \mathcal{N}\prime(A)$ by prerequisite. If $A \cap B \subseteq C \subseteq A$, $B \in \mathcal{N}(A)$, then by $A \in \mathcal{N}(A)$ $A \cap B \in \mathcal{N}(A)$, so $C \in \mathcal{N}(A)$, and $C \in \mathcal{N}\prime(A)$. $A \cap B$, $A \cap B\prime \in \mathcal{N}\prime(A) \rightarrow A \cap B \cap B\prime \in \mathcal{N}\prime(A)$, as $B \cap B\prime \in \mathcal{N}(A)$.
(2) $A \cap B \in \mathcal{N}\prime(A) \rightarrow A \cap B \in \mathcal{N}(A)$. $B \in \mathcal{N}(A)$, $B \subseteq A \rightarrow A \cap B = B \in \mathcal{N}\prime(A)$.
(3) "$\subseteq$:" Let $C \in \mathcal{N}(A)$, then $C \cap A \in \mathcal{N}\prime(A)$ by definition. "$\supseteq$:" Let $C \subseteq U$, $\exists B \in \mathcal{N}\prime(A).B \subseteq C$. As $\mathcal{N}\prime(A) \subseteq \mathcal{N}(A)$ by (2), $B \in \mathcal{N}(A)$, so $C \in \mathcal{N}(A)$. $\square$

Definition 2.3 We define the coherence conditions for a modified system $\mathcal{N}\prime := \{\mathcal{N}\prime(A) : A \subseteq U\}$, each $\mathcal{N}\prime(A)$ a filter over A:
UC': $B \in \mathcal{N}\prime(A) \rightarrow \mathcal{N}\prime(B) \subseteq \mathcal{N}\prime(A)$,
DC': $B \in \mathcal{N}\prime(A) \rightarrow \mathcal{N}\prime(A) \cap \mathcal{P}(B) \subseteq \mathcal{N}\prime(B)$,
RBC': $X \in \mathcal{N}\prime(A)$, $Y \in \mathcal{N}\prime(B) \rightarrow X \cup Y \in \mathcal{N}\prime(A \cup B)$,
SRM': $X \in \mathcal{N}\prime(A)$, $Y \subseteq A \rightarrow A - Y \in \mathcal{N}\prime(A) \vee X \cap Y \in \mathcal{N}\prime(Y)$,
GTS': $C \in \mathcal{N}\prime(A)$, $B \subseteq A \rightarrow C \cap B \in \mathcal{N}\prime(B)$.

Proposition 2.4 If $\mathcal{N}$ and $\mathcal{N}\prime$ are interdefinable as in Fact 2.2, i.e. for given $\mathcal{N}$, $\mathcal{N}\prime$ is as defined by (1) or (2), for given $\mathcal{N}\prime$, $\mathcal{N}$ is defined by (3), then:
(1) UC for $\mathcal{N} \leftrightarrow$ UC' for $\mathcal{N}\prime$,
(2) DC for $\mathcal{N} \leftrightarrow$ DC' for $\mathcal{N}\prime$,
(3) RBC for $\mathcal{N} \leftrightarrow$ RBC' for $\mathcal{N}\prime$,
(4) SRM for $\mathcal{N} \leftrightarrow$ SRM' for $\mathcal{N}\prime$,
(5) GTS for $\mathcal{N} \leftrightarrow$ GTS' for $\mathcal{N}\prime$.

Proof: We use Fact 2.2.
(1) "$\rightarrow$:" Let $B \in \mathcal{N}\prime(A) \rightarrow B \in \mathcal{N}(A) \rightarrow \mathcal{N}(A \cap B) \subseteq \mathcal{N}(A) \rightarrow \mathcal{N}\prime(B) = \mathcal{N}\prime(A \cap B) = \mathcal{N}(A \cap B) \cap \mathcal{P}(A \cap B) \subseteq \mathcal{N}(A \cap B) \cap \mathcal{P}(A) \subseteq \mathcal{N}(A) \cap \mathcal{P}(A) = \mathcal{N}\prime(A)$. "$\leftarrow$:" Let $B \in \mathcal{N}(A) \rightarrow B \cap A \in \mathcal{N}\prime(A) \rightarrow \mathcal{N}\prime(A \cap B) \subseteq \mathcal{N}\prime(A)$. Let now $C \in \mathcal{N}(A \cap B)$. Then $C \cap A \cap B \in \mathcal{N}\prime(A \cap B) \subseteq \mathcal{N}\prime(A) \subseteq \mathcal{N}(A)$, so $C \in \mathcal{N}(A)$.
(2) "$\rightarrow$:" $B \in \mathcal{N}\prime(A) \subseteq \mathcal{N}(A) \rightarrow \mathcal{N}(A) \subseteq \mathcal{N}(A \cap B) \rightarrow \mathcal{N}\prime(A) \cap \mathcal{P}(B) =_{B \subseteq A} \mathcal{N}(A) \cap \mathcal{P}(B) \subseteq \mathcal{N}(A \cap B) \cap \mathcal{P}(B) =_{B \subseteq A} \mathcal{N}(A \cap B) \cap \mathcal{P}(A \cap B) = \mathcal{N}\prime(A \cap B) = \mathcal{N}\prime(B)$. "$\leftarrow$:" Let $B \in \mathcal{N}(A) \rightarrow B \cap A \in \mathcal{N}\prime(A) \rightarrow \mathcal{N}\prime(A) \cap \mathcal{P}(A \cap B) \subseteq \mathcal{N}\prime(A \cap B)$. Let $C \in \mathcal{N}(A)$, then $C \cap B \cap A \in \mathcal{N}\prime(A) \cap \mathcal{P}(A \cap B) \subseteq \mathcal{N}\prime(A \cap B) \rightarrow C \in \mathcal{N}(A \cap B)$.

(3) "$\rightarrow$:" $X \in \mathcal{N}\prime(A)$, $Y \in \mathcal{N}\prime(B) \rightarrow X \in \mathcal{N}(A)$, $Y \in \mathcal{N}(B) \rightarrow X \cup Y \in \mathcal{N}(A) \cap \mathcal{N}(B)$ $\rightarrow X \cup Y \in \mathcal{N}(A \cup B) \rightarrow_{X \subseteq A, Y \subseteq B} X \cup Y \in \mathcal{N}\prime(A \cup B)$. "$\leftarrow$:" $C \in \mathcal{N}(A) \cap \mathcal{N}(B)$ $\rightarrow C \cap A \in \mathcal{N}\prime(A)$, $C \cap B \in \mathcal{N}\prime(B) \rightarrow C \cap (A \cup B) \in \mathcal{N}\prime(A \cup B) \subseteq \mathcal{N}(A \cup B) \rightarrow$ $C \in \mathcal{N}(A \cup B)$.
(4) "$\rightarrow$:" Let $X \in \mathcal{N}\prime(A)$, $Y \subseteq A$. Then $X \in \mathcal{N}(A)$, so $X \in \mathcal{N}(Y)$ or $\boldsymbol{C}(Y) \in \mathcal{N}(A)$, so $X \cap Y \in \mathcal{N}\prime(Y)$ or $A - Y \in \mathcal{N}\prime(A)$. "$\leftarrow$:" Let $X \in \mathcal{N}(A)$, Y arbitrary. Then $X \cap A \in \mathcal{N}\prime(A)$, and by $Y \cap A \subseteq A$ $A - (Y \cap A) = A - Y \in \mathcal{N}\prime(A)$, so $\boldsymbol{C}(Y) \in \mathcal{N}(A)$, or $X \cap Y \cap A \in \mathcal{N}\prime(Y \cap A)$, so $X \in \mathcal{N}(A \cap Y)$.
(5) "$\rightarrow$:" Let $C \in \mathcal{N}\prime(A)$, $B \subseteq A$. Then $C \in \mathcal{N}(B \cup (A - B)) = \mathcal{N}(A) \rightarrow C \in \mathcal{N}(B) \rightarrow$ $C \cap B \in \mathcal{N}\prime(B)$. "$\leftarrow$:" $C \in \mathcal{N}(A \cup B) \rightarrow (C \cap A) \cup (C \cap B) \in \mathcal{N}\prime(A \cup B) \rightarrow C \cap A \in \mathcal{N}\prime(A)$, $C \cap B \in \mathcal{N}\prime(B) \rightarrow C \in \mathcal{N}(A) \cap \mathcal{N}(B)$. $\Box$
GTS/GTS' express monotony (see Ben-David/Ben-Eliyahu), and will not be considered any further.
From now on, we work with $\mathcal{N}\prime$. Furthermore, we restrict our attention to those $\mathcal{N}\prime(A)$, where $A \neq \emptyset$.

Fact 2.5 (a) DC' and RBC' entail: $Z \in \mathcal{N}\prime(Y)$, $Z \subseteq B$, $X - B \subseteq Y - Z \rightarrow B \in \mathcal{N}\prime(X \cup B)$,
(b) RBC' $\rightarrow$ UC'.

Proof: (a) $Z \in \mathcal{N}\prime(Y) \rightarrow Z \subseteq B \cap Y \in \mathcal{N}\prime(Y)$. $B \cap Y \in \mathcal{N}\prime(Y)$, $B \in \mathcal{N}\prime(B)$ $\rightarrow_{RBC'} B = (B \cap Y) \cup B \in \mathcal{N}\prime(B \cup Y)$. Thus $B \subseteq B \cup (X - B) \in \mathcal{N}\prime(B \cup Y)$, and $\mathcal{N}\prime(B \cup Y) \cap \mathcal{P}(B \cup (X - B)) \subseteq \mathcal{N}\prime(B \cup (X - B)) = \mathcal{N}\prime(X \cup B)$ by DC'.
(b) Let $B \in \mathcal{N}\prime(A)$, $X \in \mathcal{N}\prime(B)$. $X \in \mathcal{N}\prime(B) \rightarrow_{RBC'} X \cup (A - B) \in \mathcal{N}\prime(B \cup (A - B)) = \mathcal{N}\prime(A)$. So by $B \in \mathcal{N}\prime(A)$, and $X \subseteq B \subseteq A$, $X = (X \cup (A - B)) \cap B \in \mathcal{N}\prime(A)$. $\Box$

Fact 2.6 UC', DC', RBC' entail: $A, B \in \mathcal{N}\prime(A \cup B)$, $X \subseteq (A \cup C) \cap (B \cup C) \rightarrow ($ $X \in \mathcal{N}\prime(A \cup C) \leftrightarrow X \in \mathcal{N}\prime(B \cup C)$ $)$

Proof: $A, B \in \mathcal{N}\prime(A \cup B) \rightarrow A \cap B \in \mathcal{N}\prime(A \cup B) \rightarrow_{A \in \mathcal{N}\prime(A \cup B), DC'} A \cap B \in \mathcal{N}\prime(A)$ $\rightarrow_{RBC'} C \cup (A \cap B) \in \mathcal{N}\prime(A \cup C) \rightarrow_{UC', DC'} (X \in \mathcal{N}\prime(A \cup C) \leftrightarrow X \in \mathcal{N}\prime(C \cup (A \cap B))$ $)$. Likewise, by $A \cap B \in \mathcal{N}\prime(B)$, $(A \cap B) \cup C \in \mathcal{N}\prime(B \cup C)$ and $X \in \mathcal{N}\prime(B \cup C) \leftrightarrow$ $X \in \mathcal{N}\prime(C \cup (A \cap B))$ $\Box$

2.2 The system of the author

Assume from now on that $\emptyset \notin \mathcal{N}\prime(A)$ if $A \neq \emptyset$.
We present the author's conditions on coherent sets of small subsets, motivate the conditions through the system R, and show equivalence to the system of Ben-David/Ben-Eliyahu. The reader should recall that an ideal - intended to be the set of small subsets - is the dual of an filter - intended to be the set of large subsets. In our context, an

element of the filter contains normal cases, an element of the ideal contains exceptional cases. Normal birds fly, exceptional ones like penguins not, the latter are in minority.
(P0') $\emptyset$ is small in A (if $A \neq \emptyset$).
(P1') If $A \subseteq B \subseteq C \subseteq D$, and if B is small in C, then so will be A in C and B in D, and thus A in D too.
(P2') Any finite union of small subsets is small.
(P3') If $A \subseteq B$ is small, and we take away a small C from B, then A will still be small in B-C. (We assume here for simplicity $A \subseteq B - C$; if not, by (P1), $A - C$ is small in B, and then $A - C$ will be small in $B - C$.)
(P4') If $A \subseteq B$ is small, you can take away any C from B, which is not large, and A will still be small in the rest. (Again, without loss of generality $A \subseteq B - C$.)
More formally, we consider the conditions collected in the following

Definition 2.7 Let for $A \subseteq U$ $\mathcal{I}(A) \subseteq \mathcal{P}(A)$ be given.
(P0) If $A \neq \emptyset$, then $\emptyset \in \mathcal{I}(A)$,
(P1) if $A \subseteq B \subseteq C \subseteq D$, $B \in \mathcal{I}(C)$, then $A \in \mathcal{I}(C)$ and $B \in \mathcal{I}(D)$,
(P2) if $A, B \in \mathcal{I}(C)$, then $A \cup B \in \mathcal{I}(C)$,
(P3) if $A, C \in \mathcal{I}(B)$, then $A - C \in \mathcal{I}(B - C)$,
(P4) if $A \in \mathcal{I}(B)$, $C \subseteq B$, $B - C \notin \mathcal{I}(B)$, then $A - C \in \mathcal{I}(B - C)$.

Motivation of Definition 2.7, and connection to the system R (see Definition 1.3):
Let $\alpha \mid\sim \beta$ mean that we have few exceptions, i.e. that the set of α-points where β does not hold, is a small subset of the α-points. In other words, the set of β-worlds is large in the set of α-worlds.
AND corresponds to the principle (P2).
OR can be analyzed in the same way: $\alpha \wedge \neg\beta$ is small in α, so a fortiori in $\alpha \vee \gamma$, likewise for $\gamma \wedge \neg\beta$, so their union is small too. (By (P1)+(P2).)
Cautious Monotony corresponds to the principle (P3): $\alpha \wedge \beta$ differs from α only by a small quantity.
Rational Monotony corresponds to a much stronger principle, (P4). To see this, argue as follows: $\alpha \wedge \gamma$ can either be small in α, large in α, or "medium size", i.e. neither small nor large. If it is small, then $\alpha \mid\sim \neg\gamma$, if it is large, then the same principle as for Cautious Monotony carries through, and if it is medium size, we need something stronger - the above principle.
We might add another principle, which says that disjunctions are distributed evenly unless specified otherwise, this would give the Lehmann/ Magidor rational closure:
(P5): A kind of homogenousness, which says that unless specified otherwise - thus a default principle - predicates (and formulas in general) split each other up evenly, there are no "chance" $\alpha \mid\sim \beta$, all models which have such without necessity are excluded from the consideration. This principle corresponds in a certain way to direct scepticism. Direct scepticism says that, in the presence of a contradiction, we accept neither information, so

e.g. neither $\alpha \mid\sim \beta$, nor $\alpha \mid\sim \neg\beta$, so $\alpha \wedge \beta$ and $\alpha \wedge \neg\beta$ will have "medium size". (P5) is stronger, and says that, even without a contradiction, $\alpha\wedge\beta$ and $\alpha\wedge\neg\beta$ will have "medium size", whenever possible.

2.3 Equivalence of both systems

Proposition 2.8 $\mathcal{N}\prime$ satisfies UC', DC', RBC', SRM', iff the corresponding system of ideals $\mathcal{I}$ defined by $\mathcal{I}(A) := \{X : A - X \in \mathcal{N}\prime(A)\}$ satisfies (P0)-(P4).

Proof: "$\rightarrow$:" (P0) by $A \in \mathcal{N}\prime(A)$ (P1) $A \subseteq B \subseteq C$, B small in $C \rightarrow A$ small in C by the filter properties. $B \subseteq C \subseteq D$, B small in $C \rightarrow C - B \in \mathcal{N}\prime(C)$, $D - C \in \mathcal{N}\prime(D - C)$ $\rightarrow_{RBC'}$ D-B $= (C{-}B)\cup(D{-}C) \in \mathcal{N}\prime(D)$. (P2) by the filter properties. (P3) Let $A, C \subseteq B$, $A \cap C = \emptyset$. $B - A, B - C \in \mathcal{N}\prime(B) \rightarrow (B - C) - A = (B - A) \cap (B - C) \in \mathcal{N}\prime(B)$, $(B - C) - A \in \mathcal{N}\prime(B - C)$ by DC'. (P4) $B - A \in \mathcal{N}\prime(B)$, $C \not\in \mathcal{N}\prime(B)$, $(B - C) - A = (B - A) \cap (B - C) \in \mathcal{N}\prime(B - C)$ by SRM'.
$'' \leftarrow ''$: $\mathcal{N}\prime(A)$ is a filter: $\emptyset \subseteq A$ is small by (P0), so $A \in \mathcal{N}\prime(A)$. If $B \subseteq C \subseteq A$, $B \in \mathcal{N}\prime(A)$, then by (P1), A-C is small in A, so $C \in \mathcal{N}\prime(A)$. If $B, C \in \mathcal{N}\prime(A)$, then $B \cap C \in \mathcal{N}\prime(A)$ by (P2). DC': Let $B, C \in \mathcal{N}\prime(A)$, $C \subseteq B$, then $A - B$, $A - C$ are small in A, then $B - C$ is small in A, then $B - C$ is small in B by (P3), so $C \in \mathcal{N}\prime(B)$. RBC': $X \in \mathcal{N}\prime(A)$, $Y \in \mathcal{N}\prime(B)$, so $A - X$ is small in A, thus in $A \cup B$, likewise, $B - Y$ is small in $A \cup B$, so by (P2) $(A \cup B) - (X \cup Y) \subseteq (A - X) \cup (B - Y)$ is small in $A \cup B$, so $X \cup Y \in \mathcal{N}\prime(A \cup B)$. SRM': Let $X \in \mathcal{N}\prime(A)$, $Y \subseteq A$. Then $A - X$ is small in A, so $(A - X) \cap Y$ is small in A. If $A - Y \not\in \mathcal{N}\prime(A)$, then $(A - X) \cap Y$ is small in Y by (P4), so $X \cap Y = Y - ((A - X) \cap Y) \in \mathcal{N}\prime(Y)$. $\Box$

3 RESULTS ON PARTIAL ORDERS AND COHERENT SYSTEMS OF FILTERS

Definition 3.1 (Friedman/Halpern, modified)
Let U be a set, $<$ a strict partial order on $\mathcal{P}(U)$, (i.e. $<$ is transitive, and contains no cycles). Consider the following conditions for $<$:
(B1) $A\prime \subseteq A < B \subseteq B\prime \rightarrow A\prime < B\prime$,
(B2') $A - B < A \cap B$, $A - B\prime < A \cap B\prime \rightarrow (A - B) \cup (A - B\prime) < A \cap B \cap B\prime$,
(B2) if A,B,C are pairwise disjoint, then $C < A \cup B$, $B < A \cup C \rightarrow B \cup C < A$,
(B3) $\emptyset < X$ for all $X \neq \emptyset$,
(B4) $A < B \rightarrow A < B - A$,
(B5) Let $X, Y \subseteq A$. If $A - X < X$, then $Y < A - Y$ or $Y - X < X \cap Y$.

Fact 3.2 (essentially Friedman/Halpern)
In the presence of (B1), (B2) and (B2') are equivalent.

Proof: (B2) $\rightarrow$ (B2'): Assume without loss of generality $B, B\prime \subseteq A$. Set $A\prime\prime := B \cap B\prime$, $B\prime\prime := (A - B) \cap B\prime$, $C\prime\prime := A - B\prime$. $A\prime\prime, B\prime\prime, C\prime\prime$ are pairwise disjoint. We have $B\prime\prime = (A - B) \cap B\prime \subseteq$ A-B $< A \cap B = B = (B \cap B\prime) \cup (B \cap (A - B\prime)) \subseteq (B \cap B\prime) \cup (A - B\prime) = A\prime\prime \cup C\prime\prime$. $C\prime\prime =$ A-B' $< A \cap B\prime = B\prime = A\prime\prime \cup B\prime\prime$. So by (B2), $(A - B) \cup (A - B\prime) = B\prime\prime \cup C\prime\prime < A\prime\prime = B \cap B\prime$.
(B2') $\rightarrow$ (B2): Let A,B,C be pairwise disjoint, $C < A \cup B$, $B < A \cup C$. Set $A\prime\prime := A \cup B \cup C$, $B\prime\prime := A \cup B$, $C\prime\prime := A \cup C$. Then $A\prime\prime - B\prime\prime = C < A \cup B = B\prime\prime$, $A\prime\prime - C\prime\prime = B < A \cup C = C\prime\prime$. Thus $B \cup C = (A\prime\prime - B\prime\prime) \cup (A\prime\prime - C\prime\prime) < B\prime\prime \cap C\prime\prime = A$. $\Box$

Proposition 3.3 Let $<$ on $\mathcal{P}(U)$ satisfy (B1)-(B4), and $\mathcal{N}\prime$ be a coherent system of proper filters on U (i.e. for $A \subseteq U$ $\mathcal{N}\prime(A) \neq \mathcal{P}(A)$), satisfying UC', DC', RBC'. Define for $X \neq \emptyset$ $\mathcal{N}\prime_<(X) := \{B \subseteq X : X - B < B\}$, and $A <_{\mathcal{N}\prime} B :\leftrightarrow \exists X, Y(A \subseteq X - Y, Y \subseteq B, Y \in \mathcal{N}\prime(X), Y \neq \emptyset)$. (Consequently, $A <_{\mathcal{N}\prime} B \rightarrow B \neq \emptyset$.)
Then:
(1) Setting $\mathcal{N}\prime(X) := \mathcal{N}\prime_<(X)$, $\mathcal{N}\prime(X)$ is a proper filter, and UC', DC', RBC', hold for $\mathcal{N}\prime$.
(2) Setting $< := <_{\mathcal{N}\prime}$, $<$ will be transitive, cycle-free, and satisfy (B1)-(B4).
(3) The operations are inverse: $\mathcal{N}\prime(X) = \mathcal{N}\prime_{<_{\mathcal{N}\prime}}(X)$ and $< \, = \, <_{\mathcal{N}\prime_<}$.
(4) If (B5) holds for $<$, then SRM' holds for $\mathcal{N}\prime_<$. Conversely, if SRM' holds for $\mathcal{N}\prime$, then (B5) holds for $<_{\mathcal{N}\prime}$.

Proof: Note: (a) If $A \cap B = \emptyset$, and $A <_{\mathcal{N}\prime} B$, then $B \in \mathcal{N}\prime(A \cup B)$. (b) $B\prime \subseteq A$, $B \subseteq A\prime$, $B \in \mathcal{N}\prime(A)$, $B\prime \in \mathcal{N}\prime(A\prime) \rightarrow B \cap B\prime \neq \emptyset$.
Proof: (a) By prerequisite, $\exists X, Y(A \subseteq X - Y, Y \subseteq B, Y \in \mathcal{N}\prime(X))$. $Y \in \mathcal{N}\prime(X) \rightarrow Y \cup A \in \mathcal{N}\prime(X) \rightarrow_{DC\prime} Y \in \mathcal{N}\prime(Y \cup A)$. $Y \in \mathcal{N}\prime(Y \cup A)$, $B - Y \in \mathcal{N}\prime(B - Y) \rightarrow_{RBC\prime} B \in \mathcal{N}\prime(B \cup A)$. (b) $B \in \mathcal{N}\prime(A)$, $B \subseteq A\prime \rightarrow A \cap A\prime \in \mathcal{N}\prime(A) \rightarrow_{DC\prime} B \in \mathcal{N}\prime(A \cap A\prime)$, likewise $B\prime \in \mathcal{N}\prime(A \cap A\prime)$, so $B \cap B\prime \in \mathcal{N}\prime(A \cap A\prime)$, thus $B \cap B\prime \neq \emptyset$.
(1)
For $X \neq \emptyset$, $\mathcal{N}\prime(X)$ is a proper filter: 1. $X \in \mathcal{N}\prime(X)$ by (B3). 2. Let $B \in \mathcal{N}\prime(X)$, $B \subseteq B\prime \subseteq X$, then $X - B\prime \subseteq X - B < B \subseteq B\prime$, the result follows from (B1). 3. Let $B, B\prime \in \mathcal{N}\prime(X)$, then $X - B < B$, $X - B\prime < B\prime$, so $X - (B \cap B\prime) < B \cap B\prime$ by (B2'). 4. If $\emptyset \in \mathcal{N}\prime(X)$, then $X < \emptyset$, but $\emptyset < X$ by (B3), a contradiction.
UC' follows from RBC'.
DC': Let $B \in \mathcal{N}\prime(A)$, $C \subseteq B$, $C \in \mathcal{N}\prime(A)$, so $A - B < B$, and $A - C < C$, so by $C \subseteq B \subseteq A$ $B - C \subseteq A - C < C$, thus $C \in \mathcal{N}\prime(B)$.
RBC': Let $X \in \mathcal{N}\prime(A)$, $Y \in \mathcal{N}\prime(B)$, we have to show $X \cup Y \in \mathcal{N}\prime(A \cup B)$. By prerequisite, $A - X < X$, $B - Y < Y$, we have to show $(A \cup B) - (X \cup Y) < X \cup Y$. $X \cup Y$, $A - (X \cup Y)$, $B - (Y \cup A)$ are pairwise disjoint, and $(A \cup B) - (X \cup Y) = (A - (X \cup Y)) \cup (B - (Y \cup A))$. By prerequisite, $A - (X \cup Y) < X \cup Y$ and $B - (Y \cup A) < X \cup Y$. But if C,D,E are pairwise disjoint, and $C < E$, $D < E$, then $C \cup D < E : C < E \subseteq E \cup D$, $D < E \subseteq E \cup C \rightarrow C \cup D < E$ by (B1) and (B2). Thus, $(A \cup B) - (X \cup Y) < X \cup Y$.

(2)
Transitivity: Let $A < B$, $B < C$, so $\exists X, Y$ ($A \subseteq X - Y$, $Y \subseteq B$, $Y \in \mathcal{N}'(X)$, $Y \neq \emptyset$), $\exists X', Y'$ ($B \subseteq X' - Y'$, $Y' \subseteq C$, $Y' \in \mathcal{N}'(X')$, $Y' \neq \emptyset$). We will show $Y' - (X - Y) \in \mathcal{N}'(X' \cup (X-Y))$ and $Y' - (X-Y) \neq \emptyset$, which proves $A < C$, as $A \subseteq (X' \cup (X-Y)) - (Y' - (X-Y))$, and $Y' - (X-Y) \subseteq C$. Note that, by $Y \subseteq X'$, $\mathcal{N}'(X' \cup (X-Y)) = \mathcal{N}'(X' \cup X)$. First, $Y \in \mathcal{N}'(X) \rightarrow_{RBC'} Y \cup (Y' - X) \in \mathcal{N}'(X \cup (Y' - X)) = \mathcal{N}'(X \cup Y')$. Second, if $Z \in \mathcal{N}'(X \cup Y')$, then $Z \in \mathcal{N}'(X \cup X')$: By RBC' and $Y' \in \mathcal{N}'(X')$, $X \cup Y' \in \mathcal{N}'(X \cup X')$. Thus, if $Z \in \mathcal{N}'(X \cup Y')$, then $Z \in \mathcal{N}'(X \cup X')$ by UC'. Consequently, $Y \cup (Y' - X) \in \mathcal{N}'(X \cup X')$. Third, $Y' \in \mathcal{N}'(X \cup X') : Y \in \mathcal{N}'(X)$, $X' \in \mathcal{N}'(X') \rightarrow_{RBC'} X' = X' \cup Y \in \mathcal{N}'(X \cup X')$. Thus, by $Y' \in \mathcal{N}'(X')$ and UC', $Y' \in \mathcal{N}'(X \cup X')$. Finally, $Y \cup (Y' - X)$, $Y' \in \mathcal{N}'(X \cup X')$, so $Y' \cap (Y \cup (Y' - X)) \in \mathcal{N}'(X \cup X')$, but, as $Y \cap Y' = \emptyset$, $Y' \cap (Y \cup (Y' - X)) = Y' - X = Y' - (X - Y)$. Finally, suppose $Y' - X = \emptyset$, i.e. $Y' \subseteq X$. We thus have $Y \cap Y' = \emptyset$, $Y' \subseteq X$, $Y \subseteq X'$, $Y \in \mathcal{N}'(X)$, $Y' \in \mathcal{N}'(X')$, a contradiction to (b) above.
Acyclicity: By transitivity, it suffices to show that $A < A$ is impossible. $A < A \rightarrow \exists X, Y$ ($A \subseteq X - Y$, $Y \subseteq A$, $Y \in \mathcal{N}'(X)$, $Y \neq \emptyset$) $\rightarrow Y = \emptyset$, contradiction.
(B1') holds by definition of $<$.
(B2) Let A,B,C be disjoint. If $C < A \cup B$, $B < A \cup C$, then by (a) above, $A \cup B, A \cup C \in \mathcal{N}'(A \cup B \cup C) \rightarrow A = (A \cup B) \cap (A \cup C) \in \mathcal{N}'(A \cup B \cup C)$, so $B \cup C < A$. (Note that $A \neq \emptyset : A = \emptyset \rightarrow C < B < C$, a contradiction to acyclicity.)
(B3) trivial, as $X \in \mathcal{N}'(X)$.
(B4) $A < B \rightarrow \exists X, Y$ ($A \subseteq X - Y$, $Y \subseteq B$, $Y \in \mathcal{N}'(X)$). The same X,Y will show $A < B - A$.
(3)
Let $X \neq \emptyset$. $B \in \mathcal{N}'(X) \rightarrow B \in \mathcal{N}'_{<_{\mathcal{N}'}}(X)$: $B \in \mathcal{N}'(X) \rightarrow X - B <_{\mathcal{N}'} B$ (note that $B \neq \emptyset$) $\rightarrow B \in \mathcal{N}'_{<_{\mathcal{N}'}}(X)$.
$B \in \mathcal{N}'_{<_{\mathcal{N}'}}(X) \rightarrow B \in \mathcal{N}'(X)$: $B \in \mathcal{N}'_{<_{\mathcal{N}'}}(X) \rightarrow B \subseteq X$ and $X - B <_{\mathcal{N}'} B \rightarrow$ (by (a) above) $B \in \mathcal{N}'(X)$.
$A < B \rightarrow A <_{\mathcal{N}'_<} B$: $A < B \rightarrow B \neq \emptyset$ by (B3) and acyclicity, and by (B4) $(A \cup B) - (B - A) = A < B - A \rightarrow B - A \in \mathcal{N}'_<(A \cup B)$, thus $B - A \neq \emptyset$, and $A = (A \cup B) - (B - A) <_{\mathcal{N}'_<} B$.
$A <_{\mathcal{N}'_<} B \rightarrow A < B$: $A <_{\mathcal{N}'_<} B \rightarrow \exists X, Y$ ($A \subseteq X - Y$, $Y \subseteq B$, $Y \in \mathcal{N}'_<(X)$, $Y \neq \emptyset$). Thus $Y \subseteq X$, $X - Y < Y$. Thus $A \subseteq X - Y < Y \subseteq B \rightarrow A < B$ by (B1).
(4)
Let $X \in \mathcal{N}'_<(A)$, $Y \subseteq A$. Then $A - X < X$. If $Y < A - Y$, then $A - Y \in \mathcal{N}'_<(A)$. If $Y - X < X \cap Y$, then $Y - X = Y - (X \cap Y) < X \cap Y$, and $X \cap Y \in \mathcal{N}'_<(Y)$. Let $A - X <_{\mathcal{N}'} X$, $X, Y \subseteq A$. Then, by (a), $X \in \mathcal{N}'(A)$, so by SRM', $A - Y \in \mathcal{N}'(A)$, thus $Y <_{\mathcal{N}'} A - Y$, or $X \cap Y \in \mathcal{N}'(Y)$, thus $Y - (X \cap Y) = Y - X <_{\mathcal{N}'} X \cap Y$. □
We work now with the original plausibility spaces as defined in Definition 1.10.

Proposition 3.4 (Friedman/Halpern/Schlechta)

To represent Pl's, where $Pl(X) = \bot$ for some $X \neq \emptyset$, we need now degenerate filters, where $\mathcal{N}\prime(X) = \mathcal{P}(X)$.
(1) Let D and Pl, and $\leq$ be as in Definition 1.10. Define $\mathcal{N}\prime_{\leq}(X) := \{B \subseteq X : Pl(X - B) < Pl(B)\}$ if $Pl(X) \neq \bot$, and $\mathcal{N}\prime_{\leq}(X) := \mathcal{P}(X)$ if $Pl(X) = \bot$. Then $\mathcal{N}\prime(X) := \mathcal{N}\prime_{\leq}(X)$ is a filter, and UC', DC', RBC', hold for $\mathcal{N}\prime$.
(2) Let $\mathcal{N}\prime$ be a coherent system of filters, satisfying UC', DC', RBC'. Then there is a Plausibility space D, ordered by some $\leq_{\mathcal{N}\prime}$, and $Pl : \mathcal{P}(U) \rightarrow D$, satisfying (A1)-(A3) s.t. $Pl(A) \leq_{\mathcal{N}\prime} Pl(B) :\leftrightarrow B \in \mathcal{N}\prime(A \cup B)$

Proof: (a)
We first note: If $A \subseteq B$, then $B - A \in \mathcal{N}\prime(B)$ iff ($(Pl(A) <_{\mathcal{N}\prime} Pl(B - A)$ or $\mathcal{N}\prime(B) = \mathcal{P}(B))$). For, if $B - A \in \mathcal{N}\prime(B)$, then $Pl(A) \leq_{\mathcal{N}\prime} Pl(B - A)$. If $Pl(B - A) \leq_{\mathcal{N}\prime} Pl(A)$ holds too, then $A \in \mathcal{N}\prime(B)$, so $\mathcal{N}\prime(B) = \mathcal{P}(B)$. The converse is trivial.
(1)
If $Pl(X) \neq \bot$, $\mathcal{N}\prime(X)$ is a proper filter: 1. $X \in \mathcal{N}\prime(X)$ by $Pl(\emptyset) = \bot < Pl(X)$. 2. Let $B \in \mathcal{N}\prime(X)$, $B \subseteq B\prime \subseteq X$, then $PL(X - B\prime) \leq PL(X - B) < PL(B) \leq Pl(B\prime)$. 3. Let $B, B\prime \in \mathcal{N}\prime(X)$, then $PL(X-B) < PL(B)$, $PL(X-B\prime) < PL(B\prime)$, so $PL(X-(B \cap B\prime)) < PL(B \cap B\prime)$ by (A2'). 4. $\emptyset \notin \mathcal{N}\prime(X) : \emptyset \in \mathcal{N}\prime(X) \rightarrow Pl(X) < Pl(\emptyset)$, *contradiction.*
UC' follows from RBC'.
DC': If $Pl(A) \neq \bot$: Let $B \in \mathcal{N}\prime(A)$, $C \subseteq B$, $C \in \mathcal{N}\prime(A)$, so $PL(A - B) < PL(B)$, and $PL(A - C) < PL(C)$, so by $C \subseteq B \subseteq A$ $PL(B - C) \leq PL(A - C) < PL(C)$, thus $C \in \mathcal{N}\prime(B)$. If $Pl(A) = \bot$, then by $B \subseteq A$ $Pl(B) = \bot$, too.
RBC': Let $X \in \mathcal{N}\prime(A)$, $Y \in \mathcal{N}\prime(B)$, we have to show $X \cup Y \in \mathcal{N}\prime(A \cup B)$. If $Pl(A), Pl(B) \neq \bot$: By prerequisite, $Pl(A - X) < Pl(X)$, $Pl(B - Y) < Pl(Y)$, we have to show $Pl((A \cup B) - (X \cup Y)) < Pl(X \cup Y)$. $X \cup Y$, $A - (X \cup Y)$, $B - (Y \cup A)$ are pairwise disjoint, and $(A \cup B) - (X \cup Y) = (A - (X \cup Y)) \cup (B - (Y \cup A))$. By prerequisite, $Pl(A - (X \cup Y)) < Pl(X \cup Y)$ and $Pl(B - (Y \cup A)) < Pl(X \cup Y)$. But if C,D,E are pairwise disjoint, and $Pl(C) < Pl(E)$, $Pl(D) < Pl(E)$, then $Pl(C \cup D) < Pl(E)$: $Pl(C) < Pl(E) \leq Pl(E \cup D)$, $Pl(D) < Pl(E) \leq Pl(E \cup C) \rightarrow Pl(C \cup D) < Pl(E)$ by (A1) and (A2). Thus, $Pl((A \cup B) - (X \cup Y)) < Pl(X \cup Y)$. If $Pl(A) = \bot$, $Pl(B) \neq \bot$, then $Pl(Y) \neq \bot$, and $\bot = Pl((A - X) - Y) < Pl(Y)$, $Pl((B - Y) - A) < Pl(Y)$, so by (A2) $Pl((A \cup B) - (X \cup Y)) < Pl(Y)$, so $Pl((A \cup B) - (X \cup Y)) < Pl(X \cup Y)$, and $X \cup Y \in \mathcal{N}\prime(A \cup B)$. If $Pl(A) = Pl(B) = \bot$, then by (A3) $Pl(A \cup B) = \bot$, and $\mathcal{N}\prime(A \cup B) = \mathcal{P}(A \cup B)$.
(2)
We have to define the partial order, check that it is transitive, reflexive, and antisymmetric, and verify (A1), (A2), (A3). Let $A \approx B :\leftrightarrow A, B \in \mathcal{N}\prime(A \cup B)$. Note that $A \approx \emptyset$ iff $\mathcal{N}\prime(A) = \mathcal{P}(A)$: $A \approx \emptyset \leftrightarrow \emptyset \in \mathcal{N}\prime(A) \leftrightarrow \mathcal{N}\prime(A) = \mathcal{P}(A)$. We show that $\approx$ is an equivalence relation. Obviously, $A \approx A$, $A \approx B \rightarrow B \approx A$. Moreover, if $A \approx B$, $B \approx C$, then $B, C \in \mathcal{N}\prime(B \cup C)$, so $A \in \mathcal{N}\prime(A \cup B) \leftrightarrow A \in \mathcal{N}\prime(A \cup C)$ by Fact 2.6, so $A \in \mathcal{N}\prime(A \cup C)$. Likewise, $C \in \mathcal{N}\prime(A \cup C) \leftrightarrow C \in \mathcal{N}\prime(B \cup C)$, so $C \in \mathcal{N}\prime(A \cup C)$.

Take now $D :=$ the set of $\approx$-equivalence classes [A] for $A \subseteq U$.
Define $[A] <_{\mathcal{N}'} [B]$ iff $B \in \mathcal{N}'(A \cup B)$, but not $A \in \mathcal{N}'(A \cup B)$. This is well defined: Suppose $A \approx A'$, $B \approx B'$. If $B \in \mathcal{N}'(A \cup B)$, then $B \approx B' \to B \cap B' \in \mathcal{N}'(B \cup B') \to$ (by Fact 2.6) $B \cap B' \in \mathcal{N}'(B \cup B) = \mathcal{N}'(B) \to$ (by $B \in \mathcal{N}'(A \cup B)$) $B \cap B' \in \mathcal{N}'(A \cup B)$ $\to$ (by Fact 2.6) $B \cap B' \in \mathcal{N}'(A' \cup B')$, so $B' \in \mathcal{N}'(A' \cup B')$.
We check the conditions on $\leq$: (with $[A] \leq [B]$ iff $[A] < [B]$ or $[A] = [B]$, i.e. iff $B \in \mathcal{N}'(A \cup B)$) Reflexivity: $[A] \leq [A]$ is trivial. Antisymmetry: $[A] \leq [B] \leq [A] \to A \approx B \to [A] = [B]$. Transitivity: $[A] < [B] < [C] \to B \in \mathcal{N}'(A \cup B)$, $C \in \mathcal{N}'(B \cup C)$ $\to B \cup C \in \mathcal{N}'(A \cup B \cup C) \to C \in \mathcal{N}'(A \cup B \cup C) \to C \in \mathcal{N}'(A \cup C)$. On the other hand, $A \notin \mathcal{N}'(A \cup C)$. For $A \in \mathcal{N}'(A \cup C)$, $C \in \mathcal{N}'(B \cup C) \to A \cup C \in \mathcal{N}'(A \cup B \cup C)$ $\to A \in \mathcal{N}'(A \cup B \cup C) \to A \in \mathcal{N}'(A \cup B)$, *contradiction*.
Finally, define $Pl(A) := [A]$. Take $\perp := [\emptyset]$, $T := [U]$. Then $[\emptyset] \leq_{\mathcal{N}'} [A] \leq_{\mathcal{N}'} [U]$, as $A \in \mathcal{N}'(A)$, $U \in \mathcal{N}'(U)$.
It remains to show (A1), (A2), (A3). (A1) Let $A \subseteq B$. Then $Pl(A) \leq_{\mathcal{N}'} Pl(B)$, by $B \in \mathcal{N}'(A \cup B) = \mathcal{N}'(B)$. (A2') We have to show $Pl(A \cap B) > Pl(A - B)$, $Pl(A \cap B') > Pl(A - B') \to Pl(A \cap B \cap B') > Pl(A - (B \cap B'))$. If $B, B \not\subseteq A$, consider $B^* := A \cap B$, $B'^* := A \cap B'$, so without loss of generality $B, B' \subseteq A$. Then $Pl(B) > Pl(A - B)$, $Pl(B') > Pl(A - B')$, so $B, B' \in \mathcal{N}'(A)$, and $\mathcal{N}'(A) \neq \mathcal{P}(A)$, so $B \cap B' \in \mathcal{N}'(A)$, so $Pl(B \cap B') \geq Pl(A - (B \cap B'))$, and by $\mathcal{N}'(A) \neq \mathcal{P}(A)$ $Pl(B \cap B') > Pl(A - (B \cap B'))$.
(A3) $Pl(A) = Pl(B) = \perp \to \emptyset \in \mathcal{N}'(A)$, $\emptyset \in \mathcal{N}'(B) \to \emptyset \in \mathcal{N}'(A \cup B)$ by RBC' $\to$ $\mathcal{N}'(A \cup B) = \mathcal{P}(A \cup B) \to Pl(A \cup B) = \perp$. □

Remark 3.5 We conclude with a short remark on the central property of (minimal) preferential structures (see (1) in Section 1.4.1). This property corresponds to the coherence property
(F') $A \subseteq B$, $Y \in \mathcal{N}'(A) \to (B - A) \cup Y \in \mathcal{N}'(B)$.
(F') is a consequence of RBC', and corresponds to the following $<$-property: $A \subseteq B$, $A - Y < Y \to A - Y < (B - A) \cup Y$. □

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