Some Results on Classical Preferential Models

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Abstract

We first show that a result of Kraus, Lehmann, Magidor on classical preferential models does not carry over to the general infinite case. We further show that - in the absence of all restrictions on finiteness - "logically nice" (definability preserving) classical preferential models correspond essentially to infinite conditionalisation.

1 INTRODUCTION

Semantically, Circumscription corresponds to minimal classical models : the models of a circumscribed theory are roughly those in which the extension of some predicate is minimal by set inclusion. This was generalized by Shoham (see e.g. [Sho87]) to consider models, ordered by some binary relation. On the other hand, the general proof-theoretic properties of non-monotonic logic were examined by Gabbay (see [Gab85]) et al. A confluence of both directions of research is found in [KLM90], where S.Kraus, D.Lehmann, M.Magidor have linked proof-theoretic to semantic approaches of non-monotonic logics in a number of important representation theorems. They use, however, some finiteness assumptions whose relevance is shown in Section 2, where we give an example which proves that one of their results fails in the general infinitary version. In the following Section 3 we work without any such finitary restrictions and obtain a positive result. We show the central role of infinite conditionalization, discussed in [$Mak\infty$], Section 3.4. As a matter of fact, this property almost fully characterizes "logically nice" - definability preserving - classical preferential models (Theorem 3.1). Further detailed discussions of semantic and proof-theoretic properties of non-monotonic logics can be found in [Mak89], and especially in the already quoted [$Mak\infty$].

We shall use some set theoretic results and prerequisites, in particular, the axiom of choice will be assumed to hold, and some reformulations and consequences thereof, e.g. the usual elementary cardinal arithmetic, are used without explicit reference. The technique of definition and proof by infinite recursion will also be used without further explanation. We assume familiarity with these matters, which the interested reader can find e.g. in [Jec78] or [Kun80]. We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \rightarrow \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general cartesian product, card(X) shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets. Given a class of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X} \upharpoonright X := \{< x, i > \in \mathcal{X} : x \in X\}$, so if $\mathcal{X}$ is a function f, $f \upharpoonright X$ is the usual notation for the restriction of f to a subset of its domain.

In the rest of the introduction, we collect some basic definitions and examples.

Definition 1.1 *Let $\mathcal{L}$ be a propositional language, we denote by $v(\mathcal{L})$ the set of its variables, by $M_\mathcal{L}$ the set of its classical models, ϕ etc. shall denote formulas, T etc. theories in $\mathcal{L}$ (i.e. $T \subseteq \mathcal{L}$), and $M_T \subseteq M_\mathcal{L}$ the models of T. $\overline{T} \subseteq \mathcal{L}$ will denote the closure of T under classical logic. Given some other logic, $\overline{\overline{T}}$ will denote the set of consequences of T under that logic, i.e. if the more conventional notation for the logic is $\mid\!\sim$, then $\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$. $\mathcal{D}_\mathcal{L} \subseteq \mathcal{P}(M_\mathcal{L})$ shall be the set of definable subsets of $M_\mathcal{L}$, i.e. $A \in \mathcal{D}_\mathcal{L}$ iff there is some $T \subseteq \mathcal{L}$ s.th. $A = M_T$. If the context is clear, we omit the subscript $\mathcal{L}$*

from $\mathcal{D}_{\mathcal{L}}$*. Fact 1.1 and Example 1.1 below will show that, in general, not all subsets of* $M_{\mathcal{L}}$ *are definable. For* $X \subseteq \mathcal{P}(M_{\mathcal{L}})$ *, a function* $f : X \to \mathcal{P}(M_{\mathcal{L}})$ *will be called definability preserving (dp), iff for all* $Y \in \mathcal{D}_{\mathcal{L}} \cap X f(Y) \in \mathcal{D}_{\mathcal{L}}$*. If* $\mathcal{D}_{\mathcal{L}} \subseteq X$*, then* $f : X \to \mathcal{P}(M_{\mathcal{L}})$ *defines a logic* $T \mapsto T^f$ *on* $\mathcal{L}$ *by* $T^f := \{\phi : \forall m \in f(M_T).m \models \phi\}$*. So, if f=id, then* $T^f = \overline{T}$*. Note that* $f(M_T) \subseteq M_{T^f}$ *always holds, but not necessarily* $f(M_T) = M_{T^f}$*, the latter only iff f is dp - see Example 1.1 (1) below. The logics we shall consider here will be strengthenings of classical logic, and will often be defined semantically by some such f, i.e.* $\overline{\overline{T}} := T^f \supseteq \overline{T}$*. Thus, for such f,* $f(M_T) \subseteq M_{T^f} \subseteq M_T$*.*

Fact 1.1 *1. If* $v(\mathcal{L})$ *is infinite, then* $\mathcal{D}_{\mathcal{L}} \neq \mathcal{P}(M_{\mathcal{L}})$ *2.* $\emptyset, M_{\mathcal{L}} \in \mathcal{D}_{\mathcal{L}}$ *3.* $\mathcal{D}_{\mathcal{L}}$ *contains all singletons 4.* $\mathcal{D}_{\mathcal{L}}$ *is closed under arbitrary intersections 5.* $\mathcal{D}_{\mathcal{L}}$ *is closed under finite unions. (Thus, the elements of* $\mathcal{D}_{\mathcal{L}}$ *are the closed sets of a suitable* $T_1-Topology$ *on* $M_{\mathcal{L}}$*. The reader will find a discussion of separation properties in point set topology e.g. in [Eng77] or [Kel75].)*

Proof : 1.: Let $card(v(\mathcal{L})) = \kappa \geq \omega$, then, as each $\phi \in \mathcal{L}$ is finite, $card(\mathcal{L}) = \kappa$, so $card(\mathcal{D}) \leq card(\mathcal{P}(\mathcal{L})) = 2^\kappa$, but $card(M_{\mathcal{L}}) = 2^\kappa$, so $card(\mathcal{P}(M_{\mathcal{L}})) = 2^{(2^\kappa)} > 2^\kappa$. (The argument collapses in the finite case, as then $card(v(\mathcal{L})) < \omega = card(\mathcal{L})$.) 4.: let $A_i \in \mathcal{D}, i \in I, A_i = M_{T_i}$, T:= $\bigcup\{T_i : i \in I\}$, $A := M_T$. Then $m \in \bigcap\{A_i : i \in I\} \leftrightarrow \forall i \in I.m \models T_i \leftrightarrow m \in A$. 5.: let $A = M_T$, $A' = M_{T'}$, and $B = M_{T \vee T'}$, where $T \vee T' := \{\phi \vee \psi : \phi \in T, \psi \in T'\}$. Then $m \in A \cup A' \leftrightarrow \forall \phi \in T.m \models \phi \vee \forall \psi \in T'.m \models \psi \leftrightarrow \forall \phi \vee \psi \in T \vee T'.m \models \phi \vee \psi \leftrightarrow m \in B$. □

Example 1.1 *Let* $v(\mathcal{L}) := \{p_i : i \in \omega\}$ *, and* $m_0, m_i^- \in M_{\mathcal{L}}$ *be defined by* $m_0 \models \{p_i : i < \omega\}$ *and* $m_i^- \models \{\neg p_i\} \cup \{p_j : j < \omega, i \neq j\}$*.*

(1) Let $M' := M_{\mathcal{L}} - \{m_0\}$*. We show that M' is not definable. Suppose there is* ϕ *s.th.* ϕ *holds in all* $m \in M'$*, but not in* m_0*, so* $m_0 \models \neg\phi$*, but by finiteness of* ϕ*, there is a finite fragment of* m_0*, consisting of the propositional variables in* ϕ*, which decides* ϕ*, and there is* $m \in M'$*, which coincides with*

m_0 on that finite fragment, so $m \models \neg\phi$, contradiction. (Each $m \in M_{\mathcal{L}}$ is essentially a function $f_m : v(\mathcal{L}) \to 2$, and if $a \subseteq v(\mathcal{L})$ contains the propositional variables of ϕ, then for all m,m' s.th. $f_m \upharpoonright a = f_{m'} \upharpoonright a$ we have $m \models \phi$ iff $m' \models \phi$.) So M' is not definable, and $f : \mathcal{D} \to \mathcal{P}(M_{\mathcal{L}})$ defined by

$$f(M_T) := \begin{cases} M' & \text{iff } M_T = M_{\mathcal{L}} \\ M_T & \text{otherwise} \end{cases}$$

is not definability preserving.

(2) We conclude by giving an example of a countable definable set of models, and a countable set of models which is not definable. Let $T := \{p_i \vee p_j : i,j < \omega, i \neq j\}$, $M := \{m_0\} \cup \{m_i^- : i < \omega\}, M' := \{m_i^- : i < \omega\}$. Obviously, M and M' are countable, and $M_T = M$: If $m \in M_{\mathcal{L}}$ makes more than one p_i false, T does not hold any more in m. But M' is not definable : The same argument as in (1) works, as for each finite fragment of m_0 there is $m' \in M'$ s.th. m' and m_0 agree on that fragment. □

Definition 1.2 *$\mathcal{Z} =< \mathcal{X}, \prec >$ will be called a preferential structure iff $\mathcal{X}$ is a set of pairs and $\prec$ is a binary relation on $\mathcal{X}$. We say that $\mathcal{Z}$ is transitive, irreflexive etc., iff $\prec$ is. $< y, i >$ is called a minimal element of $\mathcal{X} \upharpoonright Y$ in $\mathcal{Z}$ iff : 1. $< y, i > \in \mathcal{X} \upharpoonright Y$ and 2. there is no $< y', i' > \in \mathcal{X} \upharpoonright Y$ s.th. $< y', i' > \prec < y, i >$. Thus, $\mathcal{Z}$ defines a function $\mu_{\mathcal{Z}} : V \to V$ (V the set-theoretic universe) by $\mu_{\mathcal{Z}}(Y) := \{y :$ there is i s.th. $< y, i >$ is a minimal element of $\mathcal{X} \upharpoonright Y\}$. (Note that $\mu_{\mathcal{Z}}$ is thus a proper class, but this need not bother us.) Given a set Z, $\mu_{\mathcal{Z},Z}$ shall denote $\mu_{\mathcal{Z}} \upharpoonright \mathcal{P}(Z)$. (A short motivation for indexing : if there is just one copy of y in $\mathcal{X}$, and e.g. $y' \prec y$, then, for $y \in Y$, y will not be minimal in Y if $y' \in Y$. If we want two y' , y" necessary in Y for y not to be minimal, we need something like y' $\prec < y, 0 >$, y" $\prec < y, 1 >$. So the different $< y, i >$, $< y, j >$ encode conjunction, the different y' $\prec < y, i >$, y" $\prec < y, i >$ disjunction : in other words, we look at the product.)*

Example 1.2 *$\mathcal{Z} =< \mathcal{X}, \prec>$ will be called $\mathcal{Y}-$ smooth (terminology of [KLM90]) or $\mathcal{Y}-$ stoppered (terminology of [$Mak\infty$]) iff for all $X \in \mathcal{Y}$ and $< y, i > \in \mathcal{X} \uparrow X$, either $< y, i >$ is minimal in $\mathcal{X} \uparrow X$, or there is $< y', i' > \prec < y, i >, < y', i' >$ minimal in $\mathcal{X} \uparrow X$. In shorthand, all non-minimal elements are "killed" by minimal ones. It is an immediate and important consequence that then for $X \in \mathcal{Y}$, $\mathcal{X} \uparrow X \neq \emptyset$ implies $\mu_{\mathcal{Z}}(X) \neq \emptyset$. The following example shows that there is $\mathcal{Z}$ V-smooth, and $x \in \omega + 1$, s.th. there is X, $x \in X - \mu_{\mathcal{Z}}(X)$, but there is no such X minimal by set-inclusion : Let $\mathcal{Z} := < \mathcal{X}, \prec>$, where $\mathcal{X} := \{< \omega, i >: i < \omega\} \cup \{< m, 0 >: m < \omega\}$, and $< m, 0 > \prec < \omega, i >$ iff $i \leq m$. Consider now $\mu := \mu_{\mathcal{Z}}$, then $\omega \in X - \mu(X)$ iff $\omega \in X$ and X contains a cofinal subset of ω - but, of course, there is no minimal such X.*

Remark 1.2 *Consider $\mathcal{Z} =< \mathcal{X}, \prec>$. 1) Note that $\mathcal{X}$ and Y may be finite, $\mathcal{X} \uparrow Y \neq \emptyset$, and still $\mu_{\mathcal{Z}}(Y) = \emptyset$: Just look at $\mathcal{Z} :=< \{< m, 0 >, < m, 1 >\}, < m, 0 > \prec < m, 1 > \prec < m, 0 >>$, and Y:={m}. Such "suicidal techniques" can be used extensively for general $\prec$. 2) Note that given any Z and $\mathcal{Z}$, there is a structure $\mathcal{Z}' :=< \mathcal{X}', \prec'>$ s.th. $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$, and, for all $z \in Z$, there is $< z, i > \in \mathcal{X}'$, if we admit cycles. (Alternatively, one might use infinite descending chains, see Lemma 3.2 below.)*

Proof : Let $\mathcal{Z} = < \mathcal{X}, \prec>$ be given and $Y := \{z \in Z :$ *there* is no $< z, i > \in \mathcal{X}\} \neq \emptyset$. Define $\mathcal{Z}' :=< \mathcal{X}', \prec'>$ by $\mathcal{X}' := \mathcal{X} \cup \{< x, j > : x \in Y$, $j \in 2\}$. Let $\prec'$ agree with $\prec$ on $\mathcal{X}$, and extend by $< x, 0 > \prec' < x, 1 > \prec' < x, 0 >$ for all $x \in Y$. Note that the construction preserves transitivity, i.e. if $\prec$ was transitive, then so is $\prec'$. We show $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$. Let X be given, $x \in X$. Case 1 : $x \in Y$: Then $x \notin \mu_{\mathcal{Z}}(X)$, since there is no i s.th. $< x, i > \in \mathcal{X}$. But the only $< x, i > \in \mathcal{X}'$ are $< x, 0 >$ and $< x, 1 >$, and $< x, 0 > \prec' < x, 1 > \prec' < x, 0 >$, so $x \notin \mu_{\mathcal{Z}'}(X)$. Case 2 : $x \in Z - Y$: Note that "old" and "new" elements are not comparable, so $x \in \mu_{\mathcal{Z}}(X)$ iff $x \in \mu_{\mathcal{Z}'}(X)$. $\square$

Definition 1.3 *A preferential structure $\mathcal{M} =< \mathcal{X}, \prec>$ will be called a clas-*

sical preferential model (cpm) for $\mathcal{L}$, iff for all $< x, i > \in \mathcal{X}, x \in M_{\mathcal{L}}$. $\mathcal{M}$ will be called definability preserving (dp) iff $\mu := \mu_{\mathcal{M},M_{\mathcal{L}}} : \mathcal{P}(M_{\mathcal{L}}) \to \mathcal{P}(M_{\mathcal{L}})$ is definability preserving. By the above, $\mathcal{M}$ defines a logic on $\mathcal{L}$ by $T^{\mathcal{M}} := T^{\mu}$, i.e. $T^{\mathcal{M}} := \{\phi \in \mathcal{L} : \phi$ holds in all $m \in \mu(M_T)\}$. Note, that if $\mathcal{M}$ is dp, $\mu(M_T) = M_{T^{\mathcal{M}}}$. A logic $^{=}$ for $\mathcal{L}$ is said to be representable by a cpm, iff there is a cpm $\mathcal{M}$ for $\mathcal{L}$, s.th. for all $T \subseteq \mathcal{L}$ $T^{\mathcal{M}} = \overline{\overline{T}}$. For $< m, i > \in \mathcal{X}$, we shall abuse notation and say $< m, i > \models \phi$ iff $m \models \phi$, for $\phi \in \mathcal{L}$.

Remark : Our definition is a notational variant of e.g. the definition in [KLM90] : The function $l : X \to M_{\mathcal{L}}$ in [KLM90] has the same meaning as our indices : l need not be injective. Neither need l be onto, and we do not require for all $m \in M_{\mathcal{L}}$ some $< m, i > \in \mathcal{X}$. $\models\!\!\!\equiv$ corresponds to our "abuse of notation" just introduced.

Example 1.3 *We give two examples for cpm's which are not definability preserving, they will be used later on. Let $v(\mathcal{L}) := \{p_i : i \in \omega\}, m_0, m_0^- \in M_{\mathcal{L}}$ be defined by $m_0 \models \{p_i : i \in \omega\}, m_0^- \models \{\neg p_0\} \cup \{p_i : 0 < i < \omega\}$.*

(1) Let $\mathcal{M} :=< M_{\mathcal{L}} x\{0\}, \prec>$ where only $< m_0^-, 0 > \prec < m_0, 0 >$, i.e. just two models are comparable. Let $\mu := \mu_{\mathcal{M},M_{\mathcal{L}}}, T := \emptyset, T' := \{p_i : 0 < i < \omega\}$. We have $M_T = M_{\mathcal{L}}, \mu(M_T) = M_{\mathcal{L}} - \{m_0\}, M_{T'} = \{m_0, m_0^-\}, \mu(M_{T'}) = \{m_0^-\}$. So by the result of Example 1.1, $\mathcal{M}$ is not dp, and, furthermore, $T^{\mathcal{M}} = \overline{T}, T'^{\mathcal{M}} = \overline{\{\neg p_0\} \cup \{p_i : 0 < i < \omega\}}$, so $\neg p_0 \in (T \cup T')^{\mathcal{M}}$, but $\overline{T^{\mathcal{M}} \cup T'} = \overline{\overline{T} \cup T'} = \overline{T'}$, so$\neg p_0 \notin \overline{T'}$. (Thus, Theorem 3.1 below can fail without the dp-condition.) D.Makinson gives in $[Mak\infty]$, Section 3.4, below Observation 3.4.8, another example of a preferential model violating condition $(^{=}4)$, and definability preservation.

(2) Let $M' := M_{\mathcal{L}} - \{m_0\}$. Consider now $\mathcal{M} := < \mathcal{X}, \prec>$, where $\mathcal{X} := \{< m, 0 >: m \in M'\} \cup \{< m_0, m >: m \in M'\}$, and define $\prec$ by $< m, 0 > \prec < m_0, m >$ for all $m \in M'$. Let again $\mu := \mu_{\mathcal{M},M_{\mathcal{L}}}$, and $\overline{\overline{T}} := T^{\mathcal{M}}$. Thus, $\mu(M_{\mathcal{L}}) = M'$, so $\mathcal{M}$ is not dp , and $\mu(A) = A$ for all $A \neq M_{\mathcal{L}}$. (Note that

the only model which can be eliminated, is m_0*. But you have to kill all copies* $< m_0, m >$ *,* $m \in M'$ *, and for that you need all* $m \in M'$*.) Let now* $\overline{T} := \overline{\emptyset}$*. Then* $M_T = M_{\mathcal{L}}$ *,* $\mu(M_T) = M'$ *, and* $\overline{\overline{T}} = \overline{T}$ *by the above. If T' is such that* $\overline{T'} \neq \overline{\emptyset}$ *, then* $M_{T'} \neq M_{\mathcal{L}}$ *, and* $\mu(M_{T'}) = M_{T'}$ *, so* $\overline{T'} = \overline{\overline{T'}}$*. So, for all* $T \subseteq \mathcal{L}\overline{\overline{T}} = \overline{T}$*. (Thus condition (*$^{=}4$*) of Theorem 3.1 below is trivially true, and Theorem 3.1 may hold even if the dp-condition fails.)* □

2 SUPRACLASSICALITY + CUMULATIVITY + DISTRIBUTIVITY =≠=> CLASSICAL REPRESENTABILITY

In [KLM90], S.Kraus, D.Lehmann, M.Magidor have shown that the finitary restrictions of all supraclassical, cumulative, and distributive inference operations are representable by classical preferential model structures. Using a Lemma by David Makinson, we show in this Section (Example 2.1) that this result does not generalize to the general infinitary case.

Both Lemma 2.1 and our counterexample 2.1 are to appear in [$Mak\infty$], Section 3.4 (Lemma 3.4.9, Observation 3.4.10). The reader less familiar with transfinite ordinals can find there a more algebraic proof that our counterexample satisfies the logical properties claimed. Our technique of constructing a logic inductively by a mixed iteration of suitable length has, however, proved useful in other situations as well (see [Sch91]), moreover, it is very fast and straightforward : once you have the necessary ingredients, the machinery will run almost by itself.

Definition 2.1 *The following definitions are meanwhile standard (folklore) for non-monotonic inference operations : We say that* $^{=}$ *satisfies*
Supraclassicality iff $\overline{A} \subseteq \overline{\overline{A}}$*(*$\overline{A}$ *the classical closure)*
Cumulativity iff $A \subseteq B \subseteq \overline{\overline{A}} \rightarrow \overline{\overline{A}} = \overline{\overline{B}}$
Distributivity iff $\overline{\overline{A}} \cap \overline{\overline{B}} \subseteq \overline{\overline{\overline{\overline{A}} \cap \overline{\overline{B}}}}$ *for all* $A, B \subseteq \mathcal{L}$*.*

S.Kraus, D.Lehmann, and M.Magidor have shown that for any logic $^{=}$ for $\mathcal{L}$, which is supraclassical, cumulative, and distributive, there is a $\mathcal{D}-stoppered$ preferential model $\mathcal{M}$, s.th. for all $finite$ $T \subseteq \mathcal{L}$ $T^{\mathcal{M}} = \overline{\overline{T}}$. ([KLM90], see also [$Mak\infty$], Observation 3.4.7.) We now show that the restriction to finite T is necessary, by providing a counterexample for the infinite case. We start by quoting a Lemma by D. Makinson.

Lemma 2.1 *(Lemma and Proof due to D.Makinson.) Let a logic* $^{=}$ *on* $\mathcal{L}$ *be representable by a classical preferential model structure. Then, for all* $A \subseteq \mathcal{L}, x \in \mathcal{L}, x \notin \overline{\overline{A}}$ *there is a maximal consistent (under* $\vdash$*)* $\Delta \subseteq \mathcal{L}$ *s.th.* $\overline{\overline{A}} \subseteq \Delta, x \notin \Delta$, *and* $\overline{\overline{\Delta}} \neq \mathcal{L}$.

Proof : Let $\mathcal{M} = (\mathcal{X}, \prec)$ be a representation of $^{=}$, i.e. $\overline{\overline{A}} = A^{\mathcal{M}}$ for all $A \subseteq \mathcal{L}$. Let $A \subseteq \mathcal{L}, x \in \mathcal{L}$, and $x \notin \overline{\overline{A}}$. Then there is $< m, i >$ minimal in $\mathcal{X} \uparrow M_A$, with $m \not\models x$. Note that by minimality, $m \models \overline{\overline{A}}$. $\Delta := \{y \in \mathcal{L} : m \models y\}$ is maximal consistent, $x \notin \Delta, \overline{\overline{A}} \subseteq \Delta$, and $< m, i >$ is also minimal in $\mathcal{X} \uparrow M_\Delta$, by $M_\Delta \subseteq M_A$. Thus, $m \models \overline{\overline{\Delta}}$, and by classicality of the models, $\overline{\overline{\Delta}} \neq \mathcal{L}$. $\Box$

We now construct a supraclassical, cumulative, distributive logic, and show that the logic so defined fails to satisfy the condition of Lemma 2.1, and is thus not representable by a cpm.

Example 2.1 *Let* $v(\mathcal{L})$ *contain the propositional variables* $p_i : i \in \omega$, r. *(Note that we do not require* $\mathcal{L}$ *to be countable, we leave plenty of room for modifications of the construction!) We shall violate compactness badly "in both directions" by adding the rules (infinitely many* p_i*)*$\mid\!\sim r$ *and (infinitely many* $\neg p_i$*)*$\mid\!\sim r$. *To account for distributivity, we shall add for all* $\phi \in \mathcal{L}$ *(infinitely many* $p_i \vee \phi$*)*$\mid\!\sim r \vee \phi$ *and (infinitely many* $\neg p_i \vee \phi$*)*$\mid\!\sim r \vee \phi$. *Closing under* $\mid\!\sim$ *and classical logic* ω_1 *many times to take care of the countably infinite rules will give the result.*

The details : We define the logic $^{=}$ by a mixed iteration:
For $B \subseteq \mathcal{L}$ define $I^+_{B,\phi} := \{i < \omega : p_i \vee \phi \in \text{B}\}$, $I^-_{B,\phi} := \{i < \omega : \neg p_i \vee \phi \in$

B}.
Define now inductively
$A_0 :=$ A
for successor ordinals (α a limit or 0, $i \in \omega$) :
$A_{\alpha+2i+1} := \overline{A_{\alpha+2i}}$
$A_{\alpha+2i+2} := A_{\alpha+2i+1} \cup \{r \vee \phi : I^+_{A_{\alpha+2i+1},\phi}$ is infinite or $I^-_{A_{\alpha+2i+1},\phi}$ is infinite$\}$
for limit λ : $A_\lambda := \bigcup\{A_i : i < \lambda\}$
$\overline{\overline{A}} := A_{\omega_1}$.

We show $^=$ is as desired. Note that the defined logic is monotone.

1) $\overline{A} \subseteq \overline{\overline{A}}$ is trivial. 2) $A \subseteq B \subseteq \overline{\overline{A}} \rightarrow \overline{\overline{A}} = \overline{\overline{B}}$: 2.1) $\overline{\overline{A}} \subseteq \overline{\overline{B}}$ by monotony 2.2) $\overline{\overline{B}} \subseteq \overline{\overline{A}}$: Let $\phi \in \overline{\overline{B}}$. In deriving ϕ in $\overline{\overline{B}}$, we have used only countably many elements from B. This is seen as follows. Let β be minimal such that $\phi \in B_\beta$. ϕ can be derived from at most countably many $\phi_i \in B_{\beta-1}$(β has to be a successor ordinal). Arguing backwards, and using $\omega.\omega = \omega$ (cardinal multiplication), we see what we wanted. (This is, of course, the outline for an inductive proof.) As $B \subseteq \overline{\overline{A}}$, using regularity of ω_1, we see that there is some $\alpha < \omega_1$ s.th. all ϕ_j used in the derivation of ϕ from B are in A_α. But then $\phi \in A_{\alpha+\beta}$.

3) Distributivity : We show by induction on the derivation of a, b that $a \in \overline{\overline{A}}, b \in \overline{\overline{B}} \rightarrow a \vee b \in \overline{\overline{\overline{A} \cap \overline{B}}}$. To get started, use $A_0 \subseteq A_1 = \overline{A}$, and $a \in \overline{A}, b \in \overline{B} \rightarrow a \vee b \in \overline{A} \cap \overline{B}$. By symmetry, it suffices to consider the cases for a. Let $a_1, \ldots, a_n \vdash a$ by classical inference. By induction hypothesis, $a_1 \vee b, \ldots, a_n \vee b \in \overline{\overline{\overline{A} \cap \overline{B}}}$, but then $a \vee b \in \overline{\overline{\overline{A} \cap \overline{B}}}$, as the latter is closed under $\vdash$. Assume now $a = r \vee \phi \in A_\alpha$ has been derived from infinitely many $p_i \vee \phi (i \in I)$ in $A_{\alpha-1}$. By induction hypothesis, $p_i \vee \phi \vee b \in \overline{\overline{\overline{A} \cap \overline{B}}}$. So $p_i \vee \phi \vee b \in (\overline{A} \cap \overline{B})_{\beta_i}$ for $\beta_i < \omega_1$. Again by regularity of ω_1, all $p_i \vee \phi \vee b \in (\overline{A} \cap \overline{B})_\beta (i \in I)$ for some $\beta < \omega_1$. But then $r \vee \phi \vee b = a \vee b \in (\overline{A} \cap \overline{B})_{\beta+2}$. The case $\neg p_i \vee \phi$ is similar. □

We use the Lemma to obtain the negative result, as the logic constructed above does not satisfy the Lemma's condition :

Consider now $A := \emptyset$. Assume there is ϕ s.th. infinitely many $p_i \vee \phi \in \overline{A}$, thus there is ϕ s.th. infinitely many $p_i \vee \phi$ are tautologies. But then ϕ has to be a tautology (consider $(p_i \vee \phi) \leftrightarrow (\neg\phi \rightarrow p_i)$ and finiteness of ϕ!), thus ϕ and $\phi \vee r \in \overline{A}$. Likewise for $\neg p_i \vee \phi$. So, the rules (infinitely many $p_i \vee \phi) \mid\!\sim r \vee \phi$ etc. give nothing new, and $\overline{A} = \overline{\overline{A}}$. In particular, $r \notin \overline{\overline{A}}$.

Assume now $\Delta \subseteq \mathcal{L}$ to be maximal consistent. So Δ decides all $p_i : i \in \omega$. Thus either infinitely many p_i, or $\neg p_i$ in Δ. Thus, $r \in \overline{\overline{\Delta}}$. Hence $^=$ is not cpm representable. $\square$

Remark 2.2 *In the last step, finiteness of ϕ - i.e. all ϕ have size $< \omega$ - seems to play a decisive role. So one might be tempted to try to obtain a positive result with languages admitting infinite formulas. Yet, if the size of all formulas is less than β, a similar construction with $\beta^+ p_i' s$, and induction to β^{++}, will give the same result (β^+ the cardinal successor of β).*

We now address more positive results :

3 A REPRESENTATION THEOREM FOR PREFERENTIAL MODELS

Introduction : The main result of this section is Theorem 3.1, which shows that *definability preserving* classical preferential models are essentially characterized by infinite conditionalization, $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$, discussed in $[Mak\infty]$, Sections 2.2 and 3.4. The latter contains a detailed overview of results linking preferential models and logical properties, and the reader is referred there for a wider perspective.

The proof proceeds in two steps. Recall from the introduction that a classical preferential model for $\mathcal{L}$ defines a function $f : \mathcal{P}(M_{\mathcal{L}}) \to \mathcal{P}(M_{\mathcal{L}})$. We first characterize such f generated by a preferential structure (Proposition 3.3) and then characterize the logics corresponding to such f (Proposition 3.4).

We state the main result and then turn to the proofs.

Theorem 3.1 *Let $^=$ be a logic for $\mathcal{L}$. Then there is a definability preserving classical preferential model $\mathcal{M}$ s.th. $\overline{\overline{T}} = T^{\mathcal{M}}$*
iff
$(^=1) \overline{T} = \overline{T'} \to \overline{\overline{T}} = \overline{\overline{T'}}$

$(^=2)\overline{\overline{T}}$ *is classically closed*

$(^=3) T \subseteq \overline{\overline{T}}$

$(^=4) \overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$

for all $T, T' \subseteq \mathcal{L}$.

Moreover, given $^=$, $\mathcal{M}$ *can be chosen transitive and irreflexive.*

Remark : Example 1.3 (1) shows that " $\rightarrow$ " of Theorem 3.1 is false in general without the definability preservation condition, by failure of $(^=4)$, Example 1.3 (2) shows that there are structures $\mathcal{M}$, which are not definability preserving, but whose logic nonetheless satisfies $(^=4)$.

We first show that every preferential structure has an equivalent irreflexive one.

Lemma 3.2 *For any preferential structure* $\mathcal{Z} =< \mathcal{X}, \prec>$, *there is a preferential structure* $\mathcal{Z}' :=< \mathcal{X}', \prec'>$ *s.th.*

(1) $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$

(2) $\mathcal{Z}'$ *is irreflexive*

(3) if $\mathcal{Z}$ *is transitive, then so is* $\mathcal{Z}'$.

Proof of Lemma 3.2 : Let $\mathcal{X}' := \{< x, < i, n >> : < x, i > \in \mathcal{X}, n \in \omega\}$ and $< x', < i', n' >> \prec' < x, < i, n >>$ iff
(i) $n' > n$ and (ii) $< x', i' > \prec < x, i >$.
(1) Let Y be any set, we have to show $\mu_{\mathcal{Z}}(Y) = \mu_{\mathcal{Z}'}(Y)$. " $\subseteq$ " : Suppose $y \in \mu_{\mathcal{Z}}(Y)$, but $y \notin \mu_{\mathcal{Z}'}(Y)$. Take $< y, i > \in \mathcal{X}$ s.th. there is no $< y', i' > \in \mathcal{X} \upharpoonright Y, < y', i' > \prec < y, i >$. Consider $u :=< y, < i, 0 >> \in \mathcal{X}' \upharpoonright Y$. By $y \notin \mu_{\mathcal{Z}'}$, there is $u' :=< y', < i', n' >> \in \mathcal{X}' \upharpoonright Y, u' \prec' u$, but then $< y', i' > \prec < y, i >$, contradiction. " $\supseteq$ " : Suppose $y \in \mu_{\mathcal{Z}'}(Y)$, but $y \notin \mu_{\mathcal{Z}}(Y)$. Take $u :=< y, < i, n >> \in \mathcal{X}' \upharpoonright Y$ s.th. there is no $u' :=< y', < i', n' >> \in \mathcal{X}' \upharpoonright Y, u' \prec' u$. Then $< y, i > \in \mathcal{X} \upharpoonright Y$, so there is $< y', i' > \in \mathcal{X} \upharpoonright Y$ s.th. $< y', i' > \prec < y, i >$. But then $< y', < i', n+1 >> \prec' < y, < i, n >>$, contradiction. (2) is trivial by the condition $n' > n$. (3) Let $< x", < i", n" >> \prec' < x', < i', n' >> \prec' < x, < i, n >>$. Then $< x", i" > \prec < x', i' > \prec < x, i >$, so by transitivity of $\prec$, $< x", i" > \prec < x, i >$. Moreover, $n" > n' > n$, so $n" > n$, and thus $< x", < i", n" >> \prec' < x, < i, n >>$. □ (Lemma 3.2)

Proposition 3.3 *Let Z be any set,* $\mathcal{Y} \subseteq \mathcal{P}(Z), f : \mathcal{Y} \to \mathcal{P}(Z)$*. Then there is a preferential structure* $\mathcal{Z} =< \mathcal{X}, \prec>$ *s.th. for all* $X \in \mathcal{Y} f(X) = \mu_{\mathcal{Z}}(X)$ *iff (f1)* $f(X) \subseteq X$ *and (f2)* $X \subseteq Y \to f(Y) \cap X \subseteq f(X)$ *for all* $X, Y \in \mathcal{Y}$*. Moreover, given such f,* $\mathcal{Z}$ *can be chosen transitive and irreflexive.*

(Note that, if $\mathcal{Y}$ is closed under finite intersections, then, in the presence of (f1), (f2) is equivalent to (f2'), where (f2') $f(X) \cap Y \subseteq f(X \cap Y)$.)

Proof of Proposition 3.3 : " $\to$ " : (f1) is trivial,
(f2) : Let $\mu := \mu_{\mathcal{Z}}$, $X \subseteq Y$, and $x \in \mu(Y) \cap X$. So $x \in X \cap Y$, and there is i s.th. $< x, i >\in \mathcal{X}$, and there is no $< x', i' >\in \mathcal{X}, x' \in Y, < x', i' >\prec< x, i >$. But then there can be no such $< x', i' >$ with $x' \in X$. Consequently, $x \in \mu(X)$.

" $\leftarrow$ " : Let $\mathcal{Y}_x := \{X \in \mathcal{Y} : x \in X - f(X)\}, F_x := \Pi \mathcal{Y}_x$. (If $\mathcal{Y}_x = \emptyset, F_x = \{\emptyset\}$, and the following Claim 1 will be trivially true.) It is important to note that $F_x \neq \emptyset$ by the axiom of choice : A product of non-empty sets is non-empty.

Claim 1 : Let $X \in \mathcal{Y}$.
(1) If $x \in f(X)$, then there is $g_X \in F_x$ with $ran(g_X) \cap X = \emptyset$
(2) $x \in f(X) \leftrightarrow x \in X$ and $\exists g \in F_x.(ran(g) \cap X) = \emptyset$.

Proof : (1) : It suffices to show that for each $Y \in \mathcal{Y}_x$ $Y - X \neq \emptyset$. So let $Y \in \mathcal{Y}_x$, and suppose $Y \subseteq X$. But $x \in f(X)$ and $x \in Y - f(Y)$ by $Y \in \mathcal{Y}_x$, so $x \in f(X) \cap Y \subseteq f(Y)$ by (f2) , contradiction. (2) : " $\to$ " : By (f1), $x \in X$, and by (1), there is such g. " $\leftarrow$ " : Suppose $x \in X - f(X)$, then $X \in \mathcal{Y}_x$, so for all $g \in F_x$ $g(X) \in X$, thus $ran(g) \cap X \neq \emptyset$. $\square$ (Claim 1)

To clarify the main idea, we first define in (A) a simple ps, which describes f on $\mathcal{Y}$, and then turn in (B) to a more complicated structure to achieve transitivity.

(A)

Let $\mathcal{X} := \{< x, g >: x \in Z \wedge g \in F_x\}$, and $< x', g' >\prec< x, g >:\leftrightarrow x' \in ran(g)$, let $\mathcal{Z} :=< \mathcal{X}, \prec>$, and $\mu := \mu_{\mathcal{Z}}$.

Claim 2 (A) : For $X \in \mathcal{Y}, f(X) = \mu(X)$.

Proof : By Claim 1, (2), it suffices to show that for all $X \in \mathcal{Y}$ $x \in \mu(X) \leftrightarrow x \in X$ and $\exists g \in F_x.(ran(g) \cap X) = \emptyset$. So let $X \in \mathcal{Y}$. " $\rightarrow$ " : If $x \in \mu(X)$, then there is $< x, g >$ minimal in $\mathcal{X} \uparrow X$, so $x \in X$, and there is no $< x', g' > \prec < x, g >, x' \in X$, so there is no $x' \in ran(g)$, $x' \in X$, but then $ran(g) \cap X = \emptyset$. " $\leftarrow$ " : If $x \in X$, and there is $g \in F_x$, $ran(g) \cap X = \emptyset$, then $< x, g >$ is minimal in $\mathcal{X} \uparrow X$.□ (Claim 2 (A))

(B)

Let $I := \{< y, g, i > : y \in Z, g \in F_y,$ i=0,1$\}$, $\mathcal{X} := ZxI$, and define $< x', < y', g', i' >> \prec < x, < y, g, i >>$ iff
(a) y=y' $\wedge$ g=g' and
(b1) $x \neq y \wedge$ x=x' or (b2) x=y $\wedge x' \in ran(g)$.
Finally, let $\mathcal{Z} :=< \mathcal{X}, \prec >, \mu := \mu_{\mathcal{Z}}$.

Remarks : 1) Condition (a) makes the construction "local", we only have to consider two "layers" of Z at a time. 2) Condition (b1) makes every $x \neq y$ non-minimal in $Z_{<y,g,0>} \cup Z_{<y,g,1>}$ - where $Z_{<y,g,i>} := \{< z, < y, g, i >>: z \in Z\}$. 3) Condition (b2) makes x=y the "center of a star" in $Z_{<y,g,i>}$: any element of ran(g) prevents x from being minimal. 4) $\prec$ is transitive, as can easily be seen by examining the four possible cases.

As in case (A) we formulate and show

Claim 2 (B) : For $X \in \mathcal{Y}, f(X) = \mu(X)$.

Proof : By Claim 1, (2) again, it suffices to show that for all $X \in \mathcal{Y}$ $x \in \mu(X) \leftrightarrow x \in X$ and $\exists g \in F_x.(ran(g) \cap X) = \emptyset$. So let $X \in \mathcal{Y}$. " $\rightarrow$ " : Suppose $x \in X$ and $\forall g \in F_x.(ran(g) \cap X) \neq \emptyset$, we show that no $< x, < y, g, i >>$ is minimal in $\mathcal{X} \uparrow X$. Let $x' \in ran(g) \cap X$. If $y \neq x$, then e.g. $< x, < y, g, 1 - i >> \prec < x, < y, g, i >>$. If y=x, then $< x', < x, g, i >> \prec < x, < x, g, i >>$. " $\leftarrow$ " : If $x \notin \mu(X)$, but $x \in X$, then for all $< x, < x, g, i >>, g \in F_x$, there is $< x', < y', g', i' >> \prec < x, < x, g, i >>, x' \in X$, but then $x' \in ran(g)$. So $\forall g \in F_x.(ran(g) \cap X) \neq \emptyset$.□ (Claim 2 (B))

By Lemma 3.2, there is a transitive, irreflexive cpm $\mathcal{Z}' =< \mathcal{X}', \prec' >$ s.th. $f(X) = \mu_{\mathcal{Z}}(X) = \mu_{\mathcal{Z}'}(X)$ for all $X \in \mathcal{Y}$.□ (Proposition 3.3)

Proposition 3.4 *Consider for a logic* $^{=}$ *on* $\mathcal{L}$ *the properties*

($^=1$)$\overline{T} = \overline{T'} \rightarrow \overline{\overline{\overline{T}}} = \overline{\overline{\overline{T'}}}$

($^=2$)$\overline{\overline{\overline{T}}}$ *is classically closed*

($^=3$)$T \subseteq \overline{\overline{\overline{T}}}$

($^=4$)$\overline{\overline{\overline{T \cup T'}}} \subseteq \overline{\overline{\overline{T}} \cup T'}$

for all $T, T' \subseteq \mathcal{L}$

and for a function $f : \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ *with* $\mathcal{D}_{\mathcal{L}} \subseteq \mathcal{Y} \subseteq M_{\mathcal{L}}$ *the properties*

(f1) $f(X) \subseteq X$

(f2) $f(X) \cap Y \subseteq f(X \cap Y)$

(f3) f is definability preserving

for all $X, Y \in \mathcal{D}_{\mathcal{L}}$.

The following holds :

(a.1) If f satisfies (f1), then $^=$ *defined by* $\overline{\overline{\overline{T}}} := T^f$ *satisfies* ($^=1$) $-$ ($^=3$).

(a.2) If f satisfies (f1)-(f3), then $^=$ *defined by* $\overline{\overline{\overline{T}}} := T^f$ *satisfies* ($^=1$) $-$ ($^=4$).

(b.1) If $^=$ *satisfies* ($^=1$) $-$ ($^=3$), *then there is* $f : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ *s.th.* $\overline{\overline{\overline{T}}} = T^f$ *for all* $T \subseteq \mathcal{L}$ *and f satisfies (f1) and (f3).*

(b.2) If $^=$ *satisfies* ($^=1$) $-$ ($^=4$), *then there is* $f : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ *s.th.* $\overline{\overline{\overline{T}}} = T^f$ *for all* $T \subseteq \mathcal{L}$ *and f satisfies (f1)-(f3).*

Recall that, as $\mathcal{D}_{\mathcal{L}}$ is closed under finite intersections, in the presence of (f1), (f2) is equivalent to (f2') $X \subseteq Y \rightarrow f(Y) \cap X \subseteq f(X)$.

Remark : The following observation is due to a referee : Define for any formula ϕ $f(M_\phi) := M_{K*\phi}$, where K is a fixed theory, and $K * \phi$ is the result of revising K by ϕ in the framework of Theory Revision (see [AGM85], [Gär88], or [Mak85]). We consider the postulates (K*2) and (K*7) of Theory Revision :
(K*2) $\phi \in K * \phi$
(K*7) $K * (\phi \wedge \psi) \subseteq (K * \phi) \cup \{\psi\}$
Now $\phi \in K * \phi \leftrightarrow M_{K*\phi} \subseteq M_\phi \leftrightarrow f(M_\phi) \subseteq M_\phi$ and $K * (\phi \wedge \psi) \subseteq (K * \phi) \cup \{\psi\} \leftrightarrow M_{K*\phi} \cap M_\psi \subseteq M_{K*(\phi\wedge\psi)} \leftrightarrow f(M_\phi) \cap M_\psi \subseteq f(M_{\phi\wedge\psi}) = f(M_\phi \cap M_\psi)$. Thus, for $X = M_\phi, Y = M_\psi$, and f defined as above, (f1) is equivalent to (K*2), and (f2) to (K*7).

Proof of Proposition 3.4 : (a.1) Suppose $\overline{\overline{T}} = T^f$ for some such f, and all T. $({}^{=}1)$ If $\overline{T} = \overline{T'}$, then $M_T = M_{T'}$, so $f(M_T) = f(M_{T'})$, and $T^f = T'^f$. $({}^{=}2)$ is trivial by definition, and $({}^{=}3)$ is trivial by $f(X) \subseteq X$.

(a.2) Suppose there is f s.th. $\overline{\overline{T}} = T^f$ and f satisfies (f1)-(f3). It remains to show $({}^{=}4)$. Let now $\phi \in \overline{\overline{T \cup T'}}$, so ϕ holds in all $m \in f(M_{T \cup T'}) = f(M_T \cap M_{T'})$, so by (f2), ϕ holds in all $m \in f(M_T) \cap M_{T'}$. By (f3), $f(M_T) = M_{T^f} = M_{\overline{\overline{T}}}$, so ϕ holds in all $m \in M_{\overline{\overline{T}}} \cap M_{T'} = M_{\overline{\overline{T}} \cup T'}$, so $\overline{\overline{T}} \cup T' \models \phi$, and $\phi \in \overline{\overline{\overline{\overline{T}} \cup T'}}$.

For the necessity of some assumption additional to (f1) and (f2), the reader is referred to Example 1.3.

(b.1) Let ${}^{=}$ satisfy $({}^{=}1) - ({}^{=}3)$ for all T. We define f and show $\overline{\overline{T}} = T^f$. If $X \not\in \mathcal{D}$, set e.g. $f(X) := \emptyset$. Otherwise, if $X = M_T$ for some $T \subseteq \mathcal{L}$, set $f(X) := M_{\overline{\overline{T}}}$. If $X = M_T = M_{T'}$, then $\overline{T} = \overline{T'}$, thus $\overline{\overline{T}} = \overline{\overline{T'}}$ by $({}^{=}1)$, so $M_{\overline{\overline{T}}} = M_{\overline{\overline{T'}}}$, and f is well-defined. Moreover, f satisfies (f3), and by $({}^{=}3)$, $f(X) \subseteq X$. It remains to show $\overline{\overline{T}} = T^f$. Let now $T \subseteq \mathcal{L}$ be given. Then $\phi \in T^f :\leftrightarrow \forall m \in f(M_T).m \models \phi \leftrightarrow \forall m \in M_{\overline{\overline{T}}}.m \models \phi \leftrightarrow \overline{\overline{T}} \vdash \phi \leftrightarrow \phi \in \overline{\overline{T}}$ (as $\overline{\overline{T}}$ is classically closed).

(b.2) By (b.1), it suffices to show that the above defined f satisfies (f2), if ${}^{=}$ satisfies also $({}^{=}4)$. Suppose $X := M_T$, $Y := M_{T'}$. Let $m \in f(X) \cap Y = M_{\overline{\overline{T}}} \cap M_{T'}$, so $m \models \overline{\overline{T}} \cup T'$, and $m \models \overline{\overline{\overline{\overline{T}} \cup T'}}$, so by $({}^{=}4)$ $m \models \overline{\overline{T \cup T'}}$. As $X \cap Y = M_T \cap M_{T'} = M_{T \cup T'}$, $f(X \cap Y) = M_{\overline{\overline{T \cup T'}}}$ by (f3), so $m \in f(X \cap Y)$.□ (Proposition 3.4)

Proof of Theorem 3.1 : " $\rightarrow$ " : Let $\mathcal{M}$ be a dp cpm, then $f := \mu_{\mathcal{M},M_{\mathcal{L}}} : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ is dp, and satisfies (f1) and (f2) of Proposition 3.3 for all $X, Y \in \mathcal{P}(M_{\mathcal{L}})$. By Proposition 3.4, (a.2), the logic defined by $\overline{\overline{T}} := T^{\mathcal{M}}$ satisfies $({}^{=}1) - ({}^{=}4)$. " $\leftarrow$ " : Let ${}^{=}$ be a logic for $\mathcal{L}$ which satisfies $({}^{=}1) - ({}^{=}4)$. By Proposition 3.4, (b.2), there is $f : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ s.th. f satisfies (f1)-(f3) and for all $T \subseteq \mathcal{L}\overline{\overline{T}} = T^f$. By Proposition 3.3, for $\mathcal{Y} := \mathcal{D}$, there is a transitive irreflexive classical preferential model $\mathcal{M} =< M_{\mathcal{L}}xI, \prec>$ s.th. $f(X) = \mu_{\mathcal{M}}(X)$ for all $X \in \mathcal{D}$. But now $\overline{\overline{T}} = T^f = T^{\mathcal{M}}$ for all $T \subseteq \mathcal{L}$, and we are done. □ (Theorem 3.1)

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