Remarks on Consistency and Completeness of Circumscription

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Abstract

We discuss definable minimal models, the semantical counterpart of first order circumscription, examine the adequacy of Mott's system of circumscription and show that some completeness results of Perlis and Minker fail in Mott's system.

1 Introduction

Circumscription, introduced by McCarthy [McC1], is a technique of nonmonotonic reasoning. Since publication of [McC1], a large number of different syntactical and semantical variants of circumscription have been discussed, without affecting the central idea, however : Circumscription is a process which strengthens theories so as to minimize the number of abnormal or unwanted cases (like birds that can't fly, penguins that can etc.). If T is a (first order) theory, Circum(T) will be a theory whose models are the minimal - in a sense to be made precise - classical models of T. So minimal models are the semantics of circumscription.

The work on circumscription has seen a very active development in recent years which may be roughly classified as follows :

1. strengthening its expressive power (e.g. variable circumscription [L], prioritized circumscription [McC2], [L2])
2. comparison of semantical and syntactical versions of circumscription [M], [PM]
3. questions of computability [L2], [GPP], [G]
4. problems of consistency : there are consistent theories T such that Circum(T) may be inconsistent in some versions of circumscription, and there are theories which have (classical) models, but no minimal ones [M], [L], [EMR]
5. generalizing the semantics of minimal models and comparing circumscription with the other systems of nonmonotonic reasoning [I], [S], [Sch]

The above list is by no means intended to be complete. The present paper is of the second category.
First, we will fix some notation. In the section on definable minimal models, we note for completeness' sake that *definable* minimal models are the semantical counterpart of first order circumscription. This is more or less immediate, and has been done for domain circumscription in [Mr]. The last two sections start on Mott's ideas in [M]. Mott has modified the first order circumscription process in a way that consistent theories will stay consistent when circumscribed. (In contrast, Lifschitz [L] et al. have shown that (consistent) theories with certain syntactical properties are still consistent when circumscribed in the usual way.) In the section on adequacy, we will point out in a few words that this (desirable) feature of consistency preservation has its own drawbacks : it will loosen the tight connection between circumscription and minimal models, which is one of the attractive features of circumscription. In the last and central section of this paper, we show that some completeness results in [PM] fail in Mott's version of circumscription. The reason is very simple. In order to obtain their results, Perlis and Minker need enough definable subsets of their models. As they (implicitely) work with subsets definable from parameters, they have many more subsets at hand than Mott does, who works with plain definability (without parameters). We should note, however, that plain definability is not central to Mott's idea of consistency preserving circumscription, but it is part of his system as it stands and is published, and should as such be discussed. In conclusion, we see that the precise distinction between different forms of subsets is well worth doing in circumscription theory.
We shall assume familiarity with formal logic, the basic ideas of circumscription as expressed in [McC1] and the very beginning of set theory. For the last two sections, the reader will need a copy of [M] and [PM] at hand.

2 Notation

If not stated otherwise, we will circumscribe one unary predicate, P, without variables, in a first order theory A of some fixed language $\mathcal{L}$. For notational simplicity, we will identify models with their domains.
For any model M, let $[P]_M := \{x \in M : M \models P(x)\}$. For M, M' models of A, $M \leq M'$ means as usual $[P]_M \subseteq [P]_{M'}$ and M, M' identical otherwise. Furthermore, $M \leq_d M'$ (d for definable) iff $M \leq M'$ and there is a formula φ of $\mathcal{L}$ with $[P]_M = \{x \in M' : M' \models \varphi(x)\}$. If we compare several models M_i, we say $M_i \leq_{ud} M_j$ (for uniformly definable) iff there is a fixed φ such that for all considered pairs $M_i \leq M_j, [P]_{M_i} = \{x \in M_j : M_j \models \varphi(x)\}$. The corresponding notions of minimality will be : minimal, definable minimal (dm), uniformly definable minimal (udm), and of minimal entailment $\models_m$, $\models_{dm}$, $\models_{udm}$, i.e. $A \models_m \psi$ iff ψ is valid in every minimal model of A etc.
We follow Mott [M] and say that P admits φ in A iff P and φ have exactly the

same free variables and substituting φ for P in A (abbreviated $A(P/\varphi)$) does not result in clash of variables. We will tacitly assume that all substitutions are admissible. Further following Mott, we define the model $M_{P,\varphi}$ from M by $[P]_{M_{P,\varphi}} := \{x \in M : M \models \varphi(x)\}$, and like M otherwise. On the syntactical side, we define
$C(A, \varphi) := A(P/\varphi) \wedge \forall x(\varphi(x) \rightarrow P(x)) \rightarrow \forall x(P(x) \rightarrow \varphi(x))$
$C1(A) := \{A(P/\varphi) \wedge \forall x(\varphi(x) \rightarrow P(x)) \rightarrow \forall x(P(x) \rightarrow \varphi(x)) :$
φ a formula which P admits in A$\}$
$CM(A) := \{A(P/\varphi) \wedge \forall x(\varphi(x) \rightarrow P(x)) \rightarrow \forall x(P(x) \rightarrow \varphi(x)) :$
φ a formula which P admits in A, and which does not contain P$\}$ (This is Mott's system)
$C2(A) := \forall p(A(P/p) \wedge \forall x(p(x) \rightarrow P(x)) \rightarrow \forall x(P(x) \rightarrow p(x))),$
p a second order variable,
and write $A \vdash_{C(A,\varphi)} \psi$ for $A + C(A, \varphi) \vdash \psi$, $A \vdash_{C1} \psi$, $A \vdash_{CM} \psi$ likewise. Let $\perp$ stand for anything wrong, like $\exists x(x \neq x)$.

3 Definable minimal models

Before we start, let us make a few introductory remarks on definable subsets. The interested reader may consult any advanced book on set theory like [D] to see the general importance of this concept : e.g., it is central to Goedel's model of set theory, the constructible universe.
Simple cardinality considerations will already show the difference between "subset", "definable subset", and "subset definable with parameters":
Let $\mathcal{L}$ be any at most countable first oder language, M an uncountable $\mathcal{L}$-structure of cardinality α . Then M has $2^\alpha > \alpha$ subsets (the full power set). But M has at most ω many definable subsets, i.e. of the form $X = \{x \in M : M \models \varphi(x)\}$, as there are only countably many formulae φ in $\mathcal{L}$. Finally, M has α many subsets definable with parameters, of the form $X_{x_1 \cdots x_n} = \{x \in M : M \models \varphi(x, x_1, \cdots, x_n)\}, x_i \in M$. (To see this, consider the singletons $X_a = \{x \in M : x = a\}, a \in M$, plus some basic cardinal arithmetic).
Especially, any finite subset is definable with parameters : $X = \{x \in M : M \models x = x_1 \vee \cdots \vee x_n\}$.
In minimal models and second order circumscription, we argue with the full powerset of the models, in some universe where we do all the model theory. In definable minimal models, we argue from the inside of just those models using the language at hand, working with definable subsets - just as in first order circumscription. And it is precisely the difference between "definable" and "definable with parameters", which causes the failure of the completeness results of [PM] in Mott's system. In their basic result (in the proof of their Lemma 4.3), Perlis and Minker implicitely use definability with parameters. This gives them enough definable subsets to obtain their results. Mott, on the other hand, uses

plain definability. These facts lead to the differences between the two systems - each correct in its own way - as discussed in our last section.
We have the following

Lemma 1 *a) M is a udm model (with resp. to φ) of A iff* $M \models A + C(A, \varphi)$
b) M is a dm model of A iff $M \models A + C1(A)$
c) M is a minimal model of A iff $M \models A + C2(A)$

Proof b) " $\rightarrow$ " Let $M \models A(P/\varphi) \wedge \forall x(\varphi(x) \rightarrow P(x))$, P admit φ in A. We have to show $M \models \forall x(P(x) \rightarrow \varphi(x))$. Since $M \models \forall x(\varphi(x) \rightarrow P(x))$, $[P]_{M_{P,\varphi}} = [\varphi]_M \subseteq [P]_M$. By [M], Lemma 3.1 $M_{P,\varphi} \models A$, thus, as M is a definable minimal model of A, $[\varphi]_M = [P]_{M_{P,\varphi}} = [P]_M$ and $M \models \forall x(P(x) \rightarrow \varphi(x))$.
" $\leftarrow$ " Suppose $M' <_d M$, definable by φ . By renaming variables, we may assume that P admits φ in A. Thus $M' = M_{P,\varphi}$, and $M_{P,\varphi} \models A$. By [M], Lemma 3.1 again, $M \models A(P/\varphi)$. Thus, as $M \models C1(A)$, $M \models \forall x(P(x) \rightarrow \varphi(x))$ and $[P]_M = [P]_{M'}$, contradiction.
a) Like b)
c) This is just proposition 1 of [L2].
By first order completeness and soundness, we conclude

Corollary 2 *a)* $A \models_{udm} \psi$ *(w.r.t. φ) iff* $A \vdash_{C(A,\varphi)} \psi$
b) $A \models_{dm} \psi$ *iff* $A \vdash_{C1} \psi$

Proof a) $A \models_{udm} \psi(w.r.t.\varphi)$ iff ψ is satisfied in all φ-udm models of A iff ψ is satisfied in all models of $A + C(A, \varphi)$ iff $A + C(A, \varphi) \models \psi$ iff $A + C(A, \varphi) \vdash \psi$ iff $A \vdash_{C(A,\varphi)} \psi$.
b) the same as for a)
and

Corollary 3 *A+C1(A) is inconsistent iff A has no dm model.*

Proof : A+C1(A) is inconsistent $\longleftrightarrow A + C1(A) \vdash \perp \longleftrightarrow A \models_{dm} \perp \longleftrightarrow$ there is no dm model of A.

4 Problems of Adequacy

We now show that any weakening of the circumscription process to the preservation of consistency - a moment's reflection will suffice to show that this is indeed a weakening - results in the loss of the close connection of circumscription and minimal models. We will exemplify this with Mott's system. Let CSR abbreviate "common sense reasoning", whatever that may be. Informally speaking, we might look at questions of adequacy from three sides : Here, we will examine the adequacy of Mott's system of circumscription. By CSR's vagueness, (1) and (2) can only be approximately discussed. Mott considers his system more adequate in the sense of (2), since it will not lead to inconsistencies.

Figure 1: Adequacy

This however, is being paid for by the failure of (3) : Theorem 4.1 of [M] shows $A \vdash_{CM} \psi \rightarrow A \models_m \psi$
Proof: Let M be a minimal model of A. By theorem 4.1 of [M], $M \models CM(A)$. As $A \vdash_{CM} \psi, M \models \psi$.
The inverse fails already with our weakest system :

Lemma 4 *It is not true that* $A \models_{udm} \psi \rightarrow A \vdash_{CM} \psi$.

Proof :
This is shown by the example given in [M, page 90] (following Mott, we minimize N here) : Let $A := A_1 \wedge A_2 \wedge A_3$, with $A_1 := \exists x(Nx \wedge \forall y(Ny \rightarrow \neg x = sy))$, $A_2 := \forall x(Nx \rightarrow Nsx)$, $A_3 := \forall xy(sx = sy \rightarrow x = y)$.
As is well known, A has no udm models. (If M is a model of A, x_0 satisfying the condition A_1 , consider M', just like M, but $[N]_{M'} := [N]_M - \{x_0\}$, then $M' <_{ud} M$ and M' will be a model of A too.)
As Mott has shown, his version of Circumscription preserves consistency : A+CM(A) is consistent. Thus $A \models_{udm} \bot$, but not $A \vdash_{CM} \bot$. The point against adequacy of CM(A) in the sense of (3) can be expressed more strongly still. Let $M_1 < M_2$. If we consider minimization an important aspect of circumscription, every $x \in [P]_{M_2} - [P]_{M_1}$ should be excluded from P via circumscription. To put it differently, a version of circumscription which admits M_2 as a model, does not satisfy an essential condition of the circumscriptive procedure. In the above example, A+CM(A) is consistent, so it has a model. But, as we have seen, A has no minimal models, so there is a model $M' \leq M$ of A and $x \in [N]_M - [N]_{M'}$, which this circumscription does not exclude from N, contrary to the intuitive requirements.
The above objection to Mott's system will apply to all versions of circumscrip-

tion, which "force" consistency, of course.

5 The Role of Parameters

We show that some of the completeness results of Perlis and Minker fail in Mott's system, essentially, because Perlis and Minker work with parameters and Mott does not (see the section on definable minimal models, for introduction).
Let $\mathcal{L} = \{\mathcal{P}\}$, the language with just one predicate letter other than = , and P be unary. Let M_1 ,M_2 be models, with $domain(M_1) = domain(M_2)$, and $\{a, b, c\} \subseteq domain(M_i)$, i=1,2 and $M_1 \models P(a), \neg P(b), \neg P(c)$, $M_2 \models P(a), P(b), \neg P(c)$. Let $A = \exists x P(x)$.
Obviously, both M_1 and M_2 are models of A.

Lemma 5 *Both M_1 , M_2 are models of CM(A).*

We shall use in the proof :

Lemma 6 *Let B(x) be a formula of $\mathcal{L}$, where P does not occur, x is the only free variable, M any model, then either for all $a \in M, M \models B(a)$ or for no $a \in M, M \models B(a)$.*

Proof of Lemma 5 :
Let $\exists x Bx \wedge \forall x(Bx \rightarrow Px) \rightarrow \forall x(Px \rightarrow Bx) \in CM(A)$.
By Lemma 6, it is sufficient to distinguish two cases:
a. $\forall a \in M_i$ (i=1,2) $M_i \models B(a)$. Thus, since $M_i \models \neg P(c)$, and $M_i \models B(c)$, $M_i \models \neg\forall x(Bx \rightarrow Px)$.
b. $\forall a \in M_i(i = 1, 2), M_i \models \neg B(a)$. Thus $M_i \models \neg\exists x Bx$.
In both cases, $M_i \models \neg(\exists x Bx \wedge \forall x(Bx \rightarrow Px))$ and , consequently, $M_i \models \exists x Bx \wedge \forall x(Bx \rightarrow Px) \rightarrow \forall x(Px \rightarrow Bx)$.
Proof of Lemma 6 : Intuitively clear, because there is no way to mark anything in the domain, since equality is totally homogenous, it is an immediate consequence of a theorem on the elimination of quantifiers of model-theory, theorem 1.5.7 of [CK]. As a consequence of that result, B(v) is equivalent to a Boolean combination of v=v and $\forall v_1..v_n \exists v_0(\neg v_0 = v_1 \wedge \cdots \wedge \neg v_0 = v_n)$. In the second case, there is no occurrence of v, and in the first, any interpretation of v by a is as good as any other, so the truth is not affected by the choice of a, either it is true for all a, or for none.

All subsequent results apply to Mott's system only, and do not affect the version of circumscription as discussed in [PM].

Corollary 7 *There is a finite model M of CM(A), which is not P-minimal, contradicting the main result in the proof of Lemma 4.3 in [PM].*

Proof : Take domain $M_i = \{a, b, c\}$ and $M = M_2$. Obviously, in any P-minimal model of A, $\exists_1 x Px$ (i.e. $\exists x(Px \wedge \forall y(Py \rightarrow x = y))$) holds.

Thus

Corollary 8 *Lemma 4.3 of [PM] fails*

Proof: $A \models_m \exists_1 x P(x)$, but $M_2 \models \neg\exists_1 x Px$.

Corollary 9 *Theorem 4.4 of [PM] fails.*

Proof : Consider $A = \exists x Px \wedge \exists x_1, x_2, x_3 (x_i \neq x_j \wedge \forall x (x = x_1 \vee x = x_2 \vee x = x_3))$. This has only models of size = 3, $A \models_m \exists_1 x Px$ as we have seen above, but $\neg A \vdash_{CM} \exists_1 x Px$, since M_2 was a model of A+CM(A) and $M_2 \models \neg\exists_1 x Px$.

Corollary 10 *Corollary 4.5 of [PM] fails.*

Proof: The same example as in the proof of Cor. 9 works.

Corollary 11 *Theorem 4.7 of [PM] fails.*

Proof: Again, the same example as in the proof of Cor. 9 works, since it has the theorem $\exists x_1, x_2, x_3 \forall x (Px \rightarrow x = x_1 \vee x = x_2 \vee x = x_3)$.

6 Conclusion

We have discussed the set-theoretical notions of "subset", "definable subset", and "subset definable with parameters" and their correspondence to different systems of circumscription. Our main intention was to prove that some results on the completeness of circumscription rely heavily on the presence of enough definable subsets, and here plain definability is not sufficient, whereas definability with parameters will do.

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