

PREFERENTIAL CHOICE REPRESENTATION THEOREMS FOR BRANCHING TIME STRUCTURES

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Abstract

The idea of preferential choice is applied here to dynamic structures in two directions: 1. We show that a deontic choice function of "good" developments can be represented by a ranked, stoppered preferential relation on all developments. 2. We generalize the Katsuno/Mendelzon Update Semantics to preferences between developments and obtain a representation theorem for arbitrarily many time points.

1 INTRODUCTION

1.1 Preferential Structures

1.1.1 The Intuitive Background:

The basic idea is to interpret a primitive notion of "importance" or "value", introduced into a given language and logic, by a function which chooses the subset of "important" models of a theory or formula of that language.

In other words, we work on a set of "possible worlds", i.e. models of the underlying base logic, but do not accord the same importance or value to all such models. Given then a theory T of the base language and logic, we determine the semantical consequences of T in a structure \mathcal{M} by considering only the subset of "important" models of T : $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all important models of T in our structure. More formally, such a structure \mathcal{M} will then consist of a set M of models or possible worlds for the base logic, and a choice function f on $\mathcal{P}(M)$ - the power set of M - which, for each base theory T , singles out the set $f(M(T)) \subseteq M(T)$ of important models of T in that structure \mathcal{M} , where $M(T)$ is the set of all base models of T in \mathcal{M} . We thus define $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$.

A refinement of the idea is to work not with one subset of "maximally important" models, but with many subsets of important models, perhaps of increasing importance. This translates into the existence of several choice functions f_i in the structure \mathcal{M} , and we define $T \models_{\mathcal{M}} \phi$ iff there is some f_i such that ϕ holds in all $m \in f_i(M(T))$. This captures the intuition that we may not dispose of ideal models, but of ever better ones, which, in a sense, approximate the limit of the ideal case. Thus, each $f_i(M(T))$ may be non-empty, but $\bigcap \{f_i(M(T)) : i \in I\}$ may be empty.

Already this very abstract description makes it plausible that representation theorems for the latter approach - which I shall call the limit case - are harder to obtain than for the first variant - which I shall call, for historical reasons, the minimal case: In the latter we have to handle a possibly infinite set of choice functions, and there need not be a global f such that for all ϕ $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in f(M(T))$. In other words, we do not always have a set of "joint witnesses" for all consequences of a theory.

1.1.2 Logical Consequences:

It is evident that such consequence relations will be well-behaved with respect to the base logic - provided the latter is sound and complete for the models we have chosen as possible worlds - i.e. if T and T' are equivalent with respect to the base logic, they will have the same set of semantic consequences, and, if $T \models_{\mathcal{M}} \phi$, and ϕ implies ψ in the base logic, then also $T \models_{\mathcal{M}} \psi$. Moreover, if ϕ is a consequence of T in the base logic, then $T \models_{\mathcal{M}} \phi$, as the choice functions will choose a subset of $M(T)$. These facts hold in both the limit and the minimal version.

1.1.3 Preferential Structures:

Preferential Structures are a special case of the above, the choice is made *locally* by a binary relation \prec on the set M of base models, m is considered to be more important than m' iff $m \prec m'$ ($m \prec m'$ instead of $m' \prec m$ for historical reasons). They are thus very similar to Kripke structures, but use the relation \prec differently.

In the minimal case, we define f from \prec by $f(A) := \{a \in A : \neg \exists b \in A. b \prec a\}$.

In the limit case, the natural definition is to consider initial segments of A : $\delta_A \subseteq A$ is called an initial segment of A iff ($\delta 1$) we find some $b \in \delta_A$ below each $a \in A$: $\forall a \in A \exists b \in \delta_A (b = a \vee b \prec a)$, ($\delta 2$) δ_A is downward closed: $\forall a \in A \forall b \in \delta_A (a \prec b \rightarrow a \in \delta_A)$. Each f_i corresponds then to the choice of one such δ_A for each $A \subseteq M$.

We thus have in the minimal case $T \models_{\mathcal{M}} \phi$ iff ϕ holds in all $m \in \mu(T)$ - the set of \prec -minimal models of T in \mathcal{M} . If, for instance, $M(T)$ consists of infinite descending chains, then $\mu(T) = \emptyset$, and $T \models_{\mathcal{M}} \phi$ for any ϕ , \perp included. On the other hand, any $m \in \mu(T)$ will be a "witness" of all $\models_{\mathcal{M}}$ -consequences of T , all ϕ with $T \models_{\mathcal{M}} \phi$ will hold in such m .

In the limit case, we have $T \models_{\mathcal{M}} \phi$ iff there is some $\delta_{T,\phi} \subseteq M(T)$ which satisfies ($\delta 1$) and ($\delta 2$) with respect to $M(T)$ and such that ϕ holds in all $m \in \delta_{T,\phi}$. Thus, in the limit case, $\mu(T)$ may be empty, but if $M(T) \neq \emptyset$, we will still not have $T \models_{\mathcal{M}} \perp$, as all $\delta_{T,\phi}$ are then non-empty. It is easily seen, that if $T \models_{\mathcal{M}} \phi$ and $T \models_{\mathcal{M}} \phi'$, and \prec is transitive, then also $T \models_{\mathcal{M}} \phi \wedge \phi'$: if $\delta_{T,\phi}$ and $\delta_{T,\phi'}$ are suitable, then $\delta_{T,\phi} \cap \delta_{T,\phi'}$ will be a suitable $\delta_{T,\phi \wedge \phi'}$. Moreover, if $T \models_{\mathcal{M}} \phi$, and $M(T \cup \{\phi\}) \subseteq M(T') \subseteq M(T)$, then also $T' \models_{\mathcal{M}} \phi$.

An immediate consequence of the locality of the definition of f is a kind of upward absoluteness in the minimal case. An element, which is not minimal in A , can't be minimal in any B with $A \subseteq B$:

$$(1) A \subseteq B \rightarrow f(B) \cap A \subseteq f(A).$$

In contrast, in the general case of arbitrary f , the choice may depend on the "context", there need not be any interdependence between $f(A)$ and $f(B)$, even if $A \subseteq B$.

As a matter of fact, (1) is *the* crucial property for Minimal Preferential Structures, in the sense that any choice function which obeys (1) and the trivial property

$$(0) f(A) \subseteq A$$

can be represented by a Preferential Structure, i.e. by such a binary relation of preference (see [Sch92], Proposition 3.3). This is a very general "algebraic" characterization, the underlying set M need not consist of models, it may be just any arbitrary set.

A similar result for Limit Preferential Structures seems to be missing up to now, see [Sch94-t1], [Bou90a], [Bou90b], [Bou92] for restricted cases. (Boutilier's results are restricted in the sense that they treat finitely axiomatisable theories only, but such theories correspond exactly to clopen sets in the standard topology. Yet clopen sets can neither be entered nor left by approximation, so this seems to go somewhat against the spirit of the limit approach. See [Sch94-t1] for details).

1.1.4 Interpretation:

We have so far deliberately left open the base logic and its models in M , as well as the intuition behind the "importance" of models of the base logic.

Non-monotonic Logic: This "importance" may be read as "normality" in the case of non-monotonic logics: We are primarily interested in reasoning about the normal cases, and the preferred models are the most normal ones - where birds can fly, houses have doors etc.

As a matter of fact, Preferential Structures in their various forms provide an important and relatively well-studied group of semantics for non-monotonic logics and have proved a powerful tool for investigation, providing - via additional properties of the relation \prec - a technique of constructing

semantics of logical systems of different strengths. Limit Preferential Structures for non-monotonic logics were introduced by G.Bossu and P.Siegel in [BS85], the minimal case was first examined by Y.Shoham ([Sho87]) as a generalization of the Minimal Model Semantics for Circumscription. More or less general cases of Preferential Structures are characterized by soundness and completeness theorems in [KLM90], [LM92], [Sch92], [Sch92-n4] for the minimal case, in [Bou90a], [Bou90b], [Bou92], and [Sch94-t1] for the limit case. For an overview, see also [Mak94].

Deontic Logic: Deontic logic reasons about the morally acceptable situations, and about what ought to be done (by humans, robots etc.). Reasoning about morally acceptable actions can be split into two subquestions: Reasoning about the morally acceptable states, and reasoning about the problem of acting in a way that those states are reached. The latter question can be considered separately, at least in first approximation.

In this framework, the preferred or more important models are those which are morally more acceptable. Thus, Preferential Structures also provide a natural semantics for deontic logic, and, in fact, were examined as such before the advent of non-monotonic logics [Han69]. This was pointed out by D.Makinson in [Mak93].

In hindsight, it is no surprise that, when examining choice functions which single out some states as more important or interesting than others, a local preference by a binary relation tends to emerge. Such local preferences seem to correspond well to intuitions, and simplify the situation by making the choice context-independent.

In [Mak93], still other natural applications of Preferential Structures are discussed.

1.1.5 An Example:

Before we proceed, we give a simple example which shows that the relation $\models_{\mathcal{M}}$ defined by a Preferential Structure may indeed be a non-monotonic consequence relation.

Let \mathcal{L} be the propositional language with two variables p, q , let M consist of two (classical) models, $m \models p \wedge q$, $m' \models \neg p \wedge \neg q$, and let $m' \prec m$. Then $\emptyset \models_{\mathcal{M}} \neg q$, but $p \models_{\mathcal{M}} q$ in both the minimal and the limit definition (they

coincide for finite structures without cycles).

As is the case already in our example, not all classical models for a given language \mathcal{L} need occur in the base set M of a Preferential Structure \mathcal{M} (e.g., in our example, some $m'' \models p \wedge \neg q$ is missing). Moreover, some classical models might occur several times, even infinitely often. Take for example \mathcal{L} with one propositional variable p and consider the structure $\mathcal{M} := \langle \{ \langle m, i \rangle : i < \omega \}, \prec \rangle$ with $m \models p$, and $\langle m, i \rangle \prec \langle m, j \rangle$ iff $j < i$. Then $\mu(M) = \emptyset$, so $true \models_{\mathcal{M}} \perp$ in the minimal reading, but $true \models_{\mathcal{M}} \phi$ iff ϕ is a classical consequence of p , in the limit reading. More details and examples of logics which require several copies of classical models to be representable by preferential models can be found in [Sch92-n4].

1.1.6 Strengthenings of the Conditions for the Relation \prec :

Various additional conditions for the relation \prec have been introduced and examined for Minimal Preferential Structures.

The most natural one is perhaps transitivity.

An important condition, which results in nice properties of the semantic consequence relation $\models_{\mathcal{M}}$ is smoothness (terminology of D.Lehmann and his co-authors) or stopperedness (terminology of D.Makinson): Given a theory T , and a non-minimal model m of T , there is $m' \prec m$, which is a minimal model of T . (This condition can e.g. be violated through the existence of infinite descending chains or by non-transitive relations.) Consequently, if $M(T) \neq \emptyset$, then $\mu(T) \neq \emptyset$. The counterpart for the consequence relation $\models_{\mathcal{M}}$ is Cumulativity (see [KLM90] and [Gab85]) which says that two theories T, T' with $T \subseteq T' \subseteq \{ \phi : T \models_{\mathcal{M}} \phi \}$ have the same consequences: $T \models_{\mathcal{M}} \phi$ iff $T' \models_{\mathcal{M}} \phi$. We may read this as "normal use of Lemmas": If we have already deduced the "Lemma" ϕ from T , we neither loose nor win in terms of possible deductions by starting from $T \cup \{ \phi \}$.

As a matter of fact, again a very general algebraic representation result can be obtained: A choice function f can be represented by a smooth Minimal Preferential Structure iff it satisfies the conditions (0), (1) and

$$(2) f(A) \subseteq B \subseteq A \rightarrow f(A) = f(B)$$

and if its domain satisfies closure under finite intersections and unions (see [Sch92-n4], Theorem 1).

Another strengthening of \prec is rankedness, which may be seen as the existence of a "rotating scale with fixed origin": \prec is called ranked (on M),

iff there is an order-preserving function $f : (M, \prec) \rightarrow (X, \prec \bullet)$, where $\prec \bullet$ is a total order on X . Then two \prec -incomparable elements $m, m' \in M$ behave exactly the same way with respect to \prec : $n \prec m$ iff $n \prec m'$, and $m \prec n$ iff $m' \prec n$. The corresponding property of $\models_{\mathcal{M}}$ is Rational Monotony: If $\alpha \models_{\mathcal{M}} \gamma$, then $\alpha \wedge \beta \models_{\mathcal{M}} \gamma$ or $\alpha \models_{\mathcal{M}} \neg \beta$ (see [LM92]). General representation results are again to be found in [Sch92-n4].

1.1.7 Generalizations:

We can consider choice functions on arbitrary sets, which need not be sets of models.

We have already seen above two characterizations of such functions defined by Minimal Preferential Structures.

The strength of these algebraic representation results lies in their generality, and we can more or less easily obtain soundness and completeness results as Corollaries for non-monotonic logics with classical propositional logic as background in [Sch92] and [Sch94-t1] (which can also be read with classical predicate logic in the background), and for Plausibility Logic (a sequent calculus for a very poor language without connectives, introduced by D.Lehmann, see [Leh92a], [Leh92b], [Sch94-t2]).

But these representation results (or at least their ideas) can be used in still more general situations, where we do not compare single models, but whole "threads" of developments in dynamic situations. This is done in the present article.

In the first part (Theorem 2.8), we show that a deontic choice function of "good" developments defined in [Tho84] can be represented by a ranked, stoppered relation on all developments. Thus, we do not compare single models, but developments in a branching time structure. Again, the question of acting in a way that those preferred developments are reached (or not left), is left open, we only discuss - as R.Thomason does in [Tho84] - the "quality" of the developments, and show that again a local choice by a binary preference relation suffices, and even a very nice one.

In the second part (Theorem 3.4), we give a characterization of coupled logics which can be obtained from a preference relation on developments: Given the information S and T at time point s and time point t about a development, and a preference over developments, we examine the resulting preferred theories S' and T' , where S' and T' are the theories determined

by the end-points (i.e. models) of the preferred developments among those which pass through S- and T-models. This defines a pair of coupled logics, $\langle S, T \rangle \succsim \langle S', T' \rangle$.

1.2 Overview of the Present Article, Basic Definitions

As already said above, here we apply preferential structures to dynamic situations in two ways:

(a) We show that Thomason's choice function O of "good" branches [Tho84] of dynamic deontic choice can be represented by a very nice preferential structure.

(b) The Katsuno/Mendelzon Update Semantics [KM90] restricts comparisons to single worlds (points), and does not compare two arbitrary developments, only those originating from the same initial state. Their approach is thus in the spirit of Lewis' Counterfactual Conditional Semantics. We generalize this to compare arbitrary developments through two or arbitrarily many time points by a preferential relation, and characterize the resulting functions. More precisely - for the case of two time points - we suppose that we are given two theories, S and T in a fixed propositional language \mathcal{L} that describe the state of affairs at two times, t and t' . On this approach, worlds are identified with models. A subset Π of $M_{\mathcal{L}} \times M_{\mathcal{L}}$ (where $M_{\mathcal{L}}$ is the set of models of \mathcal{L}) represents the possible transitions from worlds to worlds. Thus, where S is an initial and T a final theory, any pair $\langle u, v \rangle \in \Pi$ such that $u \models S$ and $v \models T$ represents a possible transition. The preferential relation \prec on Π tells us which developments are the preferred ones. So, the \prec -minimal transitions $\mu(\Pi \cap (M_S \times M_T)) \subseteq \Pi \cap (M_S \times M_T)$ (M_S the set of models of S etc.) are the preferred ones, and we consider first $\{x : \exists \langle u, v \rangle \in \mu(\Pi \cap (M_S \times M_T)). x = u\}$ and $\{y : \exists \langle u, v \rangle \in \mu(\Pi \cap (M_S \times M_T)). y = v\}$, i.e. both projections $\pi_i(\mu(\Pi \cap (M_S \times M_T)))$, and then the theories generated by them: $S' := \{\phi : \forall m \in \pi_0(\mu(\Pi \cap (M_S \times M_T))). m \models \phi\}$ and $T' := \{\phi : \forall m \in \pi_1(\mu(\Pi \cap (M_S \times M_T))). m \models \phi\}$. Obviously, S' and T' are stronger than or equivalent to S and T respectively. Thus, we use \prec on $\Pi \subseteq M_{\mathcal{L}} \times M_{\mathcal{L}}$ to map a pair of theories $\langle S, T \rangle$ to another such pair $\langle S', T' \rangle$, and consider a pair of logics, which are connected through a common preferential relation. We present a soundness and completeness theorem, first for the combinatorial functions $\pi_i(\mu(\Pi \cap (M_S \times M_T)))$ (Theorem 3.1),

and then for the resulting logics (Theorem 3.4). The approach generalizes in a straightforward manner to arbitrary cartesian products (Theorem 3.2).

In the rest of the Introduction, we repeat basic definitions, taken mostly from [Sch92].

We shall use some set theoretic results and prerequisites, in particular, the axiom of choice will be assumed to hold and is used without explicit reference. We assume familiarity with these matters, which the interested reader can find e.g. in [Jec78] or [Kun80]. We use \mathcal{P} to denote the power set operator, $\prod\{X_i : i \in I\} := \{g : g : I \rightarrow \bigcup\{X_i : i \in I\}, \forall i \in I. g(i) \in X_i\}$ is the general cartesian product, \mathbb{V} shall denote the set-theoretic universe we work in - the class of all sets. Given a class of pairs \mathcal{X} , and a set X , we denote by $\mathcal{X}[X := \{ \langle x, i \rangle \in \mathcal{X} : x \in X \}]$, so if \mathcal{X} is a function f , $f[X$ is the usual notation for the restriction of f to a subset of its domain.

Definition 1.1 *Let \mathcal{L} be a propositional language, we denote by $v(\mathcal{L})$ the set of its variables, by $M_{\mathcal{L}}$ the set of its classical models. ϕ etc. shall denote formulas, T etc. theories in \mathcal{L} (i.e. $T \subseteq \mathcal{L}$), and $M_T \subseteq M_{\mathcal{L}}$ the models of T .*

$\bar{T} \subseteq \mathcal{L}$ will denote the closure of T under classical logic. Given some other logic, $\bar{\bar{T}}$ will denote the set of consequences of T under that logic, i.e. if the more conventional notation for the logic is \sim , then $\bar{\bar{T}} := \{\phi : T \sim \phi\}$.

$\mathbf{D}_{\mathcal{L}} \subseteq \mathcal{P}(M_{\mathcal{L}})$ shall be the set of definable subsets of $M_{\mathcal{L}}$, i.e. $A \in \mathbf{D}_{\mathcal{L}}$ iff there is some $T \subseteq \mathcal{L}$ such that $A = M_T$. If the context is clear, we omit the subscript \mathcal{L} from $\mathbf{D}_{\mathcal{L}}$.

It is well-known that, in general, not all subsets of $M_{\mathcal{L}}$ are definable. For a given classical language, the family of definable sets of models is closed under finite unions and arbitrary intersections, see e.g. [Sch92]. This fact is used non-trivially in [Sch92-n4] to show a representation result for smooth Minimal Preferential Structures: The representation fails for D.Lehmann's Plausibility Logic, whose family of definable sets of models fails to satisfy closure under finite unions, due to the absence of "or" on the left hand side of the sequents (see [Leh92a], [Leh92b], [Sch94-t2]).

For $X \subseteq \mathcal{P}(M_{\mathcal{L}})$, a function $f : X \rightarrow \mathcal{P}(M_{\mathcal{L}})$ will be called *definability preserving (dp)*, iff for all $Y \in \mathbf{D}_{\mathcal{L}} \cap X$ $f(Y) \in \mathbf{D}_{\mathcal{L}}$.

If $\mathbf{D}_{\mathcal{L}} \subseteq X$, then $f : X \rightarrow \mathcal{P}(M_{\mathcal{L}})$ defines a logic $T \mapsto T^f$ on \mathcal{L} by $T^f := \{\phi : \forall m \in f(M_T).m \models \phi\}$. So, if $f=id$, then $T^f = \bar{T}$. Note that $f(M_T) \subseteq M_{T^f}$ always holds, but not necessarily $f(M_T) = M_{T^f}$, the latter only iff f is dp, the reader is referred again to [Sch92].

Definition 1.2 $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ will be called a *preferential structure* iff \mathcal{X} is a set of pairs and \prec is a binary relation on \mathcal{X} . We say that \mathcal{Z} is *transitive*, *irreflexive* etc., iff \prec is. $\langle y, i \rangle$ is called a *minimal element* of $\mathcal{X}[Y]$ in \mathcal{Z} iff: 1. $\langle y, i \rangle \in \mathcal{X}[Y]$ and 2. there is no $\langle y', i' \rangle \in \mathcal{X}[Y]$ such that $\langle y', i' \rangle \prec \langle y, i \rangle$.

Thus, \mathcal{Z} defines a function $\mu_{\mathcal{Z}} : V \rightarrow V$ (V the set-theoretic universe) by $\mu_{\mathcal{Z}}(Y) := \{y : \text{there is } i \text{ such that } \langle y, i \rangle \text{ is a minimal element of } \mathcal{X}[Y]\}$. (Note that $\mu_{\mathcal{Z}}$ is thus a proper class, but this need not bother us.) Given a set Z , $\mu_{\mathcal{Z}, Z}$ shall denote $\mu_{\mathcal{Z}}[\mathcal{P}(Z)]$.

(A short motivation for indexing: if there is just one copy of y in \mathcal{X} , and e.g. $y' \prec y$, then, for $y \in Y$, y will not be minimal in Y if $y' \in Y$. If we want two y' , y'' necessary in Y for y not to be minimal, we need something like $y' \prec \langle y, 0 \rangle$, $y'' \prec \langle y, 1 \rangle$. So the different $\langle y, i \rangle$, $\langle y, j \rangle$ encode conjunction, the different $y' \prec \langle y, i \rangle$, $y'' \prec \langle y, i \rangle$ disjunction: in other words, we look at the product.)

$\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ will be called \mathcal{Y} -smooth (terminology of [KLM90]) or \mathcal{Y} -stoppered (terminology of [Mak94]) iff for all $X \in \mathcal{Y}$ and $\langle y, i \rangle \in \mathcal{X}[X]$, either $\langle y, i \rangle$ is minimal in $\mathcal{X}[X]$, or there is $\langle y', i' \rangle \prec \langle y, i \rangle$, $\langle y', i' \rangle$ minimal in $\mathcal{X}[X]$. In shorthand, all non-minimal elements are "killed" by minimal ones. It is an immediate and important consequence that then for $X \in \mathcal{Y}$, $\mathcal{X}[X] \neq \emptyset$ implies $\mu_{\mathcal{Z}}(X) \neq \emptyset$.

Definition 1.3 A preferential structure $\mathcal{M} = \langle \mathcal{X}, \prec \rangle$ will be called a classical preferential model (cpm) for \mathcal{L} , iff for all $\langle x, i \rangle \in \mathcal{X}$, $x \in M_{\mathcal{L}}$. \mathcal{M} will be called definability preserving (dp) iff $\mu := \mu_{\mathcal{M}, M_{\mathcal{L}}} : \mathcal{P}(M_{\mathcal{L}}) \rightarrow \mathcal{P}(M_{\mathcal{L}})$ is definability preserving. By the above, \mathcal{M} defines a logic on \mathcal{L} by $T^{\mathcal{M}} := T^{\mu}$, i.e. $T^{\mathcal{M}} := \{\phi \in \mathcal{L} : \phi \text{ holds in all } m \in \mu(M_T)\}$. Note, that if \mathcal{M} is dp, $\mu(M_T) = M_{T^{\mathcal{M}}}$. A logic $=$ for \mathcal{L} is said to be representable by a cpm, iff there is a cpm \mathcal{M} for \mathcal{L} , such that for all $T \subseteq \mathcal{L}$ $T^{\mathcal{M}} = \overline{\overline{T}}$. For $\langle m, i \rangle \in \mathcal{X}$, we shall abuse notation and say $\langle m, i \rangle \models \phi$ iff $m \models \phi$, for $\phi \in \mathcal{L}$.

Remark 1.1 Our definition is a notational variant of e.g. the definition in [KLM90]: The function $l : X \rightarrow M_{\mathcal{L}}$ in [KLM90] has the same meaning as our indices: l need not be injective. Neither need l be onto, and we do not require for all $m \in M_{\mathcal{L}}$ some $\langle m, i \rangle \in \mathcal{X}$. \models corresponds to our "abuse of notation" just introduced.

Fact 1.2 Let \prec be an irreflexive, binary relation on X , then the following two conditions are equivalent:

- (1) There is Ω and an irreflexive, total, binary relation \prec' on Ω and a function $f : X \rightarrow \Omega$ such that $x \prec y \leftrightarrow fx \prec' fy$ for all $x, y \in X$
- (2) Let $x, y, z \in X$ and $x \perp y$ wrt. \prec (i.e. neither $x \prec y$ nor $y \prec x$), then $z \prec x \rightarrow z \prec y$ and $x \prec z \rightarrow y \prec z$.

Proof: (1) \rightarrow (2): Let $x \perp y$, thus neither $fx \prec' fy$ nor $fy \prec' fx$, but then $fx = fy$. Let now $z \prec x$, so $fz \prec' fx = fy$, so $z \prec y$. $x \prec z \rightarrow y \prec z$ is similar.
(2) \rightarrow (1): For $x \in X$ let $[x] := \{x' \in X : x \perp x'\}$, and $\Omega := \{[x] : x \in X\}$. For $[x], [y] \in \Omega$ let $[x] \prec' [y] :\leftrightarrow x \prec y$. This is well-defined: Let $x \perp x', y \perp y'$ and $x \prec y$, then $x' \prec y'$. Obviously, \prec' is an irreflexive, total binary relation. Define $f : X \rightarrow \Omega$ by $fx := [x]$, then $x \prec y \leftrightarrow [x] \prec' [y] \leftrightarrow fx \prec' fy$. \square

Definition 1.4 Call an irreflexive, binary relation \prec on X , which satisfies (1) (equivalently (2)) above, ranked.

Fact 1.3 *If \prec on X is ranked, and free of cycles, then \prec is transitive.*

Proof: Let $x \prec y \prec z$. If $x \perp z$, then $z \prec y$, resulting in a cycle of length 2. If $z \prec x$, then we have a cycle of length 3. So $x \prec z$. \square

2 A RANKED AND SMOOTH PREFERENTIAL REPRESENTATION FOR A DEONTIC CHOICE FUNCTION

2.1 Outline

We show here that Thomason's deontic choice function O of "good" branches (see [Tho84]) can be represented by a ranked, stoppered preferential relation on all branches (Theorem 2.8). We look for a ranked stoppered (=smooth) order on the set of branches $B := \bigcup\{B_t : t \in T\}$, which describes $O_t \subseteq B_t$ as the minimal elements of B_t . So we have to find an order \prec on B such that $O_t = \mu(B_t) := \{b \in B_t : \neg\exists b' \in B_t. b' \prec b\}$, a totally ordered set $(Z, <)$, and a function $h : B \rightarrow Z$ such that $b \prec b' \leftrightarrow h(b) < h(b')$. For the construction of $(Z, <)$, we (essentially) rearrange $\{B_t : t \in T\}$ in a suitable way to a total order (Definition 5), and for $b \in B$, we let $h(b)$ be the \subseteq -largest B_t , $t \in T$ such that $b \in O_t$. For this to be defined, we close $\{B_t : t \in T\}$ under unions of \subseteq -chains, this is done in the first part of the text, up to Definition 3. Finally, we inherit the order \prec on B from $<$ via h . It is easily shown that \prec is stoppered and represents the choice function O .

We conclude by remarking that Bartha's choice of "bad" branches by validity of a special formula \mathcal{S} at branch/point pairs (see [Bar91]) can also express Thomason's O -function (Fact 2.9).

2.2 The Details

Definition 2.1 *Let $\mathcal{X} \subseteq \mathcal{P}(X)$. We say that $\mathcal{X}' \subseteq \mathcal{X}$ is a chain in \mathcal{X} iff \mathcal{X}' is totally ordered under \subseteq . We say that \mathcal{X} satisfies (tp) iff $A, B \in \mathcal{X}$*

$\rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$. Thus, if \mathcal{X} has (tp), and is ordered by $A < B :\leftrightarrow B \subset A$, it is almost tree-like.

Fact 2.1 Let $\{\mathcal{X}_i : i \in I\}$ be a collection of chains in \mathcal{X} , $X_i := \cup \mathcal{X}_i$, and let \mathcal{X} satisfy (tp), then so does $\mathcal{X}' := \mathcal{X} \cup \{X_i : i \in I\}$.

Proof: Each $A \in \mathcal{X}$ can be written as $\cup\{A\}$, so it suffices to consider unions of chains in \mathcal{X} . Let $A, B \in \mathcal{X}'$, $A = \cup \mathcal{A}$, $B = \cup \mathcal{B}$, \mathcal{A}, \mathcal{B} chains in \mathcal{X} . Suppose $A \cap B \neq \emptyset$ and that there are $a \in A - B$, $b \in B - A$. Then there are $A' \in \mathcal{A}$, $B' \in \mathcal{B}$ with $A' \cap B' \neq \emptyset$, and there are $A'' \in \mathcal{A}$, $B'' \in \mathcal{B}$ with $a \in A''$, $b \in B''$. So there are $A_0 \in \mathcal{A}$, $B_0 \in \mathcal{B}$ with $a \in A_0$, $b \in B_0$, $A_0 \cap B_0 \neq \emptyset$. As $A_0, B_0 \in \mathcal{X}$, either $A_0 \subseteq B_0$ or $B_0 \subseteq A_0$, so $a \in B$ or $b \in A$, Contradiction. \square

Definition 2.2 Let $\mathcal{X} \subseteq \mathcal{P}(X)$ and $f : \mathcal{X} \rightarrow \mathcal{P}(X)$ be such that for $A, B \in \mathcal{X}$ (1) $f(A) \subseteq A$, (2) $f(A) \neq \emptyset$ if $A \neq \emptyset$, (3) $A \subseteq B \rightarrow f(B) \cap A \subseteq f(A)$, (4) $A \subseteq B \wedge f(A) \cap f(B) \neq \emptyset \rightarrow f(A) = f(B) \cap A$. We then say that a chain $\mathcal{X}' \subseteq \mathcal{X}$ satisfies (ip) iff for $A, B \in \mathcal{X}'$ $f(A) \cap f(B) \neq \emptyset$.

Note that f is then monotone on \mathcal{X}' : $A \subseteq B \rightarrow f(A) \subseteq f(B)$ by (4) for $A, B \in \mathcal{X}'$. But f is also continuous in the following sense:

Fact 2.2 Let f be as in Definition 2.2. Let $\mathcal{X}' \subseteq \mathcal{X}$ satisfy (ip), $\cup \mathcal{X}' \in \mathcal{X}$, then $f(\cup \mathcal{X}') = \cup\{f(A) : A \in \mathcal{X}'\}$.

Proof: The case $\cup \mathcal{X}' = \emptyset$ is trivial. Let $X' := \cup \mathcal{X}'$. We first show that $f(X') \cap f(B) \neq \emptyset$ for all $B \in \mathcal{X}'$. As $f(X') \neq \emptyset$, there is $a \in f(X') \subseteq X'$, so $a \in A$ for some $A \in \mathcal{X}'$. But then $a \in f(A)$ by (3). So $f(X') \cap f(A) \neq \emptyset$, and $f(A) = A \cap f(X')$ by (4). Let $B \in \mathcal{X}'$, then there is $b \in f(A) \cap f(B)$, so $b \in f(X')$. But now by (4) for each $A \in \mathcal{X}'$ $f(A) = f(X') \cap A$, so by $X' = \cup \mathcal{X}'$ and $f(X') \subseteq X'$ $f(X') = \cup\{f(A) : A \in \mathcal{X}'\}$. \square

Fact 2.3 Let \mathcal{X} satisfy (tp) and let f be as in Definition 2.2. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$ satisfy (ip), $A := \cup \mathcal{A}$, $B := \cup \mathcal{B}$, $A \subseteq B$, $A' \in \mathcal{A}$, $B' \in \mathcal{B}$, then $f(B') \cap A' \subseteq f(A')$.

Proof: Let $x \in f(B') \cap A'$, so $A' \cap B' \neq \emptyset$. Consider $\mathbf{B}'' := \{B'' \in \mathbf{B} : B'' \supseteq B'\}$. Suppose there is $B'' \in \mathbf{B}''$ such that $A' \subseteq B''$, then, by $f(B') \subseteq f(B'')$, $x \in f(B'') \cap A'$, so $x \in f(A')$ by (3). Otherwise, as $B' \cap A' \neq \emptyset$, for all $B'' \in \mathbf{B}''$ $B'' \cap A' \neq \emptyset$, so for all $B'' \in \mathbf{B}''$ $B'' \subseteq A'$ by (tp) for \mathcal{X} . But, as $\cup \mathbf{B}'' = \cup \mathbf{B}$, $B \subseteq A' \subseteq A \subseteq B$, so $A'=B$. By Fact 2.2 $f(A') = \cup\{f(B'') : B'' \in \mathbf{B}''\}$, so $x \in f(A')$. \square

Fact 2.4 *Let \mathcal{X} satisfy (tp) and let f be as in Definition 2.2. Let $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{X}$ satisfy (ip), $\cup \mathcal{A} = \cup \mathcal{A}'$, then $\cup\{f(A) : A \in \mathcal{A}\} = \cup\{f(A') : A' \in \mathcal{A}'\}$.*

Proof: We show " \subseteq ", " \supseteq " is symmetrical. Let $a \in \cup\{f(A) : A \in \mathcal{A}\}$, so $a \in f(A) \subseteq A$ for some $A \in \mathcal{A}$. So there is $A' \in \mathcal{A}'$ such that $a \in A'$, thus $a \in f(A) \cap A'$, so $a \in f(A')$ by Fact 2.3. \square

Fact 2.5 *Let \mathcal{X} satisfy (tp), f be as in Definition 2.2, $\mathcal{Y} := \{\mathcal{A} \subseteq \mathcal{X} : \mathcal{A}$ satisfies (ip)\}, and $\mathcal{X}' := \mathcal{X} \cup \{\cup \mathcal{A} : \mathcal{A} \in \mathcal{Y}\}$. Extend f to $f' : \mathcal{X}' \rightarrow \mathcal{P}(X)$ by defining $f'(\cup \mathcal{A}) := \cup\{f(A) : A \in \mathcal{A}\}$ for any $\mathcal{A} \in \mathcal{Y}$ with $\cup \mathcal{A} \notin \mathcal{X}$. Then f' also satisfies (1)-(4) of Definition 2.2.*

Proof: By Fact 2.4, f' is well-defined. As in the proof of Fact 2.1, we may assume that all $A \in \mathcal{X}'$ are given as $\cup \mathcal{A}$, where $\mathcal{A} \subseteq \mathcal{X}$ satisfies (ip), and $f'(\cup \mathcal{A}) = \cup\{f(A') : A' \in \mathcal{A}\}$. Let $A = \cup \mathcal{A}$, $B = \cup \mathcal{B}$. (1) is trivial by (1) for f . (2) is also trivial. (3) Let $A \subseteq B$, $x \in f'(B) \cap A$. Then $x \in f(B') \cap A'$ for some $A' \in \mathcal{A}$, $B' \in \mathbf{B}$, so $x \in f(A')$ by Fact 2.3, so $x \in f'(A)$. (4) Let $A \subseteq B$, $f'(A) \cap f'(B) \neq \emptyset$, we have to show $f'(A) \subseteq f'(B)$. Let $x \in f'(A) = \cup\{f(A') : A' \in \mathcal{A}\}$. By prerequisite, there are $A'' \in \mathcal{A}$, $B'' \in \mathbf{B}$ such that $f(A'') \cap f(B'') \neq \emptyset$. By monotony of f on \mathcal{A} , there are $A_0 \in \mathcal{A}$, $B_0 \in \mathbf{B}$ such that $x \in f(A_0)$ and $f(A_0) \cap f(B_0) \neq \emptyset$. Consider $\mathbf{B}'' := \{B'' \in \mathbf{B} : B'' \supseteq B_0\}$. For $B'' \in \mathbf{B}''$ $f(A_0) \cap f(B'') \neq \emptyset$. Suppose there is $B'' \in \mathbf{B}''$ such that $A_0 \subseteq B''$, so by (4) $f(A_0) \subseteq f(B'') \subseteq f'(B)$, so $x \in f'(B)$. Otherwise, by (tp), $B = \cup \mathbf{B} = \cup \mathbf{B}'' \subseteq A_0 \subseteq A \subseteq B$, so $f'(A)=f'(B)$. \square

Definition 2.3 *We can thus extend \mathcal{X} and f to $\mathcal{Y} \subseteq \mathcal{P}(X)$, the closure under all unions which satisfy (ip) as follows:*

$\mathcal{X}_0 := \mathcal{X}$, $f_0 := f$.

In the successor step, let $\mathcal{X}_{\alpha+1} := \mathcal{X}'$, $f_{\alpha+1} :=$ the extension of f_α to \mathcal{X}' as defined and examined in Facts 2.1 and 2.5.

In the limit step, let $\mathcal{X}_\lambda := \bigcup\{\mathcal{X}_\alpha : \alpha < \lambda\}$, $f_\lambda := \bigcup\{f_\alpha : \alpha < \lambda\}$. Obviously, the limit step preserves (tp) and (1)-(4) of f .

Finally, set $\mathcal{Y} := \bigcup\{\mathcal{X}_\alpha : \alpha < \kappa\}$, $g := \bigcup\{f_\alpha : \alpha < \kappa\}$, where κ is large enough, $(\text{card}(\mathcal{P}(X)))^+$ will do.

We now work in \mathcal{Y} , with g , as just defined.

Definition 2.4 For $\emptyset \neq A \in \mathcal{Y}$, let A^+ be the largest $B \in \mathcal{Y}$ such that $A \subseteq B$, $g(A) \cap g(B) \neq \emptyset$.

This is well-defined: Let $\mathbf{B}_A := \{B' \in \mathcal{Y} : A \subseteq B' \wedge g(A) \cap g(B') \neq \emptyset\}$. Obviously, for $B', B'' \in \mathbf{B}_A$, $B' \cap B'' \neq \emptyset$, so by (tp) \mathbf{B}_A is totally ordered by \subseteq . By (4), $g(A) \subseteq g(B')$ for all $B' \in \mathbf{B}_A$, so \mathbf{B}_A has (ip), and by construction $\bigcup \mathbf{B}_A \in \mathcal{Y}$, but $A^+ = \bigcup \mathbf{B}_A$.

Moreover, the following holds:

Fact 2.6 Let $g(A) \cap g(A') \neq \emptyset$, then $A^+ = A'^+$.

Proof: Let $x \in g(A) \cap g(A')$. By $A \cap A' \neq \emptyset$, $A \subseteq A'$ or $A' \subseteq A$. Consider the case $A \subseteq A'$. Define $\mathbf{B}_A, \mathbf{B}_{A'}$ as above. Let $B' \in \mathbf{B}_{A'}$. Then $B' \supseteq A' \supseteq A$, and $g(A) \cap g(B') \neq \emptyset$: By $A' \subseteq B'$ and $g(A') \cap g(B') \neq \emptyset$, $g(A') \subseteq g(B')$, so $x \in g(A') \subseteq g(B')$. Thus $\mathbf{B}_{A'} \subseteq \mathbf{B}_A$, and $A'^+ \subseteq A^+$. Furthermore, if $B \in \mathbf{B}_A$, then $A \subseteq B$, so $A' \cap B \neq \emptyset$, thus $A' \subseteq B$ or $B \subseteq A'$. If $B \subseteq A'$, then there is $B' := A' \in \mathbf{B}_{A'}$ with $B \subseteq B'$. If $A' \subseteq B$, then $g(A') \cap g(B) \neq \emptyset$: By $B \in \mathbf{B}_A$, $g(A) \subseteq g(B)$, so $x \in g(A') \cap g(B)$. Thus $B \in \mathbf{B}_{A'}$. In any case, for $B \in \mathbf{B}_A$, there is $B' \in \mathbf{B}_{A'}$ with $B \subseteq B'$. Thus $A'^+ = \bigcup \mathbf{B}_{A'} \supseteq \bigcup \mathbf{B}_A = A^+$. \square

Definition 2.5 Let $\langle Y_i : i < \mu \rangle$ be some enumeration of \mathcal{Y} . For $\emptyset \neq Y \in \mathcal{Y}$, let $[Y] := \{Y' \in \mathcal{Y} : Y' \supseteq Y\}$. By (tp), $[Y]$ is totally ordered by \subseteq .

We note the following obvious Fact: (a) If $<$ is a strict total order on C , $<'$ a strict total order on C' , $C \cap C' = \emptyset$, then "putting C' strictly on top of C " will be a strict total order too. We will denote this ordered set by $C+C'$. (b) If $(C_i, <_i)$, $i < \lambda$ is an increasing chain of totally, strictly ordered sets, where the $<_i$ agree whenever defined, then $\bigcup\{C_i : i < \lambda\}$ is also totally, strictly ordered by $\bigcup\{<_i : i < \lambda\}$. We will denote this ordered set by $+\{C_i : i < \lambda\}$.

We now define an increasing sequence of total orders X_i :

$Z_0 := X_0 := [Y_0]$ - where $[Y_0]$ is ordered by $Y' < Y'' \iff Y'' \subset Y'$, Y_0 the first element in the above, arbitrary enumeration of \mathcal{Y} .

$Z_{i+1} := [Y_{i+1}] - X_i$ (which may be empty), $X_{i+1} := X_i + Z_{i+1}$.

For $\text{lim}(\lambda)$: $Z_\lambda := [Y_\lambda] - \bigcup\{X_i : i < \lambda\}$, $X_\lambda := (+\{X_i : i < \lambda\}) + Z_\lambda$.

$(\mathcal{X}, <)$:= $+\{X_i : i < \mu\}$ will be totally, strictly ordered, and if $x < y$ then $\neg(y \supseteq x)$, and thus also $x \subset y \rightarrow y < x$. (This is all we need.) Finally, we put a (new) element, say t , on top of \mathcal{X} .

Definition 2.6 We define now an order \prec on X , by mapping X by h into $\mathcal{Y} + \{t\}$, and inheriting the order: Let $x \in X$. Case 1: There is no $A \in \mathcal{Y}$ such that $x \in g(A)$, then $h(x)=t$. Case 2: There is $A \in \mathcal{Y}$ such that $x \in g(A)$, let then $h(x) := A^+$. By Fact 2.6, this is well-defined. Finally, set $x \prec x' \iff h(x) < h(x')$.

It remains to show that this order \prec represents g , and thus f on $\mathcal{X} \subseteq \mathcal{Y}$:

Fact 2.7 For $Y \in \mathcal{Y}$, $g(Y) = \mu(Y)$ and \prec is \mathcal{Y} -stoppered.

Proof: Let $Y \in \mathcal{Y}$, $x \in Y - g(Y)$. Case 1: For no $Y' \in \mathcal{Y}$ $x \in g(Y')$. Then $h(x)=t$, and, as there is $x' \in g(Y)$, $h(x') < h(x)$, so there is $x' \prec x$, $x' \in g(Y) \subseteq Y$, thus $x \notin \mu(Y)$. Case 2: There is $Y' \in \mathcal{Y}$ with $x \in g(Y')$. Then $Y \cap Y' \neq \emptyset$, so $Y \subseteq Y'$ or $Y' \subseteq Y$. If $Y \subseteq Y'$, then $x \in g(Y)$ by (3), contradiction. So $Y' \subseteq Y$, and by definition, $h(x) = Y'^+$, and $x \in g(Y'^+)$. So $Y \cap Y'^+ \neq \emptyset$, and $Y \subseteq Y'^+$ or $Y'^+ \subseteq Y$. But $Y \subseteq Y'^+$ again implies $x \in g(Y)$, contradiction. So $Y'^+ \subset Y \subseteq Y^+$, thus $Y'^+ > Y \geq Y^+$. But now there is $x' \in g(Y)$ such that $h(x') = Y^+ < h(x)$, so $x' \prec x$, and $x \notin \mu(Y)$.

For the converse, let $x \in g(Y)$. Suppose there is $x' \in Y$ with $x' \prec x$. x' has to be in some $g(Y')$, otherwise $h(x')=t$. So $Y \cap Y' \neq \emptyset$, thus $Y \subseteq Y'$

or $Y' \subseteq Y$. If $Y \subseteq Y'$, then $x' \in g(Y)$, but then $h(x)=h(x')$, so $x \perp x'$. So $Y' \subseteq Y$. But $x' \prec x$ implies $h(x') < h(x)$, so $Y'^+ < Y^+$, thus $\neg(Y^+ \supseteq Y'^+)$. As $Y' \subseteq Y \subseteq Y^+$ and $Y' \subseteq Y'^+$, $Y'^+ \cap Y^+ \neq \emptyset$. $Y'^+ \subseteq Y^+$ is not possible, and if $Y \subseteq Y^+ \subseteq Y'^+$, then, as $x' \in g(Y'^+) \cap Y$, $x' \in g(Y)$, so again $x \perp x'$, contradiction.

It remains to show \mathcal{Y} -stopperedness: So let $x \in Y - \mu(X) = Y - g(Y)$, $Y \in \mathcal{Y}$. But then, in Case 1 and 2 above, there is $x' \in g(X) = \mu(X)$ with $x' \prec x$. \square

We thus have shown:

Theorem 2.8 *The "good" branches of Thomason's [Tho84] can be chosen by a ranked, stoppered preferential order on the set of branches.*

More precisely (see Definition 12 in [Tho84]):

Let $\langle T, \prec, O \rangle$ be a tree-like frame for deliberative deontic tense logic, i.e.

1. $T \neq \emptyset$

2. \prec is a transitive irreflexive binary relation such that $t_1 \prec t$, $t_2 \prec t$ implies $t_1 = t_2$ or $t_1 \prec t_2$ or $t_2 \prec t_1$

3. For $t \in T$, let $B_t := \{b : t \in b \text{ and } b \text{ is a maximal subset of } T, \text{ totally ordered under } \prec\}$ (i.e. B_t is the set of (maximal) branches through t) and $O : T \rightarrow \cup\{B_t : t \in T\}$ is a "choice function of good branches" such that (a) $O_t \neq \emptyset$, (b) $O_t \subseteq B_t$, (c) $t \prec t'$, $t' \in b \in O_t \rightarrow O_{t'} = O_t \cap B_{t'}$.

Then there is a ranked, stoppered partial order on $\cup\{B_t : t \in T\}$, which represents O .

Proof: Obviously, if $B_t \cap B_{t'} \neq \emptyset$, then $B_t \subseteq B_{t'}$ or $B_{t'} \subseteq B_t$. So $\mathcal{X} := \{B_t : t \in T\}$ satisfies (tp). Properties (1) and (2) of Definition 2.2 are obvious for $f:=O$. (3) and (4): $B_{t'} \subseteq B_t$ implies $t \prec t'$. If $O_t \cap B_{t'} \neq \emptyset$, then there is $b \in O_t \cap B_{t'}$, so $t' \in b$, but then $O_{t'} = O_t \cap B_{t'}$ by (c). We extend \mathcal{X} and f to \mathcal{Y} and g as in Definition 2.3, and represent g as in Definitions 2.5 and 2.6. \square

We conclude by remarking that Bartha's choice of "bad" branches can express Thomason's choice of "good" branches too. Familiarity with [Bar91] is assumed.

Fact 2.9 *Bartha, on pages 95-96 of [Bar91], defines "bad" choices by validity of a special constant "proposition" \mathbf{S} - which is left unexplained there. Let us introduce a new propositional variable, say $*$, and set $\mathbf{S} := F\neg*$, expressing that, sometimes in the future, $*$ will not hold. So the good end-segments of branches, seen from moment m , are those where $\langle b, m \rangle \models \neg F\neg*$ - i.e., everywhere above m on b , $*$ holds. But now, if m' is below m , and b is a branch good above m' and which passes through m , then it is good above m . And if there is another branch b' good above m , then we can "cut and paste" b' onto the initial segment of b , thus $O_m = O_{m'} \cap B_m$. \square*

3 AN EXTENSION OF THE KATSUNO-MENDELZON UPDATE SEMANTICS

3.1 Introduction

The subject of Katsuno/Mendelzon's Update Semantics [KM90] is to describe/characterize reasonable modifications in a database, when learning that the world changes.

Let me begin by emphasizing the following obvious differences:

1) Learning that the world changes (or has changed) is not the same as learning new facts about a static world.

2) Reasoning about possible developments is not the same as reasoning about "normal" developments. E.g., earthquakes are geologically possible at Stuttgart in southern Germany - due to proximity of the Hohenzollerngraben -, but are certainly an unnormal event. For reasons of motivation/simplicity I shall sometimes use the word "probable" as synonym for "normal".

3) The treatment of real time databases is not the same as the treatment of databases on developments: In a real time database, it makes no sense

and is impossible to "correct" past beliefs. In a database on developments, say an archeological database on ancient mediterranean cultures, it makes sense to change beliefs about the past. As a matter of fact, the aim of archeological inquiry is to maintain a reconstruction of past events. This database of archeological beliefs is not constructed in the order in which events occurred, and in general will use inferences from later to earlier events as well as inferences from earlier to later ones. As in most areas of empirical inquiry, the process will be nonmonotonic, cases can arise in which beliefs need to be retracted in light of new evidence.

I think that the example of an archeologist trying to make "the best" of the data at hand - and, in an extension, to guide his further research - is a good example of reasoning about developments. He uses information about several time points and "laws" of nature and history to reason about possible developments fitting his information, and uses hypotheses about the normal developments to choose among these the most likely ones, and to conclude back on the probable state of the world at those time points. In the general case, the reasoning processes might be even more complicated, but this exceeds the aims of this article. A short discussion of such possible extensions is given at the end of this Section.

3.1.1 Criticism of the Katsuno/Mendelzon Approach

My critique of their approach focusses on two points:

1. I do not see why corrections of past beliefs should be excluded. Katsuno/Mendelzon say in [KM90] on page 2, 3. paragraph, last sentence: "The fact that the real worlds has changed gives us no grounds to conclude that some of the old worlds were actually not possible." Look at the following situation: I have heard that a neighbourhood dog has died. There are two candidates, dog d and e, and I am not sure which. So I think two worlds are possible, one where d is dead, the other where e is (assume that not both are dead). Now I hear dog d barking. No one reasonable would uphold the possibility that d was dead a minute ago. So even in the relatively simple context of possible developments, to be rational, we should revise our belief about the past in the light of new information about the present (or future).

2. Katsuno/Mendelzon's "adding up normality" seems to me to be too strong a simplification in some situations. In essence, it says that if, from world v, a development to world x is normal, and from world w a development

to world y is normal, then, if I believe that v or w holds, I should believe that both developments to x or y are normal. But, why can't I compare developments from different origins? Suppose I live at Naples, and read a book on Pompei last night. Moreover, my neighbour has a truck which makes a horrible noise. Now I am peacefully sitting at my morning's coffee, and hear a terrible rumbling noise. I have two options: Vesuvius has erupted again, or my neighbour has started his truck. The next moment, I feel the house shaking. If I thought only the truck possible, then I would suppose that he has smashed into my house - though a little astonished by the effect on the very stable house. But believing an eruption of Vesuvius possible, this seems to provide a far better explanation of the dramatic developments. So I shall believe in an earthquake, and neglect the second possibility as less normal. Note, however, that if we speak about the possible developments, we are fully entitled to add up all possibilities.

3.1.2 How can we patch things up?

As basic entity, assume a directed graph Δ (for developments), whose nodes are possible (static) worlds, i.e. propositional models. $w \rightarrow v$ means that (in one time step) world w can develop to world v - "can" in the sense of "it is possible". Moreover, we assume given a partial order on the set of arrows, $(w \rightarrow v) \prec (x \rightarrow y)$ means that the development from w to v is more normal than that from x to y .

Suppose we start reasoning at time t , and have a database $K(t)$ of propositional formulas. This singles out a set possible worlds (nodes) in Δ , where $K(t)$ holds, say $W(t)$. I now get information $K(t+1)$ about the state of affairs at time $t+1$. If I reason about the possible, I shall consider all worlds v as possible, which can be reached in one step from $W(t)$, and where $K(t+1)$ holds (this is as in [KM90]). But, if some world $w \in W(t)$ has no arrow into $K(t+1)$ -worlds, then this world should be excluded from $W(t)$ (it has become impossible in the light of the new information), and $K(t)$ be modified (strengthened) accordingly. Analogously, getting information $K(t-1)$, singles out those worlds where $K(t-1)$ holds, and which have an arrow leading into a $K(t)$ -world. Again, $K(t)$ -worlds, which cannot be reached from $K(t-1)$ -worlds, should be eliminated (propagating to $W(t+1)$).

This seems to be a rational approach to the possible. We also note that we can do all three steps simultaneously, by looking at the $K(\tau)$ - *worlds*,

and seeing what can be connected by suitable threads, eliminating the rest.

We turn to the probable or normal. Suppose again that we have information $K(t-1)$, $K(t)$, $K(t+1)$. Again, we can single out simultaneously $W(t-1)$, $W(t)$, $W(t+1)$. First, as in the "possible"-case, we eliminate all worlds which do not lie on some complete path (of length 2). But now, we use our partial order on arrows, which we extend suitably to paths: e.g. $\pi < \rho$ iff for all $x \rightarrow y$ in π there is $x' \rightarrow y'$ in ρ such that $x \rightarrow y < x' \rightarrow y'$, or, if the order on arrows is ranked, i.e. given by a rotating scale, we can do with one arrow better, and the rest incomparable or better, etc . . . We now choose the best paths, and accept only those worlds in $W(\tau)$ which lie on some preferred path. (Of course, if we like, we may introduce some order on worlds too, but this will not tell us anything really new.)

Again, this seems to be a rational approach.

The full picture will thus be given by a graph Δ as above, preferences on its arrows, and we consider paths of length e.g. \mathcal{Z} (integers), which we can compare in some reasonable extension of the order on arrows, we have information $K(t)$, $K(t')$, . . . , which give us "bottlenecks" $W(t)$, $W(t')$, . . . , and we further restrict $W(t)$ to $V(t)$ by considering only those nodes which lie on full paths or even on preferred paths. We generalize even further below by comparing arbitrary functions of the cartesian product, not necessarily through comparing single arrows.

Note: Directed graphs correspond to binary relations. In the formal part below, we will use the relation language.

Turning back to our archeological example, suppose we are interested in the extinction of the Minoan culture. Then we will try to find information temporarily surrounding the event, to narrow the possible or probable conjectures about the event itself. E.g. decline of other cultures before or after, geological evidence, etc.

We would also like to point out that we do not seek to justify or reason about our choice relation. It is a primitive which we assume given. Neither do we impose any (perhaps plausible) restrictions like: If the development $a \rightarrow b$ is preferred over the development $a' \rightarrow b'$, and $b \rightarrow c$ over $b' \rightarrow c'$, then $a \rightarrow b \rightarrow c$ is preferred over $a' \rightarrow b' \rightarrow c'$. We do not speak about subjective versus objective preferences either. In a probabilistic setting, D.Lewis [Lew81] addresses such questions, and an investigation of these problems about preferences could follow Lewis's arguments in a qualitative framework.

3.2 Formalizing the Problem, Soundness and Completeness

We start with two time points; our considerations generalize trivially to the arbitrary case.

So we are given $X \times Y$, and a preference on pairs $\langle x, y \rangle \prec \langle x', y' \rangle$. We are given $Z \subseteq X \times Y$ as input, where

- (1) Z may be $A \times B$, $A \subseteq X$, $B \subseteq Y$
- (2) Z may e.g. be an arbitrary union of such $A \times B$'s
- (3) Z may be an arbitrary subset of $X \times Y$. The "central machinery" will give the preferred elements $\mu(Z)$ of Z . We can ask the following questions:

- (4) What are the preferred elements of Z ?
- (5) What are the projections of $\mu(Z)$ - i.e. those states in X (Y) through which preferred paths (pairs) pass? More precisely, we look for $\mu(Z)_X := \{x \in X : \exists y \in Y. \langle x, y \rangle \in \mu(Z)\}$ and $\mu(Z)_Y := \{y \in Y : \exists x \in X. \langle x, y \rangle \in \mu(Z)\}$. (Warning: Of course, $x \in \mu(Z)_X$, $y \in \mu(Z)_Y$ does not imply $\langle x, y \rangle \in \mu(Z)$.)

We consider here the case (1) + (5).

Our approach is a generalization of the Katsuno/Mendelzon one: Forget about the first coordinate, assume that all arrows (pairs) of different origin are incomparable, and that for any point x , the arrow $x \rightarrow x$ exists and is "the best".

Definition 3.1 Fix X, Y . Let $\mathcal{Y}(X) \subseteq \mathcal{P}(X)$, $\mathcal{Y}(Y) \subseteq \mathcal{P}(Y)$.

For $A \subseteq X$, $B \subseteq Y$, let $|A \times B| := \{A' \times B' : A' \subseteq A, B' \subseteq B, A' \in \mathcal{Y}(X), B' \in \mathcal{Y}(Y)\}$. Note that $A'' \times B'' \subseteq A \times B \rightarrow |A'' \times B''| \subseteq |A \times B|$.

For $Z \subseteq X \times Y$, let the projections be $\pi_X(Z) := \{x : \exists y. \langle x, y \rangle \in Z\}$, and $\pi_Y(Z) := \{y : \exists x. \langle x, y \rangle \in Z\}$.

Given $f_X : |X \times Y| \rightarrow \mathcal{P}(X)$, $f_Y : |X \times Y| \rightarrow \mathcal{P}(Y)$, and $A \times B \subseteq X \times Y$, $a \in X$, $b \in Y$, we use the abbreviation $\phi(\langle a, b \rangle, A \times B) := \forall A' \times B' \in |A \times B| (\langle a, b \rangle \in A' \times B' \rightarrow a \in f_X(A' \times B') \wedge b \in f_Y(A' \times B'))$. Note that $A'' \times B'' \subseteq A \times B$, $\phi(\langle a, b \rangle, A \times B) \rightarrow \phi(\langle a, b \rangle, A'' \times B'')$. (This is trivial by $|A'' \times B''| \subseteq |A \times B|$.)

Recall from [Sch92], Proposition 3.3 there:

Let $\mathcal{U} \subseteq \mathcal{P}(U)$, then $f : \mathcal{U} \rightarrow \mathcal{P}(U)$ can be represented by a preferential structure iff for all $V, W \in \mathcal{U}$

- (f1) $f(V) \subseteq V$
- (f2) $V \subseteq W \rightarrow f(W) \cap V \subseteq f(V)$.

Theorem 3.1 a) Let $f : | X \times Y | \rightarrow \mathcal{P}(X \times Y)$ be determined by a preferential structure, let $f_X = \pi_X \circ f$, $f_Y = \pi_Y \circ f$ (\circ the composition of functions), then for all $A \times B \in | X \times Y |$ (1X) $f_X(A \times B) \subseteq A$, (1Y) $f_Y(A \times B) \subseteq B$, (2X) For $a \in A$ ($a \in f_X(A \times B) \leftrightarrow \exists b \in B. \phi(\langle a, b \rangle, A \times B)$), (2Y) For $b \in B$ ($b \in f_Y(A \times B) \leftrightarrow \exists a \in A. \phi(\langle a, b \rangle, A \times B)$).

b) Conversely, let $f_X : | X \times Y | \rightarrow \mathcal{P}(X)$, $f_Y : | X \times Y | \rightarrow \mathcal{P}(Y)$, and (1X) $f_X(A \times B) \subseteq A$, (1Y) $f_Y(A \times B) \subseteq B$, (2X) For $a \in A$ ($a \in f_X(A \times B) \leftrightarrow \exists b \in B. \phi(\langle a, b \rangle, A \times B)$), (2Y) For $b \in B$ ($b \in f_Y(A \times B) \leftrightarrow \exists a \in A. \phi(\langle a, b \rangle, A \times B)$), then there is $f : | X \times Y | \rightarrow \mathcal{P}(X \times Y)$ determined by a preferential structure, such that $f_X = \pi_X \circ f$, $f_Y = \pi_Y \circ f$.

Proof: We use Proposition 3.3 of [Sch92] mentioned above.

a) We show (iX), (iY) is analogous. (1X) is trivial by (f1). (2X) " \rightarrow " Let $a \in f_X(A \times B) = \pi_X(f(A \times B))$. So there is $b \in B$ with $\langle a, b \rangle \in f(A \times B)$, we show $\phi(\langle a, b \rangle, A \times B)$. So let $A' \times B' \in | A \times B | \subseteq | X \times Y |$ with $\langle a, b \rangle \in A' \times B'$, then by (f2) $\langle a, b \rangle \in f(A' \times B')$, so $a \in \pi_X(f(A' \times B')) = f_X(A' \times B')$, and $b \in \pi_Y(f(A' \times B')) = f_Y(A' \times B')$. " \leftarrow " Let $a \in A, b \in B$, and $\phi(\langle a, b \rangle, A \times B)$, so by $A \times B \in | A \times B |$, $a \in f_X(A \times B)$.

b) For $A \times B \in | X \times Y |$, define $f(A \times B) := \{ \langle a, b \rangle \in A \times B : \phi(\langle a, b \rangle, A \times B) \}$. We have to show (f1), (f2) and $f_X = \pi_X \circ f$, $f_Y = \pi_Y \circ f$ (on $| X \times Y |$). (f1) is trivial. (f2) Let $A' \times B', A \times B \in | X \times Y |$, $A' \times B' \subseteq A \times B$ and $\langle a, b \rangle \in f(A \times B) \cap A' \times B'$. We have to show $\phi(\langle a, b \rangle, A' \times B')$. But this is trivial by $\phi(\langle a, b \rangle, A \times B)$, $A' \times B' \subseteq A \times B$, and above remark. $f_X = \pi_X \circ f$: Let $A \times B \in | X \times Y |$. " \subseteq " Let $a \in f_X(A \times B)$, then by (2X), $\exists b \in B. \phi(\langle a, b \rangle, A \times B)$, thus $\langle a, b \rangle \in f(A \times B)$, so $a \in \pi_X \circ f(A \times B)$. " \supseteq " Let $a \in \pi_X \circ f(A \times B)$, so there is $b \in B$ such that $\langle a, b \rangle \in f(A \times B)$, so $\phi(\langle a, b \rangle, A \times B)$, and $a \in A, b \in B$, so by (2X) $a \in f_X(A \times B)$. $f_Y = \pi_Y \circ f$ is analogous. \square

We generalize to the arbitrary cartesian product (this is straightforward)
:

Definition 3.2 $\Pi\mathcal{S}$ will denote the cartesian product.

Fix I , $\mathcal{X} := \{X_i : i \in I\}$ and for $i \in I$ let $\mathcal{Y}_i \subseteq \mathcal{P}(X_i)$. For $\mathcal{A} = \{A_i : i \in I\}$ with $A_i \subseteq X_i$ for all $i \in I$, let $|\mathcal{A}| := \{\mathbf{B} = \{B_i : i \in I\} : \forall i \in I (B_i \subseteq A_i \wedge B_i \in \mathcal{Y}_i)\}$. Note again that for $\mathcal{E} = \{E_i : i \in I\}$ and $\Pi\mathcal{E} \subseteq \Pi\mathcal{A} \mid \mathcal{E} \subseteq |\mathcal{A}|$.

Given $f_i : | \mathcal{X} | \rightarrow \mathcal{P}(X_i)$ for all $i \in I$, $\mathcal{A} = \{A_i : i \in I\}$ with $A_i \subseteq X_i$, $g \in \Pi\mathcal{X}$, let $\phi(g, \mathcal{A}) := \forall \mathbf{B} \in |\mathcal{A}| (g \in \Pi\mathbf{B} \rightarrow \forall i \in I. g(i) \in f_i(\Pi\mathbf{B}))$. Note again that $\Pi\mathbf{B} \subseteq \Pi\mathcal{A}$, $\phi(g, \mathcal{A}) \rightarrow \phi(g, \mathbf{B})$.

For any $Z \subseteq \Pi\mathcal{X}$ let $\pi_j(Z) := \{x_j \in X_j : \exists z \in Z. z(j) = x_j\}$, the projection to the j -coordinate.

Theorem 3.2 a) Let $f : \{\Pi\mathcal{A} : \mathcal{A} \in | \mathcal{X} | \} \rightarrow \mathcal{P}(\Pi\mathcal{X})$ be defined by a preferential structure on $\Pi\mathcal{X}$ and $\forall i \in I f_i = \pi_i \circ f$, then for all $\mathcal{A} \in | \mathcal{X} |$
(1) $\forall i \in I. f_i(\Pi\mathcal{A}) \subseteq A_i$ (2) $\forall i \in I \forall a_i \in A_i (a_i \in f_i(\Pi\mathcal{A}) \leftrightarrow \exists g \in \Pi\mathcal{A} (g(i) = a_i \wedge \phi(g, \mathcal{A})))$

b) Conversely let for all $i \in I f_i : \{\Pi\mathcal{A} : \mathcal{A} \in | \mathcal{X} | \} \rightarrow \mathcal{P}(X_i)$ and for all $\mathcal{A} \in | \mathcal{X} |$
(1) $\forall i \in I. f_i(\Pi\mathcal{A}) \subseteq A_i$, (2) $\forall i \in I \forall a_i \in A_i (a_i \in f_i(\Pi\mathcal{A}) \leftrightarrow \exists g \in \Pi\mathcal{A} (g(i) = a_i \wedge \phi(g, \mathcal{A})))$ then there is $f : \{\Pi\mathcal{A} : \mathcal{A} \in | \mathcal{X} | \} \rightarrow \mathcal{P}(\Pi\mathcal{X})$ determined by a preferential structure such that $\forall i \in I f_i = \pi_i \circ f$.

Proof: We use again Proposition 3.3 of [Sch92] mentioned above. a) Fix $i \in I$. (1) is trivial by $f(Z) \subseteq Z$. (2) " \rightarrow " Let $a_i \in f_i(\Pi\mathcal{A}) = \pi_i(f(\Pi\mathcal{A}))$. So there is $g \in f(\Pi\mathcal{A})$, $a_i = g(i)$. We show $\phi(g, \mathcal{A})$. Let $\mathbf{B} \in |\mathcal{A}|$, $g \in \Pi\mathbf{B}$, so $g \in f(\Pi\mathbf{B})$ by (f2), so $\forall i \in I. g(i) \in \pi_i(f(\Pi\mathbf{B})) = f_i(\Pi\mathbf{B})$. " \leftarrow " Let $a_i \in A_i$, $g \in \Pi\mathcal{A}$ such that $g(i) = a_i \wedge \phi(g, \mathcal{A})$, but then by $\mathcal{A} \in | \mathcal{X} |$ $a_i = g(i) \in f_i(\Pi\mathcal{A})$.

b) Suppose all f_i satisfy (1) and (2). For $\mathcal{A} \in | \mathcal{X} |$, define $f(\Pi\mathcal{A}) := \{g \in \Pi\mathcal{A} : \phi(g, \mathcal{A})\}$. We have to show (f1), (f2) and $\forall i \in I. f_i = \pi_i \circ f$. (f1) is trivial. (f2) Let $\mathcal{A}, \mathbf{B} \in | \mathcal{X} |$, $\Pi\mathbf{B} \subseteq \Pi\mathcal{A}$ and $g \in f(\Pi\mathcal{A}) \cap \Pi\mathbf{B}$. We have to show

$\phi(g, \mathbf{B})$, but this is trivial by $\Pi \mathbf{B} \subseteq \Pi \mathcal{A}$ and $\phi(g, \mathcal{A})$. $f_i = \pi_i \circ f$: " \subseteq " Let $a_i \in f_i(\Pi \mathcal{A}) \subseteq A_i$, so by (2) $\exists g \in \Pi \mathcal{A}. g(i) = a_i \wedge \phi(g, \mathcal{A})$, so $g \in f(\Pi \mathcal{A})$ and $a_i = g(i) \in \pi_i(f(\Pi \mathcal{A}))$. " \supseteq " Let $a_i \in \pi_i(f(\Pi \mathcal{A})) \rightarrow \exists g \in f(\Pi \mathcal{A}). g(i) = a_i \rightarrow \exists g \in \Pi \mathcal{A}. g(i) = a_i \wedge \phi(g, \mathcal{A}) \rightarrow a_i = g(i) \in f_i(\Pi \mathcal{A})$. \square

We formulate the logical counterpart of Theorem 3.1 - the proof is easy. Fix a propositional language \mathcal{L} .

Definition 3.3 We consider two logics, $\cdot^i : \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$, $\langle S, T \rangle \mapsto \langle S, T \rangle^i \subseteq \mathcal{L}$. We say that both logics $\langle S, T \rangle^i$ are given by a function $f : \mathbf{D} \times \mathbf{D} \rightarrow \mathcal{P}(M_{\mathcal{L}} \times M_{\mathcal{L}})$ iff for all theories S, T $\langle S, T \rangle^i = \{\phi : \forall m \in \pi_i(f(M_S \times M_T)). m \models \phi\}$, where π_i is the projection on the i -th coordinate. Analogously, we say that they are given by a preferential structure on $M_{\mathcal{L}} \times M_{\mathcal{L}}$ iff f is given by such a structure.

We call a function f on pairs of models definability preserving (dp) iff for all theories S, T $\pi_1(f(M_S \times M_T)) = M_U$ and $\pi_2(f(M_S \times M_T)) = M_V$ for some theories U, V , where M_S is the set of S -models etc. Note that then $\pi_i(f(M_S \times M_T)) = M_{\langle S, T \rangle^i}$.

For U, V complete, consistent theories (cct), S, T any theories, we abbreviate $\psi(U, V, S, T) := \forall S', T' (U \vdash S' \vdash S \wedge V \vdash T' \vdash T \rightarrow U \vdash \langle S', T' \rangle^1 \wedge V \vdash \langle S', T' \rangle^2)$.

Fact 3.3 Let $\mathcal{Y}(X) = \mathcal{Y}(Y) = \mathbf{D}$, $f_X(M_S \times M_T) = M_{\langle S, T \rangle^1}$, $f_Y(M_S \times M_T) = M_{\langle S, T \rangle^2}$, U, V cct, then $\psi(U, V, S, T) \leftrightarrow \phi(\langle m_U, m_V \rangle, M_S \times M_T)$.

Proof: The Fact follows immediately from: $\langle m_U, m_V \rangle \in M_{S'} \times M_{T'} \subseteq M_S \times M_T \leftrightarrow U \vdash S' \vdash S, V \vdash T' \vdash T, m_U \in f_X(M_{S'} \times M_{T'}) = M_{\langle S', T' \rangle^1} \leftrightarrow U \vdash \langle S', T' \rangle^1, m_V \in f_Y(M_{S'} \times M_{T'}) = M_{\langle S', T' \rangle^2} \leftrightarrow V \vdash \langle S', T' \rangle^2$. \square

Theorem 3.4 Let $\langle S, T \rangle^1, \langle S, T \rangle^2$ be two logics on pairs of theories. Then $\langle S, T \rangle^i$ are given by a dp preferential structure iff

(1) $\overline{S} = \overline{S'} \wedge \overline{T} = \overline{T'} \rightarrow \langle S, T \rangle^1 = \langle S', T' \rangle^1 \wedge \langle S, T \rangle^2 = \langle S', T' \rangle^2$

(2) $\langle S, T \rangle^i$ is classically closed

(3) $\langle S, T \rangle^1 \vdash S$, $\langle S, T \rangle^2 \vdash T$

(4S) If U is a cct with $U \vdash S$, then $U \vdash \langle S, T \rangle^1$ iff there is a cct V such that $V \vdash T$ and $\psi(U, V, S, T)$

(4T) If V is a cct with $V \vdash T$, then $V \vdash \langle S, T \rangle^2$ iff there is a cct U such that $U \vdash S$ and $\psi(U, V, S, T)$.

Proof: We use Theorem 3.1. For a cct U let m_U denote its model. Note that if f is dp, U a cct, and $U \vdash \langle S, T \rangle^1$, then $m_U \in \pi_X(f(M_S \times M_T))$: $m_U \in M_{\langle S, T \rangle^1} = \pi_X(f(M_S \times M_T))$, likewise for $V \vdash \langle S, T \rangle^2$.

" \rightarrow " Note that $\pi_X \circ f = f_X$. (1) and (2) are trivial. (3) By (1X) of Theorem 3.1, $\pi_X(f(M_S \times M_T)) \subseteq M_S$, so $\langle S, T \rangle^1 \vdash S$. (4S) Let U be cct, $U \vdash S$. $U \vdash \langle S, T \rangle^1$ iff $m_U \in M_{\langle S, T \rangle^1} = \pi_X(f(M_S \times M_T)) = f_X(M_S \times M_T)$ iff (by (2X) of Thm. 3.1) there is $m_V \in M_T$. $\phi(\langle m_U, m_V \rangle, M_S \times M_T)$ iff there is a cct V such that $V \vdash T$ and $\psi(U, V, S, T)$ (by Fact 3.3). (4T) is analogous.

" \leftarrow " Let $\langle S, T \rangle^i$ satisfy (1)-(4T), define $f_X, f_Y : \mathbf{D} \times \mathbf{D} \rightarrow \mathbf{D}$ as follows: For $M_S \times M_T$, let $f_X(M_S \times M_T) := M_{\langle S, T \rangle^1}$, $f_Y(M_S \times M_T) := M_{\langle S, T \rangle^2}$. By (1), this is well-defined. We have to show (iX), (iY), of Thm. 3.1, note that then $f_X = \pi_X \circ f$, $f_Y = \pi_Y \circ f$. Moreover, we have to show $\langle S, T \rangle^1 = \{\phi : \forall m \in f_X(M_S \times M_T). m \models \phi\}$ and $\langle S, T \rangle^2 = \{\phi : \forall m \in f_Y(M_S \times M_T). m \models \phi\}$. We will only show the properties for the first coordinate, the second being analogous. (1X): Let $m \in f_X(M_S \times M_T) = M_{\langle S, T \rangle^1}$, then, by (3) $m \in M_S$. (2X): Let $m_U \in M_S$, so $U \vdash S$. Then $m_U \in f_X(M_S \times M_T) = M_{\langle S, T \rangle^1}$ iff $U \vdash \langle S, T \rangle^1$ iff (by (4S)) there is cct V , $V \vdash T$ and $\psi(U, V, S, T)$ iff (by Fact 3.3) there is $m_V \in M_T$ and $\phi(\langle m_U, m_V \rangle, M_S \times M_T)$. It remains to show $\langle S, T \rangle^1 = \{\phi : \forall m \in f_X(M_S \times M_T). m \models \phi\}$. By (2), it suffices to show $M_{\langle S, T \rangle^1} = f_X(M_S \times M_T)$, but this was the definition. \square

3.3 Extensions:

3.3.1 The First Order Case:

As already pointed out in the introduction, our basic tool, Proposition 3.3 of [Sch92], is an algebraic characterization and independent of logic. To adapt our result to the first order case (and still other logics, if we wish), we thus have to look into the material from Definition 3.3 onward only. We see that the only properties we need of the base logic \vdash and its models $M_{\mathcal{L}}$ are soundness and completeness (in the strong sense, $T \vdash T'$ iff $M(T) \subseteq M(T')$), and that complete and consistent theories have exactly one model.

The latter property obviously is not satisfied by the first order case, but it does not seem unreasonable to make the additional assumption that all models, which satisfy the same first order formulas, are treated in the same manner. With this assumption, we can work with equivalence classes of models, or one representative in each case, and the proofs go through.

3.3.2 Further Extensions:

If we look back at our motivating scenario of an archeologist, we see that we have covered only part of his reasoning. For instance, he might have discovered a fact, e.g. a burial, but is unable to date it, so he knows that T holds at some time, but is uncertain about the exact time point. (This example is due to a referee.) Obviously, our results do not cover this case. For a formal result in the style of Theorem 3.4, we first need a language to speak about developments, allowing to quantify over time points, and then a base logic to reason about such developments - which would be the base models of our language. In a second step, we would work with a preference relation on developments, and use our Proposition 3.3 of [Sch92] again to work for a formal result.

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References

- [Bar91] Paul Bartha, "Conditional Obligation and the Logic of Agency", in: "Proceedings DEON 91", J.-J.Ch.Meyer, R.J.Wieringa eds., Amsterdam
- [Bou90a] C.Boutilier, "Conditional Logics of Normality as Modal Systems", AAAI 1990, Boston, p.594
- [Bou90b] C.Boutilier, "Viewing Conditional Logics of Normality as Extensions of the Modal System S4", Toronto University, KRR-TR-90-4, Toronto, Canada, June 1990
- [Bou92] C.Boutilier, "Conditional Logics for Default Reasoning and Belief Revision", Dept. of Comp. Sc., Univ. Brit. Columbia TR 92-1, Jan. 92, Vancouver, Canada
- [BS85] G.Bossu, P.Siegel, "Saturation, Nonmonotonic Reasoning and the Closed- World Assumption", Artificial Intelligence 25 (1985) 13-63
- [Gab85] D.M.Gabbay, "Theoretical foundations for non-monotonic reasoning in expert systems". In: K.R.Apt (ed.), "Logics and Models of Concurrent Systems", Springer, Berlin, 1985, p.439-457
- [Han69] B.Hansson: "An analysis of some deontic logics", *Nous* 3, 373-398. Reprinted in R.Hilpinen ed. "Deontic Logic: Introductory and Systematic Readings". Reidel, Dordrecht 1971, 121-147
- [Jec78] T.Jech, "Set Theory", Academic Press, New York, 1978
- [KLM90] S.Kraus, D.Lehmann, M.Magidor, "Nonmonotonic reasoning, preferential models and cumulative logics", Artificial Intelligence, 44 (1-2), p.167-207, July 1990
- [KM90] H.Katsuno, A.Mendelzon, "On the Difference Between Updating a Knowledge Database and Revising It", KRR-TR-90-6, Toronto Univ., Toronto, Canada, 1990
- [Kun80] K.Kunen, "Set Theory", Amsterdam, 1980
- [Leh92a] D.Lehmann: "Plausibility Logic", Proceedings CSL91

- [Leh92b] D.Lehmann: "Plausibility Logic", Tech.Rept. TR-92-3, Feb. 1992, Hebrew University, Jerusalem 91904, Israel
- [Lew81] D.Lewis: "A Subjectivist's Guide to Objective Chance", in "Ifs", W.L.Harper, R.Stalnaker, G.Pearce (eds.), D.Reidel, Dordrecht, 1981
- [LM92] D.Lehmann, M.Magidor, "What does a conditional knowledge base entail?", *Artificial Intelligence*, 55(1), p. 1-60, May 1992
- [Mak93] D.Makinson, "Five Faces of Minimality", *Studia Logica* 52 (1993), p. 339-379
- [Mak94] D.Makinson, "General patterns in nonmonotonic reasoning", in D.Gabbay, C.Hogger, Robinson (eds.), "Handbook of Logic in Artificial Intelligence and Logic Programming", vol. III: "Nonmonotonic and Uncertain Reasoning", Oxford University Press, 1994, p. 35-110
- [Sch92] K.Schlechta: "Some Results on Classical Preferential Models", *Journal of Logic and Computation*, Oxford, Vol.2, No.6 (1992), p. 675-686
- [Sch92-n4] K.Schlechta: "Some Completeness Results for Stopped and Ranked Classical Preferential Models", Research Report RR 15, 05/94, Laboratoire d'Informatique de Marseille, URA CNRS 1787, Université de Provence, 3 pl. Victor Hugo, F-13331 Marseille Cedex 3, France,
- [Sch94-t1] K.Schlechta: "Limit Preferential Models", Research Report RR 6, 03/94, Laboratoire d'Informatique de Marseille, URA CNRS 1787, Université de Provence, 3 pl. Victor Hugo, F-13331 Marseille Cedex 3, France
- [Sch94-t2] K.Schlechta: "Completeness and Incompleteness for Plausibility Logic", Research Report RR 7, 04/94, Laboratoire d'Informatique de Marseille, URA CNRS 1787, Université de Provence, 3 pl. Victor Hugo, F-13331 Marseille Cedex 3, France,

- [Sho87] Yoav Shoham: "A semantical approach to nonmonotonic logics".
In Proc. Logics in Computer Science, p.275-279, Ithaca, N.Y., 1987
- [Tho84] Richmond H. Thomason, "Combinations of Tense and Modality",
in: "Handbook of Philosophical Logic", D.Gabbay, F.Guenthner
eds., D.Reidel, Dordrecht, 1984, Vol.II, p.135-165