# DEFEASIBLE INHERITANCE : COHERENCE PROPERTIES AND SEMANTICS

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**Abstract**

In [MS], we discussed problems of both the directly sceptical and the extensions approach to reasoning in defeasible inheritance systems. Here, we present and examine solutions to some of these problems, giving stability special attention. In addition, we present a (class of) semantics for defeasible inheritance, based on "normal" subsets.

## 1 Introduction

In our joint paper [MS], we discussed coherence problems of the sceptical as well as the extensions approach in defeasible inheritance systems. These problems are not outright contradictions, but display a "funny" or incoherent behaviour of the formalism, which does not conform to intuition.

The purpose of this paper is 1. to present and discuss solutions to these problems (Sections II and III) 2. to present a semantics for defeasible inheritance (Section IV).

The status of our solutions to the decoupling problem in the extensions approach and to the propagation of scepticism problem in the directly sceptical approach is not the same. For the first problem we have come to a solution which, as far as we see, seems to be quite robust. For the second problem, we have a strong argument that the problem is in principle insoluble. So in this case we describe several solutions to the original problem, presenting a new counterexample to each solution. The purpose is therefore to give a (technique of) approximation for situations where the failure in the Double Diamond problem is serious and the general shortcomings of the directly sceptical approach (see [MS]) are tolerable.

The concluding section will propose a semantics for non-monotonic inheritance which can handle preclusion. Our approach is based on formalizing the notion of a "normal" subset, allowing us to state e.g. "normally, all p are q". Since for preclusion, direct links are in a stronger way true than valid paths, we express this by different degrees of "normality", resulting in a many-valued semantics. Primarily, our semantics is intended for the directly sceptical approach; for extensions, we suggest a combination with possible worlds. Furthermore, the semantics can be easily generalized e.g. to take the modifications presented in Section III into account.

As each section is preceded by an introduction of its own, we refer the reader to them for a more detailed overview.

I would like to thank David Makinson for many and fruitful discussions. This paper presents the more constructive or positive side, and our earlier joint paper [MS] the more critical side of the same questions. Responsibility for the present paper rests with its author alone.

We shall assume familiarity with [T], [HTT], [THT], [MS].

In addition, in all that follows we shall use without further comment

**Fact 1.1** *Let a potential path be a monotone sequence of direct links, all positive except possibly the last one. Let* $\sigma := x_1 \ldots x_n$ *be a positive potential path. If there is no negative potential path from* $x_1$ *to some* $x_i$, $1 < i \leq n$, *then* $\sigma$ *is a valid path, thus it will be in any extension, and in any sceptically admitted set of paths.* □

This Fact will be true in all definitions we handle, and in all other sensible definitions which determine validity by reasons, not by consequences. Usually, large portions of diagrams can be handled simply by applying this fact.

All diagrams are collected in an appendix and referred to as Ex .... .

# 2 THE CONFLICTUAL DECOUPLING POSTULATE (CDP)

**Outline of Section II** In our joint paper [MS], we discussed difficulties with decoupling in the extensions approach. In this section, we shall examine some possible remedies to these problems.

In II.1, we introduce some terminology and present the general framework within which we shall work in this section.

The rest of this chapter is divided into

II.2 Three attempts to solve the CDP

II.3 The b-condition, stability for extensions

II.4 Discussion of a difficult example

II.5 Requirements to solutions of the CDP, inherent failure of stability

II.6 A solution of the CDP

These subsections are largely independent, and the reader may move directly to the central Definition 2.7 in II.6, looking up the requirements R1-R5 in II.5. Subsections II.2 and II.4 may be seen as a warning and a motivation for our somewhat complex solution, which, in the process of the examination, more or less forced itself upon us. The b-condition, defined in II.3, turned out to be nice tool for all questions of stability and can itself be used as a measure to judge the behaviour of formalisms. It will be used in Section III again. Despite the modular structure of this chapter, the reader is invited to follow us through the intricacies of diagrams and questions of stability to gain more insight into the mechanisms of inheritance. As a consolation, we promise a formal semantics for even the funniest diagrams - to be given in Section IV.

## 2.1 The general framework

**Definition 2.1** *A potential path (pp) is a monotonic sequence of arrows with at most one negative link (at the end). If* $\Gamma$ *is a net, then* $\Pi(\Gamma)$ *will be the set of potential paths in* $\Gamma$*. If* $\sigma := x \ldots y$ *is a pp, then we may write* $\sigma : x \to y$*, and* $x = dom(\sigma)$*,* $y = ran(\sigma)$*.* $\sigma\rho$ *will be the concatenation of* $\sigma$ *and* $\rho$*, beginning with* $\sigma$*. Throughout this paper, an extension will always be a grounded extension. If* $\sigma : x \to y$*, then* $\sigma$ *and* $x$ *will be called initial (modulo* $\Gamma$*), iff there is no arrow in* $\Gamma$*, going into* $x$*.* $\bigcap Ext(ensions)$ *will abbreviate "intersection of extensions".*

**Definition 2.2** *1. We call a pair* $< \tau, \rho >$ *of potential paths a conflictual decoupling pair (cdp) iff it is of the type* $\tau = x \ldots y \ldots z$*,* $\rho = y..z^-$*. 2. If* $\tau$ *and* $\rho$ *are as above,* $\tau/\rho$ *will denote the path* $x \ldots y \ldots z^-$*, i.e. we follow* $\tau$ *until the branching point* $y$*, and continue along* $\rho$*. 3. The Conflictual Decoupling*

*Postulate (CDP) says, that we need a good reason to take $\tau$ instead of $\tau/\rho$ into an extension E with $\rho\epsilon E$. 4. A solution of the CDP will be the choice of one or several such good reasons in a formal definition.*

We shall examine what such good reasons might be, and investigate the intuitive, combinatorial and logical consequences of accepting or omitting such reasons.

Broadly speaking, we can distinguish :

- positive reasons, that directly support $\tau$, e.g. a valid path parallel to $\tau$, and of the same polarity.

- negative reasons, that directly undermine $\tau/\rho$, e.g. the fact, that an initial segment of $\tau/\rho$ is not a valid path.

More formally, we shall examine definitions of extensions of nets $\Gamma$ of the following form :

E is an extension in $\Gamma$ iff $E = \Gamma \cup \{\sigma : \sigma$ is a path in $\Gamma$ and $E \models \sigma\}$ where $E \models \sigma$, $\sigma = x_1 \ldots x_{n+1}$ , $n > 1$ iff

(0) $x_1 \ldots x_n \ \epsilon$ E , $x_n x_{n+1} \ \epsilon \ \Gamma$

(1) $\sigma$ is not contradicted in E , i.e. there is no $\sigma^- = x_1 \ldots x^-_{n+1} \ \epsilon$ E

(2) $\sigma$ is not precluded in E , i.e. 1. There is no $x_i x^-_{n+1} \ \epsilon \ \Gamma$, $i < n$ (not $i \leq n$, we need more specific information !) 2. There is no $x_1 \ldots x_i y_1 \ldots y_j \ldots y_m x_{i+k}$ $\epsilon$ E with $y_j x^-_{n+1} \ \epsilon \ \Gamma$

(3) If $< \sigma, \rho >$ forms a cdp, $\rho\epsilon E$, and $\sigma$ is to be accepted, then some conditions (our abovementioned good reasons) have to be satisfied in E.

Remarks : 1) The Thomason notion of capriciousness is not related to the Conflictual Decoupling Postulate (in the sense that they can occurr independently), the former is a horizontal, the latter a vertical coherence property. 2) The CDP entails some technical difficulties, and can as such be seen in parallel to the notion of preclusion : First, examples, counterexamples, proofs and definitions tend to be more complex than those not involving the CDP. Second, as preclusion dashes any hope of general stability (because of the importance of the upper direct link in the fundamental diagram of preclusion), an intuitively satisfactory solution to the CDP destroys the hope for atomically - better : initially - stable definitions of extensions too.

## 2.2 Three attempts to solve the CDP

We examine now three attempts to solve the CDP and show how each fails. This is done to serve : 1. as a warning and to save the reader looking for his own solution, unnecessary work. 2. as a warming up exercise to make the reader familiar with the problems involved. In the following subsections, we shall proceed in a more systematic way, and finally give an intuitively appealing solution, but show that such intuitively acceptable solutions inherently destroy stability. Let, in the following, $< \tau, \rho >$ be a cdp, $\rho\epsilon E$.

Solution 1 : The choice of $\tau$ into E is permitted, if $\tau/\rho$ is precluded.

Comment : 1. In diagram Ex1, this solution is not supported by intuition. Basically, the reason is, that the solution is too local, it does not see far enough. E:= $\Gamma \cup$ { abc, abe, agf, agfe, gfe, abcd, $bed^-$ } is an extension. abcd is in (conflictual) decoupling with $bed^-$, and $abed^-$ is precluded by agfe (which has to be in any grounded extension), and the direct link fd. But, what about the quality of the preclusion ? The path agfd itself is precluded by $agd^-$, and it can't be in any extension. The best information on the relation between a and d (apart from abcd) we can get is certainly $agd^-$, which rather supports $abed^-$ instead of precluding it. In other words, the intuitive force of the preclusion of $abed^-$ by agfe and fd seems to be very weak. Exaggerating a bit, we are faced again with one of the situations that led to our original criticism.

2. The solution fails to satisfy the b-condition (see below), and thus destroys stability in a strong sense. See the discussion of Ex3.

Solution 2 : The choice of $\tau$ into E is permitted if $\tau/\rho$ has a valid preclusion, i.e. if $\tau = x \ldots y \ldots z$, $\rho = y \ldots v \ldots z^-$, then there is a preclusion $\sigma := x..u..v$, uz of $\tau/\rho$ s.th. $\sigma/uz = x \ldots uz$ is in the extension E.

Solution 3 : The choice of $\tau$ into E is permitted, if every path (not only $\tau/\rho$) contradicting $\tau$ is precluded (in E).

We discuss solutions 2 and 3 together.

At first sight, solution 2 and 3 might seem equivalent. This is not so, as example Ex2 shows.

$E := \Gamma \cup \{abc, abd, aed, afe, afed, agf, agh, agi^-, abci^-, bdi, fed, ghe^-\}$ is an extension for solution 3, but not for solution 2. At the same time, this example shows that solution 2 is not intuitively valid : Every possible positive path from a to i is precluded, so we have good reason to believe in $abci^-$.

Solution 3, however, does not satisfy the b-condition (see below), so stability fails badly.

## 2.3 The b-condition, stability for extensions

**Definition 2.3** *Let E be an extension. We say E satisfies the b-condition, iff for all a ... b ... d and a ... c ... b $\epsilon$ E, a ... c..b..d will be in E too. (We have called it the b-condition, because our diagram of the situation had the form of the letter "b".)*

As we shall later see, the b-condition is not necessarily true. Its failure is closely related to conflictual decoupling in two ways :

1. The usual definitions of extensions without taking the CDP into account will make the b-condition true (see Lemma 2.2). Rejecting paths in unforced cdp's may make it fail.

2. Careful choice of solutions of the CDP will make the b-condition true again. Consider $\Gamma$ := the "inner part" of Ex3, consisting of a,c,d,g,h and the

arrows between them, and let acg be valid. Suppose we do not see here enough information to accept the cdp $< adcg, dhg^- >$ . In particular then, acg is not considered good support for adcg. Thus, $E := \Gamma \cup \{acg, adc, adh, dhg^-\}$ is an extension which fails to satisfy the b-condition. On the other hand, if we see in acg a good positive reason, we accept adcg into E, and E satisfies the b-condition.

Failure of the b-condition has several consequences :

Remarks : 1. Stability may fail if the b-condition fails : Consider example Ex3. The point is that adc and $di^-$ preclude acgi, but not agi, if we add ag as a direct link. If adcg is a valid path, it will preclude agi. The b-condition will make adcg valid by acg !

2. Preclusion need not be 2-step transitive, if the b-condition fails : Is the preclusion of a preclusion of a preclusion of path $\tau$ a preclusion of path $\tau$ ? Not necessarily so, if the b-condition fails.

**Stability for extensions** Working with extensions, there are several ways to consider stability. We first need some definitions.

**Definition 2.4** *a) If $\sigma : x \to y$, then $\sigma^+$ is the direct link xy (or $xy^-$, if $\sigma$ is negative).*

*Fix now $\sigma$, and assume $\sigma^+ \not\epsilon \Gamma$.*

*b) $\Gamma^* := \Gamma \cup \{\sigma^+\}$*

*c) If $\tau = \sigma\rho$, then $\tau^* := \sigma^+\rho$ ("shortcut") else $\tau^* := \tau$. If $\tau = \sigma^+\rho$, then $\tau^\sim := \sigma\rho$ ("expansion") else $\tau^\sim := \tau$. ($\rho$ may be empty, if not, $\sigma$ has to be positive.)*

*d) For $E \subseteq \Pi(\Gamma)$ let $E^* := E \cup \{\sigma^+\} \cup \{\tau^* : \tau \epsilon E\}$ ("add shortcuts"). For $E \subseteq \Pi(\Gamma^*)$ let $E^\sim := \{\tau^\sim : \tau \epsilon E\}$. ("expand shortcuts")*

We use this vocabulary to formulate several versions S1-S4 of stability for extensions.

**Definition 2.5** *a) Stability S1 : Let E be an extension in $\Gamma$, $E \models \sigma$. Then $E^*$ is an extension in $\Gamma^*$ (for all $\sigma$ or all $\sigma$ of some type). Remark : As S1 for all $\sigma$ fails already by the concept of preclusion, we shall confine ourselves here to initial $\sigma$.*

*Assume now that there is an extension E' in $\Gamma$, s.th. $E' \models \sigma$.*

*b) Stability S2 : If E is an extension in $\Gamma^*$, then so is $E^\sim$ in $\Gamma$.*

*c) Stability S3 : If E is an extension in $\Gamma^*$, then there is an extension E" in $\Gamma$, s.th. $E \models xy$ iff E" $\models xy$ (for all $xy = \tau^+$, $\tau \epsilon \Pi(\Gamma)$).*

*d) Stability S4 : Assume that for all extensions E' in* $\Gamma$ $E' \models \sigma$. *Then* $\bigcap Ext_\Gamma \models xy$ *iff* $\bigcap Ext_{\Gamma^*} \models xy$

We first show in Ex4 that S2-S3 fail rather trivially.

This diagram has an extension $E := \Gamma \cup \{abd^-, cdf, afg^-, cdg\}$ and there is no extension in $\Gamma$ containing both $afg^-$ and acd. Add now $\sigma := afg^-$. $\Gamma^* = \Gamma \cup \{\sigma^+\}$ has an extension $E" := \Gamma^* \cup \{cdf, cdg, acd, acdf\}$. $E^{\sim}"$ is $\Gamma \cup \{cdf, cdg, acd, acdf, afg^-\}$, which is no extension in $\Gamma$. So, S2 and S3 are wrong (S3 as $E" \models ag^-, ad$, impossible in any extension of $\Gamma$). (We could do without ae,eg, but shall use the diagram again.)

We have not yet investigated S4, the technical problems seem, at first sight, rather complex. So we postpone this investigation, until we have presented our ideas to discussion.

We proceed to discuss S1, and call it simply stability.

One of the central problems is to handle preclusion : Let $\tau = \sigma\rho$, and $\tau$ be precluded in E. Is $\tau^* = \sigma^+\rho$ still precluded in $E^*$ ?

Remark 1 above shows that this is true, if E satisfies the b-condition. (A slightly weaker condition would suffice : we need the "detoured" path adcg only in $E^*$, not in E.) It is for this purpose that we have defined the b-condition.

We reexamine Ex3 in more detail to show how an insufficient CDP-solution will make (the b-condition) and stability fail rather badly. Explanation : adc and $di^-$ destroy acgi, $dhg^-$ may prevent adcg in such a defective solution, so adding $\sigma := acg$ as a direct link to $\Gamma$ will make a path from a to i possible. abed and ei destroy $adi^-$, $< abei, bfi^- >$ are again in conflict, and $abfi^-$ is prevented by $af^-$ (abf is no path). We can see the diagram in two ways : if we see acg not a good enough reason to accept adcg, then adding $\sigma$ will enable a path from a via g to i. If we see $af^-$ not a good reason to accept abei, then we will have in E no path from a to i. In our final definition, we will see both arguments (among others) as good reasons to accept a cdp. More precisely, the failure of stability is seen by examining $E := \Gamma \cup \{cgi, dhg^-, edc, edh, edhg^-, bed, bedc, bedh, bedhg^-, bfi^-, acg, adc, adh, abe, abed, abedc, abedh\}$ is an extension (edcg etc. are again ruled out by CDP, and $dhg^-$). So, E has no path from a to i, positive or negative. If we now consider $\Gamma^* := \Gamma \cup \{ag\}$, $E^* := E \cup \{ag\}$, there is just one additional possible path, agi, to consider. As $adcg \not\in E^*$, $E^* \models agi$. So $E^*$ is no extension. Checking that $E" := E^* \cup \{agi\}$ is an extension, is easy : First, all $\tau \epsilon E"$ are valid in E" : there are no new contradictions, the only new paths are ag and agi, so there are no new preclusions, and no new cdp's. Second, all $\tau \not\in E"$ are not valid : Condition (0) is trivial, (1) and (2) worsen with increasing E, and unless we consider ag a good reason for adcg, that stays out too, and does not endanger the validity of agi. (It is not central here, that ag is a direct link, as we could easily have put an a' between a and c, setting $\sigma := a'cg$, complicating things, however.)

The reader may find example Ex6 (a slight modification of the "double kite" Ex5) still more convincing, as we admit here all initial segments of the cdp's, in contrast to the arrow $af^-$ in Ex3. Here, $no^-$ together with aenp precludes apro, but adding $\sigma := apr$, will give apro, if aenpr is not admitted, destroying stability. The purpose of the "double kite", discussed in more detail in II.4, on the left is to destroy the pp $aeno^-$ in a rather harmless way. (If we adopt the policy R5 discussed in II.5, we will choose some path from a to o, but this is relatively subtle compared to the rather coarse questions examined here.)

In conclusion of this subsection, we present some formal results related to stability :

**Claim 2.1** *a)* $\tau^\sim \epsilon E \longleftrightarrow \tau \epsilon E^*$ *b) E contradicts* $\tau^\sim$ *iff* $E^*$ *contradicts* $\tau$ *c) If* $E^*$ *precludes* $\tau$, *E precludes* $\tau^\sim$ *d) If E satisfies the b-condition,* $\sigma \epsilon E$, *and E precludes* $\tau^\sim$, *then* $E^*$ *precludes* $\tau$.

Proof : a) " $\rightarrow$ " If $\tau^\sim = \tau$, use $E \subseteq E^*$. If $\tau = \sigma^+\rho$, and $\sigma\rho = \tau^\sim \epsilon E$, then $\tau = \tau^{\sim *} \epsilon E^*$ by definition. " $\leftarrow$ " $\tau \epsilon E^*$ is either $\rho$ or $\rho^*$ for $\rho \epsilon E$. If $\tau = \rho^*$, then $\tau^\sim = \rho$. If $\tau = \rho$, then $\tau^\sim = \rho$.

b) " $\rightarrow$ " If $\tau^\sim = \tau$, use $E \subseteq E^*$. Else, there is $\tau' \epsilon E$ contradicting $\tau^\sim$. So $\tau'$ contradicts $\tau$ too, and $\tau' \epsilon E \subseteq E^*$. " $\leftarrow$ " If $\tau' \epsilon E^*$ contradicts $\tau$, $\tau'^\sim$ contradicts $\tau^\sim$, and by a) $\tau'^\sim \epsilon E$.

c) Let $\tau = x_1 \ldots x_n$, $\tau^\sim = x_1 x_{11} \ldots x_{1k} x_2 \ldots x_i \ldots x_n$, $\sigma = x_1 x_{11} .. x_{1k} x_2$ Case 1 : $x_i x_n^- \epsilon \Gamma^*$, $i > 1$. As $\sigma$ is initial, $x_i x_n^- \epsilon \Gamma$. (For i=1, $\sigma$ contradicts $\tau$.) So $\tau^\sim$ is precluded by the same $x_i x_n^-$. Case 2 : $\rho := x_1 \ldots y_j \ldots x_i \epsilon E^*$, $y_j x_n^- \epsilon \Gamma^*$. Again, $y_j x_n^- \epsilon \Gamma$, $\rho^\sim \epsilon E$, and $\rho^\sim, y_j x_n^-$ preclude $\tau^\sim inE$.

d) Let $\tau$, $\tau^\sim$ as in c). Case 1 : $x_i x_n^- \epsilon \Gamma \subseteq \Gamma^*$. If $i \geq 2$, then $x_i x_n^-$ precludes $\tau$ by case 1 too. If $x_i$ is $x_{1i}$, $\sigma$ and $x_{1i} x_n^-$ preclude $\tau$ in $E^*$ (by case 2). Case 2 : $\rho := x_1 \ldots y_j \ldots x_i \epsilon E$ , $y_j x_n^- \epsilon \Gamma$. If $i \geq 2$, then $x_i x_n^-$ precludes $\tau$ by case 2 too. If $x_i$ is $x_{1i}$, then $\sigma$, $\rho \epsilon E$, so by the b-condition, $\rho' := x_1 \ldots y_j \ldots x_{1i} .. x_2 \ \epsilon E$, and $\rho', y_j x_n^-$ preclude $\tau$ by case 2. □

**Lemma 2.2** *If* $E = \Gamma \cup \{\sigma : E \models \sigma\}$, *where* $\models$ *is defined by condition (0)-(2) in II.1 only, then E satisfies the b-condition.*

Proof : By the above claim. □

## 2.4 Discussion of a difficult example

We come to a subtle question which does not arise in the usual definitions of extensions without unforced cdp's : Suppose we have a pp $\sigma : x \rightarrow z$ in $\Gamma$, will every extension $E \subseteq \Gamma$ satisfy $E \models xz$ or $E \models xz^-$ ? The answer is : usually yes, if not, we have at least two unforced cdp's of different polarity (see Claim 2.4). Such a situation can indeed arise, as Ex5 shows. This example plays a central role in our development, because it and its modifications are a kind of worst case, and thus good tests for all attempts to solve the CDP. We first put our above question into a (local) definition, discuss Ex5 and conclude with the abovementioned Claim 2.4.

**The c-condition** We say that an extension E satisfies the c-condition iff : If $\tau = x \ldots y \epsilon E$ is positive, yz or $yz^- \ \epsilon \ \Gamma$, then there is some $\rho := x \ldots z \epsilon E$ (positive or negative). I.e. if there is a pp x ... z, whose initial segments are in E, then $E \models xz$ or $E \models xz^-$.

We need this condition for stability. Consider the important example Ex5 (the "double kite"). It may be seen as the basic example for our kind of definitions and questions. All simple possibilities are destroyed, so it is a nice test.

Let's look at the left hand side. All negative paths $a \ldots o^-$ are impossible (precluded). The positive paths a ... o are in cdp with $bgo^-$ or $cio^-$. 1. There is no positive support for the cdp's on the left, and all positive pp's on the right are precluded. 2. The negative paths $a \ldots o^-$ are precluded only on the left, not on the right. 3. All initial segments are ok, i.e. in all extensions. Conclusion : all cdp's on the left are unforced.

The right hand side is the same, only mirrorred.

So, considering only the requirements R1-R3 (to be formulated below in II.5), we should not admit any path from a to o, and thus violate the c-condition. Adding the valid path abf as a single link, will however result in a valid positive path, as the cdp is jumped, destroying stability.

Like in the Nixon-Diamond, we may adopt the policy of choosing, and this may be seen as - in a certain sense, cf. the discussion of R5 below - in agreement with intuition, since both kites have equal strength.

Essentially the same example shows the failure of stability of any solution that conforms to intuition. This too will be discussed below in II.5.

We add two technical results, the latter a motivation for our Ex5 :

Let $l(\sigma)$ be the last but one point on a path $\sigma$, $deg_x(y)$ the usual degree of y wrt. x.

**Claim 2.3** *There are no infinite descending chains of preclusions.*

Proof : Let $\sigma$ (with some link) be a preclusion of $\tau$, then $deg_x(l(\sigma)) < deg_x(l(\tau))$. $\square$

**Claim 2.4** *Assume the CDP satisfies R2 of II.5 . Assume that E violates the c-condition at x ... y. Then there are opposite cdp's* $< \tau, \rho >$, $< \tau^-, \rho^- >$,

$\tau : x \to y$, $\tau^- : x \to y^-$ *which are not forced.*

Proof : Suppose not. If one is forced, $\tau$ or $\tau^- \epsilon E$, Contradiction. Case 1 : There is no cdp. As contradiction is impossible, $x_1 \ldots y$ has to be precluded. As there is no infinite descending chain of preclusions (by Cl.2.3), there is a smallest, which is neither contradicted nor precluded, thus in E, Contradiction. Case 2 : There is a cdp, but all cdp's are of the same polarity. Assume they are of type $< \tau : x \to y, \rho : z \to y^- >$ . We show that all negative pp's $\sigma : x \to y^-$ are precluded, contradicting R2. Let $\sigma := x \ldots uy^-$ be a negative pp. As $\sigma \not\epsilon E$, and contradiction is impossible, there may be some preclusions, and then a cdp $< \tau, \rho >$ . Considering the type, the precluding steps have to be uneven, so at least 1, and $\sigma$ is precluded. $\Box$

## 2.5 Requirements to solutions of the CDP, inherent failure of stability

We summarize our experiences gained so far into a collection of requirements to solutions of the CDP.

Consider a cdp $< \tau, \rho >$, $\rho \epsilon E$.

R1. The presence of a valid path $\tau'$ in E, parallel to $\tau$ should be considered good positive reason to choose $\tau$. This is intuitively acceptable, and (in a way) necessary to satisfy the b-condition badly needed for stability. We have, however, to take care, that not two cdp's support each other, without any further independent support.

R2. The preclusion of all $\tau^-$ in E contradicting $\tau$ should be considered good negative reason to choose $\tau$. As we have seen in example Ex1, preclusion of $\tau/\rho$ alone is insufficient.

R3. If an initial segment of $\tau/\rho$ is not in E, this should be considered good negative reason to choose $\tau$. (Together with 1., this condition indirectly supports all other cdp's $< \tau', \rho' >$ parallel to $< \tau, \rho >$ .)

Any cdp $< \tau, \rho >$ not supported by reasons R1., R2. or R3. (or some additional reason, we have not thought of) should be considered shaky. This has two consequences :

R4. If an unforced cdp $< \tau, \rho >$ is in conflict with another potential path $\sigma$, which is not in a cdp, then $\sigma$ should be chosen, as $\sigma$ is vertically less contested than $\tau$. This situation is a kind of Nixon-Diamond, where one path is in a cdp. See e.g. example Ex5a. It is this intuitive requirement which finally destroys stability - see below.

R5. If - in the absence of any $\sigma$ as discussed in R4 - two unforced cdp's $< \tau, \rho >$ and $< \tau^-, \rho^- >$ are in conflict with each other (Ex5), we can either be cautious, and choose neither (which will destroy stability in yet another way), or choose one arbitrarily, just as in the case of the Nixon-Diamond. So the last decision - i.e. satisfying requirement R5 - is more in the spirit of extensions, and technically better. There is, however, an intuitive argument against R5 : If we have just one unforced $< \tau, \rho >$, we do not take it, precise- ly because it is unforced. If we have another one, of opposite polarity, we choose one. Contrast this to the ordinary Nixon-Diamond : If we have just one half of it, we happily take it !

The CDP destroys stability

So far, we have seen how careless solutions to the CDP destroy stability. But, as already indicated, the CDP itself makes stability fail. The reason is, that a new direct link can hide a cdp, because it jumps over the branching point. Consider example Ex5a, and example $Ex5a^{*}$, adding the direct link $\sigma^{+}$ (for $\sigma := adl$). So, in the initial situation, we have to choose the simple path apo by R4, because the conflicting cdp is unforced and as such (vertically) weaker. In the new situation, we can choose the conflicting path $alo^{-}$ because the cdp is now invisible.

The cautious approach (i.e. violating R5) destroys stability in yet another way :

Consider example Ex5 again, and example $Ex5^{*}$, where we have added the direct link $\sigma^{+}$ ($\sigma := adl$). In the original case, we have no valid path from a to o, whereas by R4, we have one in the new case : $alo^{-}$, it is the best choice.

## 2.6 A solution of the CDP

We give an inductive definition of extensions, which 1. takes requirements R1 - R5 into account 2. satisfies the b-condition, so gives some stability 3. fails by R4 and the above comment to be stable 4. embodies (an extended version of) R.Thomason's idea of un-capriciousness.

Sketch :

We proceed inductively following the canonical well-founded partial order on $\Gamma$. Consider [xy], the set of all pp's from x to y, and accumulate new pp's in [xy] into E such that, considering the outline of definitions given in II.1 : (0) is decided already by induction (1) is respected (the order in which [xy] is considered, will decide which extension is taken). (2) is again decided by induction We postpone all cdp's $< \tau, \rho >$ until the end. $\rho$ will have been decided by induction already. After having finished all non-cdp's, admit the cdp's which satisfy R1-R3. We have postponed the cdp's to have enough information to decide on R1 and to respect the stronger information in non-cdp's (R4). If considered suitable, decide for cdp's of one polarity to satisfy R5. Finally, admit only those extensions, which respect horizontal coherence like Thomason's non-capriciousness.

Formal Definition :

First, the framework for induction :

**Definition 2.6** *a) For x,y points in* $\Gamma$, *let* $x < y$ *iff there is a monotone sequence of positive or negative arrows in* $\Gamma$, *beginning in x and ending in y. (This is the transitive closure of the canonical order on points in* $\Gamma$.)

*b) Let* $[x,y] := \{\sigma \epsilon \Pi(\Gamma) : \sigma : x \to y\}$ *and* $[\sigma] := [dom\sigma, ran\sigma]$.

*c) define* $< \; = \; <_{\Pi(\Gamma)}$ *on* $\Pi(\Gamma)$ *by* $\sigma < \tau$ *iff* $dom(\sigma) = dom(\tau) \wedge ran(\sigma) < ran(\tau)$ *or* $ran(\sigma) = ran(\tau) \wedge dom(\sigma) > dom(\tau)$ *or* $dom(\sigma) > dom(\tau) \wedge ran(\sigma) < ran(\tau)$ .

*For the intuition : Case 1 is for initial segments and preclusion, Case 2 is for cdp's (*$\rho \epsilon E$ *is determined before* $\tau$ *is considered, for* $< \tau, \rho >$ *a cdp), Case 3 is for transitivity.*

**Claim 2.5** $<$ *on* $\Pi(\Gamma)$ *is a well-founded partial order.*

Proof : Transitivity is trivial by case 3, and transitivity of $<$ on points. Suppose now that there is an infinite descending chain $< \sigma_i : i\epsilon\omega >$, $\sigma_i > \sigma_{i+1}$. Thus, either $< dom(\sigma_i) : i\epsilon\omega >$ is an infinite increasing chain of points in $\Gamma$, or $< ran(\sigma_i) : i\epsilon\omega >$ is an infinite descending chain of points (or both), contradicting finiteness and acyclicity of $\Gamma$. □

Thus, we can do induction on $<_{\Pi(\Gamma)}$ . Moreover, $<$ can be extended naturally to $[x,y] < [x',y']$ or $[\sigma] < [\tau]$ (and is then a strict partial order), we will not distinguish.

We can define now our solution of the CDP :

**Definition 2.7** *We call E a grounded extension of* $\Gamma$ *iff it can be constructed inductively on* $<_{\Pi(\Gamma)}$ *in the following way :*

*Suppose all* $[x',y'] \cap E$ *are defined for* $[x',y'] < [x,y]$, *and con- struct* $[x,y] \cap E$ *as follows :*

*Fix a well-order* $\prec$ *on [x,y]. (The choice of* $\prec$ *will give the different extensions.)*

*<u>First,</u> put all direct links* $\tau \epsilon [x,y] \cap \Gamma$ *into* $[x,y] \cap E$. *(This is primacy of direct links.)*

*Second,* handle all $\tau \epsilon [x, y]$, which are not in cdp's $< \tau, \rho >$ (remember, by case 2 for $<_{\Pi(\Gamma)}$, we know already whether $\rho \epsilon E$, and thus all "dangerous" cdp's) in the following conventional way : $\tau = x \ldots zy \epsilon E$ iff (0) $x \ldots z \epsilon E$ (known by case 1 of $<$), and $zy \epsilon \Gamma$ (1) $\tau$ is not yet contradicted in E, i.e. there is no $\tau^- \prec \tau$, $\tau \epsilon E$, $\tau : x \to y$. Here, the order $\prec$ determines the choice of the extension. (2) $\tau$ is not precluded in E (the usual definition), i.e. 1. There is no $uy^- \ \epsilon \ \Gamma$, $u$ on $x \ldots z$, $u \neq z$ 2. There is no $x \ldots x_i y_1 \ldots y_j \ldots y_m x_{i+k} \ \epsilon \ E$ with $y_j y^- \ \epsilon \ \Gamma$, $x_i, x_{i+k}$ on the path $x \ldots y$ (This is decided by induction, case 1 again.) (hc) A condition of horizontal coherence, discussed below.

*Third,* handle all such cdp's $< \tau, \rho >$ where $\tau$ satisfies (0)-(2). Doing cdp's at the end has two effects : first, it expresses primacy of non-cdp's if there is such a choice (R4), second, it takes independent positive information into account (R1), making the b-condition true :

*(3.1) If there is $\tau'$ parallel to $\tau$, $\tau'$ already in E, put $\tau$ into E too. (3.2) If all $\tau^- : x \to y^-$ contradicting $\tau$ are precluded in E, put $\tau$ into E (use here induction again). (3.3) If an initial segment of $\tau/\rho$ is not in E, put $\tau$, and all parallel $\tau'$ satisfying (0),(2) in cdp's into E too. (R4 is already done by considering non-cdp's first, so there is no (3.4).) (3.5) If so desired, choose one of contradictory cdp's : Put either all cdp's $\tau$ or all cdp's $\tau^-$ into E, provided they satisfy (0)-(2), respecting (hc) again.*

It remains to specify (hc). We formulate a strong condition, which may be modified when desirable:

**Definition 2.8** *(hc) Suppose $[x', y] \cap E$ is decided already. Suppose there is u s.th. $x < u < y$, $x' < u < y$, all monotone sequences of arrows from x to y and from x' to y pass through u, $[x, u] \cap E$ and $[x', u] \cap E$ consist only of positive paths. Then $\tau : x \to y \epsilon E$ iff there is $\tau' \epsilon [x', y] \cap E$ s.th. $\tau$ and $\tau'$ agree from u to y.*

To show that the above definition is not totally unstable, we prove

**Lemma 2.6** *If E is an extension defined as above, then E satisfies the b-condition.*

Proof : Assume z to be least above x s.th. $\sigma := x \ldots y \ldots vz \epsilon E$, $\eta := x \ldots u..y \epsilon E$, $\tau := x \ldots u \ldots y \ldots vz \not\epsilon E$, i.e. where the b-condition fails. (0)

By induction, $x \dots u \dots y \dots v \epsilon E$, and $vz\epsilon\Gamma$, as $\sigma\epsilon E$. (1) As $\sigma$ is composite, it can't be contradicted in E, so neither can $\tau$. (2) Suppose $\tau$ is precluded. If $\tau$ is precluded by case 1, $\sigma$ is precluded by case 1 or 2, as is easy to see. If $\tau$ is precluded by case 2, $\rho := x \dots w \dots u$ and $wz^-$, then by minimality of z $\rho u \dots y \epsilon E$, so $\sigma$ is already precluded. (hc) The hc-condition chooses only extensions, without changing their internal structure, so we need not consider it here. Suppose now that $< \tau, \rho >$ is a cdp. But, $\tau$ is supported by $\sigma$, thus in E according to (3.1), and we are done. $\Box$

# 3 THE DOUBLE DIAMOND PROBLEM

**Outline of Section III** In [MS], we criticised the Thomason definition of sceptical inheritance because of its results in the Double Diamond-case. Essentially measured by extensions, its behaviour was not intuitively correct. We shall give here refinements of the sceptical method to solve the Double Diamond and similar problems. Yet, as already mentioned in [MS], there seems to be no way to make the directly sceptical approach correspond perfectly to $\bigcap Ext$ . -

We begin with a remedy to a slight problem in the Thomason definition hinted at in our joint paper [MS]. Unfortunately, our modification and the original version differ also in the validity of statements : There is $\Gamma$ and $x, y\epsilon\Gamma$ such that $\Gamma \not\models xy$, $\Gamma \models' xy$.

Next, we discuss several solutions of the double diamond problem, show that one is initially stable, but present an (intuitive) counterexample to each, proving that our definitions are not fully satisfactory and only further approximations. Considering the arguments in [MS] for the divergence between direct scepticism and $\bigcap Ext$, this is no surprise.

We conclude this section by some en passant remarks : We show how to encode propositional logic into extensions, and thus NP-hardness of $\bigcap Ext$, and give a sketch for an ATMS-structure for extensions.

## 3.1 Preclusion and the Thomason Approach

In [MS], we presented Ex7, and showed that the Thomason definition makes the path abe valid, which is against intuition. To solve the problem, we modify the definition (Def. 3.2).

**Definition 3.1** *Let X be a set of pp. in a given diagram* $\Gamma$.

*1. A pp* $\tau = x \ \dots x \ z$ *of length* $> 1$ *is precluded in X iff a) there is a direct link* $x_i z^- \epsilon\Gamma$, $i < n$, *or b) there is a pp* $\sigma := x_1 \dots y_j \dots y_k = x_i$, $\sigma\epsilon X$, *and a direct link* $y_j z^- \epsilon\Gamma$ *s.th. either* $j < k$ *or* $i < n$.

*2. A pp* $\tau = x_1 \ldots x_n z$ *of length* $> 1$ *is contradicted in* $X$ *iff there is a pp* $\sigma := x_1 = y_1 \ldots y_k$, $\sigma \epsilon X$, *and a direct link* $y_k z^- \epsilon \Gamma$

**Definition 3.2** *We build the set of valid paths,* $V = \bigcup_{m \epsilon \omega} V_m$ *inductively, by degree :*

*Set* $V_0 := \Gamma$.

*Let* $W := \bigcup_{m' < m} V_{m'}$ *be defined and consider* $\sigma := x_1..x_n z, deg(\sigma) = m$.

*Then* $\sigma \epsilon V_m$ *iff*

*(0)* $x_1 \ldots x_n \ \epsilon \ W$ , $x_n z \ \epsilon \ \Gamma$

*(1)* $\sigma$ *is not precluded in* $W$

*(2) Every contradiction to* $\sigma$ *in* $W$ *is precluded in* $W$*.*

*Finally, set* $\Gamma \models \sigma$ *iff* $\sigma \epsilon V$*.*

Ex8 shows, that our modification and the original definition make different statements valid. In the Thomason definition, abfh is valid, as $aeh^-$ is precluded by ade and dh. So $achi^-$ is precluded, and both agi and abi are valid, thus $\Gamma \models ai$. In our definition, ach is valid too, but abfh is not, as it is precluded by aef and $eh^-$. So $achi^-$ is not precluded any more, and we have a Nixon-Diamond situation and no valid path from a to i, thus $\Gamma \not\models ai$.

## 3.2 Solutions to the double diamond problem

The basic idea for all aproaches is very simple and closely follows intuition : Coding information into presence or absence of paths is too simple. Yet, we would like to preserve some of the simplicity of the directly sceptical approach as opposed to the $\bigcap Extensions$ approach. So, we split validity into really valid, and maybe valid. Thus, we have two sets of paths over Γ, V and C, for "valid" and "contested" (and some paths will stay definitely out).

We shall examine procedures which roughly follow the considerations :

For a path to be in V :

- all positive reasons to put it in must be in V too
- all negative reasons not to put it in, may be in V or C

For a path to be in C :

- all positive reasons to put it in, may be in V or C
- all negative reasons not to put it in, must be in V.

In other words, for a path to be definitely out :

- there are no positive reasons in V or C to put it in or
- there is a negative reason in V not to put it in.

<u>Solution 1 :</u>

First, put all direct links into V.

Second, consider $\tau := x \ldots v \ldots uy$.

Put $\tau$ into V iff (V0) $x \ldots v \ldots u \epsilon V$, and $uy \epsilon \Gamma$ and (V1) $\tau$ is not precluded in $V \cup C$ and (V2) every contradiction $\tau^-$ in $V \cup C$ is precluded in V

Put $\tau$ into C iff $\tau$ is not in V and (C0) $x \ldots v \ldots u \epsilon V \cup C$, and $uy \epsilon \Gamma$ and (C1) $\tau$ is not precluded in V.

It is easy to see that this definition solves the Double Diamond, see Ex9 : apq and $asq^-$ will be in C, so apqr too. So, by (V2), $astr^-$ can't be in V. (We can iterate the procedure, if so desired, and scepticism will be preserved and carried through the entire diagram.)

Unfortunately, Solution 1 will fail in Ex10. The definition puts afcd into C (but not abcd) : The preclusion agb is in V, but not the preclusion agbc. (This makes the b-condition fail for C too !) But, there is no extension where afcd is valid, because then afc will be valid, thus abc, and agbc, so we have a preclusion. Consequently, no extension will have a path from a via d to h, which gives a preclusion (or contradiction) of ahi, so ahi is valid in all extensions, but not by our def., as afcd, afcdh are in C, so by (V1) $ahi \not\epsilon V$.

Proposed remedy, Solution 2:

Modify (C1) to

(C1') $\tau$ is not precluded in $V \cup C$.

So, we are more cautious in making a path possible.

We can now show the b-condition for V and C, and prove initial stability.

**Lemma 3.1** *a) This definition satisfies the b-condition b) This definition is initially stable.*

Proof : a) We show by simultaneous induction that V and $V \cup C$ satisfy the b-condition. Let a ... cd, a ... b ... c be given, consider $\tau := a..b..cd.$ (V0), (V2), (C0) are trivial. (V1),(C1): If the preclusion $\sigma \epsilon V \cup C$ of $\tau$ hits below c, use the b-condition for $V \cup C$ by i.h ...

b) We shall show more, first some notation. Let $\rho$ be fixed, $V \models \rho$, and $\rho^+$ added to $\Gamma$, $\rho : r \to r'$. Let $V^*$, $C^*$ be as usual, i.e. $V^* = V \cup \{\rho^+\sigma : \rho\sigma\epsilon V\}$ etc., and let $V_*$ denote $V_{\Gamma^*}$ , $C_*$ $C_{\Gamma^*}$ likewise, constructed by the same principles in $\Gamma^*$.

We prove $V^* = V_*$ and $C^* = C_*$ by simultaneous induction.

Consider now $\tau := x \ldots v \ldots uy$.

First, we show $\tau \epsilon V \to \tau \epsilon V_*$ and $\tau \epsilon C \to \tau \epsilon C_*$.

(V0),(C0): Induction. (V1),(C1): Let $\sigma := x \ldots w \ldots v \epsilon V_* \cup C_*$, $wy^-$ be a preclusion, by i.h. $\sigma \epsilon V^* \cup C^*$, so $\sigma^\sim \epsilon V \cup C$ is a preclusion of $\tau$, Contradiction.

(V2): Let $\tau^- zy^-$ be a contradiction of $\tau$ in $V_* \cup C_*$, which is not precluded in $V_*$ . So $\tau^- \epsilon V^* \cup C^*$ , and $\tau^{-\sim} zy^-$ is a contradiction of $\tau$ in $V \cup C$, thus $\tau^{-\sim} zy^-$ must be precluded in V by some $\sigma$, so $\sigma \epsilon V_*$ by i.h. .

Use the b-condition for V (remember, $\rho \epsilon V!$), to show that $\sigma$ still gives rise to a preclusion of $\tau^- zy^-$ in $V_*$, in case it hits $\tau^{-\sim}$ below r' (and $\rho$ is an initial segment of $\tau^{-\sim}$).

If $\tau \epsilon V^* - V$, then $\tau^\sim \epsilon V \subseteq V_*$, and preclusions and contradictions of $\tau$ will stay so for $\tau^\sim$. Likewise for $\tau \epsilon C^* - C$.

Next, we show $\tau \epsilon V_* \to \tau \epsilon V^*$ and $\tau \epsilon C_* \to \tau \epsilon C^*$.

Case 1 : $\tau \epsilon \Pi(\Gamma)$. (V0),(C0) are trivial. (V1),(C1): A preclusion $\sigma \epsilon V \cup C$ will by i.h. be in $V_* \cup C_*$ too. (V2): Let $\tau^- zy^-$ be a contradiction in $V \cup C$. By i.h., $\tau^-$ is in $V_* \cup C_*$, so there is a preclusion $\sigma$ of $\tau^- zy^-$ in $V_*$, so $\sigma \epsilon V^*$, and $\sigma^\sim \epsilon V$ is a preclusion of $\tau^- zy^-$.

Case 2 : $\tau \not\epsilon \Pi(\Gamma)$. (V0),(C0): trivial by i.h. . (V1),(C1) : Suppose $\tau^\sim$ is precluded by some $\sigma \epsilon V \cup C$, so $\sigma \epsilon V_* \cup C_*$ by i.h. , $\sigma$ can be turned into a preclusion of $\tau$, using the b-condition for $V_* \cup C_*$ (by i.h.), if it hits $\tau^\sim$ below r'. (V2) : Suppose there is a contradiction $\tau^- zy^-$ of $\tau^\sim$ in $V \cup C$. By i.h., $\tau^- \epsilon V_* \cup C_*$, so $\tau^- zy^-$ is a contradiction to $\tau$, and must be precluded by some $\sigma \epsilon V_*$. By i.h., $\sigma \epsilon V^*$, so $\sigma^\sim \epsilon V$ is a preclusion of $\tau^- zy^-$. □

But, here is a counterexample to that definition : Ex4. Here, $acd \epsilon C$, so $acdf \epsilon C$, so $afg^-$ contradicting aeg is precluded in C and neither in V nor in C, by our modified (C1'), so aeg is in V. But, obviously, there is an extension where $afg^-$ is true. (We can construct a similar diagram s.th. $E \vdash af$ for some extension E, but $V \cup C \not\vdash af$.)

<u>So we try something else, Solution 3 :</u>

<u>Drop C (or, say $C := \Pi(\Gamma) - V$</u>), and replace in (V1), (V2) $V \cup C$ by $\Pi(\Gamma)$ :

Put $\tau$ into V iff (V0) $x \ldots v \ldots u \epsilon V$, and $uy \epsilon \Gamma$ and (V1) $\tau$ and its initial segments are not precluded in $\Pi(\Gamma)$ and (V2) every contradiction $\tau^-$ in $\Pi(\Gamma)$ (or an initial segment thereof) is precluded in V.

This will fail too. If we try to show that it corresponds to $\bigcap Ext$, we seem to need a condition like : If $\tau$ is precluded in all extensions of E, then there is a common preclusion $\sigma$ of $\tau$ in all E. As pointed out in [MS], this is not always true, here is another example, Ex11. Suppose we switch adg on (and $aeg^-$ off). Then, adgf is on too, which gives a preclusion of acfh, thus, acfhi (and $hk^-$) is no preclusion of abikm. But, if adg is on, so is adgl. $gh^-$ destroys $gfhk^-$, so adglk gives a preclusion of abikm. Change the situation, and switch $aeg^-$ on. So, adglk is no preclusion any more, its initial segment is off. At the same time, adgf is off, so nothing prevents acfhi to preclude abikm.

<u>Remark :</u> Using preclusion as negation, we can turn any difference between two notions of validity any way round.

### 3.3 Coding propositional logic into nets, outline of an ATMS-structure

We can represent each propositional variable by a Nixon-Diamond, handle negation by preclusion, and disjunction by joining at top and bottom.

As an exercise, we encode $\neg a \vee \neg(a \vee \neg b)$, Ex12 (the formula is true iff a is false). If a is true, we have preclusion of the left hand positive path, i.e. 1-4-8-17-19-24. As there can be no negative path from 1 via 5 to 21, there is no contradiction to 1-4-6-9-15-21-22, and the right positive path from 1 to 24 is precluded too. If a is false, there is nothing to prevent the left hand positive path. So we have a (positive) path from bottom to top iff the formula is true.

**Corollary 3.2** *Deciding* $\bigcap Ext$ *of nets is NP-hard.*

**ATMS** Extensions are fully determined by deciding the generalized Nixon diamonds (which may have more than two independent vertical paths, of maybe greater length), and, if so desired, the (generalized) contradictory pairs of unforced cdp's (see R5 in Section II). Let $X := \{D_1 \dots D_n, C_1 \dots C_m\}$ be the set of these structures. Let $Y \subseteq X$, and $D_i \epsilon Y$ say that $D_i$ is decided positively etc. So labelling every pp. $\tau$ with a subset of $\mathcal{P}(X)$ will fully describe the extensions where $\tau$ is valid.

# 4 SEMANTICS

**Outline of Section IV** We formalize propositions like "$\phi$ is normally valid in c" by defining sets of normal (important enough) subsets, called $\mathcal{N} - systems$ over c, and use this technique to construct models for (almost) any notion of validity in nets. (Though, in our opinion, the extensions approach is best handled by a combination of these semantics and a possible world technique.) For more details and motivation, see the Discussion following Proposition 4.3. The first construction has one defect : we build models from the full theory, not from the axioms. To do the latter, we have to refine the method (starting in Definition 4.3) to accomodate preclusion. The basic reason is, that direct links are for preclusion in a stronger sense true than valid paths. We thus define $\mathcal{N} - families$ - decreasing sequences of $\mathcal{N} - systems$ - which give increasingly strong notions of truth and use them to construct models suitable to express preclusion. The concept of $\mathcal{N} - families$ is general enough to permit the construction of semantics for more complicated notions of validity like the ones discussed in Section III.

**Definition 4.1** *Call* $\mathcal{N}(c) \subseteq \mathcal{P}(c)$ *a* $\mathcal{N} - system$ *over c iff*

*a.* $c \epsilon \mathcal{N}(c)$

*b.* $a\epsilon\mathcal{N}(c)$, $a \subseteq b \subseteq c \rightarrow b\epsilon\mathcal{N}(c)$

*c.* $a, b\epsilon\mathcal{N}(c) \rightarrow a \cap b \neq \emptyset$ *if* $c \neq \emptyset$ *(thus,* $\emptyset \not\epsilon\mathcal{N}(c)$*)*

**Remark 4.1** *a)* $\mathcal{N}$ *stands for normal. We formalize "$\phi$ is normally valid in c" by* $\exists a\epsilon\mathcal{N}(c).\forall x\epsilon a.\phi(x)$. *b) There is nothing to prevent e.g.* $\mathcal{N}(c) = \{a \subseteq c : x\epsilon a\}$ *for some fixed* $x\epsilon c$. *This might seem pathological. Two comments : First, compare to topo- logy. Suitable choice of topology will make e.g. the function* $d : RR \rightarrow RR$,

$$d(x) = \begin{cases} 0 & \textit{iff } x \textit{ is rational} \\ 1 & \textit{otherwise} \end{cases}$$

*continous, certainly a pathological case too. Second, this x might be a very prototypical case, and thus have intuitive meaning.*

**Lemma 4.2** *Let* $\mathcal{L} \subseteq \mathcal{P}(X)$ *be such that* $A, B\epsilon\mathcal{L} \rightarrow A \cap B \neq \emptyset$. *Then* $\mathcal{N}(X) := \{A \subseteq X : \exists B\epsilon\mathcal{L}.B \subseteq A\} \cup \{X\}$ *is a* $\mathcal{N}-system$ *over X.* □

**Example 4.1** *Let* $\alpha$ *be any ordinal, and* $U := \{f : \alpha \rightarrow 2 = \{0,1\}\}$. *For* $i < \alpha$ *let* $X_i := \{f\epsilon U : f(i) = 1\}$, $X"_i := \{f\epsilon U : f(i) = 0\} = U - X_i$. *For* $j < \alpha$ *let* $I_j \subseteq \alpha$, $I"_j \subseteq \alpha - \{j\}$ *such that* $I_j \cap I"_j = \emptyset$, *and let* $\mathcal{L}_j := \{X_j \cap X_i : i\epsilon I_j\} \cup \{X_j \cap X"_i : i\epsilon I"_j\}$. *Then* $\mathcal{L}_j$ *sat. the prerequisites of Lemma 2 for* $X_j$ *and all j. Consequently,* $\mathcal{L}_j$ *so defined generates a* $\mathcal{N}-system$ *over* $X_j$ *for all* $j < \alpha$.

Proof : $\mathcal{L}_j \subseteq \mathcal{P}(X_j)$ is trivial. As $I_j \cap I"_j = \emptyset$ and $j \not\epsilon I"_j$, the function

$$f(i) = \begin{cases} 1 & \text{if i=j or } i\epsilon I_j \\ 0 & \text{otherwise} \end{cases}$$

is well-defined and in all $X_j \cap X_i$, $X_j \cap X"_i$. □

**Definition 4.2** *Let* $\Gamma$ *be a net,* $V(\Gamma)$ *its vertices. We call* $V(\Gamma)$ *the language of* $\Gamma$. *A* $\Gamma-structure$ $\Sigma$ *is a tripel consisting of a universe, subsets of the universe for the* $p\epsilon V(\Gamma)$, *and* $\mathcal{N}-systems$ *over these subsets :*

$\Sigma = < U , \{[p] : p\epsilon V(\Gamma)\} , \{\mathcal{N}(p) : p\epsilon V(\Gamma), \mathcal{N}(p)$ *a* $\mathcal{N}-system$ *over* $[p]\}$ $>$ .

*We define validity :* $\Sigma \Vert\!\models p \rightarrow q$ *iff* $[p] \cap [q]\epsilon\mathcal{N}(p)$. $\Sigma \Vert\!\models p \not\rightarrow q$ *iff* $[p] - [q]\epsilon\mathcal{N}(p)$.

*and say $\Sigma$ is a $\models -model$ of $\Gamma$ (for $\models$ a notion of validity in nets), iff for all p,q ($\Sigma \Vert\models p \rightarrow q$ iff $\Gamma \models p \rightarrow q$) and ($\Sigma \Vert\models p \not\rightarrow q$ iff $\Gamma \models p \not\rightarrow q$) .*

**Proposition 4.3** *Let $\Gamma$ be a net and $\models$ be any notion of validity of links, which 1. does not permit $\Gamma \models p \rightarrow q$ and $\Gamma \models p \not\rightarrow q$ at the same time (note the subsequent Discussion 2., however) 2. satisfies $\Gamma \models p \rightarrow p$ for all p. Then there is a (canonical) $\models -model$ $\Sigma$ for $\Gamma$.*

Proof : Let $U := \{f : V(\Gamma) \rightarrow 2\}$, $X_p := \{f \epsilon U : f(p) = 1\}$, $X"_p := \{f \epsilon U : f(p) = 0\} = U - X_p$, $I_p := \{q : \Gamma \models p \rightarrow q\}$, $I"_p := \{q : \Gamma \models p \not\rightarrow q\}$, $\mathcal{L}_p := \{X_p \cap X_q : q \epsilon I_p\} \cup \{X_p \cap X"_q : q \epsilon I"_p\}$, $\mathcal{N}(p) := \{Y \subseteq X_p : \exists Z \epsilon \mathcal{L}_p . Z \subseteq Y\} \cup \{X_p\}$.

We show that $\Sigma := <$ U , $\{X_p : p \epsilon V(\Gamma)\}$ , $\{\mathcal{N}(p) : p \epsilon V(\Gamma)\} >$ is a $\models -model$ of $\Gamma$ : As $\Gamma \models p \rightarrow p$, so $\Gamma \not\models p \not\rightarrow p$, and $p \not\epsilon I"_p$. Again by 1., $I_p \cap I"_p = \emptyset$. Arguing as in Example 4.1, we see that $\mathcal{N}(p)$ is a $\mathcal{N} - system$ over $X_p$ for all $p \epsilon V(\Gamma)$, so $\Sigma$ is a $\Gamma - structure$. We turn to validity. $\Gamma \models p \rightarrow q => q \epsilon I_p => X_p \cap X_q \epsilon \mathcal{N}(p) => \Sigma \Vert\models p \rightarrow q$ . $\Gamma \models p \not\rightarrow q => q \epsilon I"_p => X_p \cap X"_q = X_p - X_q \epsilon \mathcal{N}(p) => \Sigma \Vert\models p \not\rightarrow q$ . $\Sigma \Vert\models p \rightarrow q => X_p \cap X_q \epsilon \mathcal{N}(p)$ .

There are three cases to consider. Case 1 : $X_p \cap X_q = X_p$, then p=q, and we are done by 2. Case 2 : There is $Z \epsilon \mathcal{L}_p$, $Z \subseteq X_p \cap X_q$. Case 2.1 : $Z = X_p \cap X"_r \subseteq X_p \cap X_q$ for some $r \epsilon I"_p$. The function

$$f(x) = \begin{cases} 1 & \text{if x=p} \\ 0 & \text{otherwise} \end{cases}$$

is in $X_p \cap X"_r$ and shows that q=p, and we are done again. Case 2.2 : $Z = X_p \cap X_r \subseteq X_p \cap X_q$ for some $r \epsilon I_p$. The function

$$f(x) = \begin{cases} 1 & \text{if x=p or x=r} \\ 0 & \text{otherwise} \end{cases}$$

shows that either q=p again or r=q, thus $q \epsilon I_p$ and $\Gamma \models p \rightarrow q$ by definition.

$\Sigma \Vert\models p \not\rightarrow q => X_p - X_q = X_p \cap X"_q \epsilon \mathcal{N}(p)$ . Again we examine the above three cases. Case 1 : $X_p \cap X"_q = X_p$ can't be. Case 2.1 : $Z = X_p \cap X_r \subseteq X_p \cap X"_q$ is impossible too, as f=constant 1 shows. Case 2.2 : $Z = X_p \cap X"_r \subseteq X_p \cap X"_q$ for some $r \epsilon I"_p$. The function

$$f(x) = \begin{cases} 0 & \text{if x=r} \\ 1 & \text{otherwise} \end{cases}$$

shows that q=r, so $q \epsilon I"_p$ and $\Gamma \models p \not\rightarrow q$. □

Discussion : 1. We first tried simpler approaches like subsets of the real plane and measures. But look at Ex13. Following the simpler interpretation, we have to read : Most of $a_1$ is in c, most of $a_2$ is in $a_1$, most of $a_2$ is not in c.

So, $a_2$ has to be a lot smaller than $a_1$ is etc. In the end, some $a_i$ is very much smaller than $b_1$ (or some $b_i$ than $a_1$). But, then most of $b_1$ is in the complement of $a_i$, and we would have a negative arrow from $b_1$ to $a_i$, which simply is not there. If we took a lot of different models and intersections of true statements, we could easily run into situations like Ex11, and be unable to fully model a sceptical approach. So, taking different measures ($\mathcal{N}(p)$) for different p's in $V(\Gamma)$ seems necessary.

2. To represent directly contradictory information like the Garbage In Rule of Thomason, we can construct 2 models in the above way, much as his construction for the 4-valued monotonic case.

3. We have a 3-valued semantics : $[p] \cap [q] \epsilon \mathcal{N}(p)$, $[p] - [q] \epsilon \mathcal{N}(p)$ and neither. We would like to emphasize very strongly that the latter case is not intended to express lack of information : There simply is no decision wrt. q possible ! Otherwise, we might speculate, if $p \rightarrow q$, then $\ldots$, getting the same differences between scepticism and $\bigcap Extensions$, as discussed in Ex11. The $\bigcap Ext$ approach might be better modelled by a combination with a possible worlds semantics, where decisions in Nixon-Diamond's are modelled by a branching point.

4. Looking back to 1., we can see that the diagram of Ex13 simply makes no sense as a description of a simple situation like the common measure one. In this way, a clear semantics is more than a formally satisfying enterprise. We have shown how to give some sense to a very broad class of diagrams and definitions of validity. Using more restrictive semantics might give good characterizations of classes of simpler diagrams and notions of validity.

5. It should be easy to incorporate relations, functions etc. into our semantics, too. By the way, strict inheritance becomes the simple restriction $\mathcal{N}(c) = \{c\}$, so there is no problem here with representing strict and defeasible inheritance.

6. Our next task will be to construct a model from the axioms, not from the full theory. The lack of stability in all approaches that work with direct links in the case of preclusion necessitates a more fine-grained semantics if we want to express reasoning, not only truth, in nets : $p \rightarrow q$ as a direct link is in a stronger sense true than $p \rightarrow q$ by $p \rightarrow r \rightarrow q$. In our examination of the Double Diamond-problem, we have suggested still more discriminating formalisms, so we immediately prepare a general solution. The development is very parallel to the one already done above.

**Definition 4.3** *Let $\gamma$ be any ordinal. Call $< \mathcal{N}_i(c) : i < \gamma >$, $\mathcal{N}_i(c) \subseteq \mathcal{P}(c)$ a $\mathcal{N} - family$ over c iff*

*a. $c \epsilon \mathcal{N}_i(c)$ for all i*

*b. $a \epsilon \mathcal{N}_i(c)$, $a \subseteq b \subseteq c \rightarrow b \epsilon \mathcal{N}_i(c)$ for all $i < \gamma$*

*c. $< \mathcal{N}_i(c) : i < \gamma >$ is decreasing, i.e. $a \epsilon \mathcal{N}_i(c), j < i \rightarrow a \epsilon \mathcal{N}_j(c)$*

*d.* $a\epsilon\mathcal{N}_i(c)$, $b\epsilon\mathcal{N}_j(c) \rightarrow a \cap b \neq \emptyset$ *for all i,j , if* $c \neq \emptyset$.

**Remark 4.4** *We have formalized* $\gamma$ *degrees of normality. Condition d. says, that all* $\mathcal{N}_i(c)$ *are* $\mathcal{N}-systems$ *over c.*

**Lemma 4.5** *Let* $< \mathcal{L}_i : i < \gamma >$, $\mathcal{L}_i \subseteq \mathcal{P}(X)$ *be such that* $A, B\epsilon \bigcup_{k<\gamma} \mathcal{L}_k \rightarrow A \cap B \neq \emptyset$. *Then* $\mathcal{N}_i(X) := \{A \subseteq X : \exists B\epsilon \bigcup_{i \leq j < \gamma} \mathcal{L}_j.B \subseteq A\} \cup \{X\}$ *for* $i < \gamma$ *is a*

$\mathcal{N}-family$ *over X.* □

**Example 4.2** *Let* $\alpha$ *be any ordinal,* $U := \{f : \alpha \rightarrow 2\}$. *For* $i < \alpha$ *let* $X_i := \{f\epsilon U : f(i) = 1\}$, $X"_i := \{f\epsilon U : f(i) = 0\} = U - X_i$. *For* $j < \alpha$, $i < \gamma$ *let* $I_{j,i}, I"_{j,i} \subseteq \alpha$, $j \not\epsilon I"_{j,i}$, $(\bigcup_{i<\gamma} I_{j,i}) \cap (\bigcup_{i<\gamma} I"_{j,i}) = \emptyset$.

*Let* $\mathcal{L}_{j,i} := \{X_j \cap X_k : k\epsilon I_{j,i}\} \cup \{X_j \cap X"_k : k\epsilon I"_{j,i}\}$.

*Then* $< \mathcal{L}_{j,i} : i < \gamma >$ *satisfies the prerequisites of Lemma 5 for all* $X_j$, $j < \alpha$. *Thus, for all j,* $\mathcal{N}_{j,i}(X_j) := \{A \subseteq X_j : \exists B\epsilon \bigcup_{i \leq k < \gamma} \mathcal{L}_{j,k}.B \subseteq A\} \cup \{X_j\}$ *is a* $\mathcal{N}-family$ *of length* $\gamma$ *over* $X_j$.

Proof :
By $(\bigcup_{i<\gamma} I_{j,i}) \cap (\bigcup_{i<\gamma} I"_{j,i}) = \emptyset$ and $j \not\epsilon \bigcup_{i<\gamma} I"_{j,i}$ , the function

$$f(x) = \begin{cases} 1 & \text{if } \beta = j \text{ or } \beta\epsilon \bigcup_{i<\gamma} I_{j,i} \\ 0 & \text{otherwise} \end{cases}$$

is defined and in every element of $\bigcup_{i<\gamma} \mathcal{L}_{j,i}$ , so $\bigcup_{i<\gamma} \mathcal{L}_{j,i}$ contains no disjoint elements. □

**Definition 4.4** *A* $\Gamma, \gamma-structure$ $\Sigma$ *is a tripel, consisting of a universe, subsets of the universe for all* $p\epsilon V(\Gamma)$, *and a* $\mathcal{N}-family$ *of length* $\gamma$ *over these subsets :*

$\Sigma = < U , \{[p] : p\epsilon V(\Gamma)\} , \{< \mathcal{N}_i(p) : i < \gamma > : p\epsilon V(\Gamma), < \mathcal{N}_i(p) : i < \gamma >$ *a* $\mathcal{N}-family$ *over* $[p]\} >$ .

*We define* $\gamma$ *degrees of validity : Let* $i < \gamma$, *then* $\Sigma \Vert\!\models_i p \rightarrow q$ *iff* $[p] \cap [q]\epsilon\mathcal{N}_i(p)$ *and* $\Sigma \Vert\!\models_i p \not\rightarrow q$ *iff* $[p] - [q]\epsilon\mathcal{N}_i(p)$

*and say for a notion* $\models = <\models_i : i < \gamma >$ *of validity in nets of* $\gamma$ *degrees that* $\Sigma$ *is a* $\models -model$ *of* $\Gamma$ *iff* $\Sigma$ *is a* $\Gamma, \gamma - structure$ *and for all* $p, q\epsilon V(\Gamma)$, $i < \gamma$ $\Sigma \Vert\!\models_i p \rightarrow q$ *iff* $\Gamma \models_i p \rightarrow q$ *and* $\Sigma \Vert\!\models_i p \not\rightarrow q$ *iff* $\Gamma \models_i p \not\rightarrow q$ .

**Proposition 4.6** *Let* $\Gamma$ *be a net and* $\models = < \models_i : i < \gamma >$ *a notion of validity of links, which 1. does not permit* $\Gamma \models_i p \to q$, $\Gamma \models_j p \not\to q$ *for any p,q,i,j at the same time 2. satisfies* $\Gamma \models_i p \to p$ *for all p,i 3. is increasing, i.e. satisfies* $\Gamma \models_i p \to q => \Gamma \models_j p \to q$ *for all p,q,* $i > j$ *(likewise for negative links) Then there is a (canonical)* $\models -model$ *for* $\Gamma$.

Proof : (We follow almost verbatim the construction and proof of Prop. 4.3.) Let $U := \{f : V(\Gamma) \to 2\}$, $X_p := \{f \epsilon U : f(p) = 1\}$, $X"_p := \{f \epsilon U : f(p) = 0\}$ and for $p \epsilon V(\Gamma)$, $i < \gamma$ let $I_{p,i} := \{q : \Gamma \models_i p \to q\}$, $I"_{p,i} := \{q : \Gamma \models_i p \not\to q\}$. Let $\mathcal{L}_{p,i} := \{X_p \cap X_q : q \epsilon I_{p,i}\} \cup \{X_p \cap X"_q : q \epsilon I"_{p,i}\}$, and $\mathcal{N}_i(p) := Y \subseteq X_p : \exists Z \epsilon \bigcup_{i \leq k < \gamma} \mathcal{L}_{p,k}. Z \subseteq Y \cup X_p$.

We show that $\Sigma := <$ U , $\{X_p : p \epsilon V(\Gamma)\}$ , $\{\mathcal{N}(p) := < \mathcal{N}_i(p) : i < \gamma > : p \epsilon V(\Gamma)\} >$ is a $\models -model$ for $\Gamma$.

By $\Gamma \models_i p \to p$, and 1., we see that $p \not\epsilon \bigcup_{i < \gamma} I"_{p,i}$ . Again by 1., all $I_{p,i}$, $I"_{p,j}$ are disjoint. Thus, as in Example 4.2, $\mathcal{N}(p)$ is a $\mathcal{N} - family$ over $X_p$, and $\Sigma$ is a $\Gamma, \gamma - structure$. $\Gamma \models_i p \to q => q \epsilon I_{p,i} => X_p \cap X_q \epsilon \mathcal{N}_i(p) => \Sigma \mid\mid\models_i p \to q$ . $\Gamma \models_i p \not\to q => \Sigma \mid\mid\models_i p \not\to q$ likewise.

$\Sigma \mid\mid\models_i p \to q => X_p \cap X_q \epsilon \mathcal{N}_i(p)$ . Again, we have to examine the three cases : Case 1 : $X_p \cap X_q = X_p$, then p=q, and we are done by 2. Case 2 : There is $Z \epsilon \mathcal{L}_{p,k}$ , $i \leq k < \gamma$, $Z \subseteq X_p \cap X_q$. Case 2.1 : $Z = X_p \cap X"_r \subseteq X_p \cap X_q$ for some $r \epsilon I"_{p,k}$ , $i \leq k < \gamma$. The function

$$f(x) = \begin{cases} 1 & \text{if x=p} \\ 0 & \text{otherwise} \end{cases}$$

is in $X_p \cap X"_r$ and shows that q=p, and we are done again. Case 2.2 : $Z = X_p \cap X_r \subseteq X_p \cap X_q$ for some $r \epsilon I_{p,k}$ , $i \leq k < \gamma$. The function

$$f(x) = \begin{cases} 1 & \text{if x=p or x=r} \\ 0 & \text{otherwise} \end{cases}$$

shows that either q=p again or r=q, thus $q \epsilon I_{p,k}$ and $\Gamma \models_k p \to q$ by definition, so by 3. $\Gamma \models_i p \to q$ .

$\Sigma \mid\mid\models_i p \not\to q => X_p \cap X"_q \epsilon \mathcal{N}_i(p)$ . Again we examine the above three cases. Case 1 : $X_p \cap X"_q = X_p$ can't be. Case 2.1 : $Z = X_p \cap X_r \subseteq X_p \cap X"_q$ is impossible too, as f=constant 1 shows. Case 2.2 : $Z = X_p \cap X"_r \subseteq X_p \cap X"_q$ for some $r \epsilon I"_{p,k}$ , $i \leq k < \gamma$ . The function

$$f(x) = \begin{cases} 1 & \text{if x=r} \\ 0 & \text{otherwise} \end{cases}$$

shows that q=r, so $q \epsilon I"_{p,k}$ , and $\Gamma \models_k p \not\to q$ , so $\Gamma \models_i p \not\to q$ . □

We now have the tools to handle preclusion, again with the proviso that no $p \to q$, $p \not\to q \epsilon \Gamma$ simultaneously. But this is an inessential restriction, as we can again develop two models in tandem. We extend $\| \models_i$ to handle paths too. This, along with the notion of a preclusion-structure, is made precise in

**Definition 4.5** *Set $\gamma := 2$, w:=0, s:=1 (w for weak, s for strong). Let $\Sigma$ be a $\Gamma, 2-$ structure. We say $\Sigma$ is a $\Gamma, 2, p-structure$, iff it satisfies*

*a. $\Sigma \| \models_s p \to q => \Sigma \| \models_w p \to q$*

*b. $\Sigma \| \models_w \sigma <=> \Sigma \| \models_w \sigma^+$ (see Definition 2.4)*

*c. $\Sigma \| \models_w \sigma$, $\sigma = x_1 \ldots x_n z$ iff 0. $\Sigma \| \models_w x_1 \ldots x_n$, $\Sigma \| \models_s x_n z$ 1. For no $i < n$, $\Sigma \| \models_s x_i z^-$ and there is no $\tau = x_1 \ldots y_1 \ldots y_m \ldots x_i$, $\Sigma \| \models_w \tau$ and $\Sigma \| \models_s y_k z^-$ 2. If there is $\tau = x_1 \ldots y$, $\Sigma \| \models_w \tau$, $\Sigma \| \models_s yz^-$, then $x_1 \ldots yz^-$ is precluded (here, we mean of course the semantical counterpart, i.e. the negation of 1.) (The negative arrows are handled entirely symmetrically.)*

It is trivial to see that a $\Gamma, 2, p-structure$ such that $\Sigma \| \models_s p \to q$ iff $p \to q \epsilon \Gamma$ is a correct model of the sceptical notion of inheritance (as modified in Definition 3.2).

Resume : We have constructed a 5-valued model which gives $\Sigma \| \models_w p \to q$ iff $\Gamma \models \sigma : p \to q$, $\sigma$ not a direct link, and $\Sigma \| \models_s p \to q$ iff $p \to q \epsilon \Gamma$. So, strong validity is direct links, weak validity is valid paths (resp. their result), this distinction makes it possible to handle preclusion.

# References


[HTT] J.F.Horty, R.H.Thomason, D.S.Touretzky : A Sceptical Theory of Inheritance in Nonmonotonic Semantic Networks, Dept. Comp. Sci., Carnegie Mellon Univ., CMU-CS-87-175, October 1987 or : Proceedings AAAI-87 (1987), 358-363.

[MS] D.Makinson, K.Schlechta : On Some Difficulties in the Theory of Defeasible Inheritance Nets, Forthcoming.

[S] E. Sandewall : Non-monotonic inference rules for multiple in- heritance with exceptions, Proceedings IEEE 74 (1986), 1345-1353

[T] D.Touretzky : The Mathematics of Inheritance Systems, Los Altos/ London, 1986

[THT] J.F.Horty, R.H.Thomason, D.S.Touretzky : A Clash of Intuitions : The Current State of Nonmonotonic Multiple Inheritance Systems, IJCAI 1987