al. (1994) for the $K$-$\varepsilon$ closure. The latter (11) is an objective and strongly realizable closure, which forbids any turbulence degeneration, since:

**Proposition 1.** - System (4)-(6) associated with:

$$\phi_{ij}^\text{slow} = \theta^{-1} \left( a_1(I, II, III) R_{ij} + \frac{I b_1(I, II, III)}{II} (R^2)_{ij} + \frac{I c_1(I, II, III)}{III} (R^3)_{ij} \right)$$

is such that the product of eigenvalues of the Reynolds stress tensor remains strictly positive if

$$\theta^{-1} \left( 3 a_1(I, II, III) + \frac{(I)^2 b_1(I, II, III)}{II} + \frac{(I)(II)c_1(I, II, III)}{III} \right) \in L^\infty(\Omega \times [0, T]).$$

This result is linked with the particular form of the governing equation of the motion of the determinant $(7c)$. It is worth noting that (11) contains the "slow" part initially suggested in Lumley (1978). Moreover, this result may be extended to the "rapid" parts introduced in Shih (1985) and Fu (1987), since both of them may be rewritten in the form (13). Sufficient conditions to provide return-to-isotropy (see Hérard, 1994b), are given below (see (14), (16)):

**Proposition 2.** - System (15) associated with (11) ensures the return to isotropy process, in at least one plane of eigenvectors, since:

$$0 < \prod_{1 \leq i < j \leq 3} \varphi_{ij}(t) \leq \left( \prod_{1 \leq i < j \leq 3} \frac{\varphi_{ij}(t = 0)}{1 - \varphi_{ij}(t = 0)} \right) \exp \{ (\theta^{-1})_{\text{min}} (b_1)_{\text{max}} t \}.$$

System (4)-(6) together with (11) possesses interesting properties; recall (see Hérard, 1994a) that its non-viscous homogeneous part is hyperbolic; moreover, the following system:

$$\langle u_i \rangle, t + \langle u_j \rangle \langle u_i \rangle, j + \langle R_{ij} \rangle, j = 0, \quad \langle R_{ij} \rangle, t + \langle u_k \rangle \langle R_{ij} \rangle, k + R_{ik} \langle u_j \rangle, k + R_{jk} \langle u_i \rangle, k = 0$$

which has been derived by removing the divergence free velocity constraint and the mean pressure gradient in the mean momentum equation, and which may be rewritten in the form (21) (see (22) for the notations), is a non-conservative hyperbolic system, and is such that:

**Proposition 3.** - The one-dimensional Riemann problem associated with (21), i.e.:

$$\frac{d}{dt} W + A_x \frac{d}{dx} W = 0$$

and the initial conditions: $W(x < 0, t = 0) = W_G$; $W(x > 0, t = 0) = W_D$ admits a unique solution, if:

$$\langle u \rangle_D - \langle u \rangle_G < \sqrt{2 \left\{ \sqrt{\langle u^2 \rangle_D} + \sqrt{\langle u^2 \rangle_G} \right\}}.$$ 

Any quantity chosen among (7a), (7b), (7c) remains positive in the $(x, t)$ plane.

This result is very similar to those given in Hérard (1994b); it requires application of theoretical results from Le Floch (1988) and Colombeau (1992), since (21) has no conservative form. This result can lead to derivation of algorithms similar to the one initially detailed in Bell (1989), using a Godunov solver to deal with the convective part of (5), which is (21); furthermore, this provides discrete allowable (or realizable) values of the Reynolds stress components, when choosing a finite-volume co-located approach using an unstructured mesh, and also ensure a discrete return to isotropy (Hérard, 1995a). Some other results arising when dealing with non-isothermal flows are provided in Hérard (1994b); the maximum principle for the mean temperature no longer holds within this particular framework.