Qualitative properties of monostable pulsating fronts : exponential decay and monotonicity

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Abstract

In this paper, we prove various qualitative properties of pulsating travelling fronts in periodic media, for reaction-diffusion equations with Kolmogorov-Petrovsky-Piskunov type or general monostable nonlinearities. Besides monotonicity, the main part of the paper is devoted to the exponential behavior of the fronts when they approch their unstable limiting state. In the general monostable case, the logarithmic equivalent of the fronts is shown and for noncritical speeds, the decay rate is the same as in the KPP case. These results also generalize the known results in the homogeneous case or in the case when the equation is invariant by translation along the direction of propagation.

1 Introduction and main results

Propagation phenomena for reaction-diffusion models in heterogeneous media have been the purpose of very active research in the past recent years. We refer to [3] and [52] for surveys on this topic. This paper is the first of a series of two on qualitative properties of monostable pulsating travelling fronts in periodic media. Some existence results had been obtained recently, but little has been known about qualitative properties of these fronts when one of the limiting states is unstable. Here, we prove the monotonicity of the fronts in the time variable and the exponential decay when they approach their unstable limiting state. The exponential decay rate can be computed explicitely for KPP or general monostable nonlinearities. These issues had been left open so far. In the forthcoming paper [22], we are concerned with uniqueness, stability and estimates of the spreading speeds for KPP or general monostable pulsating fronts.

1.1 The periodic framework : main assumptions

We consider reaction-diffusion-advection equations of the type

$$\begin{cases} u_t - \nabla\cdot(A(z)\nabla u) + q(z)\cdot\nabla u = f(z,u), \quad z\in\overline{\Omega}, \\ \nu A\nabla u = 0, \quad z\in\partial\Omega, \end{cases} \tag{1.1}$$

in a smooth unbounded domain $\Omega\subset\mathbb{R}^N$. We denote by ν the outward unit normal on $\partial\Omega$. Given two vectors ξ and ξ' in $\mathbb{R}^N$ and a $N\times N$ matrix $B=(B_{ij})_{1\le i,j\le N}$ with real entries, we write

$$\xi B\xi' = \sum_{1\le i,j\le N} \xi_i B_{ij}\xi'_j.$$

Equations of the type (1.1) arise especially in combustion, population dynamics and ecological models (see e.g. [34, 45, 48]). The scalar passive quantity u typically stands for the temperature or the concentration of a species which diffuses and is transported in a periodic excitable medium.

The coefficients of (1.1) are not homogeneous in general, as well as the underlying domain Ω which may not be the whole space $\mathbb{R}^N$. In other words, the heterogeneous character arises both in the equation and in the underlying domain. As described in the book by Shigesada and Kawasaki [45], a first step to take into account the heterogeneities is to assume that the environment varies periodically. Namely, assume that there is an integer $d\in\{1,\cdots,N\}$ and d positive real numbers $L_1,\ldots,L_d$ such that

$$\begin{cases} \exists\ R\ge 0, \quad \forall\ z=(x,y)\in\Omega, \quad |y|\le R, \\ \forall\ k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}\times\{0\}^{N-d}, \quad \Omega=\Omega+k, \end{cases} \tag{1.2}$$

where

$$x=(x_1,\cdots,x_d), \quad y=(x_{d+1},\cdots,x_N), \quad z=(x,y)$$

and $|\cdot|$ denotes the euclidean norm. The domain Ω is assumed to be of class $C^{2,\alpha}$ for some $\alpha>0$. Let C be the periodicity cell defined by

$$C=\{(x,y)\in\Omega,\ x\in(0,L_1)\times\cdots\times(0,L_d)\}.$$

Domains satisfying (1.2) include the whole space $\mathbb{R}^N$, the whole space with periodic perforations, infinite cylinders with constant or periodically undulating sections, etc.

The matrix field $A(x,y)=(A_{ij}(x,y))_{1\le i,j\le N}$ is symmetric of class $C^{1,\alpha}(\overline{\Omega})$, the vector field $q(x,y)=(q_i(x,y))_{1\le i\le N}$ is of class $C^{0,\alpha}(\overline{\Omega})$ and the nonlinearity (x,y,u) $(\in\overline{\Omega}\times\mathbb{R})\mapsto f(x,y,u)$ is continuous, of class $C^{0,\alpha}$ with respect to (x,y) locally uniformly in $u\in\mathbb{R}$ and we assume that $\frac{\partial f}{\partial u}$ exists and is continuous in $\overline{\Omega}\times\mathbb{R}$. All functions A_{ij}, q_i and $f(\cdot,\cdot,u)$ (for all $u\in\mathbb{R}$) are assumed to be periodic, in the sense that they satisfy

$$w(x+k,y)=w(x,y) \text{ for all } (x,y)\in\overline{\Omega} \text{ and } k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}.$$

Furthermore, there is $\alpha_0>0$ such that

$$\sum_{1\le i,j\le N} A_{ij}(x,y)\xi_i\xi_j \ge \alpha_0|\xi|^2 \text{ for all } (x,y)\in\overline{\Omega} \text{ and } (\xi_i)_{1\le i\le N}\in\mathbb{R}^N.$$

We are given two periodic functions $p^\pm$ satisfying

$$p^-(x,y) < p^+(x,y) \ \text{ for all } (x,y) \in \overline{\Omega},$$

which are classical $C^{2,\alpha}(\overline{\Omega})$ solutions of the stationary equation

$$\begin{cases} -\nabla \cdot (A(x,y)\nabla p^\pm) + q(x,y) \cdot \nabla p^\pm = f(x,y,p^\pm) \text{ in } \overline{\Omega}, \\ \nu A(x,y) \nabla p^\pm = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Let us now list additional assumptions which will be used in some of the results below. Denote

$$\zeta(x,y) = \frac{\partial f}{\partial u}(x,y,p^-(x,y)) \tag{1.3}$$

and assume that ζ is of class $C^{0,\alpha}(\overline{\Omega})$ and that

$$\mu_0 < 0, \tag{1.4}$$

where μ_0 denotes the principal eigenvalue of the linearized operator around p^-

$$\psi \mapsto -\nabla \cdot (A(x,y)\nabla\psi) + q(x,y) \cdot \nabla\psi - \zeta(x,y)\ \psi$$

with periodicity conditions in $\overline{\Omega}$ and Neumann boundary condition $\nu A \nabla\psi = 0$ on $\partial\Omega$. The principal eigenvalue μ_0 is characterized by the existence of a positive periodic function φ in $\overline{\Omega}$ such that

$$\begin{cases} -\nabla \cdot (A(x,y)\nabla\varphi) + q(x,y) \cdot \nabla\varphi - \zeta(x,y)\varphi = \mu_0 \varphi \ \text{ in } \overline{\Omega}, \\ \nu A(x,y) \nabla\varphi = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

Assume that there is ρ such that $0 < \rho < \min_{\overline{\Omega}} (p^+ - p^-)$ and, for any classical bounded supersolution $\overline{u}$ of (1.1), that is

$$\begin{cases} \overline{u}_t - \nabla \cdot (A(x,y)\nabla\overline{u}) + q(x,y) \cdot \nabla\overline{u} \geq f(x,y,\overline{u}) \ \text{ in } \mathbb{R} \times \overline{\Omega}, \\ \nu A \nabla\overline{u} \geq 0 \ \text{ on } \mathbb{R} \times \partial\Omega, \end{cases}$$

satisfying $\overline{u} < p^+$ and $\Omega_{\overline{u}} = \{(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ \overline{u}(t,x,y) > p^+(x,y) - \rho\} \neq \emptyset$, there exists a family of functions $(\rho_\tau)_{\tau \in [0,1]}$ defined in $\overline{\Omega_{\overline{u}}}$ and satisfying

$$\begin{cases} \tau \mapsto \rho_\tau \text{ is continuous in } C^{1+\alpha/2;2+\alpha}_{t;(x,y)}\left(\overline{\Omega_{\overline{u}}}\right), \\ \tau \mapsto \rho_\tau(t,x,y) \text{ is nondecreasing for each } (t,x,y) \in \overline{\Omega_{\overline{u}}}, \\ \rho_0 = 0,\ \rho_1 \geq \rho,\ \inf\limits_{\Omega_{\overline{u}}} \rho_\tau > 0 \text{ for each } \tau \in (0,1], \\ (\overline{u} + \rho_\tau)_t - \nabla \cdot (A\nabla(\overline{u} + \rho_\tau)) + q \cdot \nabla(\overline{u} + \rho_\tau) \geq f(x,y,\overline{u} + \rho_\tau) \ \text{ in } \Omega_{\overline{u},\tau}, \\ \nu A \nabla(\overline{u} + \rho_\tau) \geq 0 \ \text{ on } (\mathbb{R} \times \partial\Omega) \cap \Omega_{\overline{u},\tau}, \end{cases} \tag{1.6}$$

where $\Omega_{\overline{u},\tau} = \{(t,x,y) \in \Omega_{\overline{u}},\ \overline{u}(t,x,y) + \rho_\tau(t,x,y) < p^+(x,y)\}$.

Lastly, we assume that there are $\beta > 0$ and $\gamma > 0$ such that the map

$$(x,y,s) \mapsto \frac{\partial f}{\partial u}(x,y,p^-(x,y)+s) \text{ is of class } C^{0,\beta}(\overline{\Omega}\times[0,\gamma]). \tag{1.7}$$

For some of our results, we shall assume that, for all $(x,y)\in\overline{\Omega}$ and $s\in[0,p^+(x,y)-p^-(x,y)]$,

$$f(x,y,p^-(x,y)+s) \ \le\ f(x,y,p^-(x,y)) \ +\ \zeta(x,y)\ s. \tag{1.8}$$

Actually, not all assumptions (1.4), (1.6), (1.7) and (1.8) are needed in all the results below. We will explain in each proposition or theorem what we really need.

Before stating the main results, let us comment here these above conditions on f and $p^{\pm}$. First, condition (1.4) means that the steady state p^- is linearly unstable with respect to periodic perturbations. This condition is satisfied in particular if $\zeta(x,y)=\frac{\partial f}{\partial u}(x,y,p^-(x,y))>0$ for all $(x,y)\in\overline{\Omega}$.

As far as the weak stability condition (1.6) is concerned, it is satisfied in particular if

$$\exists\ \rho>0,\ \forall\ (x,y)\in\overline{\Omega},\ \forall\ 0\le s\le s'\le\rho,\ f(x,y,p^+(x,y)-s')\ge f(x,y,p^+(x,y)-s). \tag{1.9}$$

Indeed, in this case, even if it means reducing ρ, we can take $\rho_\tau=\tau\rho$ for each $\tau\in[0,1]$. The stronger property (1.9) holds (and, thus, (1.6)) for instance if $\frac{\partial f}{\partial u}(x,y,p^+(x,y))<0$ for all $(x,y)\in\overline{\Omega}$. More generally, condition (1.6) holds if the stationary state p^+ is linearly stable, in the sense that the principal eigenvalue μ^+ of the linearized operator

$$\psi\mapsto -\nabla\cdot(A(x,y)\nabla\psi)+q(x,y)\cdot\nabla\psi-\frac{\partial f}{\partial u}(x,y,p^+(x,y))\ \psi \tag{1.10}$$

around p^+, with periodicity conditions in $\overline{\Omega}$ and Neumann boundary condition $\nu A\nabla\psi=0$ on $\partial\Omega$, satisfies : $\mu^+>0$. Indeed, in this case, if φ^+ denotes the principal eigenfunction of this operator such that $\min_{\overline{\Omega}}\varphi^+=1$ and if $\rho\in(0,\min_{\overline{\Omega}}(p^+-p^-))$ is chosen so that

$$\left|\frac{\partial f}{\partial u}(x,y,p^+(x,y))-\frac{\partial f}{\partial u}(x,y,p^+(x,y)-s)\right|\le\mu^+ \ \text{ for all } (x,y,s)\in\overline{\Omega}\times[0,\rho],$$

then we can take

$$\rho_\tau(t,x,y)=\tau\,\rho\,\varphi^+(x,y) \text{ for each } \tau\in[0,1].$$

Condition (1.6) is also fulfilled if, for each $(x,y)\in\overline{\Omega}$, the function

$$s\mapsto\frac{f(x,y,p^-(x,y)+s)-f(x,y,p^-(x,y))}{s} \text{ is nonincreasing in } (0,(p^+-p^-)(x,y)). \tag{1.11}$$

Indeed, in this case, we can take any ρ in $(0,\min_{\overline{\Omega}}(p^+-p^-))$ and

$$\rho_\tau(t,x,y)=\frac{\tau\rho}{\inf_{\Omega_{\overline{u}}}(\overline{u}-p^-)}\times(\overline{u}(t,x,y)-p^-(x,y)).$$

Therefore, one of the advantages of formulation (1.6) is that it includes the two different and important cases (1.9) and (1.11), which have already been considered in the literature (see the comments below after the main theorems). Notice also that property (1.11) (and, thus, (1.6)) holds for nonlinearities of the type

$$f(x,y,s) = s(\zeta(x,y) - \eta(x,y)s) \text{ with } p^- = 0, \text{ and } \eta(x,y) \geq 0 \text{ in } \overline{\Omega}.$$

Typical cases are when f depends on u only, admits two zeroes $p^- < p^+ \in \mathbb{R}$ such that $f'(p^-) > 0$ and f is nonincreasing in a left neighborhood of p^+. Assumption (1.8) reads in this case:

$$f(u) \leq f'(p^-) \times (u - p^-) \text{ for all } u \in [p^-, p^+].$$

The nonlinearities $f(u) = u(1-u)$ or $f(u) = u(1-u)^m$ with $m \geq 1$ are archetype examples with $p^- = 0$ and $p^+ = 1$ which arise in biological models (see Fisher [18], or Kolmogorov, Petrovsky and Piskunov [28]). In general, the steady states $p^\pm$ truly depend on the position (x,y) –examples will be cited below after the statement of the main theorems– and condition (1.8) can be viewed as a generalization of the Fisher-KPP assumption.

1.2 Main results

One of the main features of reaction-diffusion models is that transition waves may develop and establish a connection between two different steady states. In the periodic framework, we are concerned with pulsating travelling fronts between p^- and p^+, which are defined as follows.

Definition 1.1 *Given a unit vector $e \in \mathbb{R}^N$ whose last $N-d$ components are zero, that is $|e| = 1$ and $e \in \mathbb{R}^d \times \{0\}^{N-d}$, a pulsating front connecting p^- and p^+, travelling in the direction e with (mean) speed $c \in \mathbb{R}^*$, is a time-global classical solution $u(t,x,y)$ of* (1.1) *which can be written as*

$$u(t,x,y) = \phi(ct - x \cdot e, x, y) \quad \text{for all } (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \tag{1.12}$$

where ϕ is continuous on $\mathbb{R} \times \overline{\Omega}$ and satisfies

$$(x,y) \mapsto \phi(s,x,y) \text{ is periodic in } \overline{\Omega} \text{ for all } s \in \mathbb{R} \tag{1.13}$$

and

$$\phi(s,x,y) \underset{s \to \pm\infty}{\longrightarrow} p^\pm(x,y) \text{ uniformly in } (x,y) \in \overline{\Omega}. \tag{1.14}$$

With a slight abuse of notation, $x \cdot e$ denotes $x_1 e_1 + \cdots + x_d e_d$, where $e_1, \ldots, e_d$ are the first d components of the vector e. We are interested only in fronts such that

$$p^-(x,y) \leq u(t,x,y) \leq p^+(x,y) \text{ for all } (t,x,y) \in \mathbb{R} \times \overline{\Omega}. \tag{1.15}$$

Notice that, because of (1.14) *and the strong maximum principle, the inequalities* (1.15) *are actually strict for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$.*

The notion of pulsating travelling fronts extends that of usual travelling fronts which are invariant in the frame moving with speed c in the direction e. We will come back to the classical results about travelling fronts after the statements of the main theorems. We just notice here that formula (1.12) can be rewritten as

$$\phi(s,x,y) = u\left(\frac{s + x \cdot e}{c}, x, y\right) \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega},$$

while condition (1.13) means that

$$\forall\, k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z},\ \ \forall\, (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad u\left(t - \frac{k \cdot e}{c}, x, y\right) = u(t, x+k, y). \quad (1.16)$$

Notice also that (1.14) is equivalent to

$$\lim_{A \to +\infty} \left(\sup_{\pm(ct - x\cdot e) \geq A,\ (t,x,y) \in \mathbb{R} \times \overline{\Omega}} |u(t,x,y) - p^{\pm}(x,y)| \right) = 0.$$

We are concerned in this paper with some qualitative properties of the pulsating travelling fronts connecting p^- and p^+, such as monotonicity with respect to the variable $s = ct - x \cdot e$, bounds for the speeds, behavior of the functions ϕ when they approach the unstable limiting state p^-. These properties are of essential interest and enable us, in the second part [22], to derive uniqueness, stability and spreading type results. Under assumptions (1.4) and (1.6), the pulsating travelling fronts $\phi(s,x,y)$, in the sense of Definition 1.1, connect the unstable state p^- to the stable one p^+. As known in simpler situations, what really matters and what makes the analysis difficult is the behavior of the front near its unstable limiting state p^-. In particular, we prove here that $\phi(s,x,y)$ decays exponentially to $p^-(x,y)$ as $s \to -\infty$.

The exponential behavior of $\phi(s,x,y) - p^-(x,y)$ can be made explicit in terms of some linear operators depending on p^-, and we need a few more notations. Let $\zeta(x,y)$ be defined as in (1.3). For each $\lambda \in \mathbb{R}$, call $k(\lambda)$ the principal eigenvalue of the operator

$$L_\lambda \psi := -\nabla \cdot (A\nabla\psi) + 2\lambda e A \nabla\psi + q \cdot \nabla\psi + [\lambda \nabla \cdot (Ae) - \lambda q \cdot e - \lambda^2 eAe - \zeta]\psi \qquad (1.17)$$

acting on the set

$$E_\lambda = \{\psi \in C^2(\overline{\Omega}),\ \psi \text{ is periodic in } \overline{\Omega} \text{ and } \nu A \nabla\psi = \lambda(\nu A e)\psi \text{ on } \partial\Omega\}.$$

Note in particular that

$$k(0) = \mu_0$$

where μ_0 is given in (1.5). Let ψ_λ denote the unique positive principal eigenfunction of L_λ such that, say,

$$\|\psi_\lambda\|_{L^\infty(C)} = 1. \qquad (1.18)$$

Lastly, define

$$c^*(e) = \inf_{\lambda > 0} \left(-\frac{k(\lambda)}{\lambda} \right). \qquad (1.19)$$

This quantity turns out to be a real number, and for each $c > c^*(e)$, the number

$$\lambda_c = \min\{\lambda > 0,\ k(\lambda) + c\lambda = 0\} \tag{1.20}$$

is well-defined (see Section 2.1). Actually, for each $c > c^*(e)$, the set $F_c = \{\lambda \in (0, +\infty),\ k(\lambda) + c\lambda = 0\}$ is either the singleton $\{\lambda_c\}$, or it is made of two points $\{\lambda_c, \lambda_c^+\}$ with $\lambda_c < \lambda_c^+$, while, for $c = c^*(e)$, the set F_c is either empty or it is a singleton $\{\lambda^*\}$ (see Section 2.1 and the results below in this section).

The following proposition gathers a few basic properties which are satisfied by the pulsating fronts, even without the regularity assumption (1.7) or the KPP assumption (1.8).

Proposition 1.2 *Let $u(t, x, y) = \phi(ct - x \cdot e, x, y)$ be a pulsating front in the sense of Definition* 1.1. *Under assumption* (1.4)*, then*

$$c \geq c^*(e).$$

Under assumptions (1.4) *and* (1.6)*, then ϕ is increasing in its first variable, and $\phi_s(s, x, y) > 0$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$.*

Notice in particular that the monotonicity of ϕ with respect to s implies that u is increasing in t if $c > 0$ and decreasing if $c < 0$. Moreover, $u_t(t, x, y) > 0$ for all $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$ if $c > 0$, and $u_t(t, x, y) < 0$ if $c < 0$.

In the following theorem, we give the exact exponential behavior of the functions $\phi(s, x, y)$ as $s \to -\infty$ with the KPP assumption (1.8).

Theorem 1.3 *Let $u(t, x, y) = \phi(ct - x \cdot e, x, y)$ be a pulsating front in the sense of Definition* 1.1*, and assume that conditions* (1.4)*,* (1.7) *and* (1.8) *are satisfied. If $c > c^*(e)$, then there exists $B > 0$ such that*

$$\phi(s, x, y) - p^-(x, y) \sim B e^{\lambda_c s} \psi_{\lambda_c}(x, y) \text{ as } s \to -\infty \text{ uniformly in } (x, y) \in \overline{\Omega}. \tag{1.21}$$

If $c = c^*(e)$, then there is a unique $\lambda^* > 0$ such that $k(\lambda^*) + c^*(e)\lambda^* = 0$ and there exists $B > 0$ such that

$$\phi(s, x, y) - p^-(x, y) \sim B|s|^{2m+1} e^{\lambda^* s} \psi_{\lambda^*}(x, y) \text{ as } s \to -\infty \text{ uniformly in } (x, y) \in \overline{\Omega},$$

where $m \in \mathbb{N}$ and $2m + 2$ is the multiplicity of λ^* as a root of $k(\lambda) + c^*(e)\lambda = 0$.

Remark 1.4 In the critical case $c = c^*(e)$, the asymptotic behavior of $\phi(s, x, y) - p^-(x, y)$ is not purely exponential, but it is a power of $|s|$ times an exponential, like in the homogeneous case or in the case when the equation is invariant along the direction of propagation (see the comments in Section 1.3 below). In these cases, the multiplicity of λ^* is equal to 2 (that is $m = 0$), as it is if $\Omega = \mathbb{R}^N$ (see [41]).

The next result provides a logarithmic equivalent of $\phi(s, x, y) - p^-(x, y)$ as $s \to -\infty$ with or without the KPP assumption (1.8), but with an additional condition in the case $c > c^*(e)$.

Theorem 1.5 *Let $u(t,x,y)=\phi(ct-x\cdot e,x,y)$ be a pulsating front in the sense of Definition* 1.1*, and assume that condition* (1.4) *is satisfied.*

a) *If* (1.6) *and* (1.7) *hold and if there exists a pulsating front*

$$u'(t,x,y)=\phi'(c't-x\cdot e,x,y)$$

in the sense of Definition 1.1 *with a speed $c'<c$, then $c>c^*(e)$ and*

$$\ln(\phi(s,x,y)-p^-(x,y))\sim\lambda_c s \text{ as } s\to-\infty \text{ uniformly in } (x,y)\in\overline{\Omega}. \tag{1.22}$$

b) *If $c=c^*(e)$, then*

$$\ln(\phi(s,x,y)-p^-(x,y))\sim\lambda^* s \text{ as } s\to-\infty \text{ uniformly in } (x,y)\in\overline{\Omega}, \tag{1.23}$$

where $\lambda^*>0$ is still defined as in Theorem 1.3.

The existence of pulsating travelling fronts is known in some cases which are covered by the general assumptions of the above theorems. For instance, if

$$\begin{cases} p^-=0, \quad p^+=1, \quad f(x,y,u)>0 \text{ for all } (x,y)\in\overline{\Omega} \text{ and } u\in(0,1),\\ f(x,y,u) \text{ is nonincreasing with respect to } u \text{ in a left neighbourhood of } 1,\\ \nabla\cdot q=0 \text{ in } \overline{\Omega},\ q\cdot\nu=0 \text{ on } \partial\Omega \text{ and } \displaystyle\int_C q_i(x,y)\,dx\,dy=0 \text{ for } 1\le i\le d, \end{cases} \tag{1.24}$$

if the KPP assumption (1.8) is satisfied and if f is of class $C^{1,\beta}(\overline{\Omega}\times[0,1])$, then, given any unit vector $e\in\mathbb{R}^d\times\{0\}^{N-d}$, pulsating travelling fronts exist if and only if $c\ge c^*(e)$, where the minimal speed $c^*(e)$, as given in (1.19), is positive, see [5] (actually, this existence result has been proved under additional smoothness assumptions on the coefficients of (1.1)). Furthermore, the infimum in (1.19) is reached (see [2, 5]). Notice that conditions (1.9), and then (1.6), are satisfied with $p^+=1$. In this case, condition (1.8) reduces to

$$f(x,y,u)\le\zeta(x,y)u \ \text{ for all } (x,y)\in\overline{\Omega} \text{ and } u\in[0,1], \tag{1.25}$$

where $\zeta(x,y)=\frac{\partial f}{\partial u}(x,y,0)$, and (1.24) yields $\zeta(x,y)>0$ for all $(x,y)\in\overline{\Omega}$, whence (1.4). However, even under assumptions (1.24) and (1.25), the exact behavior of each front when it approaches its limiting unstable state (here, 0) was not known, and as it will be seen in [22], Theorem 1.3 will then provide the complete classification of all these pulsating fronts as well as several stability results. We refer to [1, 4, 5, 14, 15, 21, 25, 40, 43, 47, 53] for further existence results or applications of formulas of the type (1.19) about the dependence of the minimal speeds on the domain or on the advection, reaction, diffusion coefficients.

For nonlinearities f satisfying (1.8) and (1.24), the derivative $\zeta(x,y)=\frac{\partial f}{\partial u}(x,y,0)$ is positive everywhere, and the principal eigenvalue μ_0 given in (1.5) is necessarily negative, that is (1.4) is fulfilled. However, if ζ is not everywhere positive, μ_0 may not be negative in general. In [7], nonlinearities $f=f(x,s)$ (for $x\in\Omega=\mathbb{R}^N$) satisfying

$$\begin{cases} f(x,0)=0, \quad u\mapsto \dfrac{f(x,u)}{u} \text{ is decreasing in } u>0,\\ \exists\ M>0,\ \forall\ x\in\mathbb{R}^N,\ \forall\ u\ge M, \quad f(x,u)\le 0 \end{cases} \tag{1.26}$$

were considered, with no advection ($q = 0$). Observe that (1.26) yields in particular (1.8) and (1.11) with $p^- \equiv 0$, whence (1.6). Typical examples of such nonlinearities $f(x,u)$ are

$$f(x,u) = u(\zeta(x) - \eta(x)u),$$

where η is a periodic function such that $0 < \eta_1 \leq \eta(x) \leq \eta_2 < +\infty$ in $\mathbb{R}^N$ (see [45] for biological invasions models). Under the assumptions (1.26), the existence (and uniqueness) of a positive periodic steady state p^+ is equivalent to the condition $\mu_0 < 0$, that is (1.4) (see [6]). With the condition $\mu_0 < 0$, the existence of pulsating fronts in any direction e was proved in [7] for all (and only all) speeds $c \geq c^*(e)$, where $c^*(e)$ is still given by (1.19) (see also [26] for partial results in the one-dimensional case). However, the exponential behavior of these pulsating fronts when they approach 0 was still an open problem, even in dimension 1. The present paper gives a positive answer to this issue, in a more general setting.

Remark 1.6 In [5], assumption (1.24) on the positivity of the function f played a crucial role in the existence of pulsating fronts. In [7], assumptions (1.4) and (1.26) were essential. Notice that, in general, the existence of pulsating fronts cannot be guaranteed under assumptions (1.4), (1.6), (1.7) and (1.8). For instance, even in the homogeneous one-dimensional case $u_t - u_{xx} = f(u)$ with $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, $f(s) \leq f'(0)s$ in $[0,1]$ and f is of class $C^{1,\beta}$ in a neighbourhood of 0 (assumptions (1.4), (1.6), (1.7) and (1.8) are satisfied with $p^- = 0$ and $p^+ = 1$), there exist no fronts $u(t,x) = \phi(ct - x)$ such that $\phi(-\infty) = 0 \leq \phi \leq \phi(+\infty) = 1$ as soon as $\int_0^1 f \leq 0$ (since the speed c of any such front would have to be both positive because of the limit $\phi(-\infty) = 0$, and nonpositive because of the sign of the integral of f).

Theorem 1.3 deals with KPP case (1.8), while Theorem 1.5 is concerned with the "general monostable case". This terminology means that the fronts $u = \phi(ct - x\cdot e, x, y)$ connect two stationary states p^- and p^+, the first one being unstable and the second one being weakly stable (but it does not mean a priori that there is no other stationary state p between p^- and p^+). In the general monostable case, that is Theorem 1.5, it is worth to notice that the only knowledge of the existence of a pulsating front with a speed c' smaller than c is enough to force the exponential decay rate of the pulsating front having speed c. Actually the existence of a pulsating front with a speed $c' < c$ is a reasonable assumption. For instance, under assumptions (1.24) with $\frac{\partial f}{\partial u}(x,y,0) > 0$, even without the KPP assumption (1.25), pulsating fronts $u(t,x,y) = \phi(ct - x\cdot e, x, y)$ exist if and only if

$$c \geq c^{**}(e),$$

where the minimal speed $c^{**}(e)$ is such that $c^{**}(e) \geq c^*(e)$ and $c^*(e)$ is given in (1.19), see [2, 3]. Thus, for each pulsating front $u(t,x,y) = \phi(ct - x\cdot e, x, y)$ with $c > c^{**}(e)$, the existence of a front with a speed c' less than c is guaranteed and Theorem 1.5 provides the logarithmic decay of $\phi(s,x,y)$ as $s \to -\infty$. The existence of pulsating fronts is also known (see [44]) for the one-dimensional Allen-Cahn equation $u_t = u_{xx} + f(u)$, when f is of the bistable type between, say, -1 and 1, and the fronts connect an unstable periodic solution to the stable state 1. Any positive speed is admissible. The results of the present paper provide the exponential decay of all these fronts.

Remark 1.7 In our general framework, under the assumptions of Theorem 1.5, formulas (1.22) and (1.23) are weaker than the exponential behaviors in Theorem 1.3 with the KPP assumption (1.8). However, we conjecture that, under the assumptions of part a) of Theorem 1.5, formula (1.21) holds. So far, Theorem 1.5 provides at least the exponential decay rate in the general monostable case, which is the slowest one and the same as in the KPP case. Actually, it follows from Propositions 3.4 and 4.3 below that, under the assumptions of part a), we can be a bit more precise than (1.22), in the sense that

$$\limsup_{s\to-\infty}\Big(\max_{(x,y)\in\overline{\Omega}}\frac{\phi(s,x,y)}{e^{\lambda_c s}}\Big)<+\infty,\ \text{ and }\ \liminf_{s\to-\infty}\Big(\min_{(x,y)\in\overline{\Omega}}\frac{\phi(s,x,y)}{e^{(\lambda_c+\eta)s}}\Big)=+\infty\ \text{for each }\eta>0.$$

The only case which is not covered by our results is the general monostable case without the KPP assumption (1.8) and when the speed of the front is minimal and larger than $c^*(e)$. In this case, the fronts are pushed by their main part, instead of being pulled by their exponential tail and they are expected to have an exponential decay rate larger than λ_c (see the comments in Section 1.3).

1.3 Further comments and extensions

In this section, we relate our qualitative results to some earlier ones, starting from the simplest case of planar fronts in homogeneous media. Then, we state similar results which can be obtained with the same methods as in the present paper.

Link with some well-known results. For the homogeneous Fisher-KPP equation

$$u_t=\Delta u+f(u)\ \text{in }\mathbb{R}^N \tag{1.27}$$

with $f(0)=f(1)=0$ and $0<f(s)\le f'(0)s$ in $(0,1)$, there are planar travelling fronts

$$0<u(t,x)=\phi(ct-x\cdot e)<1$$

between $p^-=0$ and $p^+=1$ if and only if $c\ge 2\sqrt{f'(0)}$, for each unit vector e. Planar fronts $\phi(ct-x\cdot e)$ propagate with constant speed c in the direction e, their level sets are parallel hyperplanes and their shape is invariant in their moving frame. Notice that the value $c^*=c^*(e)=2\sqrt{f'(0)}$ is a particular case of formula (1.19). Actually, in this case,

$$k(\lambda)=-(\lambda^2+f'(0)),\quad \psi_\lambda=1,\quad \lambda_c=\frac{c-\sqrt{c^2-4f'(0)}}{2}\ \text{for each }c>c^*,$$

while $\lambda^*=\sqrt{f'(0)}$ and the multiplicity $2m+2$ of λ^* as a root of $k(\lambda)+2\sqrt{f'(0)}\lambda=0$ is equal to 2 (that is, $m=0$). It is immediate to see, with a phase plane analysis, that for each $c\ge c^*$, the function ϕ has to be increasing and that, if $c>2\sqrt{f'(0)}$, then $\phi(s)\sim Be^{\lambda_c s}$ as $s\to-\infty$ and, if $c=2\sqrt{f'(0)}$, then $\phi(s)\sim B|s|e^{\lambda^* s}$ as $s\to-\infty$, for some $B>0$. These behaviors can be viewed as particular cases of Theorem 1.3.

If f is simply assumed to be positive in $(0,1)$, without the KPP assumption $f(s)\le f'(0)s$, then the set of speeds of planar fronts is still of the type $[c^{**},+\infty)$, where $c^{**}\ge 2\sqrt{f'(0)}$ (this is a particular case of the result of [2, 3] which had been recalled above). Actually, if $f'(0) > 0$ and $c > c^{**}$, then $\phi(s) \sim Be^{\lambda_c s}$ as $s \to -\infty$ for some $B > 0$, which is stronger than our formula (1.22) in Theorem 1.5. But, at least, formula (1.22) provides the right logarithmic equivalent as $s \to -\infty$, with the slowest decay rate λ_c, in the general periodic monostable framework. Notice that the assumption $f'(0) > 0$ corresponds to (1.4). Similarly, if $f'(0) > 0$ and $c = c^{**} = 2\sqrt{f'(0)}$, then $\phi(s) \sim (B|s| + B')e^{\lambda^* s}$ as $s \to -\infty$ with either $B > 0$, or $B = 0$ and $B' > 0$; in this case, formula (1.23) still holds. If $c = c^{**} > 2\sqrt{f'(0)}$, then $\phi(s) \sim Be^{\lambda_+^{**} s}$ as $s \to -\infty$ with $B > 0$, where $\lambda_+^{**} = (c^{**} + \sqrt{(c^{**})^2 - 4f'(0)})/2 > \lambda_{c^{**}}$, that is ϕ decays in this case with the fastest rate. In the general periodic monostable framework, we conjecture that the same property holds for the pulsating fronts such that $c > c^*(e)$ and when there is no front with a speed smaller than c.

Equations of the type

$$u_t - \Delta u + \alpha(y)\,\frac{\partial u}{\partial x} = f(y,u),\ (x,y) \in \Omega = \mathbb{R} \times \omega, \quad \nu \cdot \nabla u = 0,\ (x,y) \in \partial\Omega \tag{1.28}$$

in straight infinite cylinders with smooth bounded sections ω and with underlying shear flows $q = (\alpha(y), 0, \dots, 0)$ have also been investigated in the past fifteen years. With KPP or monostable nonlinearities having two stationary states $p^-(y) < p^+(y)$ and none between p^- and p^+, we refer to [10] for existence results of travelling fronts

$$u(t,x,y) = \phi(ct - x, y)$$

for all speeds $c \geq c^{**}$, and their exponential behavior when $\phi \simeq p^-$, as in the homogeneous case above. Here, for each $\lambda \in \mathbb{R}$, $k(\lambda)$ is the principal eigenvalue of the operator

$$\psi = \psi(y) \mapsto -\Delta_y \psi - \left[\lambda\ \alpha(y) + \lambda^2 + \frac{\partial f}{\partial u}(y, p^-(y))\right]\psi \ \text{ in } \omega$$

with Neumann boundary conditions on $\partial\omega$. Therefore, $k(\lambda) + \lambda^2$ is concave, whence the multiplicity of λ^* as a root of $k(\lambda^*) + c^*\lambda^* = 0$ is equal to 2, where $c^* = \min_{\lambda>0}(-k(\lambda)/\lambda) = -k(\lambda^*)/\lambda^*$ is the minimal speed in the KPP case. Notice also that the equations (1.28) are invariant along the x-direction, the profiles of fronts $\phi(ct - x, y)$ are invariant in their moving frame and the equation for ϕ reduces to an elliptic problem. We refer to [10, 32, 42] for further uniqueness and stability results of the travelling fronts for problem (1.28).

Therefore, the results of the present paper generalize those which were known in the classical cases (1.27) or (1.28). In the periodic framework, under assumptions (1.24) or (1.26), they also answer some questions which had been left open so far. The results are stated here in a more general setting than (1.24) or (1.26). In particular, the nonlinearity f is not assumed to be nonnegative or to satisfy monotonicity properties. This general setting leads to additional difficulties and much more technicality.

Behavior around a stable limiting state and other types of nonlinearities. Theorems 1.3 or 1.5 were concerned with the exponential decay of $\phi(s,x,y)$ as $s \to -\infty$, that is when ϕ is close to the limiting state p^-. The unstability of p^-, namely condition (1.4), makes the analysis difficult.

The behavior of ϕ as $s \to +\infty$, that is when ϕ approches its weakly stable state limiting p^+ can also be asked. As a matter of fact, similar results concerning the exponential behavior of $\phi(s,x,y) - p^+(x,y)$ when $s \to +\infty$ can be proved, under the additional assumption

$$\mu^+ > 0,$$

where μ^+ is the principal eigenvalue of the linearized operator (1.10) around the limiting state p^+. Namely, if $\mu^+ > 0$ and if the function $(x,y,s) \mapsto \frac{\partial f}{\partial u}(x,y,p^+(x,y)-s)$ is assumed to be of class $C^{0,\beta^+}(\overline{\Omega} \times [0,\gamma^+])$ for some $\beta^+ > 0$ and $\gamma^+ > 0$, then, for any pulsating front $u(t,x) = \phi(ct - x \cdot e, x, y)$ in the sense of Definition 1.1,

$$\phi(s,x,y) - p^+(x,y) \sim -B^+ \ e^{-\lambda^+ s} \ \psi^+_{\lambda^+}(x,y) \ \text{ as } s \to +\infty, \tag{1.29}$$

uniformly in $\overline{\Omega}$, where $B^+ > 0$, $\lambda^+ > 0$ and $\psi^+_{\lambda^+}$ is periodic and positive in $\overline{\Omega}$. Here, λ^+ is the *unique* positive solution of $k^+(\lambda^+) = \lambda^+ c$, where $k^+(\lambda)$ denotes the principal eigenvalue of the operator

$$-\nabla \cdot (A\nabla\psi) - 2\lambda e A \nabla\psi + q \cdot \nabla\psi + [-\lambda\nabla \cdot (Ae) + \lambda q \cdot e - \lambda^2 eAe - \zeta^+]\psi$$

with periodicity in $\overline{\Omega}$ and boundary conditions $\nu A\nabla\psi + \lambda(\nu Ae)\psi = 0$ on $\partial\Omega$, and $\psi^+_{\lambda^+}$ denotes the principal eigenfunction with $\lambda = \lambda^+$, with normalization $\|\psi^+_{\lambda^+}\|_{L^\infty(C)} = 1$. The function $\zeta^+(x,y)$ denotes $\frac{\partial f}{\partial u}(x,y,p^+(x,y))$. The existence and uniqueness of a positive real number λ^+ satisfying $k^+(\lambda^+) = \lambda^+ c$ can be shown as in Section 2, the uniqueness uses the concavity of k^+ and the fact that

$$k^+(0) = \mu^+ > 0.$$

Furthermore, the multiplicity of λ^+ as a root of $k^+(\lambda) - \lambda c = 0$ is then always equal to 1. The proof of (1.29) would actually be much easier than those of Theorems 1.3 or 1.5 (and holds with or without any KPP assumption with respect to p^- or p^+, and whether c is critical or not) since λ^+ has multiplicity one and since comparison principles in domains of the type $[h,+\infty) \times \overline{\Omega}$ in the (s,x,y) variables can be applied, because of the stability assumption $\mu^+ > 0$.

Remember that the condition $\mu^+ > 0$ implies the weak stability property (1.6). Now, when p^+ is only assumed to be weakly stable in the sense of (1.6) (this may include the degenerate case $\mu^+ = 0$), then the exponential behavior does not hold in general and $\phi(s,x,y)$ may converge to $p^+(x,y)$ as a negative power of s as $s \to +\infty$, or even more slowly, according to the behavior of $f(x,y,u)$ when $u \simeq p^+(x,y)$. Actually, when the exponential behavior of $\phi(s,x,y) - p^-(x,y)$ is known, as in Theorem 1.3, the weak stability condition (1.6) is enough to guarantee the uniqueness of the functions ϕ up to shifts in s (see the forthcoming paper [22]), and one does not need to know the exact behavior of $\phi(s,x,y) - p^+(x,y)$ as $s \to +\infty$. Similarly, the unstability assumption (1.4) and the weak stability assumption (1.6) are enough to guarantee the strict monotonicity of $\phi(s,x,y)$ with respect to s. However, if the unstability condition (1.4) is replaced by a degenerate one (for instance, if $f = f(u)$, p^- is constant, f is positive in $(p^-, p^-+\delta]$ for some $\delta > 0$ and if $f'(p^-) = 0$), then the monotonicity of ϕ with respect to s as well as the behavior of $\phi(s,x,y) - p^-(x,y)$ as $s \to -\infty$ are unclear.

On the other hand, if both p^- and p^+ are weakly stable (that is when (1.6) is satisfied and when the unstability assumption (1.4) of the state p^- is replaced by a similar assumption as (1.6)), then comparisons principles as in Section 2 below can be stated in domains of the type $(-\infty, h] \times \overline{\Omega}$ or $[h, +\infty) \times \overline{\Omega}$ in the (s, x, y) variables, and sliding methods similar to [2] and [3] imply that the functions ϕ are increasing in s, unique up to shifts in s, and that the speed c, if any, is necessarily unique. For instance, for bistable or combustion-type nonlinearities, the speed c of usual or pulsating travelling fronts is unique and the function ϕ is then unique up to shifts. We refer to [2, 3, 8, 10, 11, 12, 16, 17, 23, 24, 31, 33, 36, 42, 46, 49, 50, 51, 52] for precise definitions and for some existence, uniqueness and further qualitative results with combustion or bistable nonlinearities, from the homogeneous to the periodic framework.

The case of time-periodic media. Finally, we mention that similar results can be established for pulsating fronts in time-periodic media with the same type of methods as in this paper. Some exponential decay results in space-time periodic media could also be derived (we refer to [35, 37] for existence results and speed estimates), but we concentrate here for simplicity on time-periodic environments. Namely, consider reaction-diffusion-advection equations of the type

$$\begin{cases} u_t - \nabla\cdot(A(t,y)\nabla u) + q(t,y)\cdot\nabla u = f(t,y,u) \text{ in } \overline{\Omega}, \\ \nu A\nabla u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.30}$$

in a smooth unbounded domain $\Omega = \{(x,y) \in \mathbb{R}^d \times \omega\}$, where ω is a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^{N-d}$. The uniformly elliptic symmetric matrix field $A(t,y) = (A_{ij}(t,y))_{1\le i,j\le N}$ is of class $C^{1,\alpha/2;1,\alpha}_{t;y}(\mathbb{R}\times\overline{\omega})$, the vector field $q(t,y) = (q_i(t,y))_{1\le i\le N}$ is of class $C^{0,\alpha/2;1,\alpha}_{t;y}(\mathbb{R}\times\overline{\omega})$ and the nonlinearity (t,y,u) $(\in \mathbb{R}\times\overline{\omega}\times\mathbb{R}) \mapsto f(t,y,u)$ is continuous, of class $C^{0,\alpha/2;0,\alpha}$ with respect to (t,y) locally uniformly in $u \in \mathbb{R}$ and we assume that $\frac{\partial f}{\partial u}$ exists and is continuous in $\mathbb{R}\times\overline{\omega}\times\mathbb{R}$. All functions A_{ij}, q_i and $f(\cdot,\cdot,u)$ (for all $u \in \mathbb{R}$) are assumed to be time-periodic, in the sense that they satisfy $w(t+T,y) = w(t,y)$ for all $(t,y) \in \mathbb{R}\times\overline{\omega}$, where $T > 0$ is fixed. We are given two time-periodic classical solutions $p^\pm$ of (1.30) satisfying

$$p^-(t,y) < p^+(t,y) \text{ for all } (t,y) \in \mathbb{R}\times\overline{\omega}.$$

Assume that $\zeta(t,y) = \frac{\partial f}{\partial u}(t,y,p^-(t,y))$ is of class $C^{0,\alpha/2;0,\alpha}_{t;y}(\mathbb{R}\times\overline{\omega})$ and that

$$\mu_0 < 0, \tag{1.31}$$

where μ_0 denotes the principal eigenvalue of the linearized operator around p^-

$$\psi(t,y) \mapsto \psi_t - \nabla\cdot(A(t,y)\nabla\psi) + q(t,y)\cdot\nabla\psi - \zeta(t,y)\ \psi$$

with time-periodicity conditions in $\mathbb{R}\times\overline{\omega}$ and Neumann boundary condition $\nu A\nabla\psi = 0$ on $\mathbb{R}\times\partial\omega$ (with a slight abuse of notations, $\nabla\psi$ denotes $(0,\ldots,0,\nabla_y\psi) \in \{0\}^d\times\mathbb{R}^{N-d}$). For some results, we shall assume that there is ρ such that $0 < \rho < \min_{\mathbb{R}\times\overline{\omega}}\,(p^+ - p^-)$ and, for any classical bounded supersolution $\overline{u}$ of (1.30) satisfying $\overline{u} < p^+$ and $\Omega_{\overline{u}} = \{\overline{u}(t,x,y) > p^+(t,y) - \rho\} \neq \emptyset$,

$$\text{there exists a family of functions } (\rho_\tau)_{\tau\in[0,1]} \text{ defined in } \overline{\Omega_{\overline{u}}} \text{ and satisfying (1.6)} \tag{1.32}$$

with $\Omega_{\overline{u},\tau} = \{(t,x,y) \in \Omega_{\overline{u}},\ \overline{u}(t,x,y) + \rho_\tau(t,x,y) < p^+(t,y)\}$. We shall also assume that there are $\beta > 0$ and $\gamma > 0$ such that the map

$$(t,y,s) \mapsto \frac{\partial f}{\partial u}(t,y,p^-(t,y)+s) \text{ is of class } C^{0,\beta}(\mathbb{R}\times\overline{\omega}\times[0,\gamma]), \tag{1.33}$$

and that, for all $(t,y) \in \mathbb{R}\times\overline{\omega}$ and $s \in [0, p^+(t,y) - p^-(t,y)]$,

$$f(t,y,p^-(t,y)+s) \ \leq\ f(t,y,p^-(t,y)) \ +\ \zeta(t,y)\, s. \tag{1.34}$$

Given a unit vector $e \in \mathbb{R}^d \times \{0\}^{N-d}$, a pulsating front connecting p^- and p^+, travelling in the direction e with mean speed $c \in \mathbb{R}^*$, is a classical solution $u(t,x,y)$ of (1.30) such that

$$\begin{cases} u(t,x,y) = \phi(ct - x\cdot e, t, y) \ \text{ for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^d\times\overline{\omega}, \\ \phi(s,t+T,y) = \phi(s,t,y) \text{ for all } (s,t,y) \in \mathbb{R}^2\times\overline{\omega}, \\ \phi(s,t,y) \underset{s\to\pm\infty}{\longrightarrow} p^\pm(t,y) \text{ uniformly in } (t,y)\in\mathbb{R}\times\overline{\omega}, \\ p^-(t,y) < \phi(s,t,y) < p^+(t,y) \text{ for all } (s,t,y) \in \mathbb{R}^2\times\overline{\omega}. \end{cases} \tag{1.35}$$

We refer to [19, 38, 39] for existence results and speed estimates of pulsating fronts for equations of the type (1.30) with time-periodic KPP nonlinearities and shear flows.

For each $\lambda \in \mathbb{R}$, still call $k(\lambda)$ the principal eigenvalue of the operator $\psi \mapsto \psi_t - \nabla\cdot(A\nabla\psi) + 2\lambda eA\nabla\psi + q\cdot\nabla\psi + [\lambda\nabla\cdot(Ae) - \lambda q\cdot e - \lambda^2 eAe - \zeta(t,y)]\psi$ with time-periodicity conditions in $\mathbb{R}\times\overline{\omega}$ and boundary conditions $\nu A\nabla\psi = \lambda(\nu Ae)\psi$ on $\mathbb{R}\times\partial\omega$, and denote by ψ_λ the unique positive principal eigenfunction such that $\|\psi_\lambda\|_{L^\infty(\mathbb{R}\times\omega)} = 1$. Define $c^*(e)$ as in (1.19) and for each $c > c^*(e)$, define $\lambda_c > 0$ as in (1.20). These quantities are well-defined real numbers.

We list below some qualitative results which can be obtained by adapting the methods of the present paper. In the sequel, $u(t,x,y) = \phi(ct - x\cdot e, x, y)$ denotes a pulsating travelling front in the sense of (1.35).

- Under assumption (1.31), then $c \geq c^*(e)$. Under assumptions (1.31) and (1.32), then $\phi_s(s,t,y) > 0$ in $\mathbb{R}^2\times\overline{\omega}$, that is $-e\cdot\nabla u(t,x,y) > 0$ in $\mathbb{R}\times\overline{\Omega}$.

- Under assumptions (1.31), (1.33) and the KPP assumption (1.34) : if $c > c^*(e)$, then there exists $B > 0$ such that

  $$\phi(s,t,y) - p^-(t,y) \sim Be^{\lambda_c s}\psi_{\lambda_c}(t,y) \text{ as } s\to-\infty \text{ uniformly in } (t,y)\in\mathbb{R}\times\overline{\omega},$$

  while if $c = c^*(e)$, then there is a unique $\lambda^* > 0$ such that $k(\lambda^*) + c^*(e)\lambda^* = 0$ and there exists $B > 0$ such that

  $$\phi(s,t,y) - p^-(t,y) \sim B|s|^{2m+1}e^{\lambda^* s}\psi_{\lambda^*}(t,y) \text{ as } s\to-\infty \text{ uniformly in } (t,y)\in\mathbb{R}\times\overline{\omega},$$

  where $m \in \mathbb{N}$ and $2m+2$ is the multiplicity of λ^* as a root of $k(\lambda) + c^*(e)\lambda = 0$.

- Under assumptions (1.31), (1.32) and (1.33), if there exists a pulsating front $u'(t,x,y) = \phi'(c't - x\cdot e, t, y)$ in the sense of (1.35) with a speed $c' < c$, then $c > c^*(e)$ and $\ln(\phi(s,t,y) - p^-(t,y)) \sim \lambda_c s$ as $s \to -\infty$ uniformly in $(t,y) \in \mathbb{R} \times \overline{\omega}$.

- Under assumption (1.31), if $c = c^*(e)$, then $\ln(\phi(s,t,y) - p^-(t,y)) \sim \lambda^* s$ as $s \to -\infty$ uniformly in $(t,y) \in \mathbb{R} \times \overline{\omega}$.

The same results can also be stated when the boundedness in y is replaced by a periodicity in y, or a mixture of periodicity and boundedness as in (1.2) (in the variable y only). These results also lead to the uniqueness for a given speed up to shifts in s in the KPP case (1.34), as well as to stability and spreading speeds estimates, as in [22].

Outline of the paper. The plan of the paper is as follows : in Section 2, we prove various qualitative properties which are satisfied by the pulsating travelling fronts, including the monotonicity in time. Sections 3 and 4 are devoted to establishing exponential lower and upper bounds, which provide in Section 5 the proofs of the main Theorems 1.3 and 1.5 on the exponential behavior when $u \simeq p^-$.

2 Monotonicity and other qualitative estimates

In this section, we establish some useful qualitative properties which are satisfied by the pulsating travelling fronts solving (1.1). In particular, we prove here the monotonicity results. Actually, we do not need the KPP assumption (1.8) or the regularity assumption (1.7). Throughout this section, we are given a unit vector $e \in \mathbb{R}^d \times \{0\}^{N-d}$ and we denote by

$$u(t,x,y) = \phi(ct - x\cdot e, x, y)$$

a pulsating travelling front with speed $c \in \mathbb{R}^*$, in the sense of Definition 1.1. We first show that we can always assume that $p^- = 0$ and $p^+ = 1$ without loss of generality. We then prove some rough estimates and the monotonicity with respect to the variable $ct - x\cdot e$.

2.1 Some preliminaries

Notice first that if we write

$$\tilde{u}(t,x,y) = \frac{u(t,x,y) - p^-(x,y)}{p^+(x,y) - p^-(x,y)}, \quad \tilde{\phi}(s,x,y) = \frac{\phi(s,x,y) - p^-(x,y)}{p^+(x,y) - p^-(x,y)}$$

and

$$\begin{cases} \tilde{q} = q - \dfrac{2A\nabla(p^+ - p^-)}{p^+ - p^-}, \\ \tilde{f}(x,y,v) = \dfrac{f(x,y,(p^+ - p^-)v + p^-) - f(x,y,p^-) + [f(x,y,p^-) - f(x,y,p^+)]\ v}{p^+ - p^-}, \end{cases}$$

where $p^\pm = p^\pm(x,y)$, then $\tilde u$ satisfies

$$\begin{cases} \tilde u_t - \nabla\cdot(A(x,y)\nabla\tilde u) + \tilde q(x,y)\cdot\nabla\tilde u = \tilde f(x,y,\tilde u) \ \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu A\nabla\tilde u = 0 \ \text{ on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{2.1}$$

and $\tilde u$ is a pulsating travelling front in the sense of Definition 1.1, with $(\tilde q, \tilde f)$ instead of (q,f), and

$$\tilde{p}^- = 0 \ \text{ and } \ \tilde{p}^+ = 1$$

instead of p^- and p^+ respectively. The constant functions $\tilde p^- < \tilde p^+$ solve $\tilde f(x,y,\tilde p^\pm) = 0$ in $\overline{\Omega}$ and are then classical periodic solutions of the stationary equation which is associated to (2.1). Furthermore, the vector field $\tilde q$ and the nonlinearity $\tilde f$ satisfy the same regularity assumptions as q and f, and if properties (1.7) and (1.8) hold with f and $p^\pm$, then they immediately hold with $\tilde f$ and $\tilde p^\pm$.

Observe also that

$$\tilde\zeta(x,y) = \frac{\partial\tilde f}{\partial v}(x,y,\tilde p^-) = \frac{\partial f}{\partial u}(x,y,p^-) + \frac{f(x,y,p^-) - f(x,y,p^+)}{p^+ - p^-}$$

and if φ is a principal eigenvalue of the operator L_0 with principal eigenvalue μ_0, then the function $\tilde\varphi = \varphi/(p^+ - p^-)$ is periodic, positive and satisfies

$$-\nabla\cdot(A\nabla\tilde\varphi) + \tilde q\cdot\nabla\tilde\varphi - \tilde\zeta\tilde\varphi = \mu_0\tilde\varphi$$

with Neumann boundary conditions $\nu A\nabla\tilde\varphi = 0$ on $\partial\Omega$, which means that μ_0 is the principal eigenvalue of the linearized equation around $\tilde p^- = 0$. In particular, if condition (1.4) holds with equation (1.1) and the stationary state p^-, then it holds with equation (2.1) and $\tilde p^-$.

Similarly, if (1.6) holds with $\rho \in (0, \min_{\overline{\Omega}}(p^+ - p^-))$, then if we define

$$\tilde\rho = \frac{\rho}{\max_{\overline{\Omega}}(p^+ - p^-)} \in (0,1)$$

and if $\overline{v}$ is a bounded classical supersolution of (2.1) such that

$$\overline{v} < 1 \ \text{ and } \ \widetilde{\Omega}_{\overline{v}} = \{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ \overline{v}(t,x,y) > 1-\tilde\rho\} \neq \emptyset,$$

then $\overline{u} := (p^+ - p^-)\,\overline{v} + p^-$ is a classical bounded supersolution of (1.1) such that $\overline{u} < p^+$ and $\Omega_{\overline{u}} \supset \widetilde{\Omega}_{\overline{v}}$, whence $\Omega_{\overline{u}} \neq \emptyset$. Let $(\rho_\tau)_{\tau\in[0,1]}$ be the family of functions associated to $\overline{u}$ and satisfying (1.6). Define $\tilde\rho_\tau = \rho_\tau/(p^+ - p^-)$ for each $\tau\in[0,1]$. It is then straightforward to check that property (1.6) then holds with $\tilde\rho_\tau$, $\overline{v}$, $\tilde q$, $\tilde f$, $\tilde p^+ = 1$ and $\widetilde{\Omega}_{\overline{v},\tau} = \widetilde{\Omega}_{\overline{v}} \cap \{\overline{v} + \tilde\rho_\tau < 1\}$ instead of ρ_τ, $\overline{u}$, q, f, p^+ and $\Omega_{\overline{u},\tau}$ respectively.

Moreover, for each $\lambda\in\mathbb{R}$, denote ψ_λ the principal eigenfunction of the operator L_λ with principal eigenvalue $k(\lambda)$, boundary conditions $\nu A\nabla\psi_\lambda = \lambda(\nu Ae)\psi_\lambda$ on $\partial\Omega$ and normalization condition (1.18). Then set

$$\tilde\psi_\lambda(x,y) = \alpha_\lambda \times \frac{\psi_\lambda(x,y)}{p^+(x,y) - p^-(x,y)},$$

where the constant $\alpha_\lambda > 0$ is such that $\|\tilde{\psi}_\lambda\|_{L^\infty(C)} = 1$. The function $\tilde{\psi}_\lambda$ is periodic in (x, y), positive, and it satisfies

$$\tilde{L}_\lambda \tilde{\psi}_\lambda := -\nabla \cdot (A \nabla \tilde{\psi}_\lambda) + 2\lambda e A \nabla \tilde{\psi}_\lambda + \tilde{q} \cdot \nabla \tilde{\psi}_\lambda + \left[\lambda \nabla \cdot (Ae) - \lambda \tilde{q} \cdot e - \lambda^2 eAe - \tilde{\zeta}\right] \tilde{\psi}_\lambda = k(\lambda) \tilde{\psi}_\lambda$$

with boundary conditions $\nu A \nabla \tilde{\psi}_\lambda = \lambda(\nu Ae)\tilde{\psi}_\lambda$ on $\partial\Omega$. In other words, $\tilde{\psi}_\lambda$ is the principal eigenfunction of $\tilde{L}_\lambda$ with principal eigenvalue $k(\lambda)$ and the same normalization condition (1.18) as ψ_λ. In particular, the quantities $c^*(e)$, λ_c and λ^* introduced in Section 1 are unchanged when problem (1.1) is replaced by (2.1), and for instance, formula (1.21) is equivalent to

$$\tilde{\phi}(s, x, y) \sim \tilde{B} e^{\lambda_c s} \tilde{\psi}_{\lambda_c}(x, y) \text{ as } s \to -\infty \text{ uniformly in } (x, y) \in \overline{\Omega},$$

where $\tilde{B} = B/\alpha_{\lambda_c} > 0$.

Lastly, notice that the monotonicity of ϕ or u with respect to s and t, and the uniqueness of these functions up to shifts in these variables, are equivalent to the same properties for $\tilde{\phi}$ and $\tilde{u}$ with respect to s and t.

As a consequence, without loss of generality, we can assume in the sequel that

$$\forall\ (x, y) \in \overline{\Omega}, \quad p^-(x, y) = 0, \quad p^+(x, y) = 1$$

and all statements in Section 1, if they hold with $p^- = 0$ and $p^+ = 1$ can then be rewritten in the general case with functions $p^\pm(x, y)$. One can then assume that $f(x, y, 0) = f(x, y, 1) = 0$ in $\overline{\Omega}$. Assumption (1.8), if it holds, is rewritten as

$$\begin{cases} \forall\ (x, y) \in \overline{\Omega}, \quad \zeta(x, y) = \dfrac{\partial f}{\partial u}(x, y, 0), \\ \forall\ (x, y) \in \overline{\Omega},\ \forall\ u \in [0, 1], \quad f(x, y, u) \le \zeta(x, y)\ u. \end{cases} \tag{2.2}$$

We then gather a few properties of the function $\lambda \mapsto k(\lambda)$, where $k(\lambda)$ denotes the principal eigenvalue of the operator L_λ defined in (1.17).

Lemma 2.1 *The function k is analytic, concave in $\mathbb{R}$ and, under assumption* (1.4),

$$c^*(e) := \inf_{\lambda > 0} \left(-\frac{k(\lambda)}{\lambda}\right) \in \mathbb{R}.$$

Furthermore, for each $c > c^*(e)$, the positive real number

$$\lambda_c = \min\{\lambda > 0,\ k(\lambda) + c\lambda = 0\}$$

is well-defined and the set

$$F_c = \{\lambda \in (0, +\infty),\ k(\lambda) + c\lambda = 0\} \tag{2.3}$$

is either the singleton $\{\lambda_c\}$, or it is equal to $\{\lambda_c, \lambda_c^+\}$ with $\lambda_c < \lambda_c^+$. The set $F_{c^*(e)}$ is either empty or it is a singleton $\{\lambda^*\}$ and if it is a singleton $\{\lambda^*\}$, then the multiplicity of λ^* as a root of $k(\lambda) + c\lambda = 0$ is equal to $2m + 2$ with $m \in \mathbb{N}$.

Proof. The analyticity of k follows from the fact that the coefficients of the operators L_λ are analytic in λ, and the eigenvalues $k(\lambda)$ are isolated, see [13, 27]. The concavity of k follows from the arguments used in Lemma 3.1 of [7]. The fact that the advection q, here, may not be zero, does not change anything.

Now, for each $\lambda \in \mathbb{R}$, the principal eigenfunction ψ_λ is positive, periodic and it satisfies

$$L_\lambda\psi_\lambda = -\nabla\cdot(A\nabla\psi_\lambda) + 2\lambda eA\nabla\psi_\lambda + q\cdot\nabla\psi_\lambda + [\lambda\nabla\cdot(Ae) - \lambda q\cdot e - \lambda^2 eAe - \zeta]\psi_\lambda = k(\lambda)\psi_\lambda$$

in $\overline{\Omega}$ with $\nu A\nabla\psi_\lambda = \lambda(\nu Ae)\psi_\lambda$ on $\partial\Omega$. Divide this equation by ψ_λ and integrate over C. It follows that

$$\begin{aligned} k(\lambda)|C| &= -\int_C \frac{\nabla\psi_\lambda A\nabla\psi_\lambda}{\psi_\lambda^2} + 2\lambda\int_C eA\frac{\nabla\psi_\lambda}{\psi_\lambda} + \int_C q\cdot\frac{\nabla\psi_\lambda}{\psi_\lambda} - \lambda\int_C q\cdot e - \lambda^2\int_C eAe - \int_C\zeta \\ &= -\int_C \left(\frac{\nabla\psi_\lambda}{\psi_\lambda} - \lambda e - \frac{A^{-1}q}{2}\right) A\left(\frac{\nabla\psi_\lambda}{\psi_\lambda} - \lambda e - \frac{A^{-1}q}{2}\right) + \frac{1}{4}\int_C qA^{-1}q - \int_C\zeta \\ &\le \frac{1}{4}\int_C qA^{-1}q - \int_C\zeta, \end{aligned}$$

where $|C|$ denotes the Lebesgue measure of C. Since $-k(0) = -\mu_0 > 0$ from (1.4), one then concludes that $\lambda \mapsto -k(\lambda)/\lambda$ is bounded from below in $(0,+\infty)$. The quantity $c^*(e)$ is then a real number.

Furthermore, for each $c > c^*(e)$, there is $\underline{\lambda} > 0$ such that $-k(\underline{\lambda})/\underline{\lambda} = c$, since $\lambda \mapsto -k(\lambda)/\lambda$ is continuous in $(0,+\infty)$ and goes to $+\infty$ as $\lambda \to 0^+$. Consequently, the positive real number λ_c is well-defined. If there are two positive real numbers λ_1 and λ_2 such that $\lambda_c < \lambda_1 < \lambda_2$ and $k(\lambda_1) + c\lambda_1 = k(\lambda_2) + c\lambda_2 = 0$, then $k(\lambda) + c\lambda = 0$ for all $\lambda \in [\lambda_c, \lambda_2]$ by concavity of k, and finally $k(\lambda)+c\lambda = 0$ for all $\lambda \in \mathbb{R}$ by analyticity of k. This leads to a contradiction, since $k(0) = \mu_0 < 0$ by assumption. Therefore, the set F_c defined in (2.3) is either the singleton $\{\lambda_c\}$, or it is equal to $\{\lambda_c, \lambda_c^+\}$ with $\lambda_c < \lambda_c^+$.

When $c = c^*(e)$, if there is $\lambda^* > 0$ such that $k(\lambda^*)+c^*(e)\lambda^* = 0$ then, by definition of $c^*(e)$, there holds $k'(\lambda^*)\lambda^* - k(\lambda^*) = 0$, whence $k'(\lambda^*) = -c^*(e)$. If there is $0 < \lambda_1 \neq \lambda^*$ such that $k(\lambda_1)+c^*(e)\lambda_1 = 0$, then, by definition of $c^*(e)$ and by concavity of the function $k(\lambda)$, one gets that $k(\lambda) + c^*(e)\lambda = 0$ for all $\lambda \in [\min(\lambda^*,\lambda_1), \max(\lambda^*,\lambda_1)]$, which leads to a contradiction as above. Hence, the positive real number λ^*, if any, such that $k(\lambda^*)+c^*(e)\lambda^* = 0$ is unique. Again by analyticity of $k(\lambda)$ and since $-k(\lambda) \ge c^*(e)\lambda$ for all $\lambda > 0$, there exists then $m \in \mathbb{N}$ such that $2m+2$ is the multiplicity of λ^* as a root of $k(\lambda) + c^*(e)\lambda = 0$, in the sense that

$$\left\{\begin{array}{l} k(\lambda^*) + c^*(e)\lambda^* = 0,\ k'(\lambda^*) + c^*(e) = 0,\ k^{(j)}(\lambda^*) = 0 \text{ for all } 2 \le j \le 2m+1, \\ k^{(2m+2)}(\lambda^*) < 0, \end{array}\right. \tag{2.4}$$

where $k^{(j)}(\lambda)$ denotes the j-th order derivative of $k(\lambda)$ with respect to λ. That completes the proof of Lemma 2.1. □

2.2 Lower bound for the speed

We prove here that the speed c of a pulsating front $u(t,x,y) = \phi(ct - x\cdot e, x, y)$ is always bounded from below by the quantity $c^*(e)$. We recall that we can assume

$$p^- = 0 \ \text{ and } p^+ = 1$$

without loss of generality. In particular, the functions u and ϕ are then positive in $\mathbb{R}\times\overline{\Omega}$. The following proposition provides the proof of the first part of Proposition 1.2.

Proposition 2.2 *Under assumption* (1.4), *the function* ϕ *satisfies*

$$0<\lambda_m:=\liminf_{s\to-\infty}\left(\min_{(x,y)\in\overline{\Omega}}\frac{\phi_s(s,x,y)}{\phi(s,x,y)}\right)\le\limsup_{s\to-\infty}\left(\max_{(x,y)\in\overline{\Omega}}\frac{\phi_s(s,x,y)}{\phi(s,x,y)}\right)=:\lambda_M<+\infty$$

and the positive real numbers λ_m *and* λ_M *satisfy*

$$k(\lambda_m)+c\lambda_m=k(\lambda_M)+c\lambda_M=0.$$

Therefore,

$$c\ge c^*(e)=\inf_{\lambda>0}\left(-\frac{k(\lambda)}{\lambda}\right).$$

Proof. The beginning of the proof follows the main lines of that of Lemma 6.5 in [2] and Lemma 3.1 in [7], we will outline it for the sake of completeness. The main difference concerns the proof of the positivity of the quantities λ_m and λ_M since weaker assumptions are made here.

From Schauder interior estimates [30], there exists $C_1>0$ such that

$$\forall\ (t,x,y)\in\mathbb{R}\times\overline{\Omega},\ |u_t(t,x,y)|+|\nabla u(t,x,y)|\le C_1\times\max_{t-1\le t'\le t,\ (x',y')\in\overline{\Omega},\ |(x',y')-(x,y)|\le 1}u(t',x',y').$$

Choose now a vector $k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}$ such that $k\cdot e/c<0$. It follows then from Krylov-Safonov-Harnack-type inequalities (see e.g. [20, 29]) that there exists $C_2>0$ such that

$$\forall\ (t,x,y)\in\mathbb{R}\times\overline{\Omega},\quad\max_{t-1\le t'\le t,\ (x',y')\in\overline{\Omega},\ |(x',y')-(x,y)|\le 1}u(t',x',y')\le C_2\times u\left(t-\frac{k\cdot e}{c},x-k,y\right).$$

Because of (1.16), one gets that

$$\sup_{(t,x,y)\in\mathbb{R}\times\overline{\Omega}}\left(\frac{|u_t(t,x,y)|}{u(t,x,y)}+\frac{|\nabla u(t,x,y)|}{u(t,x,y)}\right)\le C_1C_2<+\infty. \tag{2.5}$$

In particular, since $\phi_s(s,x,y)=u_t((s+x\cdot e)/c,x,y)/c$, the function ϕ_s/ϕ is globally bounded in $\mathbb{R}\times\overline{\Omega}$ and the quantities λ_m and λ_M defined in Proposition 2.2 are real numbers.

From (1.13), there exists a sequence (s_n,x_n,y_n) such that $(x_n,y_n)\in\overline{C}$, $s_n\to-\infty$ and

$$\frac{\phi_s(s_n,x_n,y_n)}{\phi(s_n,x_n,y_n)}\to\lambda_m\text{ as }n\to+\infty.$$

Up to extraction of a subsequence, one has $(x_n,y_n)\to(x_\infty,y_\infty)\in\overline{C}$ as $n\to+\infty$. Call

$$t_n=\frac{s_n+x_n\cdot e}{c}\text{ and }v_n(t,x,y)=\frac{u(t+t_n,x,y)}{u(t_n,x_n,y_n)}.$$

From (2.5), the functions v_n are locally bounded. They are positive and satisfy

$$\begin{cases} (v_n)_t - \nabla\cdot(A\nabla v_n) + q\cdot\nabla v_n - \dfrac{f(x,y,u(t+t_n,x,y))}{u(t+t_n,x,y)} \times v_n = 0 \ \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu A\nabla v_n = 0 \ \text{ on } \mathbb{R}\times\partial\Omega. \end{cases}$$

Since $ct_n = s_n + x_n\cdot e \to -\infty$, there holds $u(t+t_n,x,y)\to 0$ as $n\to+\infty$, locally uniformly in (t,x,y). From standard parabolic estimates, the functions v_n converge in $C^{1;2}_{t;(x,y),loc}(\mathbb{R}\times\overline{\Omega})$ (at least), up to extraction of a subsequence, to a classical solution $v_\infty \geq 0$ of

$$\begin{cases} (v_\infty)_t - \nabla\cdot(A\nabla v_\infty) + q\cdot\nabla v_\infty - \zeta(x,y)\ v_\infty = 0 \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu A\nabla v_\infty = 0 \text{ on } \mathbb{R}\times\partial\Omega. \end{cases} \tag{2.6}$$

Furthermore, $v_\infty(0,x_\infty,y_\infty)=1$, whence v_∞ is positive everywhere in $\mathbb{R}\times\overline{\Omega}$ from the strong maximum principle and Hopf lemma. On the other hand,

$$\frac{(v_n)_t(t,x,y)}{v_n(t,x,y)} = \frac{u_t(t+t_n,x,y)}{u(t+t_n,x,y)} = c\times\frac{\phi_s(c(t+t_n)-x\cdot e,x,y)}{\phi(c(t+t_n)-x\cdot e,x,y)} \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega}$$

and for all $n\in\mathbb{N}$, whence

$$w(t,x,y) := \frac{(v_\infty)_t(t,x,y)}{v_\infty(t,x,y)} \geq c\lambda_m \text{ (resp. } \leq c\lambda_m) \text{ if } c>0 \text{ (resp. if } c<0),$$

since $ct_n \to -\infty$. Actually, the function w is trapped between $\min(c\lambda_m,c\lambda_M)$ and $\max(c\lambda_m,c\lambda_M)$. But $w(0,x_\infty,y_\infty)=c\lambda_m$ from the definition of the sequence (s_n,x_n,y_n). The function w is then a classical solution of the linear parabolic equation

$$\begin{cases} w_t - \nabla\cdot(A\nabla w) - 2\dfrac{\nabla v_\infty}{v_\infty}A\nabla w + q\cdot\nabla w = 0 \ \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu A\nabla w = 0 \ \text{ on } \mathbb{R}\times\partial\Omega, \end{cases}$$

which reaches its minimum or maximum $c\lambda_m$ at $(0,x_\infty,y_\infty)$ (depending on the sign of c). From the strong maximum principle and Hopf lemma together with property (1.16) satisfied by w, it follows that $w(t,x,y)=c\lambda_m$ for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$.

In other words, v_∞ satisfies $(v_\infty)_t = c\lambda_m v_\infty$. Because of (2.6) and (1.16), v_∞ can then be written as

$$v_\infty(t,x,y) = e^{\lambda_m(ct-x\cdot e)}\ \psi(x,y),$$

where ψ is positive in $\overline{\Omega}$, periodic and satisfies

$$L_{\lambda_m}\psi = -c\lambda_m\psi \ \text{ in } \overline{\Omega} \ \text{ and } \ \nu A\nabla\psi = \lambda_m(\nu Ae)\psi \ \text{ on } \partial\Omega.$$

Therefore,

$$-c\lambda_m = k(\lambda_m).$$

Similarly, the quantity λ_M is such that $-c\lambda_M = k(\lambda_M)$. Since the function $\lambda\mapsto k(\lambda)$ is concave and since $k(0)=\mu_0<0$, it follows that λ_m and λ_M are not zero and have the same sign. But since $\phi(s,x,y)>0$ for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$ and $\phi(-\infty,\cdot,\cdot)=0$, λ_M cannot be negative. As a conclusion, λ_m and λ_M are both positive. That completes the proof of Proposition 2.2. □

2.3 Monotonicity in the variable s

We prove here that the function ϕ such that $u(t,x,y) = \phi(ct - x\cdot e, x, y)$ is increasing in its first variable, which we denote s, under assumptions (1.4) and (1.6). Before going into the proof, one needs a comparison principle for solutions which are in some sense close to $p^+ = 1$.

Lemma 2.3 *Assume that* (1.6) *holds and let* $\rho > 0$ *be given as in* (1.6). *Let* $\overline{U}$ *and* $\underline{U}$ *be respectively classical supersolution and subsolution of*

$$\begin{cases} \overline{U}_t - \nabla\cdot(A(x,y)\nabla\overline{U}) + q(x,y)\cdot\nabla\overline{U} \geq f(x,y,\overline{U}) \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu A\nabla\overline{U} \geq 0 \text{ on } \mathbb{R}\times\partial\Omega, \end{cases}$$

and

$$\begin{cases} \underline{U}_t - \nabla\cdot(A(x,y)\nabla\underline{U}) + q(x,y)\cdot\nabla\underline{U} \leq f(x,y,\underline{U}) \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu A\nabla\underline{U} \leq 0 \text{ on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{2.7}$$

such that $\overline{U} < 1$ *and* $\underline{U} < 1$ *in* $\mathbb{R}\times\overline{\Omega}$. *Assume that* $\overline{U}(t,x,y) = \overline{\Phi}(ct - x\cdot e, x, y)$ *and* $\underline{U}(t,x,y) = \underline{\Phi}(ct - x\cdot e, x, y)$, *where* $\overline{\Phi}$ *and* $\underline{\Phi}$ *are periodic in* (x,y), $c \neq 0$ *and* $e \in \mathbb{R}^d\times\{0\}^{N-d}$ *with* $|e| = 1$. *If there exists* $h\in\mathbb{R}$ *such that*

$$\begin{cases} \overline{\Phi}(s,x,y) > 1-\rho \text{ for all } s\geq h \text{ and } (x,y)\in\overline{\Omega}, \\ \overline{\Phi}(h,x,y) \geq \underline{\Phi}(h,x,y) \text{ for all } (x,y)\in\overline{\Omega}, \\ \liminf\limits_{s\to+\infty}\left[\min\limits_{(x,y)\in\overline{\Omega}}(\overline{\Phi}(s,x,y) - \underline{\Phi}(s,x,y))\right] \geq 0, \end{cases} \tag{2.8}$$

then

$$\overline{\Phi}(s,x,y) \geq \underline{\Phi}(s,x,y) \text{ for all } s\geq h \text{ and } (x,y)\in\overline{\Omega},$$

that is $\overline{U}(t,x,y) \geq \underline{U}(t,x,y)$ *for all* $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$ *such that* $ct - x\cdot e \geq h$.

Proof. Call

$$\Omega^h = \{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ ct - x\cdot e \geq h\}.$$

This set is included into the set $\Omega_{\overline{U}} = \{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ \overline{U}(t,x,y) > 1-\rho\}$. Let $(\rho_\tau)_{\tau\in[0,1]}$ be the family of functions defined in $\overline{\Omega_{\overline{U}}} \supset \overline{\Omega^h} = \Omega^h$ and satisfying (1.6) with $\overline{U}$ instead of $\overline{u}$. Set

$$\tau^* = \inf\left\{\tau\in[0,1],\ \underline{U} \leq \overline{U} + \rho_\tau \text{ in } \Omega^h\right\}.$$

Since $\rho_1 \geq \rho$, one has $\underline{U} \leq \overline{U} + \rho_1$ in Ω^h and thus $\tau^*\in[0,1]$. By continuity of the functions ρ_τ with respect to τ, one has

$$\underline{U} \leq \overline{U} + \rho_{\tau^*} \text{ in } \Omega^h. \tag{2.9}$$

Assume now that $\tau^* > 0$. Then there exist two sequences $(\tau_n)_{n\in\mathbb{N}}$ in $[0,\tau^*)$ and $(t_n,x_n,y_n)_{n\in\mathbb{N}}$ in Ω^h such that

$$\underline{U}(t_n,x_n,y_n) > \overline{U}(t_n,x_n,y_n) + \rho_{\tau_n}(t_n,x_n,y_n) \text{ for all } n, \text{ and } \tau_n\to\tau^* \text{ as } n\to+\infty. \tag{2.10}$$

Since $\liminf_{n\to+\infty} \rho_{\tau_n}(t_n, x_n, y_n) \geq \liminf_{n\to+\infty} \inf_{\Omega^h} \rho_{\tau_n} = \inf_{\Omega^h} \rho_{\tau^*} > 0$, it follows from (2.8) that, up to extraction of a subsequence,

$$s_n = ct_n - x_n \cdot e \to s^* \in (h, +\infty), \quad \text{as } n \to +\infty. \tag{2.11}$$

Write $x_n = x'_n + x''_n$ where $x'_n \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}$ and $(x''_n, y_n) \in \overline{C}$. Up to extraction of another subsequence, one can assume that $(x''_n, y_n) \to (x_\infty, y_\infty) \in \overline{C}$.

Call

$$\rho^n(t, x, y) = \rho_{\tau_n}\left(t + \frac{x'_n \cdot e}{c}, x + x'_n, y\right).$$

These functions ρ^n are uniformly bounded in $C^{1+\alpha/2;2+\alpha}_{t;(x,y)}(\Omega^h)$ and converge in $C^{1;2}_{t;(x,y),loc}(\Omega^h)$ (at least), up to extraction of a subsequence, to a function ρ^∞. Observe that

$$\overline{U}\left(t + \frac{x'_n \cdot e}{c}, x + x'_n, y\right) = \overline{U}(t, x, y) \text{ and } \underline{U}\left(t + \frac{x'_n \cdot e}{c}, x + x'_n, y\right) = \underline{U}(t, x, y) \tag{2.12}$$

since $\overline{\Phi}$ and $\underline{\Phi}$ are periodic in (x, y). Since Ω, A, q, f and $p^+ = 1$ are periodic in (x, y), it follows from (1.6) that

$$\begin{cases} (\overline{U} + \rho^n)_t - \nabla \cdot (A\nabla(\overline{U} + \rho^n)) + q \cdot \nabla(\overline{U} + \rho^n) \geq f(x, y, \overline{U} + \rho^n) & \text{in } \Omega_n, \\ \nu A\nabla(\overline{U} + \rho^n) \geq 0 \quad \text{on } (\mathbb{R} \times \partial\Omega) \cap \Omega_n, \end{cases}$$

where $\Omega_n = \Omega^h \cap \big\{(t, x, y) \in \mathbb{R} \times \overline{\Omega},\ \overline{U}(t, x, y) + \rho^n(t, x, y) < 1\big\}$. Hence

$$\begin{cases} (\overline{U} + \rho^\infty)_t - \nabla \cdot (A\nabla(\overline{U} + \rho^\infty)) + q \cdot \nabla(\overline{U} + \rho^\infty) \geq f(x, y, \overline{U} + \rho^\infty) & \text{in } \Omega_\infty, \\ \nu A\nabla(\overline{U} + \rho^\infty) \geq 0 \quad \text{on } (\mathbb{R} \times \partial\Omega) \cap \Omega_\infty, \end{cases} \tag{2.13}$$

where $\Omega_\infty = \Omega^h \cap \big\{(t, x, y) \in \mathbb{R} \times \overline{\Omega},\ \overline{U}(t, x, y) + \rho^\infty(t, x, y) < 1\big\}$.

On the other hand, for any $\varepsilon > 0$, one has $\rho_{\tau^*} \leq \rho_{\tau_n} + \varepsilon$ in Ω^h for n large enough, whence $\underline{U} \leq \overline{U} + \rho^n + \varepsilon$ in Ω^h for n large enough, from (2.9) and (2.12). By passing to the limit as $n \to +\infty$ and then $\varepsilon \to 0$, one gets that

$$\underline{U} \leq \overline{U} + \rho^\infty \text{ in } \Omega^h. \tag{2.14}$$

But (2.10) yields

$$\underline{U}\left(t_n - \frac{x'_n \cdot e}{c}, x_n - x'_n, y_n\right) > \overline{U}\left(t_n - \frac{x'_n \cdot e}{c}, x_n - x'_n, y_n\right) + \rho^n\left(t_n - \frac{x'_n \cdot e}{c}, x_n - x'_n, y_n\right)$$

and $(t_n - (x'_n \cdot e)/c, x_n - x'_n, y_n) \in \Omega^h$ for all $n \in \mathbb{N}$, while

$$x_n - x'_n = x''_n \to x_\infty \text{ and } t_n - \frac{x'_n \cdot e}{c} = \frac{s_n + x''_n \cdot e}{c} \to \frac{s^* + x_\infty \cdot e}{c} =: t_\infty$$

as $n \to +\infty$. Thus, $(t_\infty, x_\infty, y_\infty) \in \Omega^h$ and

$$\underline{U}(t_\infty, x_\infty, y_\infty) \geq \overline{U}(t_\infty, x_\infty, y_\infty) + \rho^\infty(t_\infty, x_\infty, y_\infty).$$

Therefore,
$$\underline{U}(t_\infty, x_\infty, y_\infty) = \overline{U}(t_\infty, x_\infty, y_\infty) + \rho^\infty(t_\infty, x_\infty, y_\infty)$$
from (2.14). In particular,
$$\overline{U}(t_\infty, x_\infty, y_\infty) + \rho^\infty(t_\infty, x_\infty, y_\infty) < 1. \tag{2.15}$$

Notice also that
$$ct_\infty - x_\infty \cdot e = s^* > h$$
from (2.11). Together with (2.7) and (2.13), one concludes from the strong parabolic maximum and Hopf lemma that
$$\underline{U} = \overline{U} + \rho^\infty \text{ in } \mathcal{C}, \tag{2.16}$$
where $\mathcal{C}$ is the connected component of $\Omega^\infty \cap \{t \le t_\infty\}$ containing $(t_\infty, x_\infty, y_\infty)$.

Consider first the case when $c > 0$ and call
$$\underline{t} = \frac{h + x_\infty \cdot e}{c}.$$

There holds $\underline{t} < t_\infty$ and the points (t, x_∞, y_∞) lie in Ω^h for all $t \in [\underline{t}, t_\infty]$. Because of (2.15), it follows that
$$\exists\ a > 0,\ \forall\ t \in [t_\infty - a, t_\infty], \quad \overline{U}(t, x_\infty, y_\infty) + \rho^\infty(t, x_\infty, y_\infty) < 1. \tag{2.17}$$

Call
$$t_* = \inf\left\{t \in [\underline{t}, t_\infty],\ \overline{U}(t', x_\infty, y_\infty) + \rho^\infty(t', x_\infty, y_\infty) < 1 \text{ for all } t' \in [t, t_\infty]\right\}.$$

From (2.16), one gets that
$$\underline{U}(t_*, x_\infty, y_\infty) = \overline{U}(t_*, x_\infty, y_\infty) + \rho^\infty(t_*, x_\infty, y_\infty), \tag{2.18}$$
whence
$$\overline{U}(t_*, x_\infty, y_\infty) + \rho^\infty(t_*, x_\infty, y_\infty) < 1. \tag{2.19}$$

Therefore, $t_* = \underline{t}$ and
$$\underline{\Phi}(h, x_\infty, y_\infty) = \overline{\Phi}(h, x_\infty, y_\infty) + \rho^\infty(\underline{t}, x_\infty, y_\infty).$$

But
$$\inf_{\Omega^h} \rho^n = \inf_{\Omega^h} \rho_{\tau_n} \to \inf_{\Omega^h} \rho_{\tau^*} =: \eta > 0 \text{ as } n \to +\infty,$$
whence $\inf_{\Omega^h} \rho^\infty \ge \eta$. Eventually, $\underline{\Phi}(h, x_\infty, y_\infty) > \overline{\Phi}(h, x_\infty, y_\infty)$, which contradicts the assumption (2.8).

Consider now the case when $c < 0$. The points (t, x_∞, y_∞) lie in Ω^h for all $t \le t_\infty$. Property (2.17) still holds and
$$t_* = \inf\left\{t \in (-\infty, t_\infty],\ \overline{U}(t', x_\infty, y_\infty) + \rho^\infty(t', x_\infty, y_\infty) < 1 \text{ for all } t' \in [t, t_\infty]\right\}$$

satisfies $-\infty \le t_* \le t_\infty - a < t_\infty$. If t_* is a real number, then, from (2.16), formulas (2.18) and (2.19) still hold. Therefore, $t_* = -\infty$ and

$$\forall\, t \le t_\infty, \quad \underline{U}(t, x_\infty, y_\infty) = \overline{U}(t, x_\infty, y_\infty) + \rho^\infty(t, x_\infty, y_\infty).$$

In particular, $\underline{\Phi}(s, x_\infty, y_\infty) \ge \overline{\Phi}(s, x_\infty, y_\infty) + \eta$ for all $s \ge ct_\infty - x_\infty \cdot e = s^*$, which contradicts the assumption (2.8) as $s \to +\infty$.

As a conclusion, in both cases $c > 0$ and $c < 0$, the assumption $\tau^* > 0$ is impossible. Thus, $\underline{U} \le \overline{U}$ in Ω^h. That completes the proof of Lemma 2.3. □

Remark 2.4 The proof can easily be extended to the case when $\overline{\Phi}$ and $\underline{\Phi}$ are not periodic in (x, y) anymore, under the additional assumption that, say, $\overline{U}$ and $\underline{U}$ is of class $C^{1+\alpha/2;2+\alpha}_{t;(x,y)}(\Omega^h)$. In this case, one also has to define shifted functions $\overline{U}^{\,n}$ and $\underline{U}^n$ and to pass to the limit as $n \to +\infty$ in Ω^h for a subsequence, as it was done for ρ^n.

We are now back to our main purpose, and $u(t, x, y) = \phi(ct - x \cdot e, x, y)$ denotes a pulsating travelling front with speed $c \in \mathbb{R}^*$, in the sense of Definition 1.1. The following result corresponds to the second part of Proposition 1.2.

Proposition 2.5 *Under assumptions* (1.4) *and* (1.6)*, the function $\phi(s, x, y)$ is increasing in the variable s, and $\phi_s(s, x, y) > 0$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$. In other words, $u_t(t, x, y) > 0$ in $\mathbb{R} \times \overline{\Omega}$ if $c > 0$ and $u_t(t, x, y) < 0$ if $c < 0$.*

Proof. From Proposition 2.2, there exists $\underline{s} \in \mathbb{R}$ such that

$$\phi_s(s, x, y) > 0 \text{ for all } s \le \underline{s} \text{ and for all } (x, y) \in \overline{\Omega}.$$

On the other hand, $\inf_{s \ge \underline{s}\ (x,y) \in \overline{\Omega}} \phi(s, x, y) > 0$ by continuity of ϕ and because of (1.13) and (1.14). Therefore, there exists $\Sigma \in \mathbb{R}$ such that $-\Sigma \le \underline{s}$ and

$$\forall\, \tau \ge 0,\ \forall\, s \le -\Sigma,\ \forall\, (x, y) \in \overline{\Omega}, \quad \phi(s, x, y) \le \phi(s + \tau, x, y). \tag{2.20}$$

Because of (1.14), even if it means increasing Σ, one can assume that $\Sigma > 0$ and

$$\phi(s, x, y) > 1 - \rho \ \text{ for all } s \ge \Sigma \text{ and } (x, y) \in \overline{\Omega}, \tag{2.21}$$

where $\rho > 0$ is given as in (1.6).

We will now use a sliding method as in [2] (see also [9] for elliptic versions). Take now any $\tau \ge 2\Sigma$. Thus,

$$\phi(s + \tau, x, y) > 1 - \rho \text{ for all } s \ge -\Sigma \text{ and } (x, y) \in \overline{\Omega},$$

while $\phi(-\Sigma, x, y) \le \phi(-\Sigma + \tau, x, y)$ for all $(x, y) \in \overline{\Omega}$, from (2.20). It is immediate to see that all assumptions of Lemma 2.3 are fulfilled with

$$\overline{U}(t, x, y) = u\left(t + \frac{\tau}{c}, x, y\right),\ \underline{U} = u,\ \overline{\Phi}(s, x, y) = \phi(s + \tau, x, y),\ \underline{\Phi} = \phi \text{ and } h = -\Sigma.$$

As a consequence,

$$\phi(s,x,y) \le \phi(s+\tau,x,y) \text{ for all } s \ge -\Sigma \text{ and } (x,y) \in \overline{\Omega}.$$

Together with (2.20), one gets that $\phi(s,x,y) \le \phi(s+\tau,x,y)$ for all $\tau \ge 2\Sigma$ and for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$.

Call

$$\tau^* = \inf \left\{\tau > 0,\ \phi(s,x,y) \le \phi(s+\tau',x,y) \text{ for all } \tau' \ge \tau \text{ and for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}\right\}.$$

One has that $\tau^* \in [0, 2\Sigma]$ and

$$\phi(s,x,y) \le \phi(s+\tau^*,x,y) \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Assume that $\tau^* > 0$ and define

$$z(s,x,y) = \phi(s+\tau^*,x,y) - \phi(s,x,y).$$

The function z is continuous in (s,x,y), periodic in (x,y) and nonnegative. In particular, the minimum of z over $[-\Sigma,\Sigma] \times \overline{\Omega}$ is reached and is either positive or zero.

Case 1: $\min_{(s,x,y)\in[-\Sigma,\Sigma]\times\overline{\Omega}} z(s,x,y) > 0$. Since z is actually uniformly continuous in $\mathbb{R} \times \overline{\Omega}$, there exists $\tau_* \in (0,\tau^*)$ such that

$$\phi(s,x,y) \le \phi(s+\tau,x,y) \text{ for all } \tau \in [\tau_*, \tau^*] \tag{2.22}$$

and for all $(s,x,y) \in [-\Sigma,\Sigma] \times \overline{\Omega}$. But inequality (2.22) also holds when $(s,x,y) \in (-\infty,-\Sigma] \times \overline{\Omega}$ from (2.20). It also holds for all $(s,x,y) \in [\Sigma,+\infty) \times \overline{\Omega}$ from (2.21) and Lemma 2.3 applied to $\overline{U}(t,x,y) = u(t+\tau/c,x,y)$, $\underline{U} = u$ and $h = \Sigma$. Thus,

$$\phi(s,x,y) \le \phi(s+\tau,x,y) \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega} \text{ and for all } \tau \in [\tau_*, \tau^*],$$

which contradicts the minimality of τ^*.

Case 2: $\min_{(s,x,y)\in[-\Sigma,\Sigma]\times\overline{\Omega}} z(s,x,y) = 0$. Here, the function

$$v(t,x,y) = u\left(t + \frac{\tau^*}{c}, x, y\right) - u(t,x,y)$$

is nonnegative in $\mathbb{R} \times \overline{\Omega}$ and it vanishes at a point (t^*, x^*, y^*) such that $|ct^* - x^* \cdot e| \le \Sigma$. From the strong maximum principle and Hopf lemma, the function v is then identically 0 in $(-\infty, t^*] \times \overline{\Omega}$, and then in $\mathbb{R} \times \overline{\Omega}$ by uniqueness of the Cauchy problem associated to (1.1). In particular,

$$u\left(\frac{k\tau^*}{c}, x, y\right) = u(0,x,y) \text{ for all } (x,y) \in \overline{\Omega} \text{ and for all } k \in \mathbb{Z}.$$

But, for each $(x,y) \in \overline{\Omega}$,

$$u(k\tau^*/c, x, y) \to p^{\pm} \text{ as } k \to \pm\infty$$

from (1.14) and since $\tau^* > 0$. One has then reached a contradiction, since $0 = p^- < p^+ = 1$ in $\overline{\Omega}$.

As a conclusion, $\tau^* = 0$, whence

$$u\left(t + \frac{\tau}{c}, x, y\right) \geq u(t, x, y) \text{ for all } (t, x, y) \in \mathbb{R} \times \overline{\Omega} \text{ and for all } \tau \geq 0.$$

Actually, the inequalities are strict as soon as $\tau > 0$, from the strong maximum principle, as above. Moreover, the bounded function u_t satisfies

$$\begin{cases} (u_t)_t - \nabla \cdot (A(x,y)\nabla u_t) + q(x,y) \cdot \nabla u_t = \dfrac{\partial f}{\partial u}(x,y,u)\ u_t & \text{in } \mathbb{R} \times \overline{\Omega}, \\ \nu A \nabla u_t = 0 \ \text{ on } \mathbb{R} \times \partial\Omega. \end{cases}$$

Since u_t is either nonnegative or nonpositive in $\mathbb{R} \times \overline{\Omega}$ and cannot be identically zero, it is then either positive or negative in $\mathbb{R} \times \overline{\Omega}$ from the strong parabolic maximum principle and Hopf lemma. Thus, $u_t > 0$ in $\mathbb{R} \times \overline{\Omega}$ if $c > 0$, and $u_t < 0$ in $\mathbb{R} \times \overline{\Omega}$ if $c < 0$. The function ϕ is always increasing in its first variable and $\phi_s(s, x, y) > 0$. The proof of Proposition 2.5 is now complete. □

3 Exponential lower bounds of $\phi(s, x, y)$ as $s \to -\infty$

In this section, given a pulsating travelling front

$$u(t, x, y) = \phi(ct - x \cdot e, x, y)$$

in the sense of Definition 1.1, with $p^- = 0$ and $p^+ = 1$, we shall prove that, under the various assumptions of Theorems 1.3 and 1.5, the function $\phi(s, x, y)$ cannot decay too fast as $s \to -\infty$. The proofs are based on a key-lemma and several propositions.

Lemma 3.1 *Let $\underline{u}(t, x, y) = \underline{\phi}(ct - x \cdot e, x, y)$ be a continuous function defined in $\mathbb{R} \times \overline{\Omega}$ such that $\underline{\phi}(s, x, y)$ is periodic in (x, y), $\underline{\phi}(s, x, y) < 1$ in $\mathbb{R} \times \overline{\Omega}$, and $\underline{u}$ is a classical subsolution of*

$$\begin{cases} \underline{u}_t - \nabla \cdot (A(x,y)\nabla \underline{u}) + q(x,y) \cdot \nabla \underline{u} \leq f(x, y, \underline{u}) & \text{in } \mathbb{R} \times \overline{\Omega}, \\ \nu A \nabla \underline{u} \leq 0 & \text{on } \mathbb{R} \times \partial\Omega. \end{cases}$$

Let $\overline{u}(t, x, y) = \overline{\phi}(ct - x \cdot e, x, y)$ be a continuous function defined in $\mathbb{R} \times \overline{\Omega}$ such that $\overline{\phi}(s, x, y)$ is periodic in (x, y), and assume that there is $\sigma \in \mathbb{R}$ such that

$$\underline{\phi}(\sigma, x, y) < \overline{\phi}(\sigma, x, y) \quad \textit{for all } (x, y) \in \overline{\Omega} \tag{3.1}$$

and

$$\overline{\phi}(\sigma, x, y) \leq \overline{\phi}(s, x, y) \quad \textit{for all } (s, x, y) \in [\sigma, +\infty) \times \overline{\Omega}. \tag{3.2}$$

1) *If there is $\overline{\sigma} \geq \sigma$ such that $\overline{\phi}(s, x, y) = 1$ for all $(s, x, y) \in [\overline{\sigma}, +\infty) \times \overline{\Omega}$, and if $\overline{u}$ is a classical supersolution of*

$$\begin{cases} \overline{u}_t - \nabla \cdot (A(x,y)\nabla \overline{u}) + q(x,y) \cdot \nabla \overline{u} \geq f(x, y, \overline{u}) & \text{in } \Omega', \\ \nu A \nabla \overline{u} \geq 0 & \text{on } (\mathbb{R} \times \partial\Omega) \cap \Omega', \end{cases} \tag{3.3}$$

where

$$\Omega' = \{(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ \overline{u}(t,x,y) < 1\},$$

then

$$\underline{\phi}(s,x,y) < \overline{\phi}(s,x,y) \ \ \text{for all } s \geq \sigma \text{ and } (x,y) \in \overline{\Omega}. \tag{3.4}$$

2) *Assume that* (1.6) *holds and let* $\rho > 0$ *be given as in* (1.6). *If* $\overline{\phi} = \overline{\phi}_1 + \overline{\phi}_2$, *where* $\overline{\phi}_1(s,x,y)$ *and* $\overline{\phi}_2(s,x,y)$ *are continuous in* $\mathbb{R} \times \overline{\Omega}$ *and periodic in* (x,y), *if* $\overline{\phi}_1(s,x,y) \overset{\leq}{\to} 1$ *as* $s \to +\infty$, *if* $\overline{u}_i(t,x,y) = \overline{\phi}_i(ct - x \cdot e, x, y)$ $(i = 1,2)$ *satisfy*

$$\begin{cases} (\overline{u}_1)_t - \nabla \cdot (A(x,y)\nabla \overline{u}_1) + q \cdot \nabla \overline{u}_1 \geq f(x,y,\overline{u}_1) \text{ in } \mathbb{R} \times \overline{\Omega}, & \nu A \nabla \overline{u}_1 \geq 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ \overline{u}_t - \nabla \cdot (A(x,y)\nabla \overline{u}) + q \cdot \nabla \overline{u} > f(x,y,\overline{u}) \text{ in } \Omega', & \nu A \nabla \overline{u} \geq 0 \text{ on } (\mathbb{R} \times \partial\Omega) \cap \Omega', \end{cases}$$

where

$$\overline{u} = \overline{u}_1 + \overline{u}_2 \ \ \text{and} \ \ \Omega' = \big\{(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ ct - x \cdot e \leq \overline{\sigma}\big\}$$

and $\overline{\sigma} > \sigma$ *is such that*

$$\overline{\phi}_1(s,x,y) > 1 - \rho \text{ in } [\overline{\sigma}, +\infty) \times \overline{\Omega} \text{ and } \underline{\phi}(\sigma,x,y) \leq \overline{\phi}_1(\overline{\sigma},x,y) \leq \overline{\phi}(\overline{\sigma},x,y) \text{ in } \overline{\Omega},$$

then there exists $\tau^* \in [0, \overline{\sigma} - \sigma]$ *such that*

$$\underline{\phi}(s-\tau^*,x,y) \leq \overline{\phi}(s,x,y) \text{ in } [\sigma+\tau^*, \overline{\sigma}] \times \overline{\Omega},\ \underline{\phi}(s-\tau^*,x,y) \leq \overline{\phi}_1(s,x,y) \text{ in } [\overline{\sigma}, +\infty) \times \overline{\Omega}$$

and

$$(\tau^* > 0) \Longrightarrow \left(\min_{(x,y)\in\overline{\Omega}} \left[\ \overline{\phi}_1(\overline{\sigma},x,y) - \underline{\phi}(\overline{\sigma}-\tau^*,x,y)\ \right] = 0 \right).$$

Proof. Let us first deal with part 1). Since $\underline{\phi}(s,x,y) < 1$ for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$ and since $\overline{\phi}(s,x,y) = 1$ for all $(s,x,y) \in [\overline{\sigma}, +\infty) \times \overline{\Omega}$, it follows that

$$\underline{\phi}(s-\tau,x,y) \leq \overline{\phi}(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau, +\infty) \times \overline{\Omega} \text{ and for all } \tau \geq \overline{\sigma} - \sigma \ (\geq 0).$$

Define

$$\tau^* = \inf\big\{\tau > 0,\ \underline{\phi}(s-\tau,x,y) \leq \overline{\phi}(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau, +\infty) \times \overline{\Omega}\big\}.$$

Thus, $\tau^* \in [0, \overline{\sigma} - \sigma]$ and

$$\underline{\phi}(s-\tau^*,x,y) \leq \overline{\phi}(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau^*, +\infty) \times \overline{\Omega}. \tag{3.5}$$

Our goal is to prove that $\tau^* = 0$, which would yield the conclusion. Assume by contradiction that $\tau^* > 0$. There exist two sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(s_n, x_n, y_n)_{n\in\mathbb{N}}$ such that

$$0 < \tau_n < \tau^*,\ s_n \geq \sigma + \tau_n \text{ and } \underline{\phi}(s_n - \tau_n, x_n, y_n) > \overline{\phi}(s_n, x_n, y_n) \ \text{ for all } n \in \mathbb{N}$$

and $\tau_n \to \tau^*$ as $n \to +\infty$. For the same reasons as above, the sequence (s_n) is bounded and, up to extraction of a subsequence, $s_n \to s^* \in [\sigma + \tau^*, +\infty)$ as $n \to +\infty$. Since $\underline{\phi}$ and $\overline{\phi}$ are periodic in (x,y), one can assume without loss of generality that $(x_n,y_n)\in\overline{C}$ for each n, and that $(x_n,y_n)\to(x^*,y^*)\in\overline{C}$ as $n\to+\infty$. Therefore, $\underline{\phi}(s^*-\tau^*,x^*,y^*)\geq\overline{\phi}(s^*,x^*,y^*)$, whence

$$\underline{\phi}(s^*-\tau^*,x^*,y^*)=\overline{\phi}(s^*,x^*,y^*)$$

because of (3.5). Because of (3.1) and (3.2), one has

$$\underline{\phi}(\sigma,x^*,y^*)<\overline{\phi}(\sigma,x^*,y^*)\leq\overline{\phi}(\sigma+\tau^*,x^*,y^*). \tag{3.6}$$

Thus, $s^*>\sigma+\tau^*$. Call now

$$\underline{U}(t,x,y)=\underline{\phi}(ct-x\cdot e-\tau^*,x,y)\ \text{ in } \Omega^{\sigma+\tau^*}=\{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ ct-x\cdot e\geq\sigma+\tau^*\}. \tag{3.7}$$

There holds

$$\underline{U}\leq\overline{u}\text{ in }\Omega^{\sigma+\tau^*}$$

with equality at the point

$$(t^*,x^*,y^*)=\left(\frac{s^*+x^*\cdot e}{c},x^*,y^*\right)\text{ such that }ct^*-x^*\cdot e=s^*>\sigma+\tau^*.$$

But $\overline{u}$ satisfies (3.3) in Ω' and $\underline{U}$ is a subsolution of (1.1), with

$$\overline{u}(t^*,x^*,y^*)=\underline{U}(t^*,x^*,y^*)<1.$$

From the strong maximum principle and Hopf lemma, it follows that $\overline{u}=\underline{U}$ in the connected component of $\Omega^{\sigma+\tau^*}\cap\{\overline{u}(t,x,y)<1\}\cap\{t\leq t^*\}$ containing (t^*,x^*,y^*). The end of the proof follows the same lines as that of Lemma 2.3. Namely, if $c>0$, one gets that

$$\overline{u}(t,x^*,y^*)=\underline{U}(t,x^*,y^*)\text{ for all }t\in\left[\frac{\sigma+\tau^*+x^*\cdot e}{c},t^*\right].$$

In particular, at $t=(\sigma+\tau^*+x^*\cdot e)/c$, it follows that

$$\overline{\phi}(\sigma+\tau^*,x^*,y^*)=\underline{\phi}(\sigma,x^*,y^*),$$

which is impossible from (3.6). Now, if $c<0$, then $\overline{u}(t,x^*,y^*)=\underline{U}(t,x^*,y^*)$ for all $t\leq t^*$, whence

$$\overline{\phi}(ct-x^*\cdot e,x^*,y^*)=\underline{\phi}(ct-x^*\cdot e-\tau^*,x^*,y^*)<1\text{ for all }t\leq t^*.$$

One gets a contradiction as $t\to-\infty$, since $\overline{\phi}(s,x^*,y^*)=1$ for all $s\geq\overline{\sigma}$.

As a conclusion, in both cases $c>0$ and $c<0$,

$$\tau^*=0.$$

Hence $\underline{\phi}(s,x,y)\leq\overline{\phi}(s,x,y)$ for all $(s,x,y)\in[\sigma,+\infty)\times\overline{\Omega}$. Furthermore, if there is a point $(s,x,y)\in[\sigma,+\infty)\times\overline{\Omega}$ such that $\underline{\phi}(s,x,y)=\overline{\phi}(s,x,y)$, then $s>\sigma$ from (3.1), and the last part of the above proof, which does not use the positivity of τ^*, leads to a contradiction.

Therefore, (3.4) is proved.

Let us now turn to the proof of part 2). First, for any $\tau \geq \overline{\sigma} - \sigma$ (≥ 0), one has $\underline{\phi}(\sigma, x, y) \leq \overline{\phi}_1(\sigma + \tau, x, y)$ in $\overline{\Omega}$. From the assumptions made in this part 2), one can apply Lemma 2.3 with

$$\overline{\Phi}(s,x,y) = \overline{\phi}_1(s,x,y),\ \underline{\Phi}(t,x,y) = \underline{\phi}(s-\tau,x,y) \text{ and } h = \sigma+\tau.$$

Notice especially that the limit $\lim_{s\to+\infty}\overline{\phi}_1(s,x,y) = 1$ is uniform in $(x,y) \in \overline{\Omega}$ from Dini's theorem, and then all assertions in (2.8) are satisfied. One concludes that

$$\underline{\phi}(s-\tau,x,y) \leq \overline{\phi}_1(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau,+\infty)\times\overline{\Omega} \text{ and } \tau \geq \overline{\sigma}-\sigma.$$

For all $(x,y) \in \overline{\Omega}$, define

$$\phi'(s,x,y) = \begin{cases} \overline{\phi}(s,x,y) & \text{if } \sigma \leq s < \overline{\sigma}, \\ \overline{\phi}_1(s,x,y) & \text{if } s \geq \overline{\sigma}. \end{cases}$$

Call

$$\tau^* = \inf\left\{\tau > 0,\ \underline{\phi}(s-\tau,x,y) \leq \phi'(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau,+\infty)\times\overline{\Omega}\right\}.$$

Thus, $\tau^* \in [0, \overline{\sigma}-\sigma]$ and, since $\underline{\phi}$ is continuous and ϕ' is right-continuous with respect to s (remember that $\overline{\phi}_1$ and $\overline{\phi}_2$ are continuous), it follows that

$$\underline{\phi}(s-\tau^*,x,y) \leq \phi'(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau^*,+\infty)\times\overline{\Omega}.$$

In particular,

$$\underline{\phi}(s-\tau^*,x,y) \leq \overline{\phi}_1(s,x,y) \ \text{ in } [\overline{\sigma},+\infty)\times\overline{\Omega}$$

and

$$\underline{\phi}(s-\tau^*,x,y) \leq \overline{\phi}(s,x,y) \ \text{ in } [\sigma+\tau^*,\overline{\sigma}]\times\overline{\Omega}.$$

This last property is true even if $\sigma + \tau^* = \overline{\sigma}$, since $\overline{\phi}_1(\overline{\sigma},x,y) \leq \overline{\phi}(\overline{\sigma},x,y)$ in $\overline{\Omega}$. It only remains to show that $\min_{(x,y)\in\overline{\Omega}}\big(\overline{\phi}_1(\overline{\sigma},x,y) - \underline{\phi}(\overline{\sigma}-\tau^*,x,y)\big) = 0$ if $\tau^* > 0$.

Assume by contradiction that $\tau^* > 0$ and

$$\min_{(x,y)\in\overline{\Omega}}\ \big(\overline{\phi}_1(\overline{\sigma},x,y) - \underline{\phi}(\overline{\sigma}-\tau^*,x,y)\big) > 0. \tag{3.8}$$

As a consequence, $\min_{(x,y)\in\overline{\Omega}}\ \big(\overline{\phi}(\overline{\sigma},x,y) - \underline{\phi}(\overline{\sigma}-\tau^*,x,y)\big) > 0$. On the other hand, for any $(x,y) \in \overline{\Omega}$, $\underline{\phi}(\sigma,x,y) < \overline{\phi}(\sigma,x,y) \leq \overline{\phi}(\sigma+\tau^*,x,y)$ from (3.1) and (3.2). Therefore,

$$\underline{\phi}(s-\tau^*,x,y) < \overline{\phi}(s,x,y) \text{ for all } (x,y) \in \overline{\Omega} \text{ when } s = \sigma+\tau^* \text{ or } \overline{\sigma}.$$

Since $\overline{u}(t,x,y) \geq u(t-\tau^*/c,x,y)$ for all $(t,x,y) \in \mathbb{R}\times\overline{\Omega}$ such that $ct - x\cdot e \in [\sigma+\tau^*,\overline{\sigma}]$ and since $\overline{u}$ is assumed to be a strict supersolution of (1.1) in this region, while $u(t-\tau^*/c,x,y)$ is a solution, one concludes from the strong maximum principle and Hopf lemma that

$$\overline{u}(t,x,y) > u(t-\tau^*/c,x,y) \text{ as soon as } ct - x\cdot e \in (\sigma+\tau^*,\overline{\sigma}),$$

that is $\underline{\phi}(s-\tau^*,x,y) < \overline{\phi}(s,x,y)$ for all $(s,x,y) \in (\sigma+\tau^*,\overline{\sigma}) \times \overline{\Omega}$ (notice that this part is needed only if $\sigma+\tau^* < \overline{\sigma}$). Finally,

$$\underline{\phi}(s-\tau^*,x,y) < \overline{\phi}(s,x,y) \text{ for all } (s,x,y) \in [\sigma+\tau^*,\overline{\sigma}] \times \overline{\Omega}.$$

Together with our assumption (3.8), one gets the existence of $\tau_* \in (0,\tau^*)$ such that, for all $\tau \in [\tau_*,\tau^*]$,

$$\underline{\phi}(s-\tau,x,y) \leq \overline{\phi}(s,x,y) \text{ in } [\sigma+\tau,\overline{\sigma}] \times \overline{\Omega} \text{ and } \underline{\phi}(\overline{\sigma}-\tau,x,y) \leq \overline{\phi}_1(\overline{\sigma},x,y) \text{ in } \overline{\Omega}.$$

Another application of Lemma 2.3 with $\overline{\Phi}(s,x,y) = \overline{\phi}_1(s,x,y)$, $\underline{\Phi}(s,x,y) = \underline{\phi}(s-\tau,x,y)$ and $h = \overline{\sigma}$, for any $\tau \in [\tau_*,\tau^*]$, implies that

$$\underline{\phi}(s-\tau,x,y) \leq \overline{\phi}_1(s,x,y) \text{ for all } (s,x,y) \in [\overline{\sigma},+\infty) \times \overline{\Omega}.$$

Thus $\underline{\phi}(s-\tau,x,y) \leq \phi'(s,x,y)$ for all $(s,x,y) \in [\sigma+\tau,+\infty) \times \overline{\Omega}$ and for all $\tau \in [\tau_*,\tau^*]$, which contradicts the minimality of τ^*. Therefore, (3.8) cannot hold if $\tau^* > 0$. That completes the proof of Lemma 3.1. □

We are now going to apply Lemma 3.1 to the different situations considered in Theorems 1.3 and 1.5. It will provide exponential lower bounds for the function $\phi(s,x,y) = u((s+x\cdot e)/c,x,y)$ as $s \to -\infty$. We are first concerned with the KPP case (1.8), with $c > c^*(e)$.

Proposition 3.2 *Under assumptions* (1.4) *and* (1.8), *if $c > c^*(e)$, then*

$$\liminf_{s\to-\infty} \left[\min_{(x,y)\in\overline{\Omega}} \left(\frac{\phi(s,x,y)}{e^{\lambda_c s}} \right) \right] > 0,$$

where $\lambda_c > 0$ is defined in Lemma 2.1.

Proof. Assume by contradiction that there exists a sequence $(s_n,x_n,y_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times \overline{\Omega}$ such that

$$s_n \to -\infty \text{ and } \varepsilon_n := \phi(s_n,x_n,y_n)\ e^{-\lambda_c s_n} \to 0^+ \text{ as } n \to +\infty.$$

Notice that the vector field

$$\frac{\nabla\phi(s,x,y)}{\phi(s,x,y)} = \frac{c^{-1}u_t(c^{-1}(s+x\cdot e),x,y)e + \nabla u(c^{-1}(s+x\cdot e),x,y)}{u(c^{-1}(s+x\cdot e),x,y)}$$

is bounded in $\mathbb{R} \times \overline{\Omega}$ from (2.5). Since ϕ is periodic in (x,y), there exists then a constant $C_3 > 0$ such that

$$\forall\ s \in \mathbb{R},\ \forall\ (x,y),\ (x',y') \in \overline{\Omega}, \quad \phi(s,x,y) \leq C_3\ \phi(s,x',y'). \tag{3.9}$$

Therefore, for all $n \in \mathbb{N}$ and for all $(x,y) \in \overline{\Omega}$,

$$\phi(s_n,x,y) \leq C_3\ \phi(s_n,x_n,y_n) = C_3\ \varepsilon_n\ e^{\lambda_c s_n} < \frac{2\ C_3\ \varepsilon_n}{\kappa_1}\ e^{\lambda_c s_n}\ \psi_{\lambda_c}(x,y),$$

where

$$\kappa_1 := \min_{(x,y)\in\overline{\Omega}} \psi_{\lambda_c}(x,y) > 0. \tag{3.10}$$

For each $n \in \mathbb{N}$, call $\overline{u}_n(t,x,y) = \overline{\phi}_n(ct - x\cdot e, x, y)$, where

$$\overline{\phi}_n(s,x,y) = \min\left(\frac{2\ C_3\ \varepsilon_n}{\kappa_1}\ e^{\lambda_c s}\ \psi_{\lambda_c}(x,y), 1\right).$$

Observe that $\phi(s_n,x,y) < \overline{\phi}_n(s_n,x,y)$ for all $(x,y)\in\overline{\Omega}$, that $\overline{\phi}_n(s,x,y)$ is nondecreasing with respect to s and that $\overline{\phi}_n(s,x,y) = 1$ for all $(s,x,y)\in[\overline{\sigma}_n,+\infty)\times\overline{\Omega}$, for some $\overline{\sigma}_n\in[s_n,+\infty)$. Furthermore, if $\overline{u}_n(t,x,y) < 1$, then

$$\begin{aligned}(\overline{u}_n)_t - \nabla\cdot(A\nabla\overline{u}_n) + q\cdot\nabla\overline{u}_n &= \frac{2\ C_3\ \varepsilon_n}{\kappa_1}\ (k(\lambda_c) + \lambda_c c + \zeta(x,y))\ e^{\lambda_c(ct - x\cdot e)}\ \psi_{\lambda_c}(x,y)\\ &= \zeta(x,y)\ \overline{u}_n(t,x,y)\ \geq\ f(x,y,\overline{u}_n(t,x,y))\end{aligned}$$

from (1.8) and (1.20), and $\nu A(x,y)\nabla\overline{u}_n = 0$ if $(t,x,y)\in\mathbb{R}\times\partial\Omega$.

Part 1) of Lemma 3.1 applied to $\underline{\phi} = \phi$, $\overline{\phi} = \overline{\phi}_n$ and $\sigma = s_n$ implies that

$$(0 <)\ \phi(s,x,y) < \overline{\phi}_n(s,x,y) = \min\left(\frac{2\ C_3\ \varepsilon_n}{\kappa_1}\ e^{\lambda_c s}\ \psi_{\lambda_c}(x,y), 1\right)$$

for all $(s,x,y)\in[s_n,+\infty)\times\overline{\Omega}$. Since $s_n\to-\infty$ and $\varepsilon_n\to 0^+$ as $n\to+\infty$, it follows that, for each $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$, $\phi(s,x,y) = 0$, which is impossible. Thus, the conclusion of Proposition 3.2 holds. □

It is immediate to see from the above proof that, under assumptions (1.4) and (1.8), Proposition 3.2 would still hold if λ_c were replaced by any $\lambda > 0$ such that $k(\lambda) + \lambda c = 0$. It would also hold if $c = c^*(e)$ and if λ_c is replaced by $\lambda^* > 0$, which solves $k(\lambda^*) + \lambda^* c^*(e) = 0$ (the existence of such a λ^* is given by Proposition 2.2, provided there is a pulsating front with speed $c = c^*(e)$). Actually, in the case $c = c^*(e)$, the exponential lower bound $e^{\lambda^* s}$ will not be optimal as $s\to-\infty$, as the following proposition shows.

Proposition 3.3 *Under assumptions* (1.4) *and* (1.8), *if* $c = c^*(e)$, *then there is a unique* $\lambda^* > 0$ *such that* $k(\lambda^*) + c^*(e)\lambda^* = 0$ *and*

$$\liminf_{s\to-\infty}\left[\min_{(x,y)\in\overline{\Omega}}\left(\frac{\phi(s,x,y)}{|s|^{2m+1}\ e^{\lambda^* s}}\right)\right] > 0, \tag{3.11}$$

where $m\in\mathbb{N}$ *and* $2m+2$ *is the multiplicity of* λ^* *as a root of* $k(\lambda) + c^*(e)\lambda = 0$.

Proof. First, from Proposition 2.2, it follows that there exists a positive real number $\lambda^* > 0$ such that

$$k(\lambda^*) + c^*(e)\lambda^* = 0.$$

From Lemma 2.1, λ^* is then the unique root of $k(\lambda) + \lambda c = 0$, with multiplicity $2m+2$ for some $m\in\mathbb{N}$, in the sense of (2.4).

As already underlined, the function $\lambda \mapsto k(\lambda)$ is analytic, and, because of the normalization condition (1.18) and standard elliptic estimates, the principal eigenfunctions ψ_λ of the operators L_λ are also analytic with respect to λ in the spaces $C^{2,\alpha}(\overline{\Omega})$. For each $j \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, call $\psi_\lambda^{(j)}$ the j-th order derivative of ψ_λ with respect to λ, under the convention that $\psi_\lambda^{(0)} = \psi_\lambda$. All these functions are periodic and of class C^2 in $\overline{\Omega}$. Call also $L_\lambda^{(j)}$ the operator whose coefficients are the j-th order derivatives with respect to λ of the coefficients of L_λ. In other words,

$$L_\lambda^{(0)}\psi = L_\lambda\psi,\ L_\lambda^{(1)}\psi = 2eA\nabla\psi + [\nabla\cdot(Ae) - q\cdot e - 2\lambda eAe]\psi,\ L_\lambda^{(2)}\psi = -2eAe\ \psi$$

and $L_\lambda^{(j)}\psi = 0$ for all $j \geq 3$ and for all $\psi \in C^2(\overline{\Omega})$ and $\lambda \in \mathbb{R}$. Differentiating the relation $L_\lambda\psi_\lambda - k(\lambda)\psi_\lambda = 0$ with respect to λ yields

$$\left\{\begin{array}{l} L_\lambda\psi_\lambda^{(1)} - k(\lambda)\psi_\lambda^{(1)} + 2eA\nabla\psi_\lambda + [\nabla\cdot(Ae) - q\cdot e - 2\lambda eAe]\psi_\lambda - k'(\lambda)\psi_\lambda \\ = (L_\lambda - k(\lambda))\psi_\lambda^{(1)} + (L_\lambda^{(1)} - k'(\lambda))\psi_\lambda = 0, \\ L_\lambda\psi_\lambda^{(j)} - k(\lambda)\psi_\lambda^{(j)} + j\left(2eA\nabla\psi_\lambda^{(j-1)} + [\nabla\cdot(Ae) - q\cdot e - 2\lambda eAe]\psi_\lambda^{(j-1)}\right) \\ -j\ k'(\lambda)\psi_\lambda^{(j-1)} - 2\ \mathrm{C}_j^2\ eAe\ \psi_\lambda^{(j-2)} - \displaystyle\sum_{2\leq i\leq j} \mathrm{C}_j^i\, k^{(i)}(\lambda)\, \psi_\lambda^{(j-i)} \\ = (L_\lambda - k(\lambda))\psi_\lambda^{(j)} + j(L_\lambda^{(1)} - k'(\lambda))\psi_\lambda^{(j-1)} + \mathrm{C}_j^2\, L_\lambda^{(2)}\psi_\lambda^{(j-2)} - \displaystyle\sum_{2\leq i\leq j} \mathrm{C}_j^i\, k^{(i)}(\lambda)\, \psi_\lambda^{(j-i)} \\ = 0 \quad \text{for all } j \geq 2, \end{array}\right. \tag{3.12}$$

where $C_j^i = j!/(i!(j-i)!)$ for all integers i, j such that $i \leq j$. Similarly, since $\nu A\nabla\psi_\lambda = \lambda(\nu Ae)\psi_\lambda$ on $\partial\Omega$ for all $\lambda \in \mathbb{R}$, one gets that, for all $\lambda \in \mathbb{R}$,

$$\nu A\nabla\psi_\lambda^{(j)} - \lambda(\nu Ae)\psi_\lambda^{(j)} - j\ (\nu Ae)\psi_\lambda^{(j-1)} = 0 \text{ on } \partial\Omega, \text{ for all } j \geq 1. \tag{3.13}$$

Notic that all the arguments so far have not used the KPP assumption (1.8).

Let us now prove formula (3.11). Assume by contradiction that there exists a sequence $(s_n, x_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times \overline{\Omega}$ such that

$$s_n < 0 \text{ for all } n \in \mathbb{N},\ s_n \to -\infty \text{ and } \varepsilon_n := \phi(s_n, x_n, y_n)\ |s_n|^{-(2m+1)}\ e^{-\lambda^* s_n} \to 0^+ \text{ as } n \to +\infty.$$

It follows from (3.9) and for all $n \in \mathbb{N}$ and for all $(x, y) \in \overline{\Omega}$,

$$\phi(s_n, x, y) < \frac{2\ C_3\ \varepsilon_n}{\kappa^*}\ |s_n|^{2m+1}\ e^{\lambda^* s_n}\ \psi_{\lambda^*}(x, y), \tag{3.14}$$

where $C_3 > 0$ is given in (3.9) and $\kappa^* := \min_{\overline{\Omega}}\ \psi_{\lambda^*} > 0$.

For each $n \in \mathbb{N}$, call a_n the smallest positive number such that

$$b(a_n) := \frac{C_3\ \varepsilon_n\ |s_n|^{2m+1}\ a_n^{2m+1}\ e^{\lambda^* a_n/2}}{2^{2m}\ (a_n - s_n)^{2m+1}} = 1. \tag{3.15}$$

The positive real number a_n is well-defined, since the function b is continuous on $[0,+\infty)$, vanishes at 0 and converges to $+\infty$ at $+\infty$. Observe that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to $+\infty$ as $n\to+\infty$, otherwise, up to extraction of a subsequence, it would converge to a nonnegative real number, but the left-hand side of (3.15) would then converge to 0, since $\varepsilon_n\to 0$ and $s_n\to-\infty$ as $n\to+\infty$.

Then, for each $n\in\mathbb{N}$ and $(s,x,y)\in(-\infty,a_n/2]\times\overline{\Omega}$, call

$$\left\{\begin{array}{rcl} f_n(s,x,y) & = & \displaystyle\sum_{j=1}^{2m+1}(-1)^j\ \mathrm{C}_{2m+1}^j\ (a_n-s)^{2m+1-j}\ \psi_{\lambda^*}^{(j)}(x,y),\\ g_n(s,x,y) & = & (a_n-s)^{2m+1}\ \psi_{\lambda^*}(x,y)+f_n(s,x,y)\\ & = & \displaystyle\sum_{j=0}^{2m+1}(-1)^j\ \mathrm{C}_{2m+1}^j\ (a_n-s)^{2m+1-j}\ \psi_{\lambda^*}^{(j)}(x,y),\\ h_n(s,x,y) & = & \dfrac{4\ C_3\ \varepsilon_n\ |s_n|^{2m+1}}{\kappa^*\ (a_n-s_n)^{2m+1}}\ e^{\lambda^* s}\ g_n(s,x,y). \end{array}\right.$$

For all $s\le a_n/2$ $(<a_n)$ and $(x,y)\in\overline{\Omega}$, one has

$$\frac{|f_n(s,x,y)|}{(a_n-s)^{2m+1}\ \psi_{\lambda^*}(x,y)}\le\sum_{j=1}^{2m+1}\frac{\mathrm{C}_{2m+1}^j\ \|\psi_{\lambda^*}^{(j)}\|_\infty\ (a_n-s)^{-j}}{\kappa^*}\le\sum_{j=1}^{2m+1}\frac{\mathrm{C}_{2m+1}^j\ \|\psi_{\lambda^*}^{(j)}\|_\infty\ 2^j}{\kappa^*\ a_n^j}.$$

Since $a_n\to+\infty$ as $n\to+\infty$, it follows that, for n large enough,

$$|f_n(s,x,y)|\le\frac{1}{2}\ (a_n-s)^{2m+1}\ \psi_{\lambda^*}(x,y)\text{ for all }(s,x,y)\in(-\infty,a_n/2]\times\overline{\Omega},$$

whence

$$\frac{(a_n-s)^{2m+1}\ \psi_{\lambda^*}(x,y)}{2}\le g_n(s,x,y)\le\frac{3\ (a_n-s)^{2m+1}\ \psi_{\lambda^*}(x,y)}{2}\text{ in }(-\infty,a_n/2]\times\overline{\Omega}\quad(3.16)$$

for n large enough. In particular, for n large enough and for all $(x,y)\in\overline{\Omega}$,

$$h_n(a_n/2,x,y)\ge\frac{4\ C_3\ \varepsilon_n\ |s_n|^{2m+1}}{\kappa^*\ (a_n-s_n)^{2m+1}}\times e^{\lambda^* a_n/2}\times\frac{(a_n/2)^{2m+1}\ \psi_{\lambda^*}(x,y)}{2}\ge b(a_n)=1\qquad(3.17)$$

from (3.15). On the other hand, for all $(s,x,y)\in(-\infty,a_n/2]\times\overline{\Omega}$,

$$\begin{array}{rcl}\dfrac{\partial h_n}{\partial s}(s,x,y) & = & \dfrac{4\ C_3\ \varepsilon_n\ |s_n|^{2m+1}}{\kappa_1\ (a_n-s_n)^{2m+1}}\times e^{\lambda^* s}\times\\ & & \times\left[\lambda^* g_n(s,x,y)+\displaystyle\sum_{j=0}^{2m+1}(-1)^{j+1}\ \mathrm{C}_{2m+1}^j\ (2m+1-j)\ (a_n-s)^{2m-j}\ \psi_{\lambda^*}^{(j)}(x,y)\right].\end{array}$$

Therefore, as above, there holds, for n large enough,

$$\frac{\partial h_n}{\partial s}(s,x,y)\ge\frac{4\ C_3\ \varepsilon_n\ |s_n|^{2m+1}}{\kappa^*\ (a_n-s_n)^{2m+1}}\times e^{\lambda^* s}\times\frac{\lambda^*\ (a_n-s)^{2m+1}\psi_{\lambda^*}(x,y)}{2}>0\qquad(3.18)$$

in $(-\infty, a_n/2] \times \overline{\Omega}$.

Choose now $n_0 \in \mathbb{N}$ such that (3.16), (3.17) and (3.18) hold for all $n \geq n_0$. For each $n \geq n_0$ and $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$, call

$$\overline{\phi}_n(s,x,y) = \begin{cases} \min\left(h_n(s,x,y), 1\right) & \text{if } s < a_n/2, \\ 1 & \text{if } s \geq a_n/2, \end{cases}$$

and $\overline{u}_n(t,x,y) = \overline{\phi}_n(c^*(e)t - x \cdot e, x, y)$ for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$. If follows from the above facts that $\overline{\phi}_n$ is continuous in $\mathbb{R} \times \overline{\Omega}$, nondecreasing with respect to s, periodic in (x,y), and

$$\phi(s_n, x, y) < \overline{\phi}_n(s_n, x, y) \ \text{ for all } (x,y) \in \overline{\Omega}$$

from (3.14) and (3.16).

Lastly, when $\overline{u}_n(t,x,y) < 1$, then $ct - x\cdot e < a_n/2$ and $\overline{u}_n(t,x,y) = h_n(c^*(e)t - x\cdot e, x, y) > 0$ from (3.16). Furthermore, it is straightforward to check, from the definition of h_n and from (2.4), (3.12) and (3.13) (applied at $\lambda = \lambda^*$) that, for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$ such that $\overline{u}_n(t,x,y) < 1$,

$$(\overline{u}_n)_t - \nabla \cdot (A(x,y)\nabla \overline{u}_n) + q(x,y) \cdot \nabla \overline{u}_n = \zeta(x,y)\overline{u}_n(t,x,y) \geq f(x,y,\overline{u}_n(t,x,y))$$

from (1.8), and $\nu A(x,y)\nabla \overline{u}_n = 0$ if $(t,x,y) \in \mathbb{R} \times \partial\Omega$.

Eventually, one can apply part 1) of Lemma 3.1 with $c = c^*(e)$, $\underline{\phi} = \phi$, $\overline{\phi} = \overline{\phi}_n$ and $\sigma = s_n$, for each $n \geq n_0$. Therefore, for each $n \geq n_0$ and for each $(s,x,y) \in [s_n, +\infty) \times \overline{\Omega}$,

$$(0 <)\ \phi(s,x,y) \leq \overline{\phi}_n(s,x,y).$$

In particular, for each $n \geq n_0$ and for each $(x,y) \in \overline{\Omega}$, there holds

$$0 < \phi(0,x,y) \leq h_n(0,x,y) \leq \frac{4\ C_3\ \varepsilon_n\ |s_n|^{2m+1}}{\kappa^*\ (a_n - s_n)^{2m+1}} \times \frac{3\ a_n^{2m+1}\ \ \psi_{\lambda^*}(x,y)}{2}$$

from (3.16). From (3.15), one gets that

$$0 < \phi(0,x,y) \leq \frac{3 \times 2^{2m+1}}{\kappa^* \times e^{\lambda^* a_n/2}} \to 0^+ \text{ as } n \to +\infty$$

since $\lim_{n\to+\infty} a_n = +\infty$. Thus, $\phi(0,x,y) = 0$ for all $(x,y) \in \overline{\Omega}$. We have then reached a contradiction, whence formula (3.11) follows. □

The last proposition of this section is concerned with the general monostable case, that is we do not assume the KPP assumption (1.8). Instead, we assume that there is a pulsating front with a lower speed.

Proposition 3.4 *Under assumptions* (1.4)*,* (1.6) *and* (1.7)*, if there exists a pulsating front* $u'(t,x,y) = \phi'(c't - x \cdot e, x, y)$*, in the sense of Definition* 1.1*, with a speed* $c' < c$*, then* $c > c^*(e)$ *and*

$$\forall\ \eta > 0, \quad \liminf_{s\to-\infty} \left[\min_{(x,y)\in\overline{\Omega}} \left(\frac{\phi(s,x,y)}{e^{(\lambda_c+\eta)s}}\right)\right] > 0.$$

Proof. Step 1. First, from Proposition 2.2 applied to u', it follows that $c' \geq c^*(e)$, whence $c > c^*(e)$. Moreover,

$$0 < \lambda'_m := \liminf_{s \to -\infty} \left(\min_{(x,y)\in\overline{\Omega}} \frac{\phi'_s(s,x,y)}{\phi(s,x,y)} \right) \leq \limsup_{s \to -\infty} \left(\max_{(x,y)\in\overline{\Omega}} \frac{\phi'_s(s,x,y)}{\phi(s,x,y)} \right) =: \lambda'_M < +\infty \quad (3.19)$$

and $k(\lambda'_m) + c'\lambda'_m = k(\lambda'_M) + c'\lambda'_M = 0$. Since $-k(\lambda)/\lambda \geq c > c'$ for all $\lambda \in (0, \lambda_c]$, it follows that

$$0 < \lambda_c < \lambda'_m \leq \lambda'_M.$$

Because both ϕ and ϕ' satisfy (1.14), one can assume, even if it means shifting u' and ϕ', that

$$\phi'(0,x,y) < \phi(0,x,y) \ \text{ for all } (x,y) \in \overline{\Omega}. \quad (3.20)$$

Step 2. Assume that the conclusion of the proposition does not hold, for some $\eta > 0$. Since $-k(\lambda'_M)/\lambda'_M = c' < c$, one can assume without loss of generality that $\eta > 0$ is small enough so that

$$-\frac{k(\lambda'_M + \eta)}{\lambda'_M + \eta} < c. \quad (3.21)$$

From our assumption, there exists a sequence $(s_n, x_n, y_n) \in \mathbb{R} \times \overline{\Omega}$ such that

$$s_n < 0 \text{ for all } n, \ s_n \to -\infty \text{ and } \varepsilon_n := \phi(s_n, x_n, y_n)\ e^{-(\lambda_c+\eta)s_n} \to 0^+ \ \text{ as } n \to +\infty. \quad (3.22)$$

One can also assume, without loss of generality, that $0 < \varepsilon_n < 1$ for all $n \in \mathbb{N}$. Property (3.9) yields

$$\phi(s_n, x, y) \leq C_3\ \varepsilon_n\ e^{(\lambda_c+\eta)s_n} \ \text{ for all } (x,y) \in \overline{\Omega}. \quad (3.23)$$

Step 3. We now claim that

$$\forall\ \lambda' > \lambda'_M, \quad \lim_{n \to +\infty} \varepsilon_n\ e^{(\lambda_c - \lambda')s_n} = +\infty. \quad (3.24)$$

Otherwise, there is $\lambda' > \lambda'_M$ and a constant $M_0 > 0$ such that, up to extraction of a subsequence,

$$\varepsilon_n\ e^{\lambda_c s_n} \leq M_0\ e^{\lambda' s_n}.$$

Thus,

$$\phi(s_n, x, y) < 2\ C_3\ M_0\ e^{(\lambda'+\eta)s_n} \ \text{ for all } (x,y) \in \overline{\Omega}.$$

Since $2\ C_3\ M_0\ e^{(\lambda'+\eta)s_n} \to 0^+$ as $n \to +\infty$, one can assume that this quantity is less than 1, for n large enough. On the other hand, it follows from Proposition 1.2 applied to ϕ' that the continuous function ϕ'_s is positive in $\mathbb{R} \times \overline{\Omega}$. Then, for n large enough, there exists a, unique, $\tau_n \in \mathbb{R}$ such that

$$2\ C_3\ M_0\ e^{(\lambda'+\eta)s_n} = \min_{(x,y)\in\overline{\Omega}} \phi'(s_n + \tau_n, x, y).$$

Because of (3.19), there is $\underline{s} \in \mathbb{R}$ such that $\phi'_s(s,x,y)/\phi'(s,x,y) < \lambda'$ for all $(s,x,y) \in (-\infty, \underline{s}] \times \overline{\Omega}$. Thus

$$\phi'(s,x,y) \geq \phi'(\underline{s},x,y)\ e^{\lambda'(s-\underline{s})} \geq \gamma_0\ e^{\lambda' s} \ \text{ for all } (s,x,y) \in (-\infty, \underline{s}] \times \overline{\Omega},$$

where $\gamma_0 = e^{-\lambda' \underline{s}} \times \min_{(x,y)\in\overline{\Omega}} \ \phi'(\underline{s}, x, y) > 0$. If there exists $\tau \in \mathbb{R}$ such that, up to extraction of a subsequence, $\tau_n \geq \tau$, then

$$2\ C_3\ M_0\ e^{(\lambda'+\eta)s_n} = \min_{(x,y)\in\overline{\Omega}} \ \phi'(s_n+\tau_n, x, y) \geq \min_{(x,y)\in\overline{\Omega}} \ \phi'(s_n+\tau, x, y) \geq \gamma_0\ e^{\lambda'(s_n+\tau)}$$

for n large enough, since $s_n \to -\infty$ as $n \to +\infty$. One gets a contradiction as $n \to +\infty$, since $\eta > 0$. Therefore, $\tau_n \to -\infty$ as $n \to +\infty$.

One shall now apply part 2) of Lemma 3.1 with, for n large enough, $\sigma = s_n$, $\underline{\phi} = \phi$, $\underline{u} = u$, $\overline{\phi}_2 = \overline{u}_2 = 0$ and

$$\overline{\phi}_1(s, x, y) = \phi'(s+\tau_n, x, y), \quad \overline{u}_1(t, x, y) = u'\left(\frac{ct+\tau_n}{c'}, x, y\right).$$

Indeed, notice that, for n large enough, $\phi(s_n, x, y) < \phi'(s_n+\tau_n, x, y)$ in $\overline{\Omega}$ and there exists $\overline{\sigma}_n > 0$ ($> s_n$) such that

$$\phi'(s+\tau_n, x, y) > 1-\rho \ \text{ in } [\overline{\sigma}_n, +\infty) \times \overline{\Omega} \tag{3.25}$$

and $\phi(s_n, x, y) \leq \phi'(\overline{\sigma}_n+\tau_n, x, y)$ in $\overline{\Omega}$. Furthermore, $\phi'_s > 0$ in $\mathbb{R}\times\overline{\Omega}$, which yields (3.2). Lastly, let us check that $\overline{u}_1$ is a strict supersolution of (1.1) in $\mathbb{R}\times\overline{\Omega}$. First, $\nu A\nabla\overline{u}_1 = 0$ on $\mathbb{R}\times\partial\Omega$, and, if for any $(t, x, y) \in \mathbb{R}\times\overline{\Omega}$ we call $t' = (ct+\tau_n)/c'$, then

$$\begin{aligned}
&(\overline{u}_1)_t(t,x,y) - \nabla\cdot(A(x,y)\nabla\overline{u}_1(t,x,y)) + q(x,y)\cdot\nabla\overline{u}_1(t,x,y) - f(x,y,\overline{u}_1(t,x,y))\\
&\quad = \frac{c}{c'} \times u'_t(t',x,y) - \nabla\cdot(A(x,y)\nabla u'(t',x,y)) + q(x,y)\cdot\nabla u'(t',x,y) - f(x,y,u'(t',x,y))\\
&\quad = \frac{c-c'}{c'} \times u'_t(t',x,y) > 0
\end{aligned}$$

since $c > c'$ and $c'u'_t > 0$ since $\phi'_s > 0$. Part 2) of Lemma 3.1 then implies that, for n large enough, there exists $\tau_n^* \in [0, \overline{\sigma}_n - s_n]$ such that

$$\phi(s-\tau_n^*, x, y) \leq \phi'(s+\tau_n, x, y) \ \text{ in } [s_n+\tau_n^*, +\infty)\times\overline{\Omega}$$

and $\min_{(x,y)\in\overline{\Omega}} \ (\phi'(\overline{\sigma}_n+\tau_n, x, y) - \phi(\overline{\sigma}_n-\tau_n^*, x, y)) = 0$ if $\tau_n^* > 0$.

If $0 < \tau_n^* < \overline{\sigma}_n - s_n$, then

$$u\left(t-\frac{\tau_n^*}{c}, x, y\right) \leq u'\left(\frac{ct+\tau_n}{c'}, x, y\right) \ \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega} \text{ such that } ct - x\cdot e \geq s_n+\tau_n^*$$

with equality at a point (t_n, x_n, y_n) such that $ct_n - x_n\cdot e = \overline{\sigma}_n > s_n+\tau_n^*$. Since $u(t-\tau_n^*/c, x, y)$ is a solution of (1.1) while $u'((ct+\tau_n)/c', x, y)$ is a strict supersolution, the maximum principle and Hopf lemma lead to a contradiction.

If $\tau_n^* = \overline{\sigma}_n - s_n$ (> 0), then there exists a point $(x_n, y_n) \in \overline{\Omega}$ such that

$$\phi'(\overline{\sigma}_n+\tau_n, x_n, y_n) = \phi(\overline{\sigma}_n-\tau_n^*, x_n, y_n) = \phi(s_n, x_n, y_n).$$

But the left-hand side of this last equality is larger than the fixed quantity $1-\rho>0$ from (3.25) and the definition of ρ, while the right-hand side goes to 0 since $s_n \to -\infty$. Thus, the case $\tau_n^* = \overline{\sigma}_n - s_n$ is ruled out too.

Thus, $\tau_n^* = 0$, which means that

$$\phi(s,x,y) \le \phi'(s+\tau_n,x,y) \ \text{ in } [s_n,+\infty)\times\overline{\Omega}.$$

Since $s_n \to -\infty$ and $\tau_n \to -\infty$ as $n \to +\infty$, it follows that $\phi \le 0$ in $\mathbb{R}\times\overline{\Omega}$, which is a contradiction.

As a consequence, formula (3.24) is proved.

Step 4. For each n, call μ_n the unique real number such that

$$\varepsilon_n\, e^{(\lambda_c+\eta)s_n} = \sqrt{\varepsilon_n}\, e^{\mu_n s_n}. \tag{3.26}$$

This is indeed possible since each s_n is negative. Since $0<\varepsilon_n<1$, one gets that

$$\mu_n > \lambda_c+\eta > 0 \ \text{ for all } n\in\mathbb{N}.$$

Furthermore, Step 3 implies that $\lim_{n\to+\infty} \varepsilon_n\, e^{(\lambda_c-\lambda')s_n} = +\infty$ for all $\lambda' > \lambda'_M$, that is

$$\forall\ \lambda' > \lambda'_M, \quad \lim_{n\to+\infty} e^{(2(\mu_n-\lambda_c-\eta)+\lambda_c-\lambda')s_n} = +\infty.$$

Hence, $\limsup_{n\to+\infty} \mu_n \le \lambda_c+\eta+(\lambda'-\lambda_c)/2$ for all $\lambda'>\lambda'_M$. Therefore,

$$\limsup_{n\to+\infty}\ \mu_n \le \lambda_c+\eta+\frac{\lambda'_M-\lambda_c}{2} < \lambda'_M+\eta$$

since $\lambda_c < \lambda'_M$. Thus, for n large enough, there holds

$$(\lambda_c <)\ \lambda_c+\eta < \mu_n < \lambda'_M+\eta. \tag{3.27}$$

But the function $\lambda \mapsto -k(\lambda)$ is convex, and it satisfies $-k(\lambda_c) = c\lambda_c$ and $-k(\lambda'_M+\eta) < c(\lambda'_M+\eta)$ from (3.21). Consequently, $-k(\lambda) < c\lambda$ for all $\lambda\in[\lambda_c+\eta,\lambda'_M+\eta]$, and by continuity, there exists then $k_0>0$ such that $k(\lambda)+c\lambda \ge k_0$ for all $\lambda\in[\lambda_c+\eta,\lambda'_M+\eta]$. In particular,

$$k(\mu_n)+c\mu_n \ge k_0 > 0 \ \text{ for } n \text{ large enough.} \tag{3.28}$$

Step 5. Let us now check that all assumptions of part 2) of Lemma 3.1 are fulfilled, for n large enough, with $\sigma = s_n$, $\underline{\phi} = \phi$, $\underline{u} = u$, $\overline{\phi}_1 = \phi'$, $\overline{u}_1(t,x,y) = u'(ct/c',x,y)$,

$$\overline{\phi}_2(s,x,y) = \frac{2\ C_3}{\underline{\kappa}_n}\ \sqrt{\varepsilon_n}\ e^{\mu_n s}\psi_{\mu_n}(x,y) \ \text{ and } \ \overline{u}_2(t,x,y) = \overline{\phi}_2(ct-x\cdot e,x,y),$$

where $\underline{\kappa}_n = \min_{(x,y)\in\overline{\Omega}}\ \psi_{\mu_n}(x,y)\ \in (0,1]$. Call $\overline{\phi} = \overline{\phi}_1+\overline{\phi}_2$ and $\overline{u} = \overline{u}_1+\overline{u}_2$. The functions $\underline{\phi}$, $\overline{\phi}_1$ and $\overline{\phi}_2$ are continuous, periodic in (x,y), and $\overline{\phi}_1$ and $\overline{\phi}_2$ are nondecreasing with respect to s, which yields (3.2) (actually, both $\overline{\phi}_1$ and $\overline{\phi}_2$ are increasing in s). One has $\underline{\phi}(s,x,y)<1$ in $\mathbb{R}\times\overline{\Omega}$ and $\underline{u}$ is a solution of (1.1). At s_n, there holds

$$\underline{\phi}(s_n,x,y) < 2\ C_3\ \varepsilon_n\ e^{(\lambda_c+\eta)s_n} = 2\ C_3\ \sqrt{\varepsilon_n}\ e^{\mu_n s_n} \le \overline{\phi}_2(s_n,x,y) \le \overline{\phi}(s_n,x,y)$$

for all $(x,y) \in \overline{\Omega}$, from (3.23) and (3.26). Since $\overline{\phi}_1(s,x,y) \overset{<}{\to} 1$ as $s \to +\infty$ uniformly in $(x,y) \in \overline{\Omega}$, there exists a positive number $\overline{\sigma} > 0$ (which does not depend on n) such that

$$\overline{\phi}_1(s,x,y) > 1 - \rho > 0 \;\text{ in } [\overline{\sigma}, +\infty) \times \overline{\Omega}, \tag{3.29}$$

and for n large enough,

$$\underline{\phi}(s_n,x,y) \le \overline{\phi}_1(\overline{\sigma},x,y) \le \overline{\phi}(\overline{\sigma},x,y) \;\text{ in } \overline{\Omega},$$

since $s_n \to -\infty$ as $n \to +\infty$ and $\underline{\phi}(-\infty,\cdot,\cdot) = 0$. As already underlined, the function $\overline{u}_1$ is a (strict) supersolution of (1.1).

It only remains to check that, for n large enough, $\overline{u}$ is a strict supersolution of (1.1) in the domain

$$\Omega^{\overline{\sigma}} = \{(t,x,y) \in \mathbb{R} \times \overline{\Omega},\; ct - x\cdot e \le \overline{\sigma}\}.$$

First, one has that $\nu A \nabla \overline{u} = 0$ on $\mathbb{R} \times \partial\Omega$. For any $(t,x,y) \in \Omega^{\overline{\sigma}}$,

$$0 < \overline{u}(t,x,y) \le \phi'(\overline{\sigma},x,y) + \frac{2\;C_3}{\underline{\kappa}_n}\;\sqrt{\varepsilon_n}\;e^{\mu_n\overline{\sigma}}.$$

Because of (3.27) (which holds for n large enough), the constants $\underline{\kappa}_n$ are bounded from below by a positive constant, namely there exists $\underline{\kappa} > 0$ such that

$$0 < \underline{\kappa} \le \underline{\kappa}_n \le 1 \tag{3.30}$$

for n large enough. Thus, $(2C_3/\underline{\kappa}_n)\;\sqrt{\varepsilon_n}\;e^{\mu_n\overline{\sigma}} \to 0$ as $n \to +\infty$ (remember that $\varepsilon_n \to 0^+$ as $n \to +\infty$), and $\max_{(t,x,y)\in\Omega^{\overline{\sigma}}}\;\overline{u}(t,x,y) \le 1$ for n large enough. For such n, and for any $(t,x,y) \in \Omega^{\overline{\sigma}}$, one has

$$\begin{aligned} &\overline{u}_t - \nabla\cdot(A(x,y)\nabla\overline{u}) + q(x,y)\cdot\nabla\overline{u} - f(x,y,\overline{u}) \\ &\quad = \frac{2\;C_3\;\sqrt{\varepsilon_n}}{\underline{\kappa}_n}\;(c\mu_n + k(\mu_n) + \zeta(x,y))\;e^{\mu_n(ct-x\cdot e)}\;\psi_{\mu_n}(x,y) + \frac{c-c'}{c'}\;u_t'(ct/c',x,y) \\ &\qquad\qquad + f(x,y,\overline{u}_1) - f(x,y,\overline{u}). \end{aligned} \tag{3.31}$$

From assumption (1.7) (with $p^- = 0$), and since $\frac{\partial f}{\partial u}$ is globally bounded in, say, $\overline{\Omega} \times [0,1]$, there exists $\delta > 0$ such that

$$\forall\; 0 \le \xi' \le \xi \le 1,\; \forall\;(x,y) \in \overline{\Omega},\quad f(x,y,\xi) - f(x,y,\xi') \le (\zeta(x,y) + \delta\xi^\beta) \times (\xi - \xi').$$

Thus, for n large enough,

$$f(x,y,\overline{u}_1(t,x,y)) - f(x,y,\overline{u}(t,x,y)) \ge -(\zeta(x,y) + \delta\overline{u}^\beta(t,x,y)) \times \frac{2\;C_3\;\sqrt{\varepsilon_n}}{\underline{\kappa}_n}\;e^{\mu_n(ct-x\cdot e)}\psi_{\mu_n}(x,y)$$

for all $(t,x,y) \in \Omega^{\overline{\sigma}}$. From (3.28) and (3.31), it follows that, for n large enough,

$$\begin{aligned} &\overline{u}_t - \nabla\cdot(A(x,y)\nabla\overline{u}) + q(x,y)\cdot\nabla\overline{u} - f(x,y,\overline{u}) \\ &\quad \ge 2\;C_3\;\left(k_0 - \delta\;\underline{\kappa}^{-1}\;\overline{u}^\beta(t,x,y)\right)\;\sqrt{\varepsilon_n}\;e^{\mu_n(ct-x\cdot e)}\;\psi_{\mu_n}(x,y) + \frac{c-c'}{c'}\;u_t'(ct/c',x,y) \end{aligned}$$

in $\Omega^{\overline{\sigma}}$. Fix a real number $\underline{\sigma} < \overline{\sigma}$ such that $\delta\ \underline{\kappa}^{-1}\ \phi'(s,x,y)^\beta \le k_0/2$ in $(-\infty,\underline{\sigma}] \times \overline{\Omega}$. Since

$$\max_{(s,x,y)\in(-\infty,\underline{\sigma}]\times\overline{\Omega}} \overline{\phi}_2(s,x,y) \le 2\ C_3\ \underline{\kappa}^{-1}\ \sqrt{\varepsilon_n}\ e^{\mu_n\underline{\sigma}} \to 0^+ \text{ as } n \to +\infty,$$

(because $\varepsilon_n \to 0$ and (3.27) holds), it follows that, for n large enough, $\delta\ \underline{\kappa}^{-1}\ \overline{u}^\beta(t,x,y) \le k_0$ for all (t,x,y) such that $ct - x\cdot e \le \underline{\sigma}$. Remember that $((c-c')/c')u'_t > 0$. Hence, for n large enough,

$$\overline{u}_t - \nabla\cdot(A(x,y)\nabla\overline{u}) + q(x,y)\cdot\nabla\overline{u} - f(x,y,\overline{u}) > 0 \ \text{ if } ct - x\cdot e \le \underline{\sigma}.$$

Lastly, when $\underline{\sigma} \le ct - x\cdot e \le \overline{\sigma}$,

$$\begin{aligned} &\overline{u}_t - \nabla\cdot(A(x,y)\nabla\overline{u}) + q(x,y)\cdot\nabla\overline{u} - f(x,y,\overline{u}) \\ &\qquad \ge 2\ C_3\,(k_0\ \underline{\kappa}\ e^{\mu_n\underline{\sigma}} - \delta\ \underline{\kappa}^{-1}\ e^{\mu_n\overline{\sigma}})\,\sqrt{\varepsilon_n} + \frac{c-c'}{c'}\ u'_t(ct/c',x,y). \end{aligned} \tag{3.32}$$

Since $c - c' > 0$ and since the function ϕ'_s is positive (from Proposition 1.2), continuous, and periodic in (x,y), there is $\eta_0 > 0$ such that

$$\frac{c-c'}{c'}\ u'_t(ct/c',x,y) = (c-c')\ \phi'_s(ct - x\cdot e, x, y) \ge \eta_0 > 0$$

for all (t,x,y) such that $\underline{\sigma} \le ct - x\cdot e \le \overline{\sigma}$. One concludes from (3.27) and (3.32) that, for n large enough,

$$\overline{u}_t - \nabla\cdot(A(x,y)\nabla\overline{u}) + q(x,y)\cdot\nabla\overline{u} - f(x,y,\overline{u}) \ge \frac{\eta_0}{2} > 0$$

for all (t,x,y) such that $\underline{\sigma} \le ct - x\cdot e \le \overline{\sigma}$. Eventually, for n large enough,

$$\overline{u}_t - \nabla\cdot(A(x,y)\nabla\overline{u}) + q(x,y)\cdot\nabla\overline{u} - f(x,y,\overline{u}) > 0 \ \text{ in } \Omega^{\overline{\sigma}}.$$

Step 6. As a conclusion, part 2) of Lemma 3.1 can be applied, for n large enough, with $\sigma = s_n < 0$, $\underline{\phi} = \phi$, $\overline{\phi}_1 = \phi'$, $\overline{\phi}_2(s,x,y) = (2\ C_3/\underline{\kappa}_n)\ \sqrt{\varepsilon_n}\ e^{\mu_n s}\ \psi_{\mu_n}(x,y)$ and our fixed number $\overline{\sigma} > 0$. Thus, for n large enough, there exists $\tau^*_n \in [0, \overline{\sigma} - s_n]$ such that

$$\left\{\begin{array}{lll} \phi(s-\tau^*_n,x,y) \le \phi'(s,x,y) + \dfrac{2\ C_3\ \sqrt{\varepsilon_n}}{\underline{\kappa}_n}\ e^{\mu_n s}\ \psi_{\mu_n}(x,y) & \text{in } [s_n+\tau^*_n, \overline{\sigma}] \times \overline{\Omega}, \\ \phi(s-\tau^*_n,x,y) \le \phi'(s,x,y) & \text{in } [\overline{\sigma},+\infty) \times \overline{\Omega} \end{array}\right. \tag{3.33}$$

and

$$\min_{(x,y)\in\overline{\Omega}} \ (\phi'(\overline{\sigma},x,y) - \phi(\overline{\sigma}-\tau^*_n,x,y)) = 0 \ \text{ if } \tau^*_n > 0.$$

If $\tau^*_n \to +\infty$, up to extraction of a subsequence, then there exists a sequence (x_n,y_n) in $\overline{C}$ such that

$$\phi'(\overline{\sigma},x_n,y_n) = \phi(\overline{\sigma}-\tau^*_n,x_n,y_n)$$

for n large enough. But the right-hand would then converge to 0 as $n \to +\infty$, while the left-hand side is bounded from below by $\min_{(x,y)\in\overline{\Omega}}\ \phi'(\overline{\sigma},x,y) > 0$. This case is ruled out.

Thus the sequence $(\tau_n^*)_{n\in\mathbb{N}}$ is bounded, and up to extraction of a subsequence, it converges to a real number $\tau^* \geq 0$. If $\tau^* = 0$, then, by passing to the limit as $n \to +\infty$ in (3.33) at $s = 0$, it follows from (3.30) and the limits $\lim_{n\to+\infty} \varepsilon_n = 0$ and $\lim_{n\to+\infty}(s_n + \tau_n^*) = -\infty$, that $\phi(0, x, y) \leq \phi'(0, x, y)$ in $\overline{\Omega}$. This is impossible from our normalization (3.20).

Therefore, up to extraction of a subsequence, $\tau_n^* \to \tau^* \in (0, +\infty)$ as $n \to +\infty$. For n large enough, τ_n^* is positive, and then there exists a point $(x_n, y_n) \in \overline{\Omega}$ such that $\phi(\overline{\sigma} - \tau_n^*, x_n, y_n) = \phi'(\overline{\sigma}, x_n, y_n)$. One can assume without loss of generality that $(x_n, y_n) \to (x_\infty, y_\infty) \in \overline{C}$ as $n \to +\infty$, whence

$$\phi(\overline{\sigma} - \tau^*, x_\infty, y_\infty) = \phi'(\overline{\sigma}, x_\infty, y_\infty).$$

On the other hand, as above, and since the real numbers μ_n satisfy (3.27), it follows that

$$\phi(s - \tau^*, x, y) \leq \phi'(s, x, y) \ \text{ for all } (s, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

In other words, $u(t - \tau^*/c, x, y) \leq u'(ct/c', x, y)$ for all $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$ with equality at the point $((\overline{\sigma} + x_\infty \cdot e)/c, x_\infty, y_\infty)$. But u is a solution of (1.1), while $u'(ct/c', x, y)$ is a strict supersolution, as already underlined. The maximum principle and Hopf lemma lead to a contradiction.

One concludes that our assumption (3.22) cannot be satisfied and the proof of Proposition 3.4 is now complete. □

Remark 3.5 Since the conclusion of Proposition 3.4 holds for all $\eta > 0$, it follows immediately that, under the same assumptions, then

$$\liminf_{s\to-\infty} \left[\min_{(x,y)\in\overline{\Omega}} \left(\frac{\phi(s, x, y)}{e^{(\lambda_c+\eta)s}} \right) \right] = +\infty \ \text{ for all } \eta > 0.$$

4 Exponential upper bounds of $\phi(s, x, y)$ as $s \to -\infty$

In this section, given a pulsating travelling front $u(t, x) = \phi(ct - x \cdot e, x, y)$ in the sense of Definition 1.1, we shall now construct suitable sub-solutions for ϕ in domains of the type $[\sigma, +\infty) \times \overline{\Omega}$. These estimates will then provide sharp exponential upper bounds as $s \to -\infty$. To do so, we first prove a comparison lemma, which can be viewed as the counterpart of Lemma 3.1.

Lemma 4.1 *Let $\overline{u}(t, x, y) = \overline{\phi}(ct - x \cdot e, x, y)$ be a continuous function defined in $\mathbb{R} \times \overline{\Omega}$ such that $\overline{\phi}(s, x, y)$ is periodic in (x, y) and $\overline{u}$ is a classical supersolution of*

$$\begin{cases} \overline{u}_t - \nabla \cdot (A(x, y)\nabla \overline{u}) + q(x, y) \cdot \nabla \overline{u} \geq f(x, y, \overline{u}) & \text{in } \mathbb{R} \times \overline{\Omega}, \\ \nu A \nabla \overline{u} \geq 0 & \text{on } \mathbb{R} \times \partial\Omega. \end{cases} \tag{4.1}$$

Assume that $\overline{\phi}(s, x, y) > 0$ in $\mathbb{R} \times \overline{\Omega}$ and that $\liminf_{s\to+\infty} \left(\min_{(x,y)\in\overline{\Omega}} \overline{\phi}(s, x, y) \right) \geq 1$. Let $\underline{u}(t, x, y) = \underline{\phi}(ct - x \cdot e, x, y)$ be a continuous function defined in $\mathbb{R} \times \overline{\Omega}$ such that $\underline{\phi}(s, x, y)$ is *periodic in* (x,y) *and* $\sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}}\ \underline{\phi}(s,x,y)<1$. *If there exists* $\sigma\in\mathbb{R}$ *such that* (3.1) *and* (3.2) *hold and*

$$\begin{cases} \underline{u}_t-\nabla\cdot(A(x,y)\nabla\underline{u})+q(x,y)\cdot\nabla\underline{u}\le f(x,y,\underline{u}) & \text{in } \Omega', \\ \nu A\nabla\underline{u}\le 0 \quad \text{on } (\mathbb{R}\times\partial\Omega)\cap\Omega', \end{cases} \tag{4.2}$$

where

$$\Omega'=\{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ ct-x\cdot e\ge\sigma \text{ and } \underline{u}(t,x,y)>0\},$$

then

$$\underline{\phi}(s,x,y)<\overline{\phi}(s,x,y) \quad \text{for all } s\ge\sigma \text{ and } (x,y)\in\overline{\Omega}.$$

Proof. This time, we choose to slide the function $\overline{\phi}$ with respect to s, to the left and then back to the right, and we compare it to the function $\underline{\phi}$.

First, since $\liminf_{s\to+\infty}\Big(\min_{(x,y)\in\overline{\Omega}}\ \overline{\phi}(s,x,y)\Big)\ge 1$ and $\sup_{\mathbb{R}\times\overline{\Omega}}\ \underline{\phi}<1$, it follows that there exists $\tau_0\ge 0$ such that

$$\underline{\phi}(s,x,y)\le\overline{\phi}(s+\tau,x,y) \text{ for all } (s,x,y)\in[\sigma,+\infty)\times\overline{\Omega} \text{ and for all } \tau\ge\tau_0.$$

Define

$$\tau^*=\inf\left\{\tau>0,\ \underline{\phi}(s,x,y)\le\overline{\phi}(s+\tau,x,y) \text{ for all } (s,x,y)\in[\sigma,+\infty)\times\overline{\Omega}\right\}.$$

One has $\tau^*\in[0,\tau_0]$ and

$$\underline{\phi}(s,x,y)\le\overline{\phi}(s+\tau^*,x,y) \text{ for all } (s,x,y)\in[\sigma,+\infty)\times\overline{\Omega}.$$

Assume that $\tau^*>0$. From the same reasons as above, there is then a point $(s^*,x^*,y^*)\in[\sigma,+\infty)\times\overline{\Omega}$ such that $\underline{\phi}(s^*,x^*,y^*)=\overline{\phi}(s^*+\tau^*,x^*,y^*)$. Since

$$\underline{\phi}(\sigma,x^*,y^*)<\overline{\phi}(\sigma,x^*,y^*)\le\overline{\phi}(\sigma+\tau^*,x^*,y^*) \tag{4.3}$$

from (3.1) and (3.2), one gets that $s^*>\sigma$.

Call now

$$\overline{U}(t,x,y)=\overline{\phi}(ct-x\cdot e+\tau^*,x,y)=\overline{u}\left(t+\frac{\tau^*}{c},x,y\right)$$

for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$. There holds

$$\underline{u}\le\overline{U} \text{ in } \Omega^\sigma=\{(t,x,y)\in\mathbb{R}\times\overline{\Omega},\ ct-x\cdot e\ge\sigma\}$$

with equality at the point

$$(t^*,x^*,y^*)=\left(\frac{s^*+x^*\cdot e}{c},x^*,y^*\right) \text{ such that } ct^*-x^*\cdot e=s^*>\sigma.$$

Moreover $\underline{u}(t^*,x^*,y^*)=\overline{U}(t^*,x^*,y^*)>0$. Since $\overline{U}$ still satisfies (4.1), it follows from the assumptions of Lemma 4.1 and from the strong maximum principle and Hopf lemma that

$\underline{u} = \overline{U}$ in the connected component of $\Omega^\sigma \cap \{\underline{u}(t,x,y) > 0\} \cap \{t \le t^*\}$ containing (t^*, x^*, y^*). If $c > 0$, then

$$\underline{u}(t, x^*, y^*) = \overline{U}(t, x^*, y^*) \text{ for all } t \in \left[\frac{\sigma + x^* \cdot e}{c}, t^*\right],$$

whence, at $t = (\sigma + x^* \cdot e)/c$,

$$\underline{\phi}(\sigma, x^*, y^*) = \overline{\phi}(\sigma + \tau^*, x^*, y^*),$$

which is impossible from (4.3). If $c < 0$, then $\underline{u}(t, x^*, y^*) = \overline{U}(t, x^*, y^*)$ for all $t \le t_*$, whence

$$\underline{\phi}(ct - x^* \cdot e, x^*, y^*) = \overline{\phi}(ct - x^* \cdot e + \tau^*, x^*, y^*) \text{ for all } t \le t^*.$$

As $t \to -\infty$, $\limsup_{t\to-\infty} \underline{\phi}(ct - x^* \cdot e, x^*, y^*) < 1$ while $\liminf_{t\to-\infty} \overline{\phi}(ct - x^* \cdot e + \tau^*, x^*, y^*) \ge 1$. One has again reached a contradiction.

As a conclusion, the assumption $\tau^* > 0$ cannot hold. Thus $\tau^* = 0$ and

$$\underline{\phi}(s, x, y) \le \overline{\phi}(s, x, y) \ \text{ for all } (s, x, y) \in [\sigma, +\infty) \times \overline{\Omega}.$$

Actually, the inequality is strict at $s = \sigma$, and if equality holds at point $(s, x, y) \in (\sigma, +\infty) \times \overline{\Omega}$, then the strong maximum principle and Hopf lemma lead as above to a contradiction. Thus, $\underline{\phi}(s, x, y) < \overline{\phi}(s, x, y)$ for all $(s, x, y) \in [\sigma, +\infty) \times \overline{\Omega}$ and the proof of Lemma 4.1 is complete. □

We shall now apply Lemma 4.1 to the different situations which are listed in Theorems 1.3 and 1.5, in order to get sharp exponential upper bounds for the function $\phi(s, x, y)$ as $s \to -\infty$. We first deal with the case when $c > c^*(e)$, with or without the KPP assumption (1.8). This case corresponds to both the first part of Theorem 1.3 and part a) of Theorem 1.5. Then, we shall treat separately the case when $c = c^*(e)$.

We are given a nonzero speed c such that $c > c^*(e)$, we assume that (1.4) holds (with $p^- = 0$) and we remind that $\lambda_c > 0$ is given by (1.20). We need a few more notations. First, there exists then

$$\mu > \lambda_c \ \text{ such that } \ -\frac{k(\mu)}{\mu} \ < \ c \ = \ -\frac{k(\lambda_c)}{\lambda_c}.$$

Call

$$\kappa_0 = \min_{(x,y)\in\overline{\Omega}} \ \psi_\mu(x, y) > 0 \ \text{ and } \kappa = \kappa_0 \times (k(\mu) + \mu c) > 0. \tag{4.4}$$

As done at the beginning of the proof of Proposition 2.5, it follows from (1.4) (and Proposition 2.2) that there exists $\Sigma > 0$ such that (2.20) holds, that is

$$\forall\ \tau \ge 0, \ \forall\ s \le -\Sigma, \ \forall\ (x, y) \in \overline{\Omega}, \quad \phi(s, x, y) \le \phi(s + \tau, x, y). \tag{4.5}$$

We also assume here that property (1.7) is satisfied, that is, with $p^- = 0$, the function $\frac{\partial f}{\partial u}$ is of class $C^{0,\beta}(\overline{\Omega} \times [0, \gamma])$, where β and γ are positive constants. We now fix β' such that

$$0 < \beta' \le \beta$$

and

$$\lambda_c + \lambda_c \beta' < \mu. \tag{4.6}$$

The function $\frac{\partial f}{\partial u}$ is of class $C^{0,\beta'}(\overline{\Omega} \times [0,\gamma])$. In particular, there is $\delta > 0$ such that

$$f(x,y,u) \geq \zeta(x,y)\ u - \delta\ u^{1+\beta'} \ \text{ for all } (x,y,u) \in \overline{\Omega} \times [0,\gamma]. \tag{4.7}$$

Even if it means decreasing $\gamma > 0$, one can assume without loss of generality that

$$0 < \gamma < 1. \tag{4.8}$$

Lastly, we call

$$D = \min\left(\kappa\ \delta^{-1}, \kappa_0\ \gamma^{\frac{\mu-\lambda_c}{\lambda_c}} \times \left[(\lambda_c/\mu)^{\frac{\lambda_c}{\mu-\lambda_c}} - (\lambda_c/\mu)^{\frac{\mu}{\mu-\lambda_c}}\right]^{\frac{\lambda_c-\mu}{\lambda_c}}\right) > 0. \tag{4.9}$$

All above constants are fixed in the sequel.

Corollary 4.2 *Under assumptions* (1.4), (1.7) *and* $c > c^*(e)$, *and under the above notations, if there are real numbers* σ, θ *and* ω *such that*

$$\sigma \leq -\Sigma, \quad 0 < \theta \leq 1, \quad \theta^{1+\beta'} \leq D\ \omega \tag{4.10}$$

and

$$\phi(\sigma,x,y) > \theta\ \psi_{\lambda_c}(x,y) - \omega\ \psi_\mu(x,y) \text{ for all } (x,y) \in \overline{\Omega}, \tag{4.11}$$

then

$$\phi(s,x,y) > \theta\ e^{\lambda_c(s-\sigma)}\ \psi_{\lambda_c}(x,y) - \omega\ e^{\mu(s-\sigma)}\ \psi_\mu(x,y)$$

for all $(s,x,y) \in [\sigma,+\infty) \times \overline{\Omega}$.

Proof. We are going to apply Lemma 4.1 to $\overline{u} = u$, $\overline{\phi} = \phi$,

$$\underline{\phi}(s,x,y) = \theta\ e^{\lambda_c(s-\sigma)}\ \psi_{\lambda_c}(x,y) - \omega\ e^{\mu(s-\sigma)}\ \psi_\mu(x,y),$$

$\underline{u}(t,x,y) = \underline{\phi}(ct - x\cdot e, x, y)$ and the real numbers σ which is given in (4.10) and (4.11). All assumptions related to $\overline{u}$ and $\overline{\phi}$ in Lemma 4.1 are immediately satisfied. In particular, property (3.2) holds from (4.5) and the inequality $\sigma \leq -\Sigma$. It remains to check that $\sup_{\mathbb{R}\times\overline{\Omega}}\ \underline{\phi} < 1$ and that $\underline{u}$ satisfies (4.2).

First, it is straightforward to see that, for all $s \in \mathbb{R}$ and for all $(x,y) \in \overline{\Omega}$,

$$\underline{\phi}(s,x,y) \leq \theta\ e^{\lambda_c(s-\sigma)} - \omega\kappa_0\ e^{\mu(s-\sigma)} \leq \left(\frac{\theta^\mu}{(\omega\kappa_0)^{\lambda_c}}\right)^{\frac{1}{\mu-\lambda_c}} \times \left[\left(\frac{\lambda_c}{\mu}\right)^{\frac{\lambda_c}{\mu-\lambda_c}} - \left(\frac{\lambda_c}{\mu}\right)^{\frac{\mu}{\mu-\lambda_c}}\right],$$

whence

$$\underline{\phi}(s,x,y) \leq \gamma < 1 \text{ for all } (s,x,y) \in \mathbb{R}\times\overline{\Omega} \tag{4.12}$$

from (4.8) and (4.10).

In the set $\Omega' = \{(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ ct - x \cdot e \geq \sigma \text{ and } \underline{u}(t,x,y) > 0\}$, it follows then from (1.20), (4.7) and (4.12) that

$$\begin{array}{rcl} \mathcal{L}\underline{u} & := & \underline{u}_t - \nabla \cdot (A(x,y)\nabla \underline{u}) + q(x,y) \cdot \nabla \underline{u} - f(x,y,\underline{u}(t,x,y)) \\ & = & \zeta(x,y)\ \underline{u}(t,x,y) - \omega\ (k(\mu) + \mu c)\ e^{\mu(ct-x\cdot e-\sigma)}\ \psi_\mu(x,y) - f(x,y,\underline{u}(t,x,y)) \\ & \leq & -\omega\ (k(\mu)+\mu c)\ e^{\mu(ct-x\cdot e-\sigma)}\ \psi_\mu(x,y) + \delta\ \theta^{1+\beta'}\ e^{\lambda_c(1+\beta')(ct-x\cdot e-\sigma)}. \end{array}$$

But $\lambda_c(1+\beta') \leq \mu$ from (4.6), and $ct - x \cdot e - \sigma \geq 0$ in Ω'. It then follows from (4.4) and (4.10) that, in Ω',

$$\mathcal{L}\underline{u}(t,x,y) \leq \left(-\omega\ \kappa + \delta\ \theta^{1+\beta'}\right) e^{\mu(ct-x\cdot e-\sigma)} \leq 0.$$

Moreover, $\nu A \nabla \underline{u} = 0$ on $\mathbb{R} \times \partial\Omega$.

Lemma 4.1 can then be applied, and it yields

$$\theta\ e^{\lambda_c(s-\sigma)}\ \psi_{\lambda_c}(x,y) - \omega\ e^{\mu(s-\sigma)}\ \psi_\mu(x,y) = \underline{\phi}(s,x,y) < \overline{\phi}(s,x,y) = \phi(s,x,y)$$

for all $(s,x,y) \in [\sigma, +\infty) \times \overline{\Omega}$, which is the desired conclusion. □

Proposition 4.3 *Under the same assumptions and notations as in Corollary* 4.2*, then*

$$\limsup_{s\to-\infty} \left[\max_{(x,y)\in\overline{\Omega}} \left(\frac{\phi(s,x,y)}{e^{\lambda_c s}} \right) \right] < +\infty.$$

Proof. Remember that $\mu - \lambda_c - \lambda_c\beta' > 0$ from (4.6), fix any real number m such that $m > 1$ and call

$$\beta_1 = \frac{\lambda_c \beta'}{\mu - \lambda_c - \lambda_c\beta'} > 0 \quad \text{and} \quad \beta_2 = \frac{\ln m}{\mu - \lambda_c - \lambda_c\beta'} > 0. \tag{4.13}$$

Set $\gamma_{m,0} = 1$ and, for each $n \in \mathbb{N}$, $n \geq 1$,

$$\gamma_{m,n} = (1 - m^{-1})^{-1} \times (1 - m^{-2})^{-1} \times \cdots \times (1 - m^{-n})^{-1}.$$

The sequence $(\gamma_{m,n})_{n\in\mathbb{N}}$ is increasing and it converges, as $n \to +\infty$, to the positive real number $\gamma_{m,\infty}$ defined by

$$\gamma_{m,\infty} = \prod_{n=1}^{\infty} (1 - m^{-n})^{-1}.$$

Then, fix $\eta > 0$ small enough so that

$$0 < \eta \leq \frac{1}{\gamma_{m,\infty}} \quad \text{and} \quad \beta_3 := \frac{\ln(D\kappa_1 \gamma_{m,\infty}^{-\beta'} \eta^{-\beta'})}{\mu - \lambda_c - \lambda_c\beta'} > \frac{(1+\beta_1)\beta_2}{\beta_1}, \tag{4.14}$$

where κ_1 and D are given in (3.10) and (4.9).

Lastly, choose $\sigma_0 = \sigma_0 \leq -\Sigma$ such that

$$\min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma_0,x,y)}{\psi_{\lambda_c}(x,y)} \leq \eta, \tag{4.15}$$

and define inductively σ_1, $\sigma_2, \dots$ by

$$\sigma_n - \sigma_{n-1} = \beta_1(\sigma_{n-1} - \sigma_0) + n\beta_2 - \beta_3 \tag{4.16}$$

for all $n \geq 1$. It is immediately found that, for all $n \geq 1$,

$$\begin{array}{rcl}\sigma_n - \sigma_{n-1} & = & \beta_2 \times \dfrac{(1+\beta_1)^n - 1}{\beta_1} - \beta_3 \times (1+\beta_1)^{n-1} \\ & \leq & (1+\beta_1)^{n-1} \times \dfrac{(1+\beta_1)\beta_2 - \beta_1\beta_3}{\beta_1} \;\leq\; \dfrac{(1+\beta_1)\beta_2 - \beta_1\beta_3}{\beta_1} \;=:\; \beta_4 \;<\; 0\end{array}$$

from (4.14). In particular, the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is decreasing and converges to $-\infty$ as $n \to +\infty$ (moreover, $\sigma_n - \sigma_{n-1} \sim \beta_4(1+\beta_1)^{n-1}$ and $\sigma_n \sim \beta_4(1+\beta_1)^n/\beta_1$ as $n \to +\infty$). Notice that the constants β_2, β_3, β_4, η and σ_n a priori depend on m.

Assume now, by contradiction, that there exists $n \in \mathbb{N}$ such that

$$\min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma_n, x, y)}{\psi_{\lambda_c}(x,y)} \leq \eta\ \gamma_{m,n}\ e^{\lambda_c(\sigma_n - \sigma_0)} \tag{4.17}$$

and that there exists $\sigma_{n+1} \leq \sigma < \sigma_n$ such that

$$\min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma, x, y)}{\psi_{\lambda_c}(x,y)} > \eta\ \gamma_{m,n+1}\ e^{\lambda_c(\sigma - \sigma_0)}.$$

We apply Corollary 4.2 with $\sigma(< \sigma_n \leq \sigma_0 \leq -\Sigma)$, $0 < \theta = \eta\ \gamma_{m,n+1}\ e^{\lambda_c(\sigma-\sigma_0)} \leq 1$ (from (4.14)) and $\omega = D^{-1}\ \theta^{1+\beta'} > 0$. Since

$$\phi(\sigma, x, y) > \theta\ \psi_{\lambda_c}(x,y) > \theta\ \psi_{\lambda_c}(x,y) - \omega\ \psi_\mu(x,y) \ \text{ for all } (x,y) \in \overline{\Omega},$$

one gets that

$$\phi(\sigma_n, x, y) > \theta\ e^{\lambda_c(\sigma_n - \sigma)}\ \psi_{\lambda_c}(x,y) - D^{-1}\ \theta^{1+\beta'}\ e^{\mu(\sigma_n-\sigma)}\ \psi_\mu(x,y) \ \text{ for all } (x,y) \in \overline{\Omega}. \tag{4.18}$$

Since $\mu - \lambda_c - \lambda_c\beta' > 0$ and $\sigma \geq \sigma_{n+1}$, it follows that

$$\begin{array}{rcl}\eta^{\beta'}\gamma_{m,\infty}^{\beta'}\ e^{-\lambda_c\beta'(\sigma_0-\sigma)+(\mu-\lambda_c)(\sigma_n-\sigma)} & \leq & \eta^{\beta'}\gamma_{m,\infty}^{\beta'}\ e^{-\lambda_c\beta'(\sigma_0-\sigma_{n+1})+(\mu-\lambda_c)(\sigma_n-\sigma_{n+1})} \\ & = & \eta^{\beta'}\gamma_{m,\infty}^{\beta'}\ e^{(\mu-\lambda_c-\lambda_c\beta')(\sigma_n-\sigma_{n+1})+\lambda_c\beta'(\sigma_n-\sigma_0)}.\end{array}$$

But

$$\eta^{\beta'}\gamma_{m,\infty}^{\beta'}\ e^{(\mu-\lambda_c-\lambda_c\beta')(\sigma_n-\sigma_{n+1})+\lambda_c\beta'(\sigma_n-\sigma_0)} = \frac{D\ \kappa_1}{m^{n+1}}$$

from (4.13), (4.14) and (4.16). Thus,

$$\begin{array}{rcl}\theta^{\beta'} e^{(\mu-\lambda_c)(\sigma_n-\sigma)} & = & \eta^{\beta'}\gamma_{m,n+1}^{\beta'} e^{-\lambda_c\beta'(\sigma_0-\sigma)+(\mu-\lambda_c)(\sigma_n-\sigma)} \\ & \leq & \eta^{\beta'}\gamma_{m,\infty}^{\beta'}\ e^{-\lambda_c\beta'(\sigma_0-\sigma)+(\mu-\lambda_c)(\sigma_n-\sigma)} \;\leq\; \dfrac{D\ \kappa_1}{m^{n+1}},\end{array}$$

whence

$$D^{-1}\ \theta^{1+\beta'}\ e^{\mu(\sigma_n-\sigma)} \le \frac{\kappa_1\ \theta\ e^{\lambda_c(\sigma_n-\sigma)}}{m^{n+1}}.$$

Since $\psi_\mu(x,y) \le 1$ and $\kappa_1 \le \psi_{\lambda_c}(x,y)$ for all $(x,y) \in \overline{\Omega}$, it follows that

$$D^{-1}\ \theta^{1+\beta'}\ e^{\mu(\sigma_n-\sigma)}\ \psi_\mu(x,y) \le \frac{\theta\ e^{\lambda_c(\sigma_n-\sigma)}\ \psi_{\lambda_c}(x,y)}{m^{n+1}} \ \text{ for all } (x,y) \in \overline{\Omega}.$$

One concludes from (4.18) that

$$\forall\ (x,y) \in \overline{\Omega},\ \phi(\sigma_n,x,y) > \theta\ e^{\lambda_c(\sigma_n-\sigma)}\ \psi_{\lambda_c}(x,y)\ (1-m^{-(n+1)}) = \eta\ \gamma_{m,n}\ e^{\lambda_c(\sigma_n-\sigma_0)}\ \psi_{\lambda_c}(x,y),$$

which is in contradiction with (4.17).

Therefore, if (4.17) holds, then

$$\forall\ \sigma \in [\sigma_{n+1},\sigma_n), \quad \min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma,x,y)}{\psi_{\lambda_c}(x,y)} \le \eta\ \gamma_{m,n+1}\ e^{\lambda_c(\sigma-\sigma_0)}. \tag{4.19}$$

Because of (4.15), it follows by induction that (4.19) holds for all $n \in \mathbb{N}$. Eventually, since $\gamma_{m,n} \le \gamma_{m,\infty}$ for all $n \in \mathbb{N}$ and $\sigma_n \to -\infty$ as $n \to +\infty$, one gets that

$$\forall\ \sigma \le \sigma_0, \quad \min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma,x,y)}{\psi_{\lambda_c}(x,y)} \le \eta\ \gamma_{m,\infty}\ e^{\lambda_c(\sigma-\sigma_0)}.$$

But $\phi > 0$ and $0 < \psi_{\lambda_c} \le 1$. From (3.9), one concludes that

$$\forall\ \sigma \le \sigma_0, \quad \max_{(x,y)\in\overline{\Omega}} \phi(\sigma,x,y) \le C_3\ \eta\ \gamma_{m,\infty}\ e^{\lambda_c(\sigma-\sigma_0)},$$

which completes the proof of Proposition 4.3. □

Remark 4.4 A byproduct of the proof of Proposition 4.3 is the following result: for any real number $m > 1$, there exists $\eta_m > 0$ such that for any $\eta \in (0,\eta_m)$ and $\sigma_0 \le -\Sigma$ satisfying

$$\min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma_0,x,y)}{\psi_{\lambda_c}(x,y)} \le \eta,$$

then

$$\forall\ \sigma \le \sigma_0, \quad \min_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma,x,y)}{\psi_{\lambda_c}(x,y)} \le \eta\ \gamma_{m,\infty}\ e^{\lambda_c(\sigma-\sigma_0)}.$$

Lastly, we consider the case $c = c^*(e)$, with or without the KPP assumption (1.8). Notice that in the proof of Proposition 3.3, we used the fact that all j-th order derivatives of $k(\lambda)$ at λ^* are zero for $j = 2,\ldots,2m+1$. But we did not use the fact that the derivative of order $2m+2$ is not zero (actually, it is negative). We will use this fact here to show that the lower bound obtained in Proposition 3.3 in the KPP case when $c = c^*(e)$ is actually optimal in the general monostable case.

Proposition 4.5 *Under assumptions* (1.4) *and* (1.7), *if* $c = c^*(e)$, *then*

$$\limsup_{s\to-\infty}\left[\max_{(x,y)\in\overline{\Omega}}\left(\frac{\phi(s,x,y)}{|s|^{2m+1}\ e^{\lambda^* s}}\right)\right] < +\infty, \tag{4.20}$$

where $\lambda^* > 0$ *and* $m \in \mathbb{N}$ *are the same as in Proposition* 3.3.

Proof. With the same notations as in Proposition 3.3, call, for all $(s,x,y) \in (-\infty,0]\times\overline{\Omega}$,

$$\phi_1(s,x,y) = e^{\mu(s-a)} \times \left[\sum_{j=0}^{2m+2} (-1)^j\ \mathrm{C}_{2m+2}^j\ (-s+a)^{2m+2-j}\ \psi_\mu^{(j)}(x,y)\right]$$

for some positive real numbers a and μ to be chosen later. Remember that, as soon as u is a pulsating front with speed $c^*(e)$, the numbers λ^* and m can be defined even without the KPP assumption (1.8), as already underlined in the beginning of the proof of Proposition 3.3. Let

$$u_1(t,x,y) = \phi_1(c^*(e)t - x\cdot e, x, y)$$

for all $(t,x,y) \in \mathbb{R}\times\overline{\Omega}$ such that $c^*(e)t - x\cdot e \le 0$. First, it follows from (3.13) that, if $(x,y)\in\partial\Omega$, then $\nu A(x,y)\nabla u_1(t,x,y) = 0$. On the other hand, for $s = c^*(e)t - x\cdot e \le 0$, there holds

$$\begin{aligned}
&(u_1)_t - \nabla\cdot(A(x,y)\nabla u_1) + q(x,y)\cdot\nabla u_1 - \zeta(x,y)\ u_1\\
&= e^{\mu(s-a)} \times \Bigg[\sum_{j=0}^{2m+2} (-1)^j\ \mathrm{C}_{2m+2}^j\ (-s+a)^{2m+2-j}\left(L_\mu\psi_\mu^{(j)} + \mu c^*(e)\psi_\mu^{(j)}\right)\\
&\qquad + \sum_{j=0}^{2m+1} (-1)^j\ \mathrm{C}_{2m+2}^j\ (2m+2-j)\ (-s+a)^{2m+1-j}\left(-L_\mu^{(1)}\psi_\mu^{(j)} - c^*(e)\psi_\mu^{(j)}\right)\\
&\qquad + \sum_{j=0}^{2m} (-1)^j\ \mathrm{C}_{2m+2}^j\ \frac{(2m+2-j)(2m+1-j)}{2}\ (-s+a)^{2m-j}\ L_\mu^{(2)}\psi_\mu^{(j)}\Bigg]\\
&= e^{\mu(s-a)} \times \Bigg\{\sum_{j=0}^{2m+2} (-1)^j\ \mathrm{C}_{2m+2}^j\ (-s+a)^{2m+2-j}\Big[L_\mu\psi_\mu^{(j)} + \mu c^*(e)\psi_\mu^{(j)}\\
&\qquad\qquad + j\left(L_\mu^{(1)}\psi_\mu^{(j-1)} + c^*(e)\psi_\mu^{(j-1)}\right) + \mathrm{C}_j^2\ L_\mu^{(2)}\psi_\mu^{(j-2)}\Big]\Bigg\}
\end{aligned}$$

under the convention that $\psi_\mu^{(-1)} = \psi_\mu^{(-2)} = 0$ and $\mathrm{C}_j^i = 0$ if $j < i$. From (3.12) at $\lambda = \mu$, it follows that

$$\begin{aligned}
&(u_1)_t - \nabla\cdot(A(x,y)\nabla u_1) + q(x,y)\cdot\nabla u_1 - \zeta(x,y)\ u_1\\
&\qquad = e^{\mu(s-a)} \times \left(\sum_{j=0}^{2m+2} (-1)^j\ \mathrm{C}_{2m+2}^j\ (-s+a)^{2m+2-j} R_j(x,y)\right),
\end{aligned}$$

where

$$R_j(x,y) = (k(\mu) + \mu c^*(e))\,\psi_\mu^{(j)} + j\,(k'(\mu) + c^*(e))\,\psi_\mu^{(j-1)} + \sum_{i=2}^{j}\mathrm{C}_j^i\ k^{(i)}(\mu)\ \psi_\mu^{(j-i)}.$$

Because of (2.4) and since $\psi_{\lambda^*}(x,y) \ge \kappa^* > 0$, there holds

$$R_0(x,y) \sim \frac{k^{(2m+2)}(\lambda^*)\ \psi_{\lambda^*}(x,y)}{(2m+2)!} \times (\mu-\lambda^*)^{2m+2} \ \text{ as } \mu \to \lambda^*,$$

uniformly in $(x,y) \in \overline{\Omega}$. Thus, we can fix the real number $\mu > 0$ so that

$$\lambda^* < \mu < \lambda^* + \lambda^* \beta$$

where $\beta > 0$ is given in (1.7), and, for all $(x,y) \in \overline{\Omega}$,

$$R_0(x,y) \le \frac{k^{(2m+2)}(\lambda^*)\ \psi_{\lambda^*}(x,y)}{(2m+2)!} \times \frac{(\mu-\lambda^*)^{2m+2}}{2} \le \frac{k^{(2m+2)}(\lambda^*)\ \kappa^*\ (\mu-\lambda^*)^{2m+2}}{2\ (2m+2)!} =: \kappa' < 0.$$

Call $\kappa_\mu = \min_{(x,y)\in\overline{\Omega}}\ \psi_\mu(x,y) > 0$. Then, we fix $a > 0$ so that

$$\left| \sum_{j=1}^{2m+2} (-1)^j\ \mathrm{C}_{2m+2}^j\ (-s+a)^{2m+2-j} R_j(x,y) \right| \le \frac{(-s+a)^{2m+2}\ |R_0(x,y)|}{2}$$

and

$$\left| \sum_{j=1}^{2m+2} (-1)^j\ \mathrm{C}_{2m+2}^j\ (-s+a)^{2m+2-j}\ \psi_\mu^{(j)}(x,y) \right| \le \frac{(-s+a)^{2m+2}\ \kappa_\mu}{2}$$

for all $(s,x,y) \in (-\infty,0] \times \overline{\Omega}$. Therefore,

$$\begin{cases} 0 < \dfrac{\kappa_\mu}{2} \times e^{\mu(s-a)}\ (-s+a)^{2m+2} \le \phi_1(s,x,y) \le \dfrac{3}{2} \times e^{\mu(s-a)}\ (-s+a)^{2m+2}, \\ (u_1)_t - \nabla\cdot(A(x,y)\nabla u_1) + q(x,y)\cdot\nabla u_1 - \zeta(x,y)\ u_1 \\ \qquad\qquad\qquad\qquad \le \dfrac{\kappa'}{2} \times e^{\mu(s-a)}\ (-s+a)^{2m+2} < 0 \end{cases} \tag{4.21}$$

for all $(s,x,y) = (c^*(e)t - x\cdot e, x, y) \in (-\infty,0] \times \overline{\Omega}$.

Now, for all $(s,x,y) \in (-\infty,0] \times \overline{\Omega}$, define

$$\phi_2(s,x,y) = e^{\lambda^* s} \times \left[\left(\sum_{j=0}^{2m+1} (-1)^j\ \mathrm{C}_{2m+1}^j\ (-s)^{2m+1-j}\ \psi_{\lambda^*}^{(j)}(x,y) \right) - M\ \psi_{\lambda^*}(x,y) \right],$$

where $M > 0$ is fixed so that

$$\frac{3}{2} \times e^{-\mu a}\ a^{2m+2} - \psi_{\lambda^*}^{(2m+1)}(x,y) \le M\ \kappa^* \le M\ \psi_{\lambda^*}(x,y) \ \text{ for all } (x,y) \in \overline{\Omega}, \tag{4.22}$$

and $\kappa^* = \min_{\overline{\Omega}}\ \psi_{\lambda^*} > 0$. As in Proposition 3.3, it follows from (3.12) and (3.13) applied at $\lambda = \lambda^*$ that the function

$$u_2(t,x,y) = \phi_2(c^*(e)t - x\cdot e, x, y)$$

satisfies

$$(u_2)_t - \nabla \cdot (A(x,y)\nabla u_2) + q(x,y) \cdot \nabla u_2 - \zeta(x,y)\ u_2 = 0 \tag{4.23}$$

for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$ such that $c^*(e)t - x \cdot e \leq 0$, and $\nu A \nabla u_2 = 0$ if $(x,y) \in \partial\Omega$.

For any positive real numbers b_1 and b_2, call, for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$,

$$\phi_{b_1,b_2}(s,x,y) = \begin{cases} \max\left(b_1\phi_1(s,x,y) + b_2\phi_2(s,x,y), 0\right) & \text{if } s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

It follows from the above definitions that the functions ϕ_{b_1,b_2} are periodic in (x,y), and continuous in $\mathbb{R} \times \overline{\Omega}$ as soon as $0 < b_1 \leq b_2$ (from (4.21) and (4.22)). Call $M_i = \|\phi_i\|_{L^\infty((-\infty,0]\times\overline{\Omega})} \in (0,+\infty)$ for $i = 1,2$ and observe that

$$0 \leq \phi_{b_1,b_2}(s,x,y) \leq b_1 M_1 + b_2 M_2 \ \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Owing to (4.21) and to the facts that $\mu > \lambda^*$ and $1 \geq \psi_{\lambda^*} \geq \kappa^* > 0$ in $\overline{\Omega}$, there exists $s_0 < 0$ such that

$$0 < \phi_1(s,x,y) \leq \phi_2(s,x,y) \leq 2\ e^{\lambda^* s}\ |s|^{2m+1} \ \text{ for all } (s,x,y) \in (-\infty, s_0] \times \overline{\Omega}. \tag{4.24}$$

For $0 < b_1 \leq b_2$, call

$$u_{b_1,b_2}(t,x,y) = \phi_{b_1,b_2}(c^*(e)t - x \cdot e, x, y) \ \text{ for all } (t,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Remember that, from (1.7) applied with $p^- = 0$, there exist $0 < \gamma < 1$ and $\delta > 0$ such that

$$f(x,y,v) \geq \zeta(x,y)\ v - \delta\ v^{1+\beta} \ \text{ for all } (x,y,v) \in \overline{\Omega} \times [0,\gamma]. \tag{4.25}$$

Take any

$$0 < b_1 \leq b_2 \leq \frac{\gamma}{M_1 + M_2}$$

and let $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$ be such that

$$u_{b_1,b_2}(t,x,y) > 0.$$

Then $s = c^*(e)t - x \cdot e < 0$ and $0 < u_{b_1,b_2}(t,x,y) \leq \gamma$. If $(x,y) \in \partial\Omega$, then $\nu A(x,y)\nabla u_{b_1,b_2}(t,x,y) = 0$. Furthermore, in the general case when $(x,y) \in \overline{\Omega}$, one has

$$\begin{aligned} (u_{b_1,b_2})_t &- \nabla \cdot (A(x,y)\nabla u_{b_1,b_2}) + q(x,y) \cdot \nabla u_{b_1,b_2} - f(x,y,u_{b_1,b_2}) \\ &\leq (u_{b_1,b_2})_t - \nabla \cdot (A(x,y)\nabla u_{b_1,b_2}) + q(x,y) \cdot \nabla u_{b_1,b_2} - \zeta(x,y)\ u_{b_1,b_2}(t,x,y) \\ &\quad + \delta\ (u_{b_1,b_2}(t,x,y))^{1+\beta} \\ &\leq \frac{b_1\ \kappa'}{2}\ e^{\mu(s-a)}\ (-s+a)^{2m+2} + \delta\ (u_{b_1,b_2}(t,x,y))^{1+\beta} \end{aligned} \tag{4.26}$$

from (4.21), (4.25) and (4.23). If $s = c^*(e)t - x \cdot e \leq s_0$, then (4.24) yields

$$\begin{aligned} (u_{b_1,b_2})_t &- \nabla \cdot (A(x,y)\nabla u_{b_1,b_2}) + q(x,y) \cdot \nabla u_{b_1,b_2} - f(x,y,u_{b_1,b_2}) \\ &\leq \frac{b_1\ \kappa'}{2}\ e^{\mu(s-a)}\ (-s+a)^{2m+2} + \delta\ (2\ b_2\ \phi_2(s,x,y))^{1+\beta} \\ &\leq \frac{b_1\ \kappa'}{2}\ e^{\mu(s-a)}\ (-s+a)^{2m+2} + \delta\ (4\ b_2)^{1+\beta}\ e^{\lambda^*(1+\beta)s}\ |s|^{(2m+1)(1+\beta)}. \end{aligned}$$

On the other hand, since μ was chosen so that $0 < \mu < \lambda^*(1+\beta)$, there exists a constant $M_3 > 0$ such that

$$\delta\ 4^{1+\beta}\ e^{\lambda^*(1+\beta)\xi}\ |\xi|^{(2m+1)(1+\beta)} \leq M_3 \times \frac{-\kappa'}{2} \times e^{\mu(\xi-a)}\ (-\xi+a)^{2m+2} \ \text{ for all } \xi \leq s_0$$

(remember that $\kappa' < 0$). Thus, if $s = c^*(e)t - x\cdot e \leq s_0$, then

$$(u_{b_1,b_2})_t - \nabla\cdot(A\nabla u_{b_1,b_2}) + q\cdot\nabla u_{b_1,b_2} - f(x,y,u_{b_1,b_2}) \leq \frac{\kappa'}{2} \times (b_1 - M_3\ b_2^{1+\beta}) \times e^{\mu(s-a)}(-s+a)^{2m+2}.$$

If $s_0 < s = c^*(e)t - x\cdot e < 0$, then (4.26) implies that

$$\begin{aligned}(u_{b_1,b_2})_t - \nabla\cdot(A(x,y)\nabla u_{b_1,b_2}) + q(x,y)\cdot\nabla u_{b_1,b_2} - f(x,y,u_{b_1,b_2})& \\ \leq \frac{b_1\ \kappa'}{2}\ e^{\mu(s_0-a)}\ a^{2m+2} + \delta\ (b_2(M_1+M_2))^{1+\beta}& \\ \leq \frac{\kappa'}{2} \times (b_1 - M_4\ b_2^{1+\beta}) \times e^{\mu(s_0-a)}\, a^{2m+2},&\end{aligned}$$

where

$$M_4 = 2\ \delta\ (M_1+M_2)^{1+\beta}\ |\kappa'|^{-1}\ e^{-\mu(s_0-a)}\ a^{-2m-2} > 0.$$

Call $M_5 = M_3 + M_4 > 0$. To sum up, it follows that if

$$0 < M_5\ b_2^{1+\beta} \leq b_1 \leq b_2 \leq \frac{\gamma}{M_1+M_2}, \tag{4.27}$$

then $0 \leq u_{b_1,b_2} \leq \gamma$ in $\mathbb{R}\times\overline{\Omega}$ and

$$(u_{b_1,b_2})_t - \nabla\cdot(A(x,y)\nabla u_{b_1,b_2}) + q(x,y)\cdot\nabla u_{b_1,b_2} - f(x,y,u_{b_1,b_2}) \leq 0 \text{ in } \Omega'$$

and $\nu A(x,y)\nabla u_{b_1,b_2}(t,x,y) = 0$ on $(\mathbb{R}\times\partial\Omega)\cap\Omega'$, where

$$\Omega' = \{(t,x,y) \in \mathbb{R}\times\overline{\Omega},\ u_{b_1,b_2}(t,x,y) > 0\}.$$

Lastly, fix $b_2 > 0$ small enough so that (4.27) holds with $b_1 = M_5\ b_2^{1+\beta}$. Because of (4.24) and $\phi(-\infty,x,y) = 0$ uniformly in $\overline{\Omega}$, there exists $\tau_0 \in \mathbb{R}$ such that

$$\max_{(x,y)\in\overline{\Omega}}\ \phi(s_0+\tau_0,x,y) \leq \max_{(x,y)\in\overline{\Omega}}\ (b_2\phi_2(s_0,x,y))\,. \tag{4.28}$$

Remember that $u(t,x,y) = \phi(c^*(e)t - x\cdot e, x, y)$ is a pulsating front with speed $c^*(e)$ in the sense of Definition 1.1. Notice that property (4.5) still holds for $\phi(s+\tau_0,x,y)$ when $c = c^*(e)$, from Proposition 2.2, that is there is $\Sigma > 0$ such that $\phi(s+\tau_0,x,y) \leq \phi(s+\tau+\tau_0,x,y)$ for all $\tau \geq 0$ and $(s,x,y) \in (-\infty,-\Sigma]\times\overline{\Omega}$. Assume now that there exists $\sigma < \min(-\Sigma, s_0)$ such that

$$\phi_{b_1,b_2}(\sigma,x,y) < \phi(\sigma+\tau_0,x,y) \ \text{ for all } (x,y)\in\overline{\Omega}.$$

It is then straightforward to check that all assumptions of Lemma 4.1 are satisfied with $c = c^*(e)$, $\overline{u}(t,x,y) = u(t + \tau_0/c^*(e), x, y)$, $\overline{\phi}(s,x,y) = \phi(s + \tau_0, x, y)$, $\underline{u} = u_{b_1,b_2}$, $\underline{\phi} = \phi_{b_1,b_2}$ and σ. It follows then from Lemma 4.1 that, in particular,

$$\phi_{b_1,b_2}(s_0, x, y) < \phi(s_0 + \tau_0, x, y) \text{ for all } (x,y) \in \overline{\Omega}.$$

Because of (4.24) and (4.27), one gets that

$$b_2 \phi_2(s_0, x, y) < \phi(s_0 + \tau_0, x, y),$$

which contradicts (4.28).

As a conclusion, for all $\sigma < \min(-\Sigma, s_0)$,

$$\min_{(x,y)\in\overline{\Omega}} \phi(\sigma + \tau_0, x, y) \le \max_{(x,y)\in\overline{\Omega}} \phi_{b_1,b_2}(\sigma, x, y) \le 4\ b_2\ e^{\lambda^* \sigma}\ |\sigma|^{2m+1}$$

from (4.24) and (4.27). Because of (3.9), formula (4.20) follows and the proof of Proposition 4.3 is complete. □

5 Exponential decay of $\phi(s,x,y)$ as $s \to -\infty$

This last section is devoted to the proofs of Theorems 1.3 and 1.5 about the exponential behavior or logarithmic equivalent of $\phi(s,x,y)$ as $s \to -\infty$.

Proof of Theorem 1.3. We assume that conditions (1.4), (1.7) and (1.8) are fulfilled. We will distinguish the cases when $c > c^*(e)$ or $c = c^*(e)$.

First case: $c > c^*(e)$. From Propositions 3.2 and 4.3, and the fact that $0 < \kappa_1 \le \psi_{\lambda_c} \le 1$ in $\overline{\Omega}$, one has

$$0 < B := \liminf_{s\to-\infty} \left(\min_{(x,y)\in\overline{\Omega}} \frac{\phi(s,x,y)}{e^{\lambda_c s}\ \psi_{\lambda_c}(x,y)} \right) \le \limsup_{s\to-\infty} \left(\max_{(x,y)\in\overline{\Omega}} \frac{\phi(s,x,y)}{e^{\lambda_c s}\ \psi_{\lambda_c}(x,y)} \right) =: B' < +\infty. \tag{5.1}$$

One shall now prove that $B = B'$, that is $\phi(s,x,y) \sim B e^{\lambda_c s} \psi_{\lambda_c}(x,y)$ as $s \to -\infty$.

Pick any $\varepsilon > 0$. There exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to -\infty$ as $n \to +\infty$ and

$$\min_{(x,y)\in\overline{\Omega}} \frac{\phi(s_n,x,y)}{\psi_{\lambda_c}(x,y)} \le \left(B + \frac{\varepsilon}{2}\right) e^{\lambda_c s_n} \text{ for all } n \in \mathbb{N}.$$

Under the notations of Proposition 4.3, choose $m > 1$ such that

$$\left(B + \frac{\varepsilon}{2}\right) \gamma_{m,\infty} \le B + \varepsilon.$$

This is indeed possible since $\gamma_{m,\infty} \to 1$ as $m \to +\infty$. Since $s_n \to -\infty$ as $n \to +\infty$, it follows from Remark 4.4 that there exists $n_0 \in \mathbb{N}$ such that

$$s_n \le -\Sigma \text{ and } 0 < \left(B + \frac{\varepsilon}{2}\right) e^{\lambda_c s_n} < \eta_m \text{ for all } n \ge n_0,$$

whence, for $n = n_0$,

$$\forall\ s \le s_{n_0}, \quad \min_{(x,y)\in\overline{\Omega}} \frac{\phi(s,x,y)}{\psi_{\lambda_c}(x,y)} < \Big(B + \frac{\varepsilon}{2}\Big)\, e^{\lambda_c s_{n_0}}\ \gamma_{m,\infty}\ e^{\lambda_c (s-s_{n_0})} \le (B+\varepsilon)\ e^{\lambda_c s}.$$

By definition of B, and since $\varepsilon > 0$ was arbitrary, one gets that

$$\min_{(x,y)\in\overline{\Omega}} \frac{\phi(s,x,y)}{e^{\lambda_c s}\ \psi_{\lambda_c}(x,y)} \to B \text{ as } s \to -\infty. \tag{5.2}$$

On the other hand, by definition of B', there is a sequence $(s'_n, x'_n, y'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$ such that

$$s'_n \to -\infty \ \text{ and } \ \frac{\phi(s'_n, x'_n, y'_n)}{e^{\lambda_c s'_n}\ \psi_{\lambda_c}(x'_n, y'_n)} \to B' \ \text{ as } \ n \to +\infty.$$

Since ϕ and ψ_{λ_c} are periodic in (x,y), one can assume without loss of generality that $(x'_n, y'_n) \in \overline{C}$ for all n, and that, up to extraction of a subsequence, $(x'_n, y'_n) \to (x'_\infty, y'_\infty) \in \overline{C}$ as $n \to +\infty$. Call

$$u_n(t,x,y) = \frac{\phi(ct - x\cdot e + s'_n, x, y)}{e^{\lambda_c (ct - x\cdot e + s'_n)}\ \psi_{\lambda_c}(x,y)} = \frac{u(t + s'_n/c, x, y)}{e^{\lambda_c (ct - x\cdot e + s'_n)}\ \psi_{\lambda_c}(x,y)}.$$

Since $L_{\lambda_c}\psi_{\lambda_c} + \lambda_c c \psi_{\lambda_c} = 0$ in $\overline{\Omega}$ and $\nu A \nabla \psi_{\lambda_c} = \lambda_c (\nu A e)\psi_{\lambda_c}$ on $\partial\Omega$, it follows from (1.1) that the functions u_n satisfy

$$\begin{array}{r}(u_n)_t - \nabla\cdot(A\nabla u_n) + 2\lambda_c eA\nabla u_n - 2\dfrac{\nabla\psi_{\lambda_c}}{\psi_{\lambda_c}} A\nabla u_n + q\cdot\nabla u_n \\ +\zeta\ u_n - \dfrac{f(x,y,u(t+s'_n/c,x,y))}{u(t+s'_n/c,x,y)}\ u_n = 0 \ \text{ in } \mathbb{R}\times\overline{\Omega},\end{array} \tag{5.3}$$

with $\nu A\nabla u_n = 0$ on $\mathbb{R}\times\partial\Omega$. From (5.1), and since $s'_n \to -\infty$, the functions (u_n) are locally bounded, and $u(t + s'_n/c, x, y) \to 0$ as $n \to +\infty$, locally uniformly in (t,x,y). From standard parabolic estimates, the functions u_n converge in $C^{1;2}_{t;(x,y),loc}(\mathbb{R}\times\overline{\Omega})$, up to extraction of a subsequence, to a function u_∞ satisfying

$$(u_\infty)_t - \nabla\cdot(A\nabla u_\infty) + 2\lambda_c eA\nabla u_\infty - 2\frac{\nabla\psi_{\lambda_c}}{\psi_{\lambda_c}} A\nabla u_\infty + q\cdot\nabla u_\infty = 0 \ \text{ in } \mathbb{R}\times\overline{\Omega}, \tag{5.4}$$

with $\nu A\nabla u_\infty = 0$ on $\mathbb{R}\times\partial\Omega$. From (5.1), the function u_∞ is trapped between B and B', that is $B \le u_\infty \le B'$ in $\mathbb{R}\times\overline{\Omega}$. Moreover, $u_\infty(x'_\infty\cdot e/c, x'_\infty, y'_\infty) = B'$ from the choice of the sequence (s'_n, x'_n, y'_n). The strong maximum principle and Hopf lemma imply that $u_\infty(t,x,y) = B'$ for all $t \le x'_\infty\cdot e/c$ and $(x,y)\in\overline{\Omega}$, and then for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$ by uniqueness of the Cauchy problem associated to (5.4). As a consequence,

$$\frac{\phi(s+s'_n, x, y)}{e^{\lambda_c (s+s'_n)}\ \psi_{\lambda_c}(x,y)} = u_n\left(\frac{s + x\cdot e}{c}, x, y\right) \to B' \ \text{ as } n \to +\infty,$$

locally uniformly in $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$. Since both ϕ and ψ_{λ_c} are periodic in (x,y), it follows that

$$\frac{\phi(s'_n, x, y)}{e^{\lambda_c s'_n}\ \psi_{\lambda_c}(x,y)} \to B' \ \text{ as } n \to +\infty, \text{ uniformly in } (x,y)\in\overline{\Omega}.$$

Because of (5.2), one concludes that $B = B'$, which yields

$$\frac{\phi(s,x,y)}{e^{\lambda_c s}\ \psi_{\lambda_c}(x,y)} \to B \ \text{ as } s \to -\infty, \text{ uniformly in } (x,y) \in \overline{\Omega}.$$

Second case: $c = c^*(e)$. From Propositions 3.3 and 4.5, and the fact that $0 < \kappa^* \le \psi_{\lambda^*} \le 1$ in $\overline{\Omega}$, one has

$$\begin{aligned} 0 < B^* := \liminf_{s\to-\infty} \left(\min_{(x,y)\in\overline{\Omega}} \frac{\phi(s,x,y)}{|s|^{2m+1}\ e^{\lambda^* s}\ \psi_{\lambda^*}(x,y)} \right) \\ \le \limsup_{s\to-\infty} \left(\max_{(x,y)\in\overline{\Omega}} \frac{\phi(s,x,y)}{|s|^{2m+1}\ e^{\lambda^* s}\ \psi_{\lambda^*}(x,y)} \right) =: B^{*\prime} < +\infty. \end{aligned} \tag{5.5}$$

One shall now prove that $B^* = B^{*\prime}$.

Fix any $\varepsilon \in (0, B^{*\prime})$. There exists a sequence $(s_k)_{k\in\mathbb{N}}$ such that $s_k < 0$ for all $k \in \mathbb{N}$, $s_k \to -\infty$ as $k \to +\infty$ and

$$\max_{(x,y)\in\overline{\Omega}} \frac{\phi(s_k,x,y)}{\psi_{\lambda^*}(x,y)} \ge \left(B^{*\prime} - \frac{\varepsilon}{3}\right)\ |s_k|^{2m+1}\ e^{\lambda^* s_k} \ \text{ for all } k \in \mathbb{N}. \tag{5.6}$$

Call

$$h(s,x,y) = \left(B^{*\prime} - \frac{2\,\varepsilon}{3}\right) \times e^{\lambda^* s} \times \left[\sum_{j=0}^{2m+1} (-1)^j\ \mathrm{C}_{2m+1}^j\ (a-s)^{2m+1-j}\ \psi_{\lambda^*}^{(j)}(x,y)\right]$$

for all $(s,x,y) \in (-\infty, 0] \times \overline{\Omega}$, and

$$\overline{\phi}(s,x,y) = \begin{cases} \min\left(h(s,x,y), 1\right) & \text{if } s < 0 \\ 1 & \text{if } s \ge 0, \end{cases}$$

for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$, where $a > 0$ is chosen large enough so that $\overline{\phi}$ is continuous in $\mathbb{R} \times \overline{\Omega}$ and nondecreasing with respect to s (hence, it is positive in $\mathbb{R} \times \overline{\Omega}$). As in the proof of Proposition 3.3, the function $\overline{u}(t,x,y) = \overline{\phi}(c^*(e)t - x\cdot e, x, y)$ is a solution of (1.1) in the domain where $\overline{u}(t,x,y) < 1$.

Assume now that there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n < s_0$ for all $n \in \mathbb{N}$, $\sigma_n \to -\infty$ as $n \to +\infty$, and

$$\phi(\sigma_n,x,y) < \overline{\phi}(\sigma_n,x,y) \ \text{ for all } n \in \mathbb{N} \text{ and } (x,y) \in \overline{\Omega}.$$

Thus, for each $n \in \mathbb{N}$, part 1) of Lemma 3.1 can be applied with $c = c^*(e)$, $\underline{u} = u$, $\underline{\phi} = \phi$, $\sigma = \sigma_n$, $\overline{\sigma} = 0$, $\overline{\phi}$ and $\overline{u}$. In particular, $\phi(s,x,y) < \overline{\phi}(s,x,y)$ for all $n \in \mathbb{N}$ and for all $(s,x,y) \in [\sigma_n, +\infty) \times \overline{\Omega}$. But there exists a sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that $\sigma_n \le s_{k_n}$ for all $n \in \mathbb{N}$ and $k_n \to +\infty$ as $n \to +\infty$. Since

$$\phi(s_{k_n},x,y) < \left(B^{*\prime} - \frac{2\,\varepsilon}{3}\right) \times e^{\lambda^* s_{n_k}} \times \left[\sum_{j=0}^{2m+1} (-1)^j\ \mathrm{C}_{2m+1}^j\ (a-s_{k_n})^{2m+1-j}\ \psi_{\lambda^*}^{(j)}(x,y)\right]$$

for all $n \in \mathbb{N}$ and $(x,y) \in \overline{\Omega}$, one gets that

$$\max_{(x,y)\in\overline{\Omega}} \frac{\phi(s_{k_n}, x, y)}{|s_{n_k}|^{2m+1} \ e^{\lambda^* s_{n_k}} \ \psi_{\lambda^*}(x,y)} < B^{*\prime} - \frac{\varepsilon}{3}$$

for n large enough, which is in contradiction with (5.6).

Therefore, there exists $M < s_0$ such that

$$\max_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma, x, y)}{\overline{\phi}(\sigma, x, y)} \geq 1 \ \text{ for all } \sigma \leq M.$$

Owing to the definition of $\overline{\phi}$, one gets that

$$\max_{(x,y)\in\overline{\Omega}} \frac{\phi(\sigma, x, y)}{|\sigma|^{2m+1} \ e^{\lambda^* \sigma} \ \psi_{\lambda^*}(x,y)} \geq B^{*\prime} - \varepsilon \ \text{ for all } \sigma \leq M',$$

for some $M' \leq M$. Since $\varepsilon > 0$ can be arbitrary small, one concludes from the definition of $B^{*\prime}$ that

$$\max_{(x,y)\in\overline{\Omega}} \frac{\phi(s, x, y)}{|s|^{2m+1} \ e^{\lambda^* s} \ \psi_{\lambda^*}(x,y)} \to B^{*\prime} \ \text{ as } s \to -\infty. \tag{5.7}$$

On the other hand, by definition of B^*, there is a sequence $(s_n, x_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$ such that $s_n < 0$ for all $n \in \mathbb{N}$,

$$s_n \to -\infty \ \text{ and } \ \frac{\phi(s_n, x_n, y_n)}{|s_n|^{2m+1} \ e^{\lambda^* s_n} \ \psi_{\lambda^*}(x_n, y_n)} \to B^* \ \text{ as } \ n \to +\infty.$$

As in case 1, one can assume that, up to extraction of a subsequence, $(x_n, y_n) \to (x_\infty, y_\infty) \in \overline{\Omega}$ as $n \to +\infty$. Call

$$u_n(t,x,y) = \frac{\phi(c^*(e)t - x\cdot e + s_n, x, y)}{|s_n|^{2m+1} \ e^{\lambda^*(c^*(e)t - x\cdot e + s_n)} \ \psi_{\lambda^*}(x,y)} = \frac{u(t + s_n/c^*(e), x, y)}{|s_n|^{2m+1} \ e^{\lambda^*(c^*(e)t - x\cdot e + s_n)} \ \psi_{\lambda^*}(x,y)}.$$

The functions u_n satisfy (5.3) with $c^*(e)$, λ^* and s_n instead of c, λ_c and s'_n respectively, and $\nu A \nabla u_n = 0$ on $\mathbb{R}\times\partial\Omega$. From (5.5), and since $s_n \to -\infty$, the functions (u_n) are locally bounded, and $u(t + s_n/c^*(e), x, y) \to 0$ as $n \to +\infty$, locally uniformly in (t,x,y). Up to extraction of a subsequence, the functions u_n converge in $C^{1;2}_{t;(x,y),loc}(\mathbb{R}\times\overline{\Omega})$ to a function u_∞ satisfying (5.4) in $\mathbb{R}\times\overline{\Omega}$ with λ^* instead of λ_c, and $\nu A \nabla u_\infty = 0$ on $\mathbb{R}\times\partial\Omega$. From (5.5), $B^* \leq u_\infty \leq B^{*\prime}$ in $\mathbb{R}\times\overline{\Omega}$. Moreover, $u_\infty(x_\infty\cdot e/c^*(e), x_\infty, y_\infty) = B^*$ from the choice of the sequence (s_n, x_n, y_n). The strong maximum principle and Hopf lemma imply that $u_\infty(t,x,y) = B^*$ for all $t \leq x_\infty \cdot e/c^*(e)$ and $(x,y) \in \overline{\Omega}$, and then for all $(t,x,y) \in \mathbb{R}\times\overline{\Omega}$. As a consequence,

$$\frac{\phi(s + s_n, x, y)}{|s_n|^{2m+1} \ e^{\lambda^*(s+s_n)} \ \psi_{\lambda^*}(x,y)} = u_n\left(\frac{s + x\cdot e}{c^*(e)}, x, y\right) \to B^* \ \text{ as } n \to +\infty,$$

locally uniformly in $(s,x,y) \in \mathbb{R}\times\overline{\Omega}$. Therefore,

$$\frac{\phi(s_n, x, y)}{|s_n|^{2m+1} \ e^{\lambda^* s_n} \ \psi_{\lambda^*}(x,y)} \to B^* \ \text{ as } n \to +\infty, \ \text{uniformly in } (x,y) \in \overline{\Omega}.$$

Because of (5.7), one concludes that $B^* = B^{*\prime}$, which implies that

$$\frac{\phi(s,x,y)}{|s|^{2m+1}\ e^{\lambda^* s}\ \psi_{\lambda^*}(x,y)} \to B^* > 0 \ \text{ as } s \to -\infty, \text{ uniformly in } (x,y) \in \overline{\Omega}.$$

That completes the proof of Theorem 1.3. □

Proof of Theorem 1.5. Part a) follows immediately from Propositions 3.4 and 4.3. Now, for part b), if $c = c^*(e)$, then there is a unique $\lambda^* > 0$ such that $k(\lambda^*) + c^*(e)\lambda^* = 0$ (from the first paragraph in the proof of Proposition 3.3, which does not use the KPP assumption (1.8)). Then Proposition 2.2 implies that $\phi_s(s,x,y)/\phi(s,x,y) \to \lambda^*$ as $s \to -\infty$, uniformly in $(x,y) \in \overline{\Omega}$. Formula (1.23) follows. □

Remark 5.1 The proof of the first part of Theorem 1.3 (the KPP case with $c > c^*(e)$) could have been done another way by choosing first a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\max_{(x,y)\in\overline{\Omega}} (\phi(\sigma_n,x,y)/\psi_{\lambda_c}(x,y)) \geq (B' - \varepsilon)\ e^{\lambda_c \sigma_n}$. By using the arguments of the proof of Proposition 3.2 and the KPP assumption (1.8), it would follow that $\max_{(x,y)\in\overline{\Omega}} (\phi(s,x,y)/\psi_{\lambda_c}(x,y)) \geq (B' - \varepsilon)\ e^{\lambda_c s}$ for all $s \leq \sigma_n$, whence

$$\lim_{s\to-\infty} \left[\max_{(x,y)\in\overline{\Omega}} \left(\frac{\phi(s,x,y)}{e^{\lambda_c s}\ \psi_{\lambda_c}(x,y)} \right) \right] = B'.$$

The rest of the proof can be adapted and implies that $B = B'$.

But the strategy we chose in the proof of the first part of Theorem 1.3 is motivated by the fact that it would work in the general monostable case, under the assumptions of part a) of Theorem 1.5, provided we knew that B is positive in (5.1).

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