Nonlinear methods for linear equations

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Roma, april 2008
Some results on linear elliptic (or parabolic) equations with Stampacchia’s methods

Using these methods for the study of numerical schemes (properties of the approximate solutions, convergence of schemes. . . )
Positivity

\( \Omega \) bounded open set of \( \mathbb{R}^d \) with a Lipschitz continuous boundary.

\( A : \Omega \to M_d(\mathbb{R}) \) sym. pos. def., uniformly (\( A\xi.\xi \geq \alpha \xi.\xi \) with some \( \alpha > 0 \)), with coefficients in \( L^\infty(\Omega) \).

\( f \in L^2(\Omega) \).

\[-\text{div}(A\nabla u) = f, \text{ in } \Omega,\]

\[u = 0, \text{ on } \partial \Omega.\]

\( f \leq 0 \text{ a.e. } \Rightarrow u \leq 0 \text{ a.e.} \)
Positivity, proof

\[ u \in H^1_0(\Omega), \quad \int_{\Omega} A \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx \quad \text{for all } v \in H^1_0(\Omega). \]

\[ f \leq 0. \]

Taking \( u^+ \) as test function (possible since \( u^+ \in H^1_0(\Omega) \)):

\[ \alpha \| \nabla u^+ \|_2 \leq \int_{\Omega} A \nabla u^+ \cdot \nabla u^+ = \int_{\Omega} A \nabla u \cdot \nabla u^+ = \int_{\Omega} fu^+ \leq 0. \]

Then, \( \nabla u^+ = 0 \) a.e. and \( u^+ = 0 \) a.e., \( u \leq 0 \) a.e...

Property used: \( \nabla u^+ = 1_{u>0} \nabla u = 1_{u\geq0} \nabla u \) a.e.
Nonlinear tool (Stampacchia)

\[ \varphi : \mathbb{R} \to \mathbb{R}, \text{ Lipschitz continuous function such that } \varphi(0) = 0. \]

\[ u \in H^1_0(\Omega). \text{ Then, } \varphi(u) \in H^1_0(\Omega) \text{ and} \]

\[ \nabla \varphi(u) = \varphi'(u) \nabla u \text{ a.e..} \]

Example: \( \varphi(s) = s^+, \nabla u^+ = 1_{u>0} \nabla u = 1_{u \geq 0} \nabla u \text{ a.e.} \)

Indeed, it is possible to use only regular function \( \varphi \) \( (C^1 \text{ functions}) \).
Maximum principle

\[-\text{div}(A \nabla u) = 0, \text{ in } \Omega,\]

\[u = g, \text{ on } \partial \Omega.\]

\[g \in C(\partial \Omega, \mathbb{R}) \cap H^1_0(\partial \Omega).\]

Then \(a = \inf g \leq u \leq \max g = b.\)

Proof: Take \((u - b)^+\) and \((u - a)^-\) as test functions in the weak formulation.
Bounded solutions (Stampacchia)

\[-\text{div}(A\nabla u) = f, \text{ in } \Omega,\]

\[u = 0, \text{ on } \partial \Omega.\]

\(f \in H^{-1}(\Omega).\) Existence and uniqueness of \(u\) solution to:

\(u \in H^1_0(\Omega), \int_{\Omega} A\nabla u \cdot \nabla v dx = \langle f, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \forall v \in H^1_0(\Omega).\)

Question: \(u \in L^\infty(\Omega) (d \geq 2)?\)

Answer:

\(\Rightarrow\) Yes if it exists \(p > \frac{d}{2}\) such that \(f \in L^p(\Omega)\) (and \(\langle f, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_{\Omega} fv dx\)).

\(\Rightarrow\) Yes if it exists \(p > d\) such that \(f \in W^{-1,p}(\Omega)\).

NB: \(L^{p/2}(\Omega) \subset W^{-1,p}(\Omega)\) for \(p > d\).
Bounded solutions, proof (1)

Let $p > d$ s.t. $f \in W^{-1,p}(\Omega)$. Then, it exists $F \in (L^p(\Omega))^d$ s.t. $f = \text{div} F$. One has:

$$u \in H^1_0(\Omega), \quad \int_{\Omega} A \nabla u \cdot \nabla v \, dx = \int_{\Omega} F \cdot \nabla v \, dx$$

for all $v \in H^1_0(\Omega)$.

Let $k \in \mathbb{R}^*_+$. Take $v = \psi(u) = (u - k)^+ - (u + k)^-$ ($\psi$ is nondecreasing). One has $\nabla \psi(u) = 1_{A_k} \nabla u$ a.e. with $A_k = \{|u| \geq k\}$ and:

$$\int_{A_k} A \nabla u \cdot \nabla u \, dx = \int_{A_k} F \cdot \nabla u \, dx.$$

Then, with Cauchy-Schwarz and Hölder inequalities ($p/2$ and its conjugate):

\[
\alpha \|\nabla u\|_{L^2(A_k)} \leq C_1 \|f\|_{W^{-1,p} \text{mes}(A_k)^{1/2-1/p}}.
\]
Bounded solutions, proof (2)

Using Sobolev imbedding ($W_0^{1,1}(\Omega) \subset L^{d/(d-1)}(\Omega)$) and Cauchy-Schwarz again:

$$\text{mes}(A_h) \leq \frac{C_2 \| f \|_{W^{-1,p}}^\gamma}{h - k} \text{mes}(A_k)^\beta, \text{ for } 0 \leq k < h,$$

with $\gamma = d/(d - 1)$ and $\beta = \frac{p-1}{p} \frac{d}{d-1} > 1$ (since $p > d$).

Since $\beta > 1$, this gives (with a little tricky computation) $\text{mes}(A_h) = 0$ si $h \geq C_3 \| f \|_{W^{-1,p}}$. Then:

$$\| u \|_\infty \leq C_3 \| f \|_{W^{-1,p}}.$$

A further developpement of this proof leads to $u \in C(\overline{\Omega})$ and finally to the Hölder continuity of $u$. 
Existence of a solution for $f$ “measure”

$$-\text{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

$f$ is a measure on $\Omega$ ($f \in (C(\Omega))'$).

First method: duality method (Stampacchia, 1965)

Second method: passing to the limit on approximate solutions

Main difficulty: obtain estimates on $u$ only depending of the $L^1$—norm of $f$ (with $f \in L^2$).
Existence of a solution for $f$ “measure”, proof (1)

$$u \in H^1_0(\Omega), \quad \int_\Omega A \nabla u \cdot \nabla v dx = \int_\Omega fv dx \quad \text{for all } v \in H^1_0(\Omega).$$

For $\theta > 1$, one defines $\varphi$:

$$\varphi(s) = \int_0^s \frac{1}{(1 + |t|)^\theta} dt; \quad s \in \mathbb{R}.$$ 

Taking $v = \varphi(u) \in H^1_0(\Omega)$ leads to:

$$\int_\Omega \frac{|\nabla u|^2}{(1 + |u|)^\theta} dx \leq C_\theta \|f\|_1,$$

with $C_\theta = \int_0^\infty \frac{1}{(1+|t|)^\theta} dt < \infty$. 
Existence de solution with $f$ "mesure", proof (1)

$u \in H_0^1(\Omega)$, $\int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} fv dx$ for all $v \in H_0^1(\Omega)$.

For $\theta > 1$, one defines $\varphi$:

$$\varphi(s) = \int_0^s \frac{1}{(1 + |t|)^\theta} dt; \ s \in \mathbb{R}.$$ 

taking $v = \varphi(u) \in H_0^1(\Omega)$ leads to:

$$\int_{\Omega} |\nabla \varphi(u)|^2 dx = \int_{\Omega} \frac{|\nabla u|^2}{(1 + |u|)^\theta} dx \leq C_\theta \|f\|_1,$$

with $\varphi(s) = \int_0^s \sqrt{\varphi'(t)} dt$. 
Existence of a solution for $f$ “measure”, proof (2)

Using Hölder Inequality, Sobolev imbedding and $\theta$ close 1, one obtains, for $q < \frac{d}{d-1}$:

$$
\int_\Omega |\nabla u|^q dx \leq C_q ||f||_{L^1}.
$$

Passing to the limit on a sequence of approximate solutions (corresponding to regular second members converging towards $f$), one obtains existence of a solution (in the distribution sense) if $f$ is a measure.

This solution belongs to $W^{1,q}_0(\Omega)$ for all $q < \frac{d}{d-1}$. 
Convection-diffusion without coercivity

\[- \text{div} A \nabla u + \text{div} (wu) = f \text{ in } \Omega, \]
\[u = 0 \text{ on } \partial \Omega,\]

with \( w \in C(\overline{\Omega})^d \) and \( f \in L^2(\Omega) \) (or \( f \) measure on \( \Omega \)).

Existence and uniqueness of a solution.

Main step: \textit{a priori} estimates on meas(\( \{|u| \geq k\} \)) (this measure goes to 0 as \( k \to \infty \)).

(then, one obtains an \( H^1_0(\Omega) \)–estimate and existence follows with a topological degree argument. Uniqueness is a consequence of an existence result for the dual problem.)
Convection-diffusion without coercivity, proof (1)

\[ u \in H^1_0(\Omega), \]
\[ \int_{\Omega} A \nabla u \cdot \nabla v \, dx - \int_{\Omega} uw \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \text{ for all } v \in H^1_0(\Omega). \]

Idea of Lucio Boccardo.  
Taking \( v = \varphi(u) \) with \( \varphi(s) = \int_0^s \frac{1}{(1+|s|)^2} \) (\( \theta = 2 \)):

\[ \alpha \int_{\Omega} \frac{|\nabla u|^2}{(1 + |u|)^2} \, dx \leq \|f\|_1 + \int_{\Omega} \frac{|w||u||\nabla u|}{(1 + |u|)^2} \, dx \]
\[ \leq \|f\|_1 + \|w\|_\infty \int_{\Omega} \frac{|\nabla u|}{1 + |u|} \, dx, \]

with \( \|w\|_\infty = \sup_{x \in \Omega} |w(x)| < \infty. \)
and, with Young Inequality:

\[
\int_\Omega |\nabla \ln (1 + |u|)|^2 \, dx = \int_\Omega \frac{|\nabla u|^2}{(1 + |u|)^2} \, dx \leq C(\alpha, \|f\|_1, \|w\|_\infty).
\]

\[
|\nabla \ln (1 + |u|)| = |\nabla \phi(u)|.
\]

\[
\phi(s) = \int_0^s \sqrt{\varphi'(t)} \, dt = \int_0^s \frac{1}{1 + |t|} \, dt.
\]

Since \( \ln(1 + |u|) \in H^1_0(\Omega) \), one deduces an estimate on \( \ln(1 + |u|) \) in \( L^2(\Omega) \) and then an estimate on \( \text{meas}\{ |u| \geq k \} \).
Positivity for convection-diffusion without coercivity

\[-\text{div} A \nabla u + \text{div}(wu) = f \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial \Omega,\]

with \( w \in C(\overline{\Omega})^d \) and \( f \in L^2(\Omega) \) (or \( f \) measure on \( \Omega \)).

Then \( f \geq 0 \text{ a.e. implies } u \geq 0 \text{ a.e.} \)
(Or \( f \leq 0 \text{ a.e. implies } u \leq 0 \text{ a.e.} \))
Positivity for conv.-diff. without coercivity, proof (1)

\[ u \in H^1_0(\Omega), \]
\[ \int_{\Omega} A\nabla u \cdot \nabla v dx - \int_{\Omega} uw \cdot \nabla v dx = \int_{\Omega} fv dx, \text{ for all } v \in H^1_0(\Omega). \]

Assume \( f \leq 0 \text{ a.e.} \)

Taking \( v = T_\varepsilon(u^+) = \begin{cases} 0 & \text{if } u \leq 0 \\ u & \text{if } 0 < u < \varepsilon \\ \varepsilon & \text{if } u \geq \varepsilon \end{cases} \)
leads to

\[ \alpha \| \nabla T_\varepsilon u^+ \|_2^2 - \int_{\Omega} uw \cdot \nabla T_\varepsilon(u^+) \leq 0. \]

Then, with Cauchy-Schwarz Inequality,

\[ \alpha \| \nabla T_\varepsilon u^+ \|_2^2 \leq \varepsilon a_\varepsilon \| \nabla T_\varepsilon u^+ \|_2, \]

with \( a_\varepsilon^2 = \int_{0<u<\varepsilon} |w|^2 dx \to 0, \text{ as } \varepsilon \to 0. \)
Positivity for conv.-diff. without coercivity, proof (2)

\[ \alpha \| \nabla T_\varepsilon u^+ \|_2^2 \leq \varepsilon a_\varepsilon \| \nabla T_\varepsilon u^+ \|_2 \]

gives

\[ \| \nabla T_\varepsilon u^+ \|_1 \leq C_1 \| \nabla T_\varepsilon u^+ \|_2 \leq C_2 \varepsilon a_\varepsilon . \]

Then, with Sobolev Inequality,

\[ \varepsilon \text{meas}\{ u > \varepsilon \}^{\frac{1}{1^*}} \leq \| T_\varepsilon(u^+) \|_{1^*} \leq C_3 \| \nabla T_\varepsilon u^+ \|_1 \leq C_4 \varepsilon a_\varepsilon , \]

and we obtain \( \text{meas}\{ u > \varepsilon \} \rightarrow 0 \), as \( \varepsilon \rightarrow 0 \), that is \( u \leq 0 \) a.e..

Similar ideas are in Gilbarg-Trudinger and Boccardo-G-Murat.
Stampacchia methods with FV schemes

Finite Volumes schemes with the so-called “admissible meshes for $A$”.

1. positivity, maximum principle
2. Bounded solutions, Hölder continuous solutions: Thomas Rey thesis
3. “Measure” data: G-Herbin
4. Convection-diffusion without coercivity (and with measure data) : Droniou-G-Herbin

“Non admissible” meshes ?
Stampacchia methods with FE schemes

\( \mathcal{M} \) is a mesh of \( \Omega \), with triangles \((d = 2)\) or tetrahedra \((d = 3)\).

\[ H = \{ u \in C(\bar{\Omega}); u|_K \in P^1 \} \].

\[ H_0 = \{ u \in H; u = 0 \text{ on } \partial \Omega \}. \]

\[ u_{\mathcal{M}} \in H_0, \]

\[ \int_{\Omega} A \nabla u_{\mathcal{M}} \cdot \nabla v \, dx \left( - \int_{\Omega} u_{\mathcal{M}} w \cdot \nabla v \, dx \right) = T(v), \text{ for all } v \in H_0. \]

\[ T(v) = \int_{\Omega} fv \, dx \text{ (examples 1, 2, 4)} \]

\[ T(v) = \int_{\Omega} v df \text{ (examples 3, 4)} \]

Difficulty : \( u \in H_0 \nRightarrow u^+, (u - b)^+, \psi(u), \varphi(u), T_\varepsilon(u^+) \in H_0. \)
Choice of the test function

Idea: take as test function the interpolate of the test function of the “continuous” case.

If \( v \in C(\overline{\Omega}) \), \( \Pi_M(v) \in H \) and \( \Pi_M(v) = v \) at the vertices of the mesh.

Denoting by \( V \) the set of vertices of the mesh:
\[
\Pi_M(v) = \sum_{K \in V} v(K) \phi_K,
\]
where \( \phi_K \) is the basis function associated to \( K \).
re-writing the scheme (1)

\[ u_M \in H_0, \]

\[ \int_{\Omega} A \nabla u_M \cdot \nabla v dx = \int_{\Omega} fv dx, \text{ for all } v \in H_0. \]

With \( u_M = \sum_{K \in V} u_K \phi_K \) and \( v = \sum_{L \in V} v_L \phi_L \), this gives

\[ \sum_{K \in V} \sum_{L \in V} (-T_{K,L}) u_K v_L = \int_{\Omega} fv dx, \]

with \( T_{K,L} = -\int_{\Omega} A \nabla \phi_K \cdot \nabla \phi_L dx \).

or, since \( \sum_{L \in V} T_{K,L} = 0 \),

\[ \sum_{K \in V} \sum_{L \in V} (-T_{K,L})(u_K)(v_L - v_K) = \int_{\Omega} fv dx, \]
re-writing the scheme (2)

and, finally,

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (u_K - u_L)(v_K - v_L) = \int f v dx.$$  

Taking $v = \Pi_M \varphi(u)$ leads to:

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int f v dx.$$
positivity (1)

\[ f \leq 0 \text{ a.e.} \]

\[ \varphi(s) = s^+. \]

\[
\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int f v dx,
\]

yields:

\[
\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(u_K^+ - u_L^+) = \int f v dx,
\]

If \( T_{K,L} \geq 0 \), one has:

\[
\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)^2 \leq \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(u_K^+ - u_L^+),
\]
positivity (2)

\[ \sum_{(K,L) \in \mathcal{V}^2} T_{K,L}(u_K^+ - u_L^+)(u_K^+ - u_L^+) \leq \int fvdx \leq 0, \]

then:

\[ \int_{\Omega} A \nabla \Pi_{\mathcal{M}} u^+ \cdot \nabla \Pi_{\mathcal{M}} u^+ dx = \sum_{(K,L) \in \mathcal{V}^2} T_{K,L}(u_K^+ - u_L^+)^2 = 0, \]

from which one deduces \( u^+ = 0 \).
nondecreasing function $\varphi$

Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ Lipschitz continuous and nondecreasing. Define $\phi$ by $\phi(s) = \int_0^s \sqrt{\varphi'(t)} \, dt$.

For $a, b \in \mathbb{R}$, one has (thanks to Cauchy-Schwarz Inequality):

$$(\phi(a) - \phi(b))^2 \leq (a - b)(\varphi(a) - \varphi(b)).$$

then, IF $T_{K,L} \geq 0$ (for all $(K, L)$), one has:

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (\phi(u_K) - \phi(u_L))^2 \leq \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (u_K - u_L)(\varphi(u_K) - \varphi(u_L)).$$

$$\int_{\Omega} A \nabla \Pi M \phi(u) \cdot \nabla \Pi M \phi(u) = \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (\phi(u_K) - \phi(u_L))^2.$$
For $\theta > 1$, define $\varphi$:

$$\varphi(s) = \int_0^s \frac{1}{(1 + |t|)^\theta} dt; \ s \in \mathbb{R}.$$  

Taking $v = \Pi_M \varphi(u) \in H_0$:

$$\int_{\Omega} |\nabla \Pi_M \varphi(u)|^2 \, dx \leq C_{\theta} \|f\|_1,$$

with $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$. 
Take $v = \Pi_M \varphi(u)$ with $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2} \ (\theta = 2)$.

If the mesh size is small enough (or using an “upwinding” for convection part), one obtains an $H^1_0(\Omega)$--estimate on $\Pi_M \ln(1 + |u|) \in H^1_0(\Omega)$, then, an estimate on $\ln(1 + |u|)$ in $L^2(\Omega)$ and finally, as in the “continuous” case, an estimate on $\text{meas}(\{|u| \geq k\})$. 
Conclusion

If $T_{K,L} \geq 0$, for all $K, L$, the methods of Stampacchia can be used for the study of numerical schemes (EF and VF)...

They give the desired properties on the approximate solution in Examples 1 and 2 (positivity, $L^\infty$–bound), Estimates and Convergence of the approximate solution in Examples 3 and 4 (measure data and convection-diffusion without coercivity).
Ongoing work

Without the condition $T_{K,L} \geq 0$, it seems not easy to use the methods of Stampacchia...

For Finite Volumes schemes with a “non admissible” meshes, a possible solution is (perhaps) to discretize this elliptic linear problem with a nonlinear scheme taking some $T_{K,L}(u)$ depending on the approximate solution, that is a scheme under the form:

$$
\sum_{(K,L) \in (V)^2} T_{K,L}(u)(u_K - u_L)(v_K - v_L) = T(v),
$$

and with $T_{K,L}(u) \geq 0$, for all $K, L$. 