Compactness of approximate solutions
(for some evolution PDEs with diffusion)

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Kρητη, September 2010

with coauthors...

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- Robert Eymard, Raphaèle Herbin (discrete setting, 2000)
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Example (coming from RANS model for turbulent flows)

\[
\frac{\partial u}{\partial t} + \nabla \cdot (v u) - \Delta u = f \text{ in } \Omega \times (0, T),
\]

\[
u = 0 \text{ on } \partial \Omega \times (0, T),
\]

\[
u(\cdot, 0) = u_0 \text{ in } \Omega.
\]

\(\Omega\) is a bounded open subset of \(\mathbb{R}^d\) \((d = 2 \text{ or } 3)\) with a Lipschitz continuous boundary

\(\nu \in C^1(\overline{\Omega} \times [0, T], \mathbb{R})\)

\(u_0 \in L^1(\Omega)\) \((\text{or } u_0 \text{ is a Radon measure on } \Omega)\)

\(f \in L^1(\Omega \times (0, T))\) \((\text{or } f \text{ is a Radon measure on } \Omega \times (0, T))\)

with possible generalization to nonlinear problems.

Non smooth solutions.
Example, motivation

For this example, we have two objectives:

1. Existence of weak solution and (strong) convergence of “continuous approximate solutions”, that is solutions of the continuous problem with regular data converging to $f$ and $u_0$.

2. Existence of weak solution and (strong) convergence of the approximate solutions given by a full discretized problem.

In both case, we want to prove strong compactness of a sequence of approximate solutions. This is the main subject of this talk.
Continuous approximation

\((f_n)_{n \in \mathbb{N}}\) and \((u_{0,n})_{n \in \mathbb{N}}\) are two sequences of regular functions such that

\[
\int_0^T \int_\Omega f_n \varphi \, dx \, dt \to \int_0^T \int_\Omega f \varphi \, dx \, dt, \quad \forall \varphi \in C_\infty(\Omega \times (0, T), \mathbb{R}),
\]

\[
\int_\Omega u_{0,n} \varphi \, dx \to \int_\Omega u_0 \varphi \, dx, \quad \forall \varphi \in C_\infty(\Omega, \mathbb{R}).
\]

For \(n \in \mathbb{N}\), it is well known that there exist \(u_n\) solution of the regularized problem

\[
\partial_t u_n + \text{div}(\nu u_n) - \Delta u_n = f_n \quad \text{in } \Omega \times (0, T),
\]

\(u_n = 0\) on \(\partial \Omega \times (0, T),\)

\(u_n(\cdot, 0) = u_{0,n}\) in \(\Omega.\)

One has, at least, \(u_n \in L^2((0, T), H^1_0(\Omega)) \cap C([0, T], L^2(\Omega))\) and \(\partial_t u_n \in L^2((0, T), H^{-1}(\Omega))\).
Continuous approximation, steps of the proof of convergence

1. Estimate on $u_n$ (not easy). One proves that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^q((0, T), W^{1,q}_0(\Omega))$ for all $1 \leq q < \frac{d + 2}{d + 1}$.

   (This gives, up to a subsequence, weak convergence in $L^q(\Omega \times (0, T))$ of $u_n$ to some $u$ and then, since the problem is linear, that $u$ is a weak solution of the problem with $f$ and $u_0$.)

2. Strong compactness of the sequence $(u_n)_{n \in \mathbb{N}}$

3. Regularity of the limit of the sequence $(u_n)_{n \in \mathbb{N}}$.

4. Passage to the limit in the approximate equation (easy).
Classical Lions’ lemma

$X$, $B$, $Y$ are three Banach spaces such that

- $X \subseteq B$ with compact embedding,
- $B \subseteq Y$ with continuous embedding.

Then, for any $\varepsilon > 0$, there exists $C_\varepsilon$ such that, for $w \in X$,

$$
\|w\|_B \leq \varepsilon \|w\|_X + C_\varepsilon \|w\|_Y.
$$

Example: $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_{\ast}^{-1,1}(\Omega) = (W_0^{1,\infty}(\Omega))^\prime$. As usual, we identify an $L^1$-function with the corresponding linear form on $W_0^{1,\infty}(\Omega)$. 
Classical Lions’ lemma, another formulation

$X$, $B$, $Y$ are three Banach spaces such that, $X \subset B \subset Y$,

- If $(\|w_n\|_X)_{n \in \mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in $B$.
- If $w_n \to w$ in $B$ and $\|w_n\|_Y \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists $C_\varepsilon$ such that, for $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + C_\varepsilon \|w\|_Y.$$ 

The hypothesis $B \subset Y$ is not necessary.
Classical Lions’ lemma, a particular case, simpler

$B$ is a Hilbert space and $X$ is a Banach space $X \subset B$. We define on $X$ the dual norm of $\| \cdot \|_X$, with the scalar product of $B$, namely

$$\| u \|_Y = \sup \{(u/v)_B, \ v \in X, \|v\|_X \leq 1 \}.$$ 

Then, for any $\varepsilon > 0$ and $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon} \|w\|_Y.$$ 

The proof is simple since

$$\|u\|_B = (u/u)_B^{\frac{1}{2}} \leq (\|u\|_Y \|u\|_X)^{\frac{1}{2}} \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon} \|w\|_Y.$$ 

Compactness of $X$ in $B$ is not needed here (but this compactness is needed for Aubin-Simon’ Lemma, next slide...).
Aubin-Simon’ Compactness Lemma

\( X, B, Y \) are three Banach spaces such that

- \( X \subset B \) with compact embedding,
- \( B \subset Y \) with continuous embedding.

Let \( T > 0 \) and \((u_n)_{n \in \mathbb{N}}\) be a sequence such that

- \((u_n)_{n \in \mathbb{N}}\) is bounded in \( L^1((0, T), X) \),
- \((\partial_t u_n)_{n \in \mathbb{N}}\) is bounded in \( L^1((0, T), Y) \).

Then there exists \( u \in L^1((0, T), B) \) such that, up to a subsequence, \( u_n \to u \) in \( L^1((0, T), B) \).

Example: \( X = W_0^{1,1}(\Omega) \), \( B = L^1(\Omega) \), \( Y = W_*^{-1,1}(\Omega) \).
Aubin-Simon’ Compactness Lemma, another formulation

\( X, B, Y \) are three Banach spaces such that, \( X \subset B \subset Y \),

- If \( (\|w_n\|_{X})_{n \in \mathbb{N}} \) is bounded, then, up to a subsequence, there exists \( w \in B \) such that \( w_n \rightarrow w \) in \( B \).
- If \( w_n \rightarrow w \) in \( B \) and \( \|w_n\|_Y \rightarrow 0 \), then \( w = 0 \).

Let \( T > 0 \) and \( (u_n)_{n \in \mathbb{N}} \) be a sequence such that

- \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( L^1((0, T), X) \),
- \( (\partial_t u_n)_{n \in \mathbb{N}} \) is bounded in \( L^1((0, T), Y) \).

Then there exists \( u \in L^1((0, T), B) \) such that, up to a subsequence, \( u_n \rightarrow u \) in \( L^1((0, T), B) \).

Example: \( X = W^{1,1}_0(\Omega) \), \( B = L^1(\Omega) \), \( Y = W^{-1,1}_*(\Omega) \).
Continuous approx., compactness of the sequence \( (u_n)_{n\in\mathbb{N}} \)

\( u_n \) is solution of the continuous problem with data \( f_n \) and \( u_{0,n} \).

\( X = W_{0}^{1,1}(\Omega), \ B = L^1(\Omega), \ Y = W_\ast^{-1,1}(\Omega). \)

In order to apply Aubin-Simon’ lemma we need

\( \begin{align*}
\uparrow \quad & (u_n)_{n\in\mathbb{N}} \text{ is bounded in } L^1((0, T), X), \\
\uparrow \quad & (\partial_t u_n)_{n\in\mathbb{N}} \text{ is bounded in } L^1((0, T), Y).
\end{align*} \)

The sequence \( (u_n)_{n\in\mathbb{N}} \) is bounded in \( L^q((0, T), W_{0}^{1,q}(\Omega)) \) (for \( 1 \leq q < (d + 2)/(d + 1) \)) and then is bounded in \( L^1((0, T), X) \), since \( W_{0}^{1,q}(\Omega) \) is continuously embedded in \( W_\ast^{1,1}(\Omega). \)

\( \partial_t u_n = f_n - \text{div}(v u_n) - \Delta u_n. \) Is \( (\partial_t u_n)_{n\in\mathbb{N}} \) bounded in \( L^1((0, T), Y) \) ?
Continuous approx., Compactness of the sequence \((u_n)_{n \in \mathbb{N}}\)

Bound of \((\partial_t u_n)_{n \in \mathbb{N}}\) in \(L^1((0, T), W_{-1,1}^1(\Omega))\)?
\[\partial_t u_n = f_n - \text{div}(\nu u_n) - \Delta u_n.\]

- \((f_n)_{n \in \mathbb{N}}\) is bounded in \(L^1(0, T), L^1(\Omega))\) and then in \(L^1((0, T), W_{-1,1}^1(\Omega))\), since \(L^1(\Omega)\) is continuously embedded in \(W_{-1,1}^1(\Omega)\),
- \((\text{div}(\nu u_n))_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), W_{-1,1}^1(\Omega))\) since \((\nu u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), (L^1(\Omega))^d)\) and \(\text{div}\) is a continuous operator from \((L^1(\Omega))^d\) to \(W_{-1,1}^1(\Omega)\),
- \((\Delta u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), W_{-1,1}^1(\Omega))\) since \((u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), W_0^{1,1}(\Omega))\) and \(\Delta\) is a continuous operator from \(W_0^{1,1}(\Omega)\) to \(W_{-1,1}^1(\Omega)\).

Finally, \((\partial_t u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), W_{-1,1}^1(\Omega))\).
Aubin-Simon’ lemma gives (up to a subsequence) \(u_n \to u\) in \(L^1((0, T), L^1(\Omega))\).
Regularity of the limit

\[ u_n \to u \text{ in } L^1(\Omega \times (0, T)) \text{ and } (u_n)_{n \in \mathbb{N}} \text{ bounded in } L^q((0, T), W^{1,q}_0(\Omega)) \text{ for } 1 \leq q < \frac{d + 2}{d + 1}. \] Then

\[ u_n \to u \text{ in } L^q(\Omega \times (0, T))) \text{ for } 1 \leq q < \frac{d + 2}{d + 1}, \]

\[ \nabla u_n \to \nabla u \text{ weakly in } L^q(\Omega \times (0, T))^d \text{ for } 1 \leq q < \frac{d + 2}{d + 1}, \]

\[ u \in L^q((0, T), W^{1,q}_0(\Omega)) \text{ for } 1 \leq q < \frac{d + 2}{d + 1}. \]

Remark: \( L^q((0, T), L^q(\Omega)) = L^q(\Omega \times (0, T)) \)

An additional work is needed to prove the strong convergence of \( \nabla u_n \) to \( \nabla u \).
Full approximation, FV scheme (or dG scheme)

Space discretization: Admissible mesh $\mathcal{M}$. Time step: $k \ (Nk = T)$

$$T_{K,L} = \frac{m_{K,L}}{d_{K,L}}$$

size($\mathcal{M}$) = $\sup\{\text{diam}(K), K \in \mathcal{M}\}$

Unknowns: $u_k^{(p)} \in \mathbb{R}, \ K \in \mathcal{M}, \ p \in \{1, \ldots, N\}$.

Discretization: Implicit in time, upwind for convection, classical 2-points flux for diffusion. (Well known scheme.)
Full approximation, approximate solution

- $H_M$ the space of functions from $\Omega$ to $\mathbb{R}$, constant on each $K$, $K \in \mathcal{M}$.
- The discrete solution $u$ is constant on $K \times ((p - 1)k, pk)$ with $K \in \mathcal{M}$ and $p \in \{1, \ldots, N\}$.
  
  \[ u(\cdot, t) = u^{(p)} \text{ for } t \in ((p - 1)k, pk) \text{ and } u^{(p)} \in H_M. \]

- Discrete derivatives in time, $\partial_{t,k} u$, defined by:

  \[ \partial_{t,k} u(\cdot, t) = \partial_{t,k}^{(p)} u = \frac{1}{k} (u^{(p)} - u^{(p-1)}) \text{ for } t \in ((p - 1)k, pk), \]

  for $p \in \{2, \ldots, N\}$ (and $\partial_{t,k} u(\cdot, t) = 0$ for $t \in (0, k)$).
Full approximation, steps of the proof of convergence

Sequence of meshes and time steps, \((\mathcal{M}_n)_{n \in \mathbb{N}}\) and \(k_n\).
\[
\text{size}(\mathcal{M}_n) \to 0, \quad k_n \to 0, \quad \text{as } n \to \infty.
\]
For \(n \in \mathbb{N}, \ u_n\) is the solution of the FV scheme.

1. Estimate on \(u_n\).
2. **Strong compactness of the sequence** \((u_n)_{n \in \mathbb{N}}\).
3. Regularity of the limit of the sequence \((u_n)_{n \in \mathbb{N}}\).
4. Passage to the limit in the approximate equation.
Discrete norms

Admissible mesh: $\mathcal{M}$.

$u \in H_\mathcal{M}$ (that is $u$ is a function constant on each $K$, $K \in \mathcal{M}$).

- $1 \leq q < \infty$. Discrete $W^{1,q}_0$-norm:

$$
\|u\|_{1,q,\mathcal{M}}^q = \sum_{\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L} m_\sigma d_\sigma \left| \frac{u_K - u_L}{d_\sigma} \right|^{q} + \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}_K} m_\sigma d_\sigma \left| \frac{u_K}{d_\sigma} \right|^{q}
$$

- $q = \infty$. Discrete $W^{1,\infty}_0$-norm: $\|u\|_{1,\infty,\mathcal{M}}^q = \max\{M_i, M_e, M\}$

with

$$
M_i = \max\left\{ \left| \frac{u_K - u_L}{d_\sigma} \right|, \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L\right\},
$$

$$
M_e = \max\left\{ \left| \frac{u_K}{d_\sigma} \right|, \sigma \in \mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}_K \right\},
$$

$$
M = \max\{|u_K|, K \in \mathcal{M}\}.
$$
Discrete dual norms

Admissible mesh: $M$.
For $r \in [1, \infty]$, $\| \cdot \|_{-1,r,M}$ is the dual norm of the norm $\| \cdot \|_{1,q,M}$ with $q = r / (r - 1)$. That is, for $u \in H_M$,

$$
\| u \|_{-1,r,M} = \max \left\{ \int_\Omega uv \, dx, \ v \in H_M, \| v \|_{1,q,M} \leq 1 \right\}.
$$

Example: $r = 1$ ($q = \infty$).
Full discretization, estimate on the discrete solution

For $1 \leq q < (d + 2)/(d + 1)$, the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^q((0, T), W_{q,n})$, where $W_{q,n}$ is the space $H_{\mathcal{M}_n}$, endowed with the norm $\| \cdot \|_{1,q,\mathcal{M}_n}$. That is

$$\sum_{p=1}^{N_n} k \| u_n^{(p)} \|_{1,q,\mathcal{M}_n}^q \leq C.$$
Discrete Lions’ lemma

\( B \) is a Banach space, \( (B_n)_{n \in \mathbb{N}} \) is a sequence of finite dimensional subspaces of \( B \). \( \| \cdot \|_{X_n} \) and \( \| \cdot \|_{Y_n} \) are two norms on \( B_n \) such that:

- If \( (\|w_n\|_{X_n})_{n \in \mathbb{N}} \) is bounded, then, up to a subsequence, there exists \( w \in B \) such that \( w_n \to w \) in \( B \).
- If \( w_n \to w \) in \( B \) and \( \|w_n\|_{Y_n} \to 0 \), then \( w = 0 \).

Then, for any \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that, for \( n \in \mathbb{N} \) and \( w \in B_n \)

\[
\|w\|_B \leq \varepsilon \|w\|_{X_n} + C_\varepsilon \|w\|_{Y_n}.
\]

Example: \( B = L^1(\Omega) \). \( B_n = H_{\mathcal{M}_n} \) (the finite dimensional space given by the mesh \( \mathcal{M}_n \)). We have to choose \( \| \cdot \|_{X_n} \) and \( \| \cdot \|_{Y_n} \).
Proof by contradiction. There exists $\varepsilon > 0$ and $(w_n)_{n \in \mathbb{N}}$ such that, for all $n$, $w_n \in B_n$ and

$$\|w_n\|_B > \varepsilon \|w_n\|_X + C_n \|w_n\|_Y,$$

with $\lim_{n \to \infty} C_n = +\infty$.

It is possible to assume that $\|w_n\|_B = 1$. Then $(\|w_n\|_X)_{n \in \mathbb{N}}$ is bounded and, up to a subsequence, $w_n \to w$ in $B$ (so that $\|w\|_B = 1$). But $\|w_n\|_Y \to 0$, so that $w = 0$, in contradiction with $\|w\|_B = 1$. 
**Discrete Aubin-Simon’ Compactness Lemma**

\( B \) a Banach, \((B_n)_{n\in\mathbb{N}}\) family of finite dimensional subspaces of \( B \).
\( \| \cdot \|_{X_n} \) and \( \| \cdot \|_{Y_n} \) two norms on \( B_n \) such that:

- If \((\| w_n \|_{X_n})_{n\in\mathbb{N}}\) is bounded, then, up to a subsequence, there exists \( w \in B \) such that \( w_n \to w \) in \( B \).

- If \( w_n \to w \) in \( B \) and \( \| w_n \|_{Y_n} \to 0 \), then \( w = 0 \).

\( X_n = B_n \) with norm \( \| \cdot \|_{X_n} \), \( Y_n = B_n \) with norm \( \| \cdot \|_{Y_n} \).

Let \( T > 0, k_n > 0 \) and \((u_n)_{n\in\mathbb{N}}\) be a sequence such that

- for all \( n \), \( u_n(\cdot, t) = u_n^{(p)}(\cdot) \in B_n \) for \( t \in ((p-1)k_n, pk_n) \)
- \((u_n)_{n\in\mathbb{N}}\) is bounded in \( L^1((0, T), X_n) \).
- \((\partial_t, k_n u_n)_{n\in\mathbb{N}}\) is bounded in \( L^1((0, T), Y_n) \).

Then there exists \( u \in L^1((0, T), B) \) such that, up to a subsequence, \( u_n \to u \) in \( L^1((0, T), B) \).

Example: \( B = L^1(\Omega) \). \( B_n = H_{\mathcal{M}_n} \). What choice for \( \| \cdot \|_{X_n}, \| \cdot \|_{Y_n} \)?
Full approx., compactness of the sequence \((u_n)_{n \in \mathbb{N}}\)

\(u_n\) is solution of the fully discretized problem with mesh \(\mathcal{M}_n\) and time step \(k_n\).

\[ B = L^1(\Omega), \ B_n = H_{\mathcal{M}_n}, \]
\[ \| \cdot \|_{X_n} = \| \cdot \|_{1,1,\mathcal{M}_n}, \ \| \cdot \|_{Y_n} = \| \cdot \|_{-1,1,\mathcal{M}_n} \]

In order to apply the discrete Aubin-Simon’ lemma we need to verify the hypotheses of the discrete Lions’ lemma and that

- \((u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), X_n)\),
- \((\partial_{t,k_n}u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), Y_n)\).

The sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in \(L^q((0, T), W_{q,n}(\Omega))\) (for \(1 \leq q < (d + 2)/(d + 1)\)) and then is bounded in \(L^1((0, T), X_n)\) since \(\| \cdot \|_{1,1,\mathcal{M}_n} \leq C_q \| \cdot \|_{1,q,\mathcal{M}_n}\) for \(q > 1\).

Using the scheme, it is quite easy to prove (similarly to the continuous approximation) that \((\partial_{t,k_n}u_n)_{n \in \mathbb{N}}\) is bounded in \(L^1((0, T), Y_n)\).
Full approx., Compactness of the sequence $(u_n)_{n \in \mathbb{N}}$

It remains to verify the hypotheses of the discrete Lions’ lemma.

- If $w_n \in H_{M_n}$, $(\|w_n\|_{1,1,M_n})_{n \in \mathbb{N}}$ is bounded, there exists $w \in L^1(\Omega)$ such that $w_n \rightarrow w$ in $L^1(\Omega)$?
  
  Yes, this is classical now...

- If $w_n \in H_{M_n}$, $w_n \rightarrow w$ in $L^1(\Omega)$ and $\|w_n\|_{1,1,M_n} \rightarrow 0$, then $w = 0$? Yes... Proof:
  
  Let $\varphi \in W^{1,\infty}_0(\Omega)$ and its “projection” $\pi_n \varphi \in H_{M_n}$. One has
  
  $\|\pi_n \varphi\|_{1,\infty,M_n} \leq \|\varphi\|_{W^{1,\infty}(\Omega)}$ and then

  $$|\int_{\Omega} w_n(\pi_n \varphi) dx| \leq \|w_n\|_{1,1,M_n} \|\varphi\|_{W^{1,\infty}(\Omega)} \rightarrow 0,$$

  and, since $w_n \rightarrow w$ in $L^1(\Omega)$ and $\pi_n \varphi \rightarrow \varphi$ uniformly,

  $$\int_{\Omega} w_n(\pi_n \varphi) dx \rightarrow \int_{\Omega} w \varphi dx.$$

  This gives $\int_{\Omega} w \varphi dx = 0$ for all $\varphi \in W^{1,\infty}_0(\Omega)$ and then $w = 0$ a.e.
Regularity of the limit

As the continuous approximation, 
\( u_n \to u \) in \( L^1(\Omega \times (0, T)) \) and \( (u_n)_{n \in \mathbb{N}} \) bounded in 
\( L^q((0, T), W^{q,n}(\Omega)) \) for \( 1 \leq q < \frac{(d+2)}{(d+1)} \). Then

\[
u_n \to u \text{ in } L^q(\Omega \times (0, T))) \text{ for } 1 \leq q < \frac{d + 2}{d + 1},
\]

\[
u \in L^q((0, T), W^{1,q}_0(\Omega)) \text{ for } 1 \leq q < \frac{(d + 2)}{(d + 1)}.
\]