One-dimensional symmetry of bounded entire solutions of some elliptic equations

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Abstract

This paper is about one-dimensional symmetry properties for some entire and bounded solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$. We consider solutions u such that $-1 < u < 1$ and $u(x_1, \cdots, x_n) \to \pm 1$ as $x_n \to \pm\infty$, uniformly with respect to $x_1, \cdots, x_{n-1}$. Under some conditions on f, we prove that the solutions only depend on the variable x_n. We also discuss more general elliptic operators. The qualitative properties then strongly depend on the coefficients of the operator. These results extend to higher dimensions and to more general operators a result of Ghoussoub and Gui [21] proved for $n \leq 3$.

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1 Introduction

This article is devoted to the classification of the functions u which are solutions of the following semilinear elliptic equation

$$\Delta u + f(u) = 0 \ \text{ in } \mathbb{R}^n \tag{1.1}$$

and which satisfy $|u| \leq 1$ together with the asymptotic conditions

$$u(x', x_n) \underset{x_n \to \pm\infty}{\longrightarrow} \pm 1 \text{ uniformly in } x' = (x_1, \cdots, x_{n-1}). \tag{1.2}$$

The given function $f = f(u)$ is Lipschitz-continuous in $[-1, 1]$. Clearly, for (1.1)-(1.2) to have a solution, f has to be such that $f(\pm 1) = 0$. Here, we assume furthermore that there exists $\delta > 0$ such that

$$f \text{ is non-increasing on } [-1, -1+\delta] \text{ and on } [1-\delta, 1]; \quad f(\pm 1) = 0. \tag{1.3}$$

We will prove that any solution u of the multidimensional equation (1.1) with the limiting conditions (1.2) has one-dimensional symmetry :

Theorem 1 *Let u be a solution of (1.1)-(1.2) such that $|u| \leq 1$. Then, $u(x', x_n) = u_0(x_n)$ where u_0 is a solution of*

$$\begin{cases} u_0'' + f(u_0) &= 0 \qquad \text{in } \mathbb{R} \\ \quad u_0(\pm\infty) &= \pm 1, \end{cases} \tag{1.4}$$

and u is increasing with respect to x_n. In particular, the existence of a solution u of (1.1)-(1.2) such that $|u| \leq 1$ implies the existence of a solution u_0 of (1.4). Lastly, this solution u is unique up to translations of the origin.

For the one-dimensional problem, we refer to [5], [11], [18] or [23]. For the low dimensions case $n = 2, 3$ (assuming also that f is C^1), the same result had been obtained by Ghoussoub and Gui [21]. Their method relies on spectral properties of some Schrödinger operators and is different from the one we use in this paper in any dimension n. We have recently learned that a similar result to Theorem 1 has been proved independently by Barlow, Bass and Gui [7] using a very different method relying on probabilistic arguments.

Let us point out that Theorem 1 is related to a more difficult question, known as a conjecture of De Giorgi :

Conjecture (De Giorgi) [20] *If* u *is a solution of* $\Delta u + u - u^3 = 0$ *such that* $|u| \leq 1$ *in* $\mathbb{R}^n$, $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$ *for all* $x' \in \mathbb{R}^{n-1}$ *and* $\frac{\partial u}{\partial x_n} > 0$, *then there exists a vector* $a \in \mathbb{R}^{n-1}$ *and a function* $u_1 : \mathbb{R} \to \mathbb{R}$ *such that* $u(x', x_n) = u_1(a \cdot x' + x_n)$ *in* $\mathbb{R}^n$.

In the particular case where $f = u - u^3$, we see that this conjecture is stronger than Theorem 1 in the sense that, for the conjecture of De Giorgi, the limits as $x_n \to \pm\infty$ are only simple in x' whereas they are uniform in x' for Theorem 1.

In fact, for a general nonlinearity f, the conjecture of De Giorgi has been proved in dimension $n = 2$ by Ghoussoub and Gui [21] (see also a presentation of Berestycki, Caffarelli, Nirenberg [10]), and, very recently, it has been proved in dimension $n = 3$ by Ambrosio and Cabre [3]. See also earlier work by Modica and Mortola [24] for dimension 2, and by Caffarelli, Garofalo and Segala [15] for general inequalities related to this problem.

Recently, some new results in higher dimensions have been obtained by Farina [17] and Barlow, Bass and Gui [7]. Farina proves one-dimensional symmetry for the solutions of (1.1) provided that they minimize a certain energy in a cylinder $\omega \times \mathbb{R}$ included in $\mathbb{R}^n$. Barlow, Bass and Gui, with probabilistic arguments, derive this symmetry result from a Liouville type theorem, assuming monotonicity in a cone of directions. We also refer to the papers of Berestycki, Caffarelli, Nirenberg [10] and Barlow [6] about the connection between spectral properties of Schrödinger operators and the conjecture of De Giorgi.

However, the conjecture of De Giorgi, in its general form, remains open in dimensions greater than 3.

Let us now turn to more general semilinear elliptic equations of the type:

$$Lu + g(x_n, u) = 0 \quad \text{in } \mathbb{R}^n \tag{1.5}$$

where

$$Lu = a_{ij}(x)\partial_{ij}u + b_j(x)\partial_j u$$

(here we have used standard summation conventions). This operator is not necessarily self-adjoint. We assume that the coefficients $a_{ij}(x)$, $b_j(x)$ are continuous functions and that

$$\exists c_0' \geq c_0 > 0,\ \forall x \in \mathbb{R}^n,\ \forall \xi \in \mathbb{R}^n, \quad c_0|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq c_0'|\xi|^2. \tag{1.6}$$

Here, it is natural to ask whether the one-dimensional symmetry still holds for the solutions of (1.5), (1.2) with a general elliptic operator L instead of the Laplace operator. Nothing has been known so far about this problem, even in low dimensions. The following two Theorems show that the qualitative results actually depend on the structure of the coefficients a_{ij} and b_j.

In the following results, $g(x_n, u)$ is required to be defined and continuous on $\mathbb{R} \times [-1, 1]$ and to satisfy the conditions :

$$g \text{ is non-decreasing in } x_n, \tag{1.7}$$

$$\forall x_n \in \mathbb{R}, \quad g(x_n, \pm 1) = 0, \tag{1.8}$$

$$\begin{array}{c} \exists \delta > 0 \text{ such that } (x_n, s) \mapsto g(x_n, s) \text{ is non-increasing in } s \\ \text{on } \mathbb{R} \times [-1, -1+\delta] \ \cup \ \mathbb{R} \times [1-\delta, 1], \end{array} \tag{1.9}$$

$$\exists C_0 > 0,\ \forall x_n \in \mathbb{R},\ \forall s, \tilde{s} \in [-1, 1], \quad |g(x_n, \tilde{s}) - g(x_n, s)| \le C_0 |\tilde{s} - s|. \tag{1.10}$$

We first consider the case where the coefficients a_{ij} and b_j are constant; we prove the same symmetry result as in Theorem 1 :

Theorem 2 *Assume that L and g satisfy (1.6) and (1.7)-(1.10) and assume that the coefficients a_{ij}, b_j, $i, j = 1, \cdots, n$ are constant. Let u be a solution of (1.5), (1.2) such that $|u| \le 1$. Then, $u(x', x_n) = u_0(x_n)$ where u_0 is a solution of*

$$\left\{ \begin{array}{rll} a_{nn} u_0'' + b_n u_0' + g(x_n, u_0) & = 0 & \text{in } \mathbb{R} \\ u_0(\pm\infty) & = \pm 1 & \end{array} \right. \tag{1.11}$$

and u is increasing with respect to x_n. In particular, the existence of a solution u of (1.5), (1.2) such that $|u| \le 1$ implies the existence of a solution u_0 of (1.11). Furthermore, this solution u is unique up to translations of the origin and if g is increasing in x_n, then u is unique.

For general operators with non constant coefficients, however, this symmetry property does not hold. For example, it is natural to ask if a solution of the equation

$$\Delta u + b(x_1) \partial_{x_1} u - c \partial_{x_2} u + f(u) = 0 \ \text{ in } \mathbb{R}^2 \tag{1.12}$$

together with the uniform limiting conditions (1.2) actually satisfies $u = u(x_2)$ (and therefore the term $b(x_1)\partial_{x_1} u$ drops). This is not the case as the following counter-example in dimension 2 shows :

Theorem 3 *There exist some real numbers c, some functions $f(s)$ fulfilling the assumptions of Theorem 1 and some continuous functions $b(x_1)$ such that the two-dimensional equation (1.12) together with the uniform limiting conditions (1.2), admits both a planar solution u_0 and infinitely many non-planar solutions (i.e. solutions whose level sets are not parallel lines).*

Remark 1.1 It is natural to ask whether the one-dimensional symmetry holds or not if the coefficients of the operator only depend on x_n. Recently, Alessio, Jeanjean and Montecchiari [2] have actually proved the existence of solutions, which satisfy (1.2) and which do not depend on x_n only, for some equations of the type

$$a(x_n)\Delta u + f(u) = 0 \ \text{ in } I\!R^n.$$

Lastly, whereas Theorems 1 and 2 state symmetry properties for the solutions of some elliptic equations in $I\!R^n$, the following Theorem, which can be proved in the same way as Theorems 1 and 2 (*see* section 4), deals with the case of the half space $I\!R^n_+ = \{x_n > 0\}$.

Theorem 4 *Let L satisfy (1.6) and let the coefficients a_{ij}, b_j, $i, j = 1, \cdots, n$ be constant. Assume that the function $(x_n, s) \mapsto g(x_n, s)$ is defined and continuous on $[0,\infty) \times [0,1]$ and satisfies :*

$$g \text{ is non-decreasing in } x_n, \tag{1.13}$$

$$\forall x_n \geq 0, \quad g(x_n, 1) = 0,$$

$$\exists \delta > 0, \text{ such that } (x_n, s) \mapsto g(x_n, s) \text{ is non-increasing in } s \text{ on } [0,+\infty) \times [1-\delta, 1],$$

$$\exists C_0 > 0, \ \forall x_n \in [0,+\infty), \ \forall \tilde{s}, s \in [0,1], \quad |g(x_n, \tilde{s}) - g(x_n, s)| \leq C_0 |\tilde{s} - s|,$$

$$g(0,0) \geq 0. \tag{1.14}$$

Let $u \in C(\overline{I\!R^n_+})$ be a solution of

$$Lu + g(x_n, u) = 0 \ \text{ in } I\!R^n_+ \tag{1.15}$$

satisfying $0 \leq u \leq 1$ together with the following boundary and limiting conditions

$$\begin{cases} u = 0 & \text{on } \{x_n = 0\} \\ \lim_{x_n \to +\infty} u(x', x_n) = 1 & \text{uniformly in } x' = (x_1, \cdots, x_{n-1}) \in I\!R^{n-1}. \end{cases} \tag{1.16}$$

Then, $u(x',x_n)=u_0(x_n)$ *where* u_0 *is a solution of*

$$\left\{\begin{array}{ll} a_{nn}u_0''+b_nu_0'+g(x_n,u_0)=0 & \textit{in } (0,+\infty)\\ u_0(0)=0,\ u_0(+\infty)=1 & \end{array}\right. \tag{1.17}$$

and u *is increasing in* x_n*. In particular, the existence of a solution* u *of (1.15)-(1.16) such that* $0\le u\le 1$ *implies the existence of a solution* u_0 *of (1.17). Lastly, this solution* u *is unique.*

This Theorem extends to more general operators and equations a result of Clément and Sweers ([16]) who also considered the case of uniform limits as $x_n\to+\infty$:

Theorem (Clément-Sweers) [16] *Let* $f\in C^{1,\gamma}$ *for some* $\gamma\in(0,1)$ *satisfy :*

$$\exists\rho_1<1 \textit{ such that } f(\rho_1)=f(1)=0 \textit{ and } f>0 \textit{ in } (\rho_1,1),$$

$$\forall\rho\in[0,1),\quad \int_\rho^1 f(s)ds>0,$$

$$\exists\delta>0 \textit{ such that } f'\le 0 \textit{ in } (1-\delta,1).$$

Let $u\in C^2(I\!R^n_+)\cap C(\overline{I\!R^n_+})$ *be a solution of*

$$\Delta u+f(u)=0 \textit{ in } I\!R^n_+$$

which satisfies $0\le u<1$ *in* $I\!R^n_+$ *together with (1.16). Then* $u(x',x_n)=u_0(x_n)$ *where* u_0 *is a solution of*

$$\left\{\begin{array}{ll} u_0''+f(u_0)=0 & \textit{in } (0,+\infty)\\ u_0(0)=0,\ u_0(+\infty)=1 & \end{array}\right.$$

and u *is monotonic in* x_n*.*

The method to prove this Theorem is different from the one we use in this paper. It relies on comparisons with suitable one-dimensional sub- and super-solutions and on shooting type arguments.

Other problems in half spaces have been considered by Angenent [4] and Berestycki, Caffarelli and Nirenberg [8], [10] where no assumption is imposed on the limiting behaviour of u as $x_n\to+\infty$. These symmetry results can also be thought of as extensions of the Gidas, Ni and Nirenberg [19] symmetry result for spheres.

The main device to prove Theorems 1 and 2 (and also Theorem 4) is the sliding method, which has been developed by Berestycki and Nirenberg [12] and has been used in various works of Berestycki, Caffarelli and Nirenberg [8], [9], [10]. For another semilinear elliptic equation of the type (1.5) in $\mathbb{R}^n$ with conical limiting conditions, Bonnet, Hamel and Monneau have also applied this method to state some monotonicity and uniqueness results (*see* [14], [22]).

2 Proof of Theorem 1

The proof uses a sliding method and a version of the maximum principle in unbounded domains.

Let us start by stating the following comparison result which directly follows from Lemma 1 in [9] (based on the maximum principle) :

Lemma 2.1 ([9]) *Let f be a Lipschitz-continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1, u_2 are solutions of*

$$\Delta u_i + f(u_i) = 0 \quad \textit{in } \Omega$$

and are such that $|u_i| \leq 1$ ($i = 1, 2$). Assume furthermore that

$$u_2 \geq u_1 \quad \textit{on } \partial\Omega.$$

and that either

$$u_2 \geq 1 - \delta \quad \textit{in } \Omega$$

or

$$u_1 \leq -1 + \delta \quad \textit{in } \Omega.$$

If $\Omega \subset \mathbb{R}^n$ is an open connected set such that $\mathbb{R}^n \backslash \overline{\Omega}$ contains an infinite open connected cone, then $u_2 \geq u_1$ in Ω.

Here this result will be applied for domains which are half spaces.

Let us now consider a solution u of (1.1)-(1.2) such that $|u| \leq 1$ and let f satisfy (1.3). We are first going to prove that u is increasing in any direction $\nu = (\nu_1, \cdots, \nu_n)$ such that $\nu_n > 0$.

In order to do so, for any $t \in \mathbb{R}$, we define the function u^t by : $u^t(x) = u(x + t\nu)$.

From (1.2), there exists a real $a > 0$ such that $u(x', x_n) \geq 1 - \delta$ for all $x' \in \mathbb{R}^{n-1}$ and $x_n \geq a$ and $u(x', x_n) \leq -1 + \delta$ for all $x' \in \mathbb{R}^{n-1}$ and $x_n \leq -a$. For any $t \geq \frac{2a}{\nu_n}$, the functions u and u^t are such that

$$\begin{cases} u^t(x', x_n) & \geq 1 - \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \geq -a \\ u(x', x_n) & \leq -1 + \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \leq -a \\ u^t(x', -a) & \geq u(x', -a) \quad \text{for all } x' \in \mathbb{R}^{n-1} \end{cases} \tag{2.1}$$

Consequently, u and u^t fulfill the assumptions of Lemma 2.1 in both $\Omega = \mathbb{R}^{n-1} \times (-\infty, -a)$ and $\Omega = \mathbb{R}^{n-1} \times (-a, +\infty)$. Therefore, it follows that $u^t \geq u$ in $\mathbb{R}^n$.

Let us now decrease t. We claim that $u^t \geq u$ for all $t > 0$. Indeed, define $\tau = \inf \{t > 0,\ u^t \geq u \text{ in } \mathbb{R}^n\}$. By continuity, we see that $u^\tau \geq u$ in $\mathbb{R}^n$. Let us now argue by contradiction and suppose that $\tau > 0$. Two cases may occur :

case 1 : suppose that

$$\inf_{\mathbb{R}^{n-1} \times [-a,a]} (u^\tau - u) > 0. \tag{2.2}$$

From standard elliptic estimates, u is globally Lipschitz-continuous. Hence, there exists a real η_0 small enough, which can be chosen smaller than τ, such that for all $\tau \geq t > \tau - \eta_0$, one has

$$u^t(x', x_n) - u(x', x_n) > 0 \ \text{ for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \in [-a, a]. \tag{2.3}$$

Since $u \geq 1 - \delta$ in $\mathbb{R}^{n-1} \times [a, +\infty)$, it follows that

$$u^t(x', x_n) \geq 1 - \delta \text{ for all } x' \in \mathbb{R}^{n-1},\ x_n \geq a \text{ and for all } t > 0.$$

We may now apply Lemma 2.1 in the two half spaces $\Omega^+ = \{x_n > a\}$ and $\Omega^- = \{x_n < -a\}$. We then infer that, for all $\eta \in [0, \eta_0]$, $u^{\tau-\eta}(x', x_n) \geq u(x', x_n)$ for all $x' \in \mathbb{R}^{n-1}$ and for all $x_n \in (-\infty, -a) \cup (a, +\infty)$ and so for all $x_n \in \mathbb{R}$ owing to (2.3). This is in contradiction with the minimality of τ. Hence, (2.2) is ruled out.

case 2 : suppose that

$$\inf_{\mathbb{R}^{n-1} \times [-a,a]} (u^\tau - u) = 0. \tag{2.4}$$

Then there exists a sequence $(x^k)_{k \in \mathbb{N}} \in \mathbb{R}^{n-1} \times [-a, a]$ such that $u^\tau(x^k) - u(x^k) \to 0$ as $k \to \infty$. Set $u_k(x) = u(x^k + x)$. By standard elliptic estimates and the Sobolev injections, up to extraction of a subsequence, the functions u_k approach locally a solution u_∞ of (1.1) as $k \to \infty$. We have $u^\tau_\infty(0) = u_\infty(0)$ and $u^\tau_\infty \geq u_\infty$ because $u^\tau_k \geq u_k$ for any $k \in I\!N$. The function $z = u^\tau_\infty - u_\infty$ satisfies :

$$\begin{cases} \Delta z + c(x) z = 0 & \text{in } I\!R^n \\ z \geq 0 & \text{in } I\!R^n \\ z(0) = 0 & \end{cases} \tag{2.5}$$

for some bounded function $c(x)$ defined by

$$c(x) = \frac{f(u^\tau_\infty(x)) - f(u_\infty(x))}{u^\tau_\infty(x) - u_\infty(x)}$$

if $u^\tau_\infty(x) \neq u_\infty(x)$ and, say, $c(x) = 0$ if $u^\tau_\infty(x) = u_\infty(x)$. The strong maximum principle yields that $z \equiv 0$. This means that $u_\infty(x) \equiv u_\infty(x + \tau\nu)$. Letting $\xi = \tau\nu$, we see that u_∞ is periodic with respect to the vector ξ. Recalling that $-a \leq x^k_n \leq a$, we see that the function u_∞ also satisfies the uniform limiting conditions (1.2). Hence, since $\xi_n > 0$, the function u_∞ cannot be ξ-periodic. So case 2 with (2.4) is ruled out too.

Therefore, we have proved that $\tau = 0$. The function u is then increasing in any direction $\nu = (\nu_1, \cdots, \nu_n)$ such that $\nu_n > 0$. From the continuity of ∇u, we deduce that $\partial_\nu u \geq 0$ for any ν such that $\nu_n = 0$. If $\nu_n = 0$, by taking ν and $-\nu$, we find that $\partial_\nu u = 0$. Since this is true for all ν with $\nu_n = 0$, this implies that $u(x) = u(x_n)$.

Since the solutions of (1.4) are unique up to translations, it then follows that the solutions u of (1.1)-(1.2) such that $|u| \leq 1$ are unique up to translations of the origin. The proof of Theorem 1 is complete. □

3 More general elliptic operators

In this section, we consider solutions u with $|u| \leq 1$ of more general equations

$$Lu + g(x_n, u) = 0$$

where L is a general linear elliptic second-order operator with no zero-order term :

$$Lu = a_{ij}\partial_{ij}u + b_j\partial_j u.$$

We treat separately the case of constant coefficients where symmetry holds (Theorem 2) and the case of non-constant coefficients where the symmetry may be lost (Theorem 3).

3.1 Constant coefficients

Proof of Theorem 2. Assume that L and g satisfy (1.6) and (1.7)-(1.10) and assume that the coefficients $a_{ij}, b_j,\ i,j = 1, \cdots, n$, are constant. Let us consider a solution u of (1.5), (1.2) such that $|u| \leq 1$. As in Theorem 1, we shall prove that the function u depends on x_n only.

The scheme of the proof will be similar to that of Theorem 1, apart from the fact that, instead of the maximum principle stated in Lemma 2.1 for the Laplace operator, we shall use an extended version of the maximum principle for general second-order elliptic operators in infinite slab type domains.

We are going to prove that u is increasing in any direction $\nu = (\nu_1, \cdots, \nu_n)$ such that $\nu_n > 0$. For any $t \in \mathbb{R}$, let u^t be the function : $u^t(x) = u(x + t\nu)$.

We first observe that, for all $t \geq 0$, the function u^t is a super-solution for (1.5). Indeed, for all $t \geq 0$ and for all $x \in \mathbb{R}^n$, one has :

$$\begin{aligned} Lu^t + g(x_n, u^t) &= Lu(x+t\nu) + g(x_n, u(x+t\nu)) \\ &\leq Lu(x+t\nu) + g(x_n + t\nu_n, u(x+t\nu)) \ \text{ by (1.7)} \\ &\leq 0. \end{aligned} \tag{3.1}$$

Next, as in section 2, there exists a real a such that for any $t \geq \frac{2a}{\nu_n}$,

$$\begin{cases} u^t(x', x_n) \geq 1 - \delta & \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \geq -a \\ u(x', x_n) \leq -1 + \delta & \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \leq -a \\ u^t(x', -a) \geq u(x', -a) & \text{for all } x' \in \mathbb{R}^{n-1} \end{cases} \tag{3.2}$$

We now want to say that $u^t \geq u$ in $\mathbb{R}^n$. To this end, we use the following version of the maximum principle in infinite slab type domains for general second-order elliptic operators :

Lemma 3.1 *Let w be a function satisfying*

$$\mathcal{L}w \leq 0 \quad \text{in } \Omega = \mathbb{R}^{n-1} \times (b, c)$$

where $b, c \in \mathbb{R}$ and where

$$\mathcal{L}u = \alpha_{ij}(x)\partial_{ij}u + \beta_j(x)\partial_j u + \gamma(x)u.$$

Assume that the coefficients $\alpha_{ij}(x)$, $\beta_j(x)$ are uniformly continuous in $\overline{\Omega}$ and that the α_{ij} satisfy (1.6). Assume furthermore that

$$-C \leq \gamma(x) \leq 0 \quad \textit{for all } x \in \Omega$$

for some positive real number C. The function w is required to be continuous in $\overline{\Omega}$ and to satisfy

$$\mathcal{L}w \in L^\infty(\Omega)$$

and
$$m \leq w \leq M \quad \textit{in } \Omega$$

for some $m,\ M \in I\!\!R$.

If $w \geq 0$ on $\partial\Omega$, then $w \geq 0$ in Ω.

Postponing the proof of the above Lemma, let us conclude the proof of Theorem 2.

Let us first prove that $u^t \geq u$ in $I\!\!R^{n-1} \times (-a, +\infty)$ for all $t \geq \frac{2a}{\nu_n}$. Set $z = u^t - u$. Owing to (3.2), we already know that $z \geq 0$ on $I\!\!R^{n-1} \times \{-a\}$. We are now going to show that $z \geq 0$ in $I\!\!R^{n-1} \times (-a, +\infty)$.

Owing to (3.1) and (1.10), the function z satisfies

$$Lz + c(x)z \leq 0 \quad \text{in } I\!\!R^{n-1} \times (-a, +\infty)$$

for some bounded function $c(x)$ defined by

$$c(x) = \frac{g(x_n, u^t(x)) - g(x_n, u(x))}{u^t(x) - u(x)}$$

if $u^t(x) \neq u(x)$ and, say, $c(x) = 0$ if $u^t(x) = u(x)$.

Set $\gamma(x) = \min\,(c(x), 0)$. If $x \in I\!\!R^{n-1} \times (-a, +\infty)$ is such that $z(x) \leq 0$, then $1 - \delta \leq u^t(x) \leq u(x)$, whence, owing to (1.9), one has $c(x) \leq 0$ and $\gamma(x) = c(x)$. If $z(x) \geq 0$, then

$$Lz + \gamma(x)z \leq Lz + c(x)z \leq 0.$$

Therefore, it follows that

$$Lz + \gamma(x)z \leq 0 \quad \text{in } I\!\!R^{n-1} \times (-a, +\infty) \tag{3.3}$$

where the function $\gamma(x)$ is bounded and non-negative in $I\!\!R^{n-1} \times (-a, +\infty)$.

We are now going to apply Lemma 3.1 in slabs of the type

$$\Omega_h = \mathbb{R}^{n-1} \times (-a, h)$$

with $h > -a$.

Owing to (1.2), there exists a function $\varepsilon(h) \geq 0$ such that $z(x', h) \geq -\varepsilon(h)$ for all $x' \in \mathbb{R}^{n-1}$ and $\varepsilon(h) \to 0$ as $h \to +\infty$. Choose any $h > -a$ and set

$$w = z + \varepsilon(h).$$

The function w is bounded and, from standard elliptic estimates, it is continuous in $\overline{\Omega}$. Setting $\mathcal{L} = L + \gamma(x)$, one has

$$\begin{aligned} \mathcal{L}w &= Lz + \gamma(x)z + \gamma(x)\varepsilon(h) \quad \text{in } \Omega_h \\ &\leq \gamma(x)\varepsilon(h) \quad \text{by (3.3)} \\ &\leq 0 \end{aligned}$$

since $\gamma \leq 0$ and $\varepsilon(h) \geq 0$. Furthermore, owing to the definition of w,

$$\mathcal{L}w = -g(x_n + t\nu_n, u(x + t\nu)) + g(x_n, u(x)) + \gamma(x)w \quad \in L^\infty(\Omega_h)$$

because g, γ and w are bounded (the boundedness of g resorts to (1.8) and (1.10)).

Lemma 3.1 can then be applied to the function w and the operator $\mathcal{L}$ in Ω_h. One has $w \geq 0$ on $\partial\Omega_h$. Therefore, it follows that $w \geq 0$ in Ω_h. By passing to the limit $h \to +\infty$ and recalling that $w = u^t - u + \varepsilon(h)$, one concludes that

$$u^t(x', x_n) \geq u(x', x_n) \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \geq -a.$$

Similarly, one could show that

$$u^t(x', x_n) \geq u(x', x_n) \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \leq -a,$$

whence $u^t \geq u$ in $\mathbb{R}^n$.

Define $\tau = \inf \{t > 0,\ u^t \geq u \text{ in } \mathbb{R}^n\}$. By arguing as in the proof of Theorem 1, it then follows that $\tau = 0$. More precisely, if we suppose that $\tau > 0$, then, under the same notations as in the proof of Theorem 1, case 1 is ruled out. Moreover, case 2 is ruled out too. Indeed, if case 2 occurs, one can then assume that, up to extraction of a subsequence, $x_n^k \to \overline{x}_n \in [-a, a]$ and the functions $u_k(x) = u(x + x^k)$ approach a function u_∞ solving

$$Lu_\infty + g(x_n + \overline{x}_n, u_\infty) = 0 \text{ in } \mathbb{R}^n.$$

As we did in (3.1), the function u_∞^τ satisfies $Lu_\infty^\tau + g(x_n + \overline{x}_n, u_\infty^\tau) \leq 0$. Eventually, $z = u_\infty^\tau - u_\infty$ verifies

$$\begin{cases} Lz + c(x)z & \leq 0 \text{ in } \mathbb{R}^n \\ z & \geq 0 \text{ in } \mathbb{R}^n \\ z(0) & = 0 \end{cases}$$

for some bounded function c. The impossibility of case 2 follows then, as in the proof of Theorem 1, from the strong maximum principle and from the uniform limiting conditions (1.2).

Hence, u is increasing in any direction ν such that $\nu_n > 0$. This implies that $u = u(x_n)$ and that u is a solution of (1.11). The same sliding method also allows us to conclude that, if $u(x_n)$ and $v(x_n)$ are two solutions of (1.11), then there exists a real number τ such that $u(x_n + \tau) = v(x_n)$ for all $x_n \in \mathbb{R}$. The function $v(x_n)$ then satisfies

$$\begin{cases} a_{nn}v'' + b_n v' + g(x_n, v) & = 0 \\ a_{nn}v'' + b_n v' + g(x_n + \tau, v) & = 0. \end{cases}$$

Therefore, if g is increasing in x_n, it follows that $\tau = 0$ whence $u = v$. □

Let us now turn to the

Proof of Lemma 3.1. Let $\mathcal{L}$ and w fulfill the assumptions of Lemma 3.1. Suppose that

$$\inf_\Omega w = -\lambda < 0.$$

Then there exists a sequence $(x^k)_{k\in\mathbb{N}} \in \mathbb{R}^{n-1} \times (b, c)$ such that $w(x^k) \to -\lambda$ as $k \to \infty$. From standard elliptic estimates, the function w is globally Lipschitz-continuous in $\overline{\Omega}$. Recalling that $w \geq 0$ on $\partial\Omega$, there exists then $\varepsilon > 0$ such that, up to extraction of a sub-sequence,

$$x_n^k \to \overline{x}_n \in [b + \varepsilon, c - \varepsilon] \text{ as } k \to \infty. \quad (3.4)$$

Set

$$w^k(x', x_n) = w(x + x'^k, x_n)$$

and $\alpha^k_{ij}(x', x_n) = \alpha_{ij}(x' + x'^k, x_n)$, $\beta^k_j(x', x_n) = \beta_j(x' + x'^k, x_n)$, $\gamma^k(x', x_n) = \gamma(x' + x'^k, x_n)$ for all $(x', x_n) \in \Omega$. The functions w^k satisfy

$$\begin{aligned} \alpha^k_{ij}\partial_{ij}w^k + \beta^k_j\partial_j w^k &\leq -\gamma^k w^k \quad \text{in } \Omega \\ &\leq -\gamma^k w^k - \gamma^k\lambda \ \text{ since } \gamma^k \leq 0 \text{ and } \lambda \geq 0 \\ &\leq C\ (w^k + \lambda). \end{aligned}$$

since $w^k + \lambda \geq 0$ and $-\gamma^k \leq C$. Up to extraction of sub-sequences, from Ascoli's Theorem, the functions α_{ij}, β_j locally converge to some functions $\overline{\alpha}_{ij}$, $\overline{\beta}_j$ and from standard elliptic estimates, the functions w^k locally approach a function $\overline{w}$ as $k \to +\infty$. By passing to the limit $k \to \infty$, the function $z = \overline{w} + \lambda$ satisfies

$$Mz - Cz \leq 0 \ \text{ in } \Omega$$

where $M = \overline{\alpha}_{ij}\partial_{ij} + \overline{\beta}_j\partial_j$.

Owing to the definition of λ, one has $z \geq 0$ in Ω. Furthermore, from (3.4), it follows that $z(0, \overline{x}_n) = 0$ with $\overline{x}_n \in [b + \varepsilon, c - \varepsilon]$. The strong maximum principle then yields that

$$z = \overline{w} + \lambda \equiv 0 \ \text{ in } \Omega. \tag{3.5}$$

On the other hand, since w is globally Lipschitz-continuous, there exists a real number $\delta > 0$ such that, say, $w(x', x_n) \geq -\lambda/2$ for all $x' \in \mathbb{R}^{n-1}$ and $b \leq x_n \leq b + \delta$. As a consequence, $z \geq \lambda/2 > 0$ in $\mathbb{R}^{n-1} \times [b, b + \delta]$. This is ruled out by (3.5) and the proof of the Lemma is complete. □

Let us now observe that Theorem 2 does not hold in general if, instead of the uniform limiting conditions (1.2), we only assume that $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ for each $x' \in \mathbb{R}^{n-1}$.

Consider the equation

$$\Delta u - c\partial_{x_2}u + f(u) = 0 \ \text{ in } \mathbb{R}^2 \tag{3.6}$$

with

$$u(x_1, x_2) \to \pm 1 \ \text{ as } x_2 \to \pm\infty, \text{ pointwise, for all } x_1 \in \mathbb{R} \tag{3.7}$$

Let us further assume that

$$\frac{\partial u}{\partial x_2} > 0 \ \text{ in } \mathbb{R}^2 \tag{3.8}$$

Here, c is a constant parameter and f is some C^1 function. The limits in (3.7) are only pointwise and are not required to be uniform. When $c = 0$, it follows from the result of Ghoussoub and Gui [21] that u is a function of one variable only.

This does not hold for (3.6)-(3.8) as soon as $c \neq 0$. Indeed, Bonnet and Hamel [14] have constructed for some particular function f and for some $c > 0$ a solution u such that

$$\begin{cases} u(\lambda\vec{k}) \underset{\lambda\to+\infty}{\longrightarrow} -1 \text{ for all } \vec{k} = (\cos\varphi, \sin\varphi) \text{ with } -\dfrac{\pi}{2} - \alpha < \varphi < -\dfrac{\pi}{2} + \alpha \\ u(\lambda\vec{k}) \underset{\lambda\to+\infty}{\longrightarrow} +1 \text{ for all } \vec{k} = (\cos\varphi, \sin\varphi) \text{ with } -\dfrac{\pi}{2} + \alpha < \varphi < \dfrac{3\pi}{2} - \alpha \end{cases}$$

for each angle $\alpha \in (0, \pi/2]$. Such a solution cannot have one-dimensional symmetry (with level sets being parallel lines). This problem arises in the modelling of Bunsen burner flames (*see* [14] and [22] for details).

Therefore, from this example we learn that, for some functions $f(u)$, the De Giorgi's conjecture cannot be extended to elliptic operators with nonzero first-order terms, even in dimension 2.

3.2 Non-constant coefficients

Our goal in this section is to prove Theorem 3. More precisely, we are going to prove that for an equation of the type (1.12) :

$$\Delta u + b(x_1)\partial_{x_1} u - c\partial_{x_2} u + f(u) = 0 \text{ in } \mathbb{R}^2$$

together with the limiting conditions (1.2), there exist both a solution depending on x_2 only and infinitely many non-planar solutions, *i.e.* solutions whose level sets are not parallel lines.

The construction is somewhat involved and technical. It first relies on the choice of special types of functions $b(x_1)$ and f. Next, we construct a family of non-planar solutions of (1.12), (1.2) that are between suitably chosen sub- and super-solutions.

Let us first state the type of b and f we consider. We choose a continuous function $x_1 \mapsto b(x_1)$ such that, for some $\xi \in \mathbb{R}$ and $\chi_0 > 0$, the function

$$\chi(x_1) = \int_\xi^{x_1} e^{-\int_0^y b(s)ds} dy \text{ verifies } \chi(\pm\infty) = \pm\chi_0. \tag{3.9}$$

A constant function $b(x_1) \equiv b_0$ does not fulfill this condition. In contrast, all the functions of the type $b(x_1) = \alpha \tanh x_1 + \beta$ (with $\alpha > |\beta|$) or of the type $b(x_1) = \alpha x_1 + \beta$ (with $\alpha > 0$ and $\beta \in I\!R$) fulfill this condition.

The function f will be chosen as to satisfy the following conditions :

$$f \in C^1([-1,1]), \quad f(\pm 1) = 0, \tag{3.10}$$

$$\exists\, \theta \in (-1,1) \text{ such that } f \leq 0 \text{ in } [-1,\theta], \quad f \geq 0 \text{ in } [\theta, 1], \tag{3.11}$$

and either

$$f \leq 0 \text{ in } [-1,\theta], \quad f > 0 \text{ in } (\theta,1), \quad \int_{-1}^{1} f(s)ds > 0, \tag{3.12}$$

or

$$f < 0 \text{ in } (-1,\theta), \quad f \geq 0 \text{ in } [\theta,1], \quad \int_{-1}^{1} f(s)ds < 0, \tag{3.13}$$

or

$$f < 0 \text{ in } (-1,\theta), \quad f > 0 \text{ in } (\theta,1). \tag{3.14}$$

Assume furthermore that if f is positive somewhere in $[-1,1]$, then

$$\inf_{\{f(v)>0\}} f'(v) = f'(1) < 0 \tag{3.15}$$

and that if f is negative somewhere in $[-1,1]$, then

$$\inf_{\{f(v)<0\}} f'(v) = f'(-1) < 0. \tag{3.16}$$

On the one hand, the condition (3.12) includes the case where f has an ignition temperature profile ($f \equiv 0$ in $[-1,\theta]$ and $f > 0$ in $(\theta,1)$). On the other hand, the case (3.14) corresponds to the so-called bistable profile.

From [18], [23], there exist a unique real c, whose sign is that of $\int_{-1}^{1} f(s)ds$, and a function $z(x_2)$ solving the one-dimensional problem :

$$\left\{ \begin{array}{rl} z'' - cz' + f(z) & = 0 \text{ in } I\!R \\ z(\pm\infty) & = \pm 1 \end{array} \right. \tag{3.17}$$

The solution z of (3.17) is unique up to translations and is increasing. Furthermore, it has the following asymptotic behaviour as $x_2 \to \pm\infty$ (*see* [5], [13], [18]) :

$$\left\{ \begin{array}{rr} z(x_2) = & -1 + Ce^{\lambda x_2} + o(e^{\lambda x_2}) \\ z'(x_2) = & C\lambda e^{\lambda x_2} + o(e^{\lambda x_2}) \end{array} \right. \text{ as } x_2 \to -\infty \tag{3.18}$$

$$\begin{cases} z(x_2) = & 1 - \tilde{C}e^{-\mu x_2} + o(e^{-\mu x_2}) \\ z'(x_2) = & \tilde{C}\mu e^{-\mu x_2} + o(e^{-\mu x_2}) \end{cases} \quad \text{as } x_2 \to +\infty \tag{3.19}$$

where

$$\lambda = \frac{\sqrt{c^2 - 4f'(0)} + c}{2}, \quad \mu = \frac{\sqrt{c^2 - 4f'(1)} - c}{2} \tag{3.20}$$

and $C, \tilde{C}$ are two positive constants. Under the assumptions (3.12)-(3.16), one can see that λ and μ are always positive.

Theorem 3 will be a consequence of the following

Proposition 3.2 *Under the previous assumptions, for any $a \in (-1, 1)$, there exist functions $\psi^+(x_1)$ and $\psi^-(x_1)$ such that*

(i) $\psi^- \leq \psi^+$,

(ii) the function

$$\overline{u}_a(x_1, x_2) = z(x_2 + \psi^+(x_1))$$

is a super-solution of (1.12) and the function

$$\underline{u}_a(x_1, x_2) = z(x_2 + \psi^-(x_1))$$

is a sub-solution of (1.12),

(iii) ψ^+ and ψ^- are increasing if $a > 0$ and decreasing if $a < 0$; if $a = 0$, then $\psi^+ \equiv \psi^- \equiv 0$,

(iv) $\psi^+(-\infty) = \psi^-(-\infty) \in I\!R$ and $\psi^+(+\infty) = \psi^-(+\infty) \in I\!R$,

(v) $l_- = l_-(a) := \psi^\pm(-\infty)$ is decreasing with respect to a and $l_+ = l_+(a) := \psi^\pm(+\infty)$ is increasing.

Remark 3.3 Since the function z is increasing, the assertion (i) implies that

$$\underline{u}_a(x_1, x_2) \leq \overline{u}_a(x_1, x_2) \ \text{ for all } (x_1, x_2) \in I\!R^2.$$

Remark 3.4 In the case where f is positive somewhere, one can show that this last Proposition is still true if the assumption (3.15) is replaced with $f'(1) < 0$. To this end, we approximate f in $L^\infty([-1, 1])$ norm by a sequence of functions satisfying (3.15). In the case where f is negative somewhere, Proposition 3.2 is also true if (3.16) is replaced with $f'(-1) < 0$.

Postponing the proof of this Proposition, let us first state two preliminary Lemmas and conclude the proof of Theorem 3.

Lemma 3.5 *If a function* $u(x_1, x_2)$ *is such that* $\underline{u}_a \leq u \leq \overline{u}_a$ *with* $a \neq 0$, *then* u *is not a function of* x_2 *only. Moreover, it is not a planar function (i.e. a function whose level sets are parallel lines).*

Proof. Assume first that there exists a function $x_2 \mapsto v(x_2)$ such that $u(x_1, x_2) = v(x_2)$ for all $(x_1, x_2) \in I\!R^2$. Owing to the definitions of $\underline{u}_a$ and $\overline{u}_a$, one has

$$z(x_2 + \psi^-(x_1)) \leq v(x_2) \leq z(x_2 + \psi^+(x_1)) \text{ for all } (x_1, x_2) \in I\!R^2.$$

Choose $x_2 = 0$ and take the limits $x_1 \to \pm\infty$. By the assertion (iv) of Proposition 3.2, it then follows that $v(0) = z(l_-) = z(l_+)$. Since z is increasing, one finds that $l_- = l_+$. This is ruled out by (iii).

Assume now that there exist a function $t \mapsto v(t)$ and two reals α and β such that $u(x_1, x_2) = v(\alpha x_1 + \beta x_2)$ for all $(x_1, x_2) \in I\!R^2$. Then

$$z(x_2 + \psi^-(x_1)) \leq v(\alpha x_1 + \beta x_2) \leq z(x_2 + \psi^+(x_1)) \text{ for all } (x_1, x_2) \in I\!R^2.$$

From what precedes, only the case $\alpha \neq 0$ remains to be treated. Choose now $x_1 = \gamma x_2$ where $\gamma = -\frac{\beta}{\alpha}$. One has

$$z(x_2 + \psi^-(\gamma x_2)) \leq v(0) \leq z(x_2 + \psi^+(\gamma x_2)) \text{ for all } x_2 \in I\!R.$$

Since the functions $\psi^\pm$ are bounded and $z(\pm\infty) = \pm 1$, the limits as $x_2 \to \pm\infty$ imply that $v(0) = -1$ and $v(0) = 1$. This is impossible. □

Lemma 3.6 *If two functions* $u(x_1, x_2)$ *and* $v(x_1, x_2)$ *are such that* $\underline{u}_b \leq u$ *and* $\overline{u}_a \geq v$ *with* $a \neq b$, *then* $u \neq v$.

Proof. Assume that $u \equiv v$ and write $\overline{u}_a$ and $\underline{u}_b$ as $\overline{u}_a(x_1, x_2) = z(x_2 + \psi_a^+(x_1))$ and $\underline{u}_b(x_1, x_2) = z(x_2 + \psi_b^-(x_1))$. One then has

$$z(x_2 + \psi_b^-(x_1)) \leq u(x_1, x_2) = v(x_1, x_2) \leq z(x_2 + \psi_a^+(x_1)) \text{ for all } (x_1, x_2) \in I\!R^2.$$

Therefore, since z is increasing, it follows that

$$\psi_b^-(x_1) \leq \psi_a^+(x_1) \text{ for all } x_1 \in I\!R.$$

By taking the limit as $x_1 \to -\infty$, one finds that $l_-(b) \leq l_-(a)$. By (v), this implies that $a \leq b$. Similarly, the limit as $x_1 \to +\infty$ yields that $a \geq b$.

Eventually, $a = b$. This is in contradiction with the assumption $a \neq b$ and the proof of the Lemma is complete. □

Proof of Theorem 3. Choose any $a \in (-1, 1)$ and, under the notations of Proposition 3.2, consider the functions ψ^+, ψ^- and $\overline{u}_a$, $\underline{u}_a$. By Remark 3.3, we know that $\underline{u}_a \leq \overline{u}_a$. Since $\underline{u}_a$ and $\overline{u}_a$ are respectively sub- and super-solutions for (1.12), there exists then a solution u_a of (1.12) such that $\underline{u}_a \leq u_a \leq \overline{u}_a$, *i.e.*

$$z(x_2 + \psi^-(x_1)) \leq u_a(x_1, x_2) \leq z(x_2 + \psi^+(x_1)) \text{ for all } (x_1, x_2) \in \mathbb{R}^2.$$

Owing to (iv), the functions ψ^+ and ψ^- are bounded. As a consequence, the function u_a still satisfies the uniform limiting conditions (1.2). Therefore, for each $a \in (-1, 1)$, there exists a solution u_a of (1.12), (1.2). If $a = 0$, we simply have $u_0 = z$.

By Lemma 3.5, the function u_a is not planar if $a \neq 0$. By Lemma 3.6, one has $u_a \neq u_b$ if $a \neq b$. Hence, equation (1.12) together with the limiting conditions (1.2) has a family of solutions u_a parametrized by $a \in (-1, 1)$ which are different one another and which are not planar for $a \neq 0$. □

Let us now turn to the proof of Proposition 3.2.

Proof of Proposition 3.2. Choose a real $a \in (-1, 1)$. By definition, the function $\chi(x_1)$ is increasing and it then satisfies $|\chi(x_1)| < \chi_0$ for all $x_1 \in \mathbb{R}$. We can then consider the functions

$$\begin{cases} \psi^- & = \psi_a^-(x_1) & = -\dfrac{1}{\mu} \ln(1 - a\dfrac{\chi(x_1)}{\chi_0}) \\ \psi^+ & = \psi_a^+(x_1) & = \dfrac{1}{\lambda} \ln(1 - \alpha\dfrac{\chi(x_1)}{\chi_0}) + \beta \end{cases}$$

where the positive real numbers λ and μ have been defined in (3.20) and where

$$\begin{aligned} \alpha & = \alpha_a & = \tanh\left(-\frac{\lambda}{\mu} \tanh^{-1}(a)\right) \in (-1, 1) \\ \beta & = \beta_a & = -\frac{1}{\mu} \ln(1 + a) - \frac{1}{\lambda} \ln(1 + \alpha). \end{aligned}$$

Proof of (iii). If $a = 0$, the conclusion is obvious. Take now $a > 0$. One has

$$(\psi^-)'(x_1) = \frac{a}{\mu\chi_0} \frac{\chi'(x_1)}{1 - a\frac{\chi(x_1)}{\chi_0}} > 0 \text{ for all } x_1 \in \mathbb{R}$$

since a, μ and χ_0 are positive and the function χ is increasing. As far as the function ψ^+ is concerned, one has

$$(\psi^+)'(x_1) = -\frac{\alpha}{\lambda\chi_0}\frac{\chi'(x_1)}{1-\alpha\frac{\chi(x_1)}{\chi_0}} \quad \text{for all } x_1 \in I\!R.$$

Like a, μ and χ_0, the real number λ is positive. Therefore, α is negative and ψ^+ is increasing.

The case $a < 0$ can be treated similarly.

Proof of (iv). It is straightforward owing to the definitions of $\psi^{\pm}$ and to the fact that $\chi(\pm\infty) = \pm\chi_0$.

Proof of (v). We have $l_-(a) = -\frac{1}{\mu}\ln(1+a)$ and $l_+(a) = -\frac{1}{\mu}\ln(1-a)$. Since μ is positive, this yields *(v)*.

Proof of (i). The case $a = 0$ is obvious. Choose now $a \neq 0$ and define

$$v(x_1) = \psi^+(x_1) - \psi^-(x_1)$$

Part *(iv)* says that $v(\pm\infty) = 0$. To prove that v is non-negative in $I\!R$, it is then sufficient to show that $v'(x_1)$ is positive in an interval of the type $(-\infty, \gamma)$ and negative in $(\gamma, +\infty)$. A straightforward calculation leads to

$$v'(x_1) = A(x_1)B(x_1) \quad \text{for all } x_1 \in I\!R$$

where

$$A(x_1) = \frac{\chi'(x_1)}{\lambda\mu\chi_0}\frac{1}{1-a\frac{\chi(x_1)}{\chi_0}}\frac{1}{1-\alpha\frac{\chi(x_1)}{\chi_0}} > 0 \quad \text{for all } x_1 \in I\!R$$

and where

$$B(x_1) = -(a\lambda + \alpha\mu) + a\alpha(\lambda+\mu)\frac{\chi(x_1)}{\chi_0} \quad \text{for all } x_1 \in I\!R.$$

The product $a\alpha$ is always negative whatever the sign of a may be. Moreover, remember that λ and μ are positive and that χ is increasing. Hence, the function B is (strictly) decreasing. If B did not change sign, then v would be monotone and then identically 0. That would yield $v' \equiv 0$ and $B \equiv 0$.

The latter is impossible since B is decreasing. Hence, the function B changes sign. Since it is decreasing, there exists then a real γ such that $B(x_1) > 0$ in $(-\infty, \gamma)$ and $B(x_1) < 0$ in $(\gamma, +\infty)$. The conclusion follows.

Proof of (ii). Choose $a \in (-1, 1)$ and consider the function

$$\underline{u}_a(x_1, x_2) = z(x_2 + \psi^-(x_1)).$$

Owing to its definition, it is easy to check that the function $\psi = \psi^-$ is a solution of the following ordinary differential equation

$$\mu\psi'^2 - \psi'' - b(x_1)\psi' = 0. \tag{3.21}$$

Set $I(u) := \Delta u + b(x_1)\partial_{x_1} u - c\partial_{x_2} u + f(u)$. We have

$$\begin{array}{rl} I(\underline{u}_a) & = (1+\psi'^2)z'' + (-c + \psi'' + b\psi')z' + f(z) \\ & = (1+\psi'^2)(cz' - f(z)) \\ & \qquad +(-c+\mu\psi'^2)z' + f(z) \quad \text{by (3.17) and (3.21)} \\ & = -\psi'^2 f(z) + (\mu + c)\psi'^2 z' \\ & = -\left(\dfrac{f(z)}{z'} + \dfrac{f'(1)}{\mu}\right)\psi'^2 z' \ \ \text{since } \mu^2 + c\mu + f'(1) = 0 \end{array}.$$

We now claim that

$$\frac{f(z(y))}{z'(y)} + \frac{f'(1)}{\mu} \le 0 \ \text{ for all } y \in I\!R. \tag{3.22}$$

Indeed, first, the function $v(y) = \dfrac{f(z(y))}{z'(y)}$ satisfies

$$v' = v^2 - cv + f'(z).$$

If the supremum of v were reached at a point $b \in I\!R$, then

$$\frac{f(z(b))}{z'(b)} = v(b) = \frac{c \pm \sqrt{c^2 - 4f'(z(b))}}{2}.$$

Owing to (3.10) and (3.11), one always has $f'(1) \le 0$. Therefore, if $f(z(b)) \le 0$, then $v(y) \le v(b) \le 0$ for all $y \in I\!R$ and the claim (3.22) follows.

Let us now consider the case where $f(z(b)) > 0$. By the definition of μ and by (3.15), it follows that

$$v(b) \leq \frac{c + \sqrt{c^2 - 4f'(1)}}{2} = -\frac{f'(1)}{\mu}.$$

Moreover $\limsup_{y \to -\infty} v(y) \leq 0$ owing to (3.11) and $z(-\infty) = -1$. On the other hand, $v(+\infty) = \frac{-f'(1)}{\mu} > 0$ by (3.19). Consequently, we have $\sup_{\mathbb{R}} v \leq \frac{-f'(1)}{\mu}$. This yields (3.22).

This implies that $I(\underline{u}_a) \geq 0$ in $\mathbb{R}^2$, that is to say that $\underline{u}_a$ is a sub-solution of (1.12).

Similarly, we can show that the function $\overline{u}_a$ is a super-solution of (1.12). The proof of Proposition 3.2 is complete. □

Remark 3.7 This counter-example shows that there are infinitely many non-planar solutions u_a to the equation (1.12). We can see that for any $a \neq 0$ these solutions are not symmetric with respect to any vertical axis $\{x_1 = b\}$. In fact, we conjecture that $u_0 = z$ is the unique solution which is symmetric with respect to a vertical axis.

For an equation of the type (1.12) :

$$\Delta u + b(x_1)\partial_{x_1} u - c\partial_{x_2} u + f(u) = 0 \text{ in } \mathbb{R}^2,$$

and for some functions f, as we said earlier, there are non-planar solutions with $b \equiv 0$ and $c \neq 0$ satisfying $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ for each $x' \in \mathbb{R}^{n-1}$.

If uniform limits (1.2) are satisfied, then one knows from Theorem 2 that any solution u has one-dimensional symmetry whenever b is constant. Nevertheless, Theorem 3 shows that this symmetry property does not hold for some non-constant and yet bounded functions b and some functions f. More precisely, the non-planar solutions u_a of (1.12) we have constructed are such that, say for $a > 0$,

$$z_-(x_2) := z(x_2 + l_-) \leq u(x_1, x_2) \leq z_+(x_2) := z(x_2 + l_+)$$

and

$$\begin{cases} u(x_1, x_2) \underset{x_2 \to \pm\infty}{\longrightarrow} \pm 1 \text{ uniformly in } x_1 \\ u(x_1, x_2) \underset{x_1 \to \pm\infty}{\longrightarrow} z_\pm(x_2) \end{cases} \tag{3.23}$$

where $l_- < l_+$ and $z_\pm$ are solutions of (3.17). The profile of a function safisfying these properties is drawn in Figure 1.

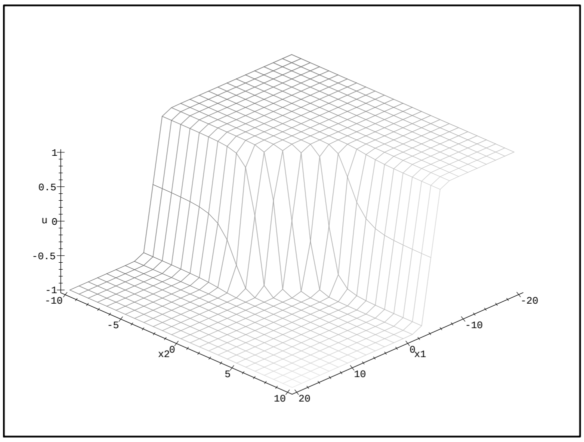

Figure 1: *Profile of a function* $u(x_1, x_2)$ *satisfying (3.23)*

Recently, similar results have been proved for different equations by Alessio, Jeanjean and Montecchiari [2] and Alama, Bronsard and Gui [1]. Alessio, Jeanjean and Montecchiari, with methods based on hamiltonian systems, have proved the existence of non-planar functions $u(x_1, x_2)$ satisfying the same kind of limits as in (3.23) and solving the equation

$$-\Delta u + a(x_2)W'(u) = 0 \text{ in } \mathbb{R}^2$$

for some functions $a(x_2)$ which are positive and periodic. Here W is a multiple well potential. Alama, Bronsard and Gui [1], with energy methods, have proved the existence of non-planar solutions $U = (u_1, u_2)$ for a system of two equations of the type

$$-\Delta U + \nabla W\ (U) = 0,\ x = (x_1, x_2) \in \mathbb{R}^2$$

satisfying asymptotic limiting conditions as $x_1, x_2 \to \pm\infty$ similar to (3.23). There, $W : \mathbb{R}^2 \to \mathbb{R}$ is also a multiple well potential.

Let us now consider the De Giorgi's non-linearity $f(u) = u - u^3$. It satisfies the conditions (3.10)-(3.11), (3.14) and (3.15)-(3.16). Furthermore, $\int_{-1}^{1} f(s)ds = 0$. The unique speed c that is a solution of (3.17) is then equal to 0. Now choose a function $b(x_1)$ satisfying (3.9). As a consequence of the preceding results, the bi-dimensional equation

$$\Delta u + b(x_1)\partial_{x_1} u + f(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{3.24}$$

together with the uniform limiting conditions (1.2), admits both a planar solution and infinitely many non-planar solutions. The same result obviously holds in any dimension $n \geq 2$ by considering the same equation (3.24) in $\mathbb{R}^n$ and choosing special solutions of the type $v(x_1, \cdots, x_n) = u(x_1, x_2)$. As a conclusion, in any dimension $n \geq 2$ and even if uniform limits (1.2) are required, the De Giorgi's conjecture cannot be extended, for a class of non-constant functions $b(x_1)$ (including some bounded functions), to equations of the type (3.24) involving the additional first-order term $b(x_1)\partial_{x_1} u$.

4 Half space case

Let L and g satisfy the assumptions of Theorem 4 and let $u \in C(\overline{\mathbb{R}^n_+})$ be a solution of (1.15)-(1.16). As in the proofs of Theorems 1 and 2, we are going to prove that u is increasing in any direction $\nu = (\nu_1, \cdots, \nu_n)$ such that $\nu_n > 0$. For any $t \geq 0$, we define the function u^t in $\{x_n \geq -t\nu_n\}$ by $u^t(x) = u(x + t\nu)$.

As we did in (3.1), one has, for any $t \geq 0$:

$$Lu^t + g(x_n, u^t) \leq 0 \quad \text{in } \{x_n > -t\nu_n\} \supset \mathbb{R}^n_+. \tag{4.1}$$

Owing to (1.16), there exists a real $a > 0$ such that $u(x', x_n) \geq 1 - \delta$ for all $x' \in \mathbb{R}^{n-1}$ and $x_n \geq a$. For all $t \geq \frac{a}{\nu_n}$, the function u^t is then such that

$$\begin{cases} u^t(x', x_n) & \geq 1 - \delta \qquad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and } x_n \geq 0 \\ u^t(x', 0) & \geq 0 = u(x', 0) \quad \text{for all } x' \in \mathbb{R}^{n-1}. \end{cases}$$

As we did in the proof of Theorem 2, using especially Lemma 3.1, it then follows that $u^t \geq u$ in $\mathbb{R}^n_+$.

Let us now decrease t. We claim that $u^t \geq u$ in $I\!R^n_+$ for all $t > 0$. Define $\tau = \inf \{t > 0,\ u^t \geq u \text{ in } I\!R^n_+\}$. By continuity, we see that $u^\tau \geq u$ in $\overline{I\!R^n_+} = \{x_n \geq 0\}$. Let us now argue by contradiction and suppose that $\tau > 0$. Two cases may occur :

case 1 : suppose that

$$\inf_{I\!R^{n-1}\times[0,a]} (u^\tau - u) > 0.$$

In this case, as in the proof of Theorem 1, there would exist a real $\eta_0 \in (0, \tau)$ such that $u^t \geq u$ in $I\!R^n_+$ for all $t \in [\tau - \eta_0, \tau]$. This would be in contradiction with the minimality of τ.

case 2 : suppose that

$$\inf_{I\!R^{n-1}\times[0,a]} (u^\tau - u) = 0.$$

Then there exists a sequence $(x^k)_{k\in I\!N} \in I\!R^{n-1} \times [0, a]$ such that $u^\tau(x^k) - u(x^k) \to 0$ as $k \to \infty$. Up to extraction of a subsequence, two sub-cases may occur :

sub-case 2-1 : suppose that $x^k_n \to \overline{x}_n \in (0, a]$ as $k \to \infty$. This sub-case shall be ruled out as case 2 in the proof of Theorem 2.

More precisely, the functions $u_k(x', x_n) = u(x' + x'^k, x_n)$ would then approach locally in $I\!R^n_+$ a function u_∞ solving

$$Lu_\infty + g(x_n, u_\infty) = 0 \ \text{ in } I\!R^n_+.$$

The function u^τ_∞ satisfies $Lu^\tau_\infty + g(x_n, u^\tau_\infty) \leq 0$ in $I\!R^n_+$. Furthermore, $u^\tau_\infty \geq u_\infty$ in $I\!R^n_+$ and $u^\tau_\infty(0, \overline{x}_n) = u_\infty(0, \overline{x}_n)$. From the strong maximum principle, it then follows that $u^\tau_\infty \equiv u_\infty$ in $I\!R^n_+$. The function u_∞ is then periodic with respect to the vector $\xi = \tau\nu$.

From elliptic regularity theory, the function u is globally Lipschitz-continuous in $\overline{I\!R^n_+}$. Since u satisfies (1.16) and since the u_k are obtained from u by shifting it with respect to the x'-variables, it follows that the function u_∞ satisfies (1.16) too. Hence, since $\xi_n > 0$, it cannot be ξ-periodic.

sub-case 2-2 : suppose that $x^k_n \to 0$ as $k \to \infty$. Since $u = 0$ on $\{x_n = 0\}$ and u is globally Lipschitz-continuous in $\{x_n \geq 0\}$, it then follows that

$$u(x^k + \tau\nu) \to 0 \ \text{ as } k \to \infty.$$

Set $u_k(x) = u(x + x^k)$. This function is defined in $\{x_n \geq -x^k_n\} \supset \{x_n \geq 0\}$. By standard elliptic estimates, up to extraction of a subsequence, the (nonnegative) functions u_k approach locally in $\{x_n > 0\}$ a function $u_\infty \geq 0$ as

$k \to \infty$. We have $u_\infty(\tau\nu) = 0$. Furthermore, as we did in (3.1) or (4.1) and since $x_n^k \geq 0$, one has

$$Lu_k(x) + g(x_n, u_k(x)) \leq 0 \text{ for all } x' \in I\!R^{n-1} \text{ and } x_n > -x_n^k.$$

As a consequence, one has, for all $x' \in I\!R^{n-1}$ and $x_n > -x_n^k$,

$$\begin{aligned} Lu_k(x) + g(x_n, u_k(x)) - g(x_n, 0) &\leq -g(x_n, 0) \\ &\leq 0 \text{ by (1.13) and (1.14).} \end{aligned}$$

Finally, there exists then a bounded function $c(x)$ such that

$$Lu_\infty + cu_\infty \leq 0 \text{ in } I\!R^n_+ = \{x_n > 0\}.$$

Since u_∞ is non-negative and vanishes at the interior point $\tau\nu \in I\!R^n_+$, the strong maximum principle implies that $u_\infty \equiv 0$ in $I\!R^n_+$. Recalling that $0 \leq x_n^k \leq a$, we see that the function u_∞ is such that $u_\infty(x', x_n) \to 1$ as $x_n \to +\infty$ (uniformly in $x' \in I\!R^{n-1}$). So sub-case 2-2 is ruled out too.

Consequently, $\tau = 0$ and, as in the proof of Theorem 1, the function u then depends on x_n only and solves (1.17).

Lastly, if $u(x_n)$ and $v(x_n)$ are two solutions of (1.17), then the previous proof implies that we simultaneously have $u \geq v$ and $v \geq u$. As a conclusion, the solution u of (1.15)-(1.16) is unique and the proof of Theorem 4 is complete.

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