THE EQUATION $-\Delta u + |u|^{\alpha-1} u = f$, FOR $0 \leq \alpha \leq 1$

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1. INTRODUCTION

Let $\alpha > 0$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $n \geq 1$. We consider the following problem:

\[
\begin{align*}
-\Delta u + |u|^{\alpha-1} u &= f \\
\text{in } \mathcal{D}'(\mathbb{R}^n), u &\in L^1_{\text{loc}}(\mathbb{R}^n),
\end{align*}
\]

\[
|u|^{\alpha-1} u \in L^1_{\text{loc}}(\mathbb{R}^n). \tag{1}
\]

In [1], Brezis showed the following result.

THEOREM [1]. Let $1 < \alpha < \infty$. For every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, there exists a unique $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying (1). Moreover, if $f \geq 0$ a.e., then $u \geq 0$ a.e.

In this theorem, no limitation on the growth at infinity of the data $f$ is required for the existence of a solution $u$, and $u$ is unique without prescribing its behaviour at infinity. The proof uses a device introduced by Baras and Pierre [2] which provides a local estimate of the solutions of (1).

The preceding result fails if $0 < \alpha \leq 1$. Indeed, it is enough to consider, for $n = 1$, the Cauchy problem:

\[
\begin{align*}
u' &= \mu' \\
u(0) &= 0, \quad u(0) = u_0
\end{align*} \tag{2}
\]

One sees that for any $\mu > 0$ (2) has a global positive solution in $\mathbb{R}$, so that no uniqueness is expected even for positive solutions of (1).

Let $0 < \alpha < 1$. We now briefly explain our plan and results: In Section 2 (theorem 1), we prove that if $f \geq 0$ and (1) has a positive solution, then (1) has a "minimal" positive solution $u$ (i.e. $0 \leq u \leq v$ for any other positive solution $v$).

In Section 3, we focus on the problem of existence for $0 < \alpha < 1$ of positive solutions of (1) for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $f \geq 0$. This problem is closely related to the following (porous medium equation):

\[
\text{for } m > 1, \text{ find } u(x, t) \geq 0 \text{ such that } \left( \frac{\partial u}{\partial t} \right) = \Delta(u^m), u(x, 0) = u_0(x). \tag{3}
\]

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Indeed, the "implicit Euler scheme" associated with (3) is:
\[ u(t + h) - u(t) - hAu^m(t + h) = 0, \quad u(t + h) \geq 0, \quad u(t) \geq 0, \quad h > 0. \]

Setting \( u(t + h) = \ell(t) = \ell(t)^{-\alpha} A^\alpha, \) where \( \alpha = m^{-1}, \) we get:
\[ -\Delta u + \ell(t) = h^\alpha A^\alpha u(t), \quad u(t) \geq 0, \quad \ell(t) \geq 0. \]

We obtain precisely equation (1) with \( f = \ell^{\alpha}(t) = \ell(t). \)

Our main result is the following (Theorem 2, Section 3): We set \( B_R = \{ x \in \mathbb{R}^n, \ |x| < R \}. \) For any \( 0 < \alpha < 1, \) there exists \( c_1 > 0 \) and \( c_2 > 0 \) depending only on \( n \) and \( \alpha \) such that for \( f \in L^\infty_{loc} (\mathbb{R}^n), f \geq 0, \) one has:
\[ \left( \limsup_{R \to \infty} R^{-(\alpha + 1 - \alpha)} \int_{B_R} f \, dx < c_1 \right) \Rightarrow \text{The equation has a solution}. \]

(1) has a positive solution \( \Rightarrow \left( \limsup_{R \to \infty} R^{-(\alpha + 1 - \alpha)} \int_{B_R} f \, dx < c_2 \right). \]

Let us compare this result with the results obtained for the porous medium equation. It is known (cf. Bénilan, Crandall and Pierre [3]) that if \( u_0 \in L^\infty_{loc} (\mathbb{R}^n), u_0 \geq 0, \) one has:
\[ \left( \lim_{R \to \infty} R^\alpha \int_{B_R} u_0 \, dx < \infty \right) \Rightarrow \text{The equation has a solution}. \]

The converse of (8) is proved in [4], (cf. also Dahlberg and Kenig [5]). It is straightforward to see that our result proves that for \( u_0 \in L^\infty_{loc} (\mathbb{R}^n), u_0 \geq 0, \) the porous medium equation (3) has a local solution if and only if the "implicit Euler scheme" (5) (with \( \ell = 0, \alpha = m^{-1} \)) has a positive solution \( u(h) \) for \( h > 0 \) small enough.

It may be asked if it is possible to have \( c_1 = c_2 \) in (6) and (7). In fact, we can prove, for \( n = 1, \) that no inequality of the kind \( \limsup_{R \to \infty} R^\alpha \int_{B_R} h(x) u(x) \, dx \leq C \) may be expected to be a necessary and sufficient condition in order to have positive solutions of (1) for \( f \in L^\infty_{loc}. \)

Many of the results stated above can be generalized: we give in Section 4 a proof that Brezis' result ([11]) holds with any odd function \( g(u), \) instead of \( |u|^{m-1} u, \) which verifies: \( g \) is convex on \( \mathbb{R}^n, \) increasing, and
\[
\int_0^\infty \left( \int_0^t g(s) \, ds \right)^{-1/2} \, ds < +\infty.
\]

This kind of condition already arises in Osserman [6], who studies the inequality \( \Delta u \geq f(u). \)

For the case \( \alpha < 1, \) our result can probably be extended to functions \( q(u), \) instead of \( |u|^{m-1} u, \) with more general conditions.

It may be asked whether equation (1) with \( 0 < \alpha < 1 \) has a solution for any \( f \in L^\infty_{loc} (\mathbb{R}^n). \) In fact we can prove that it has always infinitely many solutions (of course without condition of positivity). The proof involves different tools and will be done elsewhere [13].

In Section 5 we return to the case \( f \equiv 0, u \geq 0, \) and deal with a characterization in some weighted Banach space of the minimal positive solution of (1). We obtain that if this solution verifies \( u(x) \mid x^{-2(\alpha - 1)} \to 0 \) as \( |x| \to +\infty \) (that is the case if \( f(x) \mid x^{-2(\alpha - 1)} \to 0, \) then every other

The equation \( -\Delta u + |u|^{m-1} u = f, \) for \( 0 < \alpha < 1 \) has a positive solution.

\[
\limsup_{R \to \infty} R^{-(\alpha + 1 - \alpha)} \int_{B_R} u \, dx > 0.
\]

We prove also by an example that this characterization is not available if \( f(x) \mid x^{-2(\alpha - 1)} \to 0. \)

**Notation.** In what follows, \( \alpha \) denotes always a real number such that \( 0 < \alpha < 1. \) If \( \Omega \) is an open set of \( \mathbb{R}^n, \) we often write: \( f_s u \, dx \) for \( \int f_s u(x) \, dx, \) where \( dx \) is the Lebesgue measure in \( \mathbb{R}^n. \) By \( \mathbb{B}^n = \{ x \in \mathbb{R}^n, \ |x| < R \} \) we mean: \( x \in \mathbb{R}^n, \ |x| \leq R. \) We usually set \( s = |x| - R, \) \( B_s = \{ x \in \mathbb{R}^n, \ |x| < R \}, \) and \( f_s u \, dx = f_s u_0 \, ds, \) where \( ds \) is the usual superficial measure on \( S. \) If \( \Omega \) is an open set, \( W^{\infty} (\Omega) (k \in \mathbb{N}, \rho > 0) \) is the Sobolev space of all measurable functions \( \Omega \) derivatives in the distribution sense \( D' u \) verify \( D' u \in L^q (\Omega) \) for \( 0 < q \leq \rho, \) \( W^{k, q} (\Omega) \) denotes the set of all \( C^k \) functions with compact support in \( \Omega. \) \( W^{k, q} \) is the closure of \( \mathcal{D} (\Omega) \) in \( W^{k, q} (\Omega), \) and \( W^{k, q} \) denotes the set of all functions \( u \) such that \( \int u \in L^q (\Omega) \) for any \( q \in (0, \infty). \) For the Sobolev inclusions, we refer to Adams [8], and for the regularity properties of the Laplacian, to Agmon, Douglis and Nirenberg [9]. Finally for \( f \) and \( u \in L^\infty_{loc}, \) \( -\Delta u + u^s = f \in \mathcal{D}' (\Omega) \) means that for any \( \varphi \in \mathcal{D} (\Omega) \) one has: \( \int f_s u \varphi \, dx + \int f_s \varphi \, dx = \int \varphi \, dx. \) This makes sense, since \( u \in L^1_{loc} \to u^s \in L^\infty_{loc} \) for \( \alpha < 1. \) The support of a function \( f \) will be denoted by \( sup f. \)

### 2. Preliminary Results

Let \( 0 < \alpha < 1, f \in L^\infty_{loc} (\mathbb{R}^n) \) with \( f \equiv 0. \) Consider the problem:

\( \left( \begin{array}{c} \Delta \alpha + u^s = f \end{array} \right) \in \mathcal{D}' (\Omega). \)

We shall define that \( v \) is a supersolution of \( \left( \begin{array}{c} \Delta \alpha + u^s = f \end{array} \right) \in \mathcal{D}' (\Omega). \)

We now prove the following result:

**Theorem 1.** Let \( 0 < \alpha < 1, f \in L^\infty_{loc} (\mathbb{R}^n) \) with \( f \equiv 0. \) If \( \left( \begin{array}{c} \Delta \alpha + u^s = f \end{array} \right) \in \mathcal{D}' (\Omega). \)

In fact, theorem 1 is a consequence of the maximum principle. But the lack of regularity generates some difficulties, which we solve by using the following lemma (proved in the Appendix):

**Lemma 1.** Let \( u \in W^{1, L^\infty (B_r)} \), \( -\Delta u \in L^1 (B_r), \) and \( u \leq 0 \) on \( \partial B_r. \) If \( P \in C^1 \cap L^\infty (\mathbb{R}), \) with \( P = 0 \) on \( \mathbb{R}^n \) and \( P > 0, \) then

\[ \int \Delta u P (u) \, dx \geq 0. \]

**Proof of theorem 1.** Let \( u \in L^\infty_{loc} (\mathbb{R}^n) \) be such that \( u \geq 0 \) and \( -\Delta u + u^s = f \), with \( g \in L^1_{loc} (\mathbb{R}^n). \) We shall prove the existence of a solution \( u \) of (P) with \( u \leq 0. \) For this purpose,
define the sequence \((u_k)\) of solutions of:

\[
\begin{aligned}
-\Delta u_k + |u_k|^{p-1} u_k &= f \quad \text{in } \mathcal{D}'(B_k), \\
\varphi_k &\in W^{1,1}_0(B_k).
\end{aligned}
\]

The existence, uniqueness of \(u_k\) and the fact that \(u_k \geq 0\) (since \(f \geq 0\)) are proved in Brezis and Strauss [10]. (Another proof can be found in [11]). Note that \(u_k - v\) verifies the assumptions of lemma 1 (with \(B = B_R = B_k\)). Let \(P\) be as in lemma 1. Multiplying by \(P(u_k - v)\) the inequality

\[-\Delta(u_k - v) + |u_k|^{p-1} u_k - |v|^{p-1} v \leq 0\]

and then integrating yields:

\[
\int_B -\Delta (u_k - v) P(u_k - v) \, dx + \int_B (|u_k|^p - |v|^p) P(u_k - v) \, dx \leq 0.
\]

Thus by lemma 1:

\[
\int_B (|u_k|^p - |v|^p) P(u_k - v) \, dx \leq 0,
\]

and, choosing \(P > 0\) on \(\mathbb{R}^n\), this gives \(P(u_k - v) \leq 0\ a.e.\), and then \(u_k \leq v\ a.e.\) on \(B\). By the same argument, one obtains \(u_k \leq u_{k+1}\) on \(B_k\).

We conclude that \((u_k)\) (and, therefore, \((u_k^+)\)) converges in \(L^1(\mathbb{R}^n)\). Passing to the limit in the distribution sense in \((P)\) yields that \(u\) is solution of \((P)\). Moreover \(0 \leq u \leq v\), and we obtain \(u \leq w\) for any solution \(w\) of \((P)\), since \(w\) is also a supersolution.

3. THE MAIN THEOREM

THEOREM 2. Let \(0 < \alpha < 1\). There exists \(c_1 > 0\) and \(c_2 > 0\) (depending only on \(n\) and \(\alpha\)) such that for \(f \in L^1_{\text{loc}}(\mathbb{R}^n), f \geq 0\), one has:

1. if \(\limsup_{R \to 0} R^{-n-(2n-1)\alpha} \int_{B_R} f \, dx < c_1\), then \((P)\) has a solution (and therefore a minimal solution in the sense of theorem 1);
2. if \((P)\) has a solution then \(\limsup_{R \to 0} R^{-n-(2n-1)\alpha} \int_{B_R} f \, dx < c_2\).

3.1. Proof of part (1) of theorem 2

We first state the following lemma ("compact support lemma") which is an essential tool in our proof. Roughly speaking this lemma is contained in [12]. For a sake of completeness we give a proof of this lemma in the appendix (cf. lemma A.3).

We set for \(a, b \in \mathbb{R}, \Omega(a, b) = \{x | a \leq |x| < b\}.

COMPACT SUPPORT LEMMA (Appendix—lemma A.3). Let \(0 < \alpha < 1\). For all \(0 \leq a < b\) and for all \(e > 0\) there exists \(c(e, a, b) > 0\) such that if

\[
f \in L^1_{\text{loc}}(\mathbb{R}^n), f \geq 0, \text{Supp } f \subset \Omega(a, b), \int f \, dx \leq c(e, a, b),
\]

then the problem

\[
\begin{aligned}
-\Delta u + u^{\alpha} &= f \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \\
u &\in L^1(\mathbb{R}^n), u \geq 0, \text{Supp } u \subset \Omega(a - \varepsilon, b + \varepsilon)
\end{aligned}
\]

has a (unique) solution.

**Remark 1.** Let \(R > 0\). The function \(u\) is a solution of \((P')\) if and only if \(\hat{u}(\cdot) = R^{n-1-\alpha} u(\cdot/R)\) is a solution of \((P')\) with \(\hat{f}(\cdot) = R^{-n+(2n-1)\alpha} f(\cdot/R)\), \(\hat{a} = a R\), \(\hat{b} = b R\), \(\hat{\varepsilon} = \varepsilon R\), instead of \(f, a, b, \varepsilon\).

(The proof of this remark is straightforward.)

**Remark 2.** From the compact support lemma and remark 1, one can easily deduce that for any \(f \in L^1(\mathbb{R}^n)\) with compact support, there exists a (unique) \(u \in L^1(\mathbb{R}^n)\) with compact support and such that \(-\Delta u + u^{\alpha} = f\) in \(\mathcal{D}'(\mathbb{R}^n)\) (cf. theorem 6.1 of [12]).

Using this compact support lemma we are going to prove the existence of a supersolution for \((P)\), which, by theorem 1, is enough to ensure the existence of a solution for \((P)\).

To this end, we set, for \(k \in \mathbb{N}\),

\[
\Omega_k = \Omega(2^k, 2^{k+1}), \quad \Omega_k^* = \Omega(0, 2^k).
\]

Let \(g \in L^1_{\text{loc}}(\mathbb{R}^n), g \geq 0\), and such that for some \(K > 0\), \(\limsup_{R \to 0} R^{-n-(2n-1)\alpha} \int_{B_R} g \, dx < K\). Then one has for some \(K' > 0\),

\[
\int_{\Omega_k} g \, dx \leq K R^{n+(2n-1)\alpha} + K'.
\]

Thus, for \(k_0 > 1\) large enough, we get

\[
\begin{aligned}
&\int_{\Omega_k} g \, dx \leq 2K (2^{k_0})^{n+(2n-1)\alpha}, \\
&\int_{\Omega_k} g \, dx \leq 2^{n+(2n-1)\alpha} K \cdot (2^{k_0})^{n+(2n-1)\alpha}, \quad \text{for } k \geq k_0.
\end{aligned}
\]

If we assume that \(K \leq c_1 = 2^{-n-(2n-1)\alpha} \inf(c(1/2, 1, 2), c(1/2, 0, 1))\), we deduce from (11) and (12) (by using the compact support lemma and remark 1) the existence of \((v_k, k \geq k_0 - 1)\) such that

\[
\begin{aligned}
v_k &= 0, \quad v_k \in L^1(\mathbb{R}^n), \\
-\Delta v_k + v_k^{\alpha} &= g^{1, \Omega_k} \quad \text{if } k \geq k_0 \\
-\Delta v_k + v_k^{\alpha} &= g^{1, \Omega_k} \quad \text{if } k = k_0 - 1 \\
\text{Supp } v_k \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}, \quad k \geq k_0 \\
\text{Supp } v_{k_0-1} \subset \Omega_{k_0} \cup \Omega_{k_0}.
\end{aligned}
\]
Thus one sees that the series
\[ \sum_{i=\ell_0-1}^{\ell_0} u_i, \quad \text{and} \quad \sum_{i=\ell_0-1}^{\ell_0} v_i \]
converge a.e. and in \( L^1_{\text{loc}}(\mathbb{R}^n) \). Furthermore, as \((a^x + b^x + c^x) \leq 3(a + b + c)^x \) for all \( a, b, c \geq 0 \), one has
\[ \left( \sum_{i=\ell_0-1}^{\ell_0} v_i \right)^x \leq 3 \left( \sum_{i=\ell_0-1}^{\ell_0} v_i \right)^x. \] (14)

Setting
\[ w = \sum_{i=\ell_0-1}^{\ell_0} v_i, \]
we then deduce (with (13) and (14)):
\[ -\Delta w + 3w^\alpha \geq g \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n) \]
\[ w \geq 0 \quad w \in L^1_{\text{loc}}(\mathbb{R}^n). \] (15)

With \( z = 3^{1/\alpha - 1} w \), this gives
\[ -\Delta z + z^\alpha \geq 3^{1/\alpha - 1}, \quad z \in L^1_{\text{loc}}(\mathbb{R}^n), \quad z \geq 0. \]

Setting now \( g = 3^{1/\alpha - 1} f \), we deduce from (16) that if
\[ \lim_{R \to +\infty} R^{-1} \left( \int_{B_R} f \, dx \right) \leq c_1 = 3^{1/\alpha - 1} c_1', \]
then \( z \) is a supersolution of (P). This proves part (1) of theorem 2.

3.2. Proof of part (2) of theorem 2
Let \( u \) be a solution of (P). We first give a formal proof, that is assuming that \( u \) is regular enough. We then shall justify the calculations.

Integrating on \( B_{s_1} = B(0, s) \) the relation \(-\Delta u + u^\alpha = f\), and then integrating by parts, we get:
\[ -\int_{B_{s_1}} \frac{du}{\alpha} + \int_{B_{s_1}} u^\alpha = \int_{B_{s_1}} f. \] (17)

We set
\[ w(t) = \frac{1}{t^\alpha - 1} \int_{B_t} u \, dx, \]
and obtain:
\[ -t^{-1} w'(t) + \int_{B_t} u^\alpha = \int_{B_t} f \geq 0. \] (18)
We conclude from (22) and (23) that:

\[
\limsup_{r \to 0} r^{-\sigma-2(\alpha/\alpha-\alpha)} f \leq c_2,
\]

with \(c_2\) depending only on \(\alpha\) and \(n\).

To justify the calculations above, it is enough to have \(u \in C^1 \cap W^{2,2}(\mathbb{R}^n)\). This is not the case, but let \(\varphi_k\) be a sequence of mollifiers. One has

\[
-\Delta (u \ast \varphi_k) + u^\alpha \ast \varphi_k = f \ast \varphi_k,
\]

and \(u \ast \varphi_k \in C(\mathbb{R}^n)\),

\[
\begin{align*}
& u^\alpha \ast \varphi_k \to u^\alpha \quad \text{in } L^1_{loc}, \\
& f \ast \varphi_k \to f \quad \text{in } L^1_{loc}, \\
& u \ast \varphi_k \to u \quad \text{in } W^{1,1}_{loc} \text{ and,} \\
& \text{then: } u \ast \varphi_k \to u \quad \text{in } L^1(S_r) \text{ for any sphere } S_r.
\end{align*}
\]

The relations (17), (18) have now a sense with \(u \ast \varphi\) instead of \(u\); \(u^\alpha \ast \varphi\) instead of \(u^\alpha\), \(f \ast \varphi\) instead of \(f\). Then (19) and (20) are justified by letting \(k \to +\infty\) and using (24).

4. A GENERALIZATION OF BREZIS' RESULT [1]

THEOREM 3. Let \(g : \mathbb{R}^+ \to \mathbb{R}\) be continuous odd function, convex on \(\mathbb{R}^+\) and verifying

\[
g(s)s \geq 0, \quad \forall s \in \mathbb{R},
\]

\[
\int_0^\infty \frac{dt}{G(t)^{1/2}} < \infty, \quad \text{where } G(s) = \int_s^\infty g(t) \, dt.
\]  

Then for any \(f \in L^1_{loc}(\mathbb{R}^n)\) \((n \geq 1)\), there exists a unique function \(u\) solution of

\[
\begin{align*}
-\Delta u + g(u) &= f \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \\
u \in L^1_{loc}(\mathbb{R}^n), \quad g(u) \in L^1_{loc}(\mathbb{R}^n).
\end{align*}
\]

Moreover, if \(f \geq 0\) a.e., then \(u \geq 0\) a.e.

As in Brezis [1], the preceding result can be generalized in the following way.

THEOREM 4. Let \(h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) be a Carathéodory function such that

\[
h(x, s) \geq |g(s)|, \quad \forall s \in \mathbb{R}, \quad \text{a.e. in } x \in \mathbb{R}^n,
\]

where \(g\) verifies the assumptions of theorem 3, and:

\[
\sup_{|x| < r} |h(x, s)| \in L^1_{loc}(\mathbb{R}^n) \quad \text{for any } r \in \mathbb{R}.
\]

Then for every \(f \in L^1_{loc}(\mathbb{R}^n)\) there exists at least a solution of

\[
\begin{align*}
-\Delta u + h(\cdot, u) &= f \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \\
u \in L^1_{loc}(\mathbb{R}^n), \quad h(\cdot, u) \in L^1_{loc}(\mathbb{R}^n).
\end{align*}
\]

(30)

(We omit here the proof of theorem 4, which essentially is the same as in [1].)

Remark 3. The results of theorems 3 and 4 are optimal in the following sense: Let \(g : \mathbb{R}^+ \to \mathbb{R}^+\) be increasing, with \(g(0) = 0\), and assume that \(g\) does not verify (26), that is:

\[
\int_0^\infty \left( \int_0^t g(t) \, dt \right)^{-1/2} \, dt = \infty.
\]

Then one sees easily that for any \(u_0 > 0\), the O.D.E.

\[
\begin{align*}
-\Delta u + g(u) &= 0 \\
u_0 = u_0, \quad u'(0) = 0
\end{align*}
\]

has a positive solution in \(C(\mathbb{R}, \mathbb{R})\). Thus we have infinitely many solutions, and no local estimate on them.

4.1. A local estimate for proving theorem 3

LEMMA 2. Let \(g\) verify the assumptions of theorem 3 and \(0 < R < R'\). Let \(f \in L^1(\mathbb{R}^n)\) and \(u \in L^1_{loc}(\mathbb{R}^n)\) such that

\[
\begin{align*}
-\Delta u + g(u) &= f \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \\
u \in L^1_{loc}(\mathbb{R}^n), \quad g(u) \in L^1_{loc}(\mathbb{R}^n).
\end{align*}
\]

(32)

Then:

\[
\int_{\mathbb{R}^n} |g(u)| \leq C \left( 1 + \int_{\mathbb{R}^n} |f| \right)
\]

(33)

where \(C\) only depends on \(g\), \(R\) and \(R'\).

In the case \(g(u) = |u|^{r-1} u, \quad a > 1\), the proof in [1] is based on a tool of Baras and Pierre [2] using Hölder's inequality. The following lemma will provide a generalization of this tool to a class of convex functions.

LEMMA 3. Let \(g : \mathbb{R}^+ \to \mathbb{R}^+\) be a continuous odd function, convex on \(\mathbb{R}^+\) and verifying \(g(0) = 0\) and \(\int_{\mathbb{R}^+} (\int_0^t g(s) \, ds)^{-1/2} \, ds < \infty\). Then for any \(C > 0\) there exists a convex function \(\varphi \in C(\mathbb{R}^+, \mathbb{R}^+\) verifying \(\varphi(0) = \varphi'(0) = 0\) and

\[
\varphi''(x) + \varphi'(x) \leq \frac{1}{2C} g(t) \varphi(x) + 2
\]

(34)

for any \(x \in [0, 1]\) and \(t \in \mathbb{R}^+\). \((\varphi\) depends only on \(g\) and \(C\).)
Proof of lemma 3. Set:

\[ k(y) = \left( g^{-1}(\frac{4C}{y}) \right)^{-1} \quad \text{for } y > 0 \quad \text{and } k(0) = 0. \]  

(35)

We seek for a solution of the Cauchy problem:

\[ \begin{aligned}
\varphi''(x) &= k(\varphi(x)), \\
\varphi(0) &= \varphi'(0) = 0, \\
\varphi(x) &> 0 \quad \text{for } x \in [0, \alpha],
\end{aligned} \]

(36)

Such a nontrivial solution exists since (26) implies

\[ \int_0^x \left( \int_0^s k(r) \, dr \right)^{-1/2} \, ds < \infty. \]

Indeed, (37) allows to define a solution \((\varphi, \alpha)\) of (36) by:

\[ \begin{aligned}
\int_0^x \left( \int_0^s k(r) \, dr \right)^{-1/2} \, ds = \sqrt{2x} \quad \text{for } x \in [0, \alpha], \\
\alpha &= \min \left( \frac{1}{\sqrt{2}} \left( \int_0^1 \left( \int_0^s k(r) \, dr \right)^{-1/2} \, ds \right)^1, \quad 1 \right).
\end{aligned} \]

(37)

(For a proof of (26) \(\Rightarrow\) (37), see lemma 4 below.)

By (35) one has

\[ \frac{k(y)}{y} = \frac{1}{4C} \frac{g(1/k(y))}{1/k(y)}, \]

and, since \(t \to \frac{g(t)}{t}\) is nondecreasing we get:

\[ \frac{k(y)}{y} \leq \frac{1}{4C} \frac{g(t)}{t} \quad \text{for any } t, y \in \mathbb{R}^+ \quad \text{such that } tk(y) > 1. \]

Thus,

\[ tk(y) \leq \frac{1}{4C} g(t)y + 1 \quad \text{for any } t, y \geq 0. \]

(38)

Setting \(y = \varphi(x)\) and using (36) yields

\[ \varphi''(x) \leq \frac{1}{4C} g(t)\varphi(x) + 1 \quad \text{for } x \in [0, \alpha]. \]

Extending \(\varphi\) to \([0, 1]\) by:

\[ \varphi(x) = \varphi(\alpha) + (x - \alpha)\varphi'(\alpha) + \frac{(x - \alpha)^2}{2} \varphi''(\alpha), \]

we get:

\[ \varphi''(x) \leq \frac{1}{4C} g(t)\varphi(x) + 1 \quad \text{for } x \in [0, 1]. \]

(39)

Note that \(\varphi''\) is nondecreasing on \([0, 1]\). Thus

\[ \varphi''(x) \leq \int_0^x \varphi''(t) \, dt \leq \varphi''(x) \quad \text{for } x \in [0, 1], \]

and, therefore, (34) is a consequence of (39).

Proof of lemma 2. Multiplying (32) by \(\text{sgn } u\) and using Kato's inequality we get:

\[ -\Delta |u| + |g(u)| = |f|\quad \text{in } B_R. \]

For any \(\psi \in \mathcal{H}(B_R)\) with \(\psi \equiv 0\), we then have:

\[ -\int_{B_R} |u| \Delta \psi + \int_{B_R} |g(u)| \psi \leq \int_{B_R} |f| \psi. \]

(40)

By a density argument, (40) remains true as \(\psi \in C_c^2(B_R)\) and \(\psi \equiv 0\).

Applying now (40) with \(\varphi \circ \psi\) instead of \(\psi\), where \(\varphi\) is given by lemma 2 and \(\psi = 1\) on \(B_R\), \(0 \leq \psi \leq 1\) in \(B_R\). We obtain:

\[ -\int_{B_R} |u| (\varphi''(\psi)) |\nabla \psi|^2 + \varphi'(\psi) \Delta \psi) \psi + \int_{B_R} |g(u)| \varphi(\psi) \leq \int_{B_R} |f| \varphi(\psi). \]

Thus,

\[ \int_{B_R} |g(u)| \varphi(\psi) \leq C \int_{B_R} |u| (\varphi''(\psi) + \varphi'(\psi)) \psi + \int_{B_R} |f| \varphi(\psi), \]

where \(C\) only depends on \(\psi\). Using (34):

\[ \int_{B_R} |g(u)| \varphi(\psi) \leq \frac{1}{\epsilon} \int_{B_R} |g(u)| \varphi(\psi) + C_1 + C_2 \int_{B_R} |f|, \]

where \(C_1\) and \(C_2\) only depend on \(\psi, R, R', g\).

The use of (28) gives finally (33). (Note that \(\psi\) may be chosen in order to depend only on \(R\) and \(R'\).)

It remains to prove the relation (26) \(\Rightarrow\) (37):

**Lemma 4.** Let \(g : \mathbb{R}^+ \to \mathbb{R}^+\) be a continuous convex function and \(k : \mathbb{R}^+ \to \mathbb{R}^+\) defined by \(k(s) = \left[ g^{-1}(c/s) \right]^{-1}\), where \(c\) is some positive constant. Then

\[ \int_0^\infty \left( \int_0^t g(s) \, ds \right)^{-1/2} \, dt < \infty \quad \text{implies } \int_0^t \left( \int_0^s k(x) \, dx \right)^{-1/2} < \infty. \]

**Proof of lemma 4.** \(g\) and \(k\) being nondecreasing, one has

\[ \frac{1}{2} g \left( \frac{t}{2} \right) \leq \int_0^t g(s) \, ds \leq g(t). \]
and the same inequalities for \( k \). Thus
\[
\int^\infty \left( \int_0^t g(s) \, ds \right)^{-1/2} \, dt < \infty \Leftrightarrow \int^\infty \frac{dt}{t^{1/2} g(t)^{1/2}} < \infty, \tag{41}
\]
and
\[
\int^t_0 g(s) \, ds \Leftrightarrow \int^\infty \frac{dt}{t^{1/2} k(t)^{1/2}} < \infty \Leftrightarrow \int^\infty \frac{dt}{t^{1/2} k(t)^{1/2}} < \infty
\]
\[
\int_0^t \frac{(k^{-1}(1/\beta))^{1/2}}{t^{1/2}} \, dt < \infty \Leftrightarrow \int^\infty \frac{u^{1/2} g(u)}{g(u)^{1/2}} < \infty \quad \text{(where } t = \frac{1}{g(u)} \text{)}
\]
\[
\Rightarrow \left[ -2 u^{1/2} g(u)^{-1/2} \right] + \int^\infty \frac{du}{u^{1/2} g(u)^{1/2}} < \infty
\]
(integrating by parts).

Since \( g \) is convex, \( (g(t))/t \) is increasing, and then \( \int^\infty t^{-1/2} g(t)^{-1/2} < \infty \) implies \( (g(t))/t \to +\infty \) as \( t \to +\infty \).

Thus,
\[
\left[ -2 u^{1/2} g(u)^{-1/2} \right] + \int^\infty \frac{du}{u^{1/2} g(u)^{1/2}} < \infty \Leftrightarrow \int^\infty \frac{du}{u^{1/2} g(u)^{1/2}} < \infty.
\]

This gives lemma 4.

4.2. Existence of solutions for theorem 3

The proof is now the same as in Brezis [1]. Instead of the Baras–Pierre estimate, we use lemma 1. So we just recall the main steps of the proof.

(1) For any \( n \in \mathbb{N}^0 \) there exists \( u_n \in W^{1,1}_0(B_1) \) solution of
\[
\begin{align*}
-\Delta u_n + g(u_n) &= f \quad \text{in } D'(B_1) \\
g(u_n) &\in L^1(B_1).
\end{align*}
\]

(2) By lemma 2, \( (g(u_n))_n \) is bounded in \( L^1_\text{loc}(\mathbb{R}^n) \) and then \( \Delta u_n \) and \( u_n \) are bounded in \( L^1_\text{loc}(\mathbb{R}^n) \) (indeed, \( |g(t)| \equiv t/\beta \) for some \( \beta > 0 \) and \( r \) large enough). Using the same method as in [1], we conclude that \( u_n \to u \) and \( g(u_n) \to g(u) \) for some \( u \in L^1_\text{loc}(\mathbb{R}^n) \). Thus, \( u \) is solution of (30). This gives the "existence" of theorem 3.

4.3. End of the proof of theorem 3

(1) Lemma 5. Let \( g \) be as in theorem 3. There exists a family of functions \( (u_R)_{R \geq R_0} \) such that:

(i) \( u_R \in C^2(B_R), u_R \geq 0, u_R(x) \to +\infty \) as \( |x| \to R \);
(ii) \( u_R \to 0 \) in \( L^\infty_\text{loc}(\mathbb{R}^n) \) as \( R \to +\infty \);
(iii) \( -\Delta u_R + g(u_R) > 0 \).

Proof. Set \( H_n(u) = \int^u_0 \left( G(t) - G(u_0) \right)^{-1/2} \, dt \), where \( G(t) = \int^t_0 g(s) \, ds \), and
\[
\begin{align*}
H_n(u_0)(x) &= H_n(u_0)(\sqrt{2} |x|),
\end{align*}
\]

The equation \(-\Delta u + |u|^{-1} u = f\), for \( 0 < \alpha < 1 \)

with
\[
R(u_0) = \int^\infty \left( G(t) - G(u_0) \right)^{-1/2} \, dt, \quad u_0 > 0.
\]

One sees easily that \( -\Delta u_{R_0} + g(u_{R_0}) > 0 \). Since \( g \) is convex, \( \int^\infty G(t)^{-1/2} \, dt = +\infty \), and therefore \( R(u_0) \to \infty \) as \( u_0 \to 0 \). Thus, we may define \( u_R \) for any \( R > R_0 \) with \( R_0 \) large enough.

The property (i) is clear, and to prove (ii), note that
\[
\sqrt{2} |x| = \int^u_{u_0} \left( G(t) - G(u_0) \right)^{-1/2} \, dt \quad \text{for } R = R(u_0).
\]

As \( R \to +\infty \), one has by (42) \( u_0 \to 0 \). As \( u_0 \to 0 \), (43) shows that \( u_R(x) \to 0 \) uniformly in \( x \) as \( x \) remains in a bounded set.

(2) If \( u \) is a solution of (27), then \( f \equiv 0 \) implies \( u \equiv 0 \).

Proof. Here again we follow the method of [1], using the comparison functions constructed in lemma 5.

One has by (iii):
\[
-\Delta(u - u_R) + g(u) - g(u_R) \leq 0 \quad \text{in } B_R.
\]

Thus by Kato's inequality
\[
-\Delta(u - u_R) \leq 0.
\]

Again by Kato's inequality, we get \( -\Delta u^- \leq 0 \) from \(-\Delta u^+ + g(u)^+ \leq 0 \). That implies \( u^- \in L^\infty_\text{loc}(\mathbb{R}^n) \). Then (i) yields \( u - u_R \to 0 \) near \( \partial B_R \), and this jointly to (44) gives \( u \equiv u_R \) a.e.

(3) The solution \( u \) of (27) is unique. Let \( u \) and \( v \) be two solutions of (27). Then
\[
-\Delta(u - v) + g(u) - g(v) = 0.
\]

Thus by Kato's inequality:
\[
-\Delta(u - v) + |g(u) - g(v)| \leq 0.
\]

Note that if \( u \) and \( v \) have the same sign,
\[
|g(u) - g(v)| = \int^u v \, g'(t) \, dt = \int^u |g'(t)| \, dt = g(|u - v|),
\]
and if \( u_0 \leq 0 \) and, for instance, \( |v| \geq |u| \):
\[
|g(u) - g(v)| = |g(u) + g(v)| \geq |g(v)| \geq |g(\frac{|u| + |v|}{2})| = |g(\frac{v - u}{2})|.
\]

We then get:
\[
|g(u) - g(v)| \geq |g((u - v)/2)|, \quad \forall u, v \in \mathbb{R}.
\]

Thus,
\[
-\Delta(u - v) + g\left(\frac{|u - v|}{2}\right) \leq 0.
\]

By the preceding result (2) applied with the function \( t \mapsto g(t/2) \), we get \( |u - v| \leq 0 \), that is \( u = v \) a.e.
5. REMARKS ON THE BEHAVIOUR OF POSITIVE SOLUTIONS OF (1) AT INFINITY

We saw in theorem 1 that if (1) has a positive solution for \( f \geq 0 \), it has also a minimal positive solution. An interesting question is as to whether this minimal solution can be distinguished from the others by its behaviour at infinity. (That is the case if \( a = 1 \).) We first give a positive answer to this question as the behaviour of \( f \) at infinity is subcritical (i.e. \( f(x)|x|^{-2a/(1-a)} \to 0 \)).

**Theorem 5.** Let \( u \) be the minimal positive solution of (1) for some \( f \geq 0 \), and assume that \( |x|^{-2/(1-a)}u(x) \to 0 \) as \( |x| \to +\infty \).

Then for every other positive solution \( v \) of (1) one has:

\[
\limsup_{r \to +\infty} r^{-\gamma(1-a)} \int_{B_r} v > 0.
\]

**Remarks.** (1) If \( |x|^{-2/(1-a)}u(x) \to 0 \) at infinity, one sees easily that the supersolution constructed in the existence proof of theorem 2, and, therefore, the minimal solution \( u \), verify \( |x|^{-2/(1-a)}u(x) \to 0 \) as \( |x| \to +\infty \). Thus, as \( |x|^{-2/(1-a)}f(x) \to 0 \), we get a simple characterization of the minimal positive solution of (1).

(2) If instead of \( |x|^{-2/(1-a)}u(x) \to 0 \) one only has \( r^{-\gamma(1-a)} \int_{B_r} u \to 0 \), the result of theorem 4 seems to fail.

**Lemma 6.** Let \( w \in L^1_{loc}(\mathbb{R}^n) \) verifying \( |\Delta w| \leq u^2 \). Then there exists a constant \( C \) depending only on \( a \) such that:

\[
\forall u \geq 0, \left( \limsup_{r \to +\infty} r^{-\gamma(1-a)} \int_{B_r} w \leq a \Rightarrow \limsup_{r \to +\infty} |x|^{-2/(1-a)}w(x) \leq C a \right).
\]

**Proof.** Assume that \( \limsup_{r \to +\infty} r^{-\gamma(1-a)} \int_{B_r} u < a \). Then for \( r > r_0 \) large enough one has \( r^{-\gamma(1-a)} \int_{B_r} u \leq a \).

Set now \( u_r(x) = r^{-\gamma(1-a)}u(rx) \). Then one sees immediately that \( u_r \) verifies:

\[
|\Delta u_r| \leq u_r^2 \quad \text{and} \quad \int_{B_1} u_r \leq a.
\]

Using classical interior estimates and a straight bootstrap argument, the relations (46) imply that:

\[
|u_r|_{L^2(B_2)} \leq C a \quad \text{with} \ C \text{ depending only on } a.
\]

Thus, for \( |x| \leq r/2 \), \( u(x) = r^{\gamma(1-a)} u_r(x/r) \leq C a r^{\gamma(1-a)} \).

We conclude that \( u(x) \leq C r^{\gamma(1-a)} a |x|^{2/(1-a)} \).

**Proof of theorem 5.** Let \( u \) be a minimal positive solution of (1) such that \( u(x)|x|^{-2/(1-a)} \to 0 \) as \( |x| \to +\infty \), and \( u + w \) another positive solution. Thus, \( \Delta w = (u + w)^a - u^a \geq 0 \).

The equation \( -\Delta u + w^a = f \), for \( 0 \leq a < 1 \)

Assume by contradiction that \( r^{-\gamma(1-a)} \int_{B_r} w \to 0 \) as \( r \to +\infty \).

Since (47) implies \( |\Delta w| \leq u^2 \), we obtain by lemma 7 that \( |x|^{-2/(1-a)}w(x) \to 0 \) as \( |x| \to +\infty \).

If \( a, b \geq 0 \), one verifies easily that there exists a constant \( C \) depending only on \( a \) such that:

\[
(a + b)^a \geq C \inf(\alpha u + b, u^a).
\]

Applying this relation to \( u \) and \( w \) and taking into account that \( u(x)|x|^{-2/(1-a)} \) and \( w(x)|x|^{-2/(1-a)} \to 0 \) as \( |x| \to +\infty \), we obtain:

\[
\forall \varepsilon > 0, \varepsilon_0, \quad |x| \geq \varepsilon_0 \Rightarrow (u + w)^a - u^a \geq Ce^{-\varepsilon} |x|^{-2} w.
\]

In what follows, we make formal calculations, assuming \( w \) as smooth as necessary. We shall justify them later. Integrating (47) on \( B \), and using (48) yields:

\[
\int_{B_R} \frac{\partial w}{\partial n} \geq \int_{B_R} (u + w)^a - u^a + Ce^{-\varepsilon} \int_{B_R} |x|^{-2} w.
\]

Setting \( C_0 = \int_{B_R} (u + w)^a - u^a \) if \( R \) is large enough, we get

\[
\rho^{-\gamma(1-a)} \int_{S_{\rho}} w \geq C_0 + Ce^{-\varepsilon} \int_{S_{\rho}} |x|^{-2} w.
\]

Set \( \tilde{w}(r) = \rho^{-\gamma(1-a)} \int_{S_{\rho}} w \). We obtain:

\[
r^{-\gamma(1-a)}\tilde{w}(r) \geq C_0(e) + Ce^{-\varepsilon} \int_{r_0}^r t^{-\gamma(1-a)} \tilde{w}(t) \, dt \quad \text{for } r \geq r_0(e).
\]

By Gronwall’s lemma, this implies that for any \( k > 0 \), \( r^{-\gamma(1-a)}\tilde{w}(r) \to +\infty \) as \( r \to +\infty \), which contradicts the assumption \( |x|^{-2/(1-a)}w(x) \to 0 \) as \( |x| \to +\infty \).

We now justify the above formal calculations: as in 3.2, we can use a sequence of mollifiers \( \varphi_k \); we can write relations (47)–(48) with \( w \ast \varphi_k \) instead of \( w \) and \( (u + w)^a - u^a \ast \varphi_k \) instead of \( (u + w)^a - u^a \). With these modifications, (49) and (50) remain true. Call (50) the modified relation. Let now \( k \to +\infty \). Since \( w \in W^{1,0}_{loc}(\mathbb{R}^n) \), \( w \in A.C. (\mathbb{R}^n, \mathbb{R}) \), and passing to the limit in (50) yields (51) in the classical sense.

The following proposition proves that if \( f \) has the limit behaviour at infinity which ensures existence of positive solutions, the result of theorem 4 is not available.

**Proposition 1.** There exists two positive constants \( a \) and \( b \) such that every positive solution of

\[
-u^a + w^a = cr^b \quad \text{for } 0 \leq a < 1
\]

verifies

\[
a < \limsup_{r \to +\infty} |r|^{-2/(1-a)} u(r) < b.
\]

**Proof.** (1) By theorem 2, (51) admits positive solutions for \( c \) small enough. Thus the preceding proposition is nonempty. Assume now by contradiction that the left inequality in...
(52) is false. We get

$$\limsup_{r \to +\infty} r^{-\alpha} u(r) = 0,$$

and by (1) : $u(r) = -cr^{2\alpha/(1-\alpha)}$ as $r \to \infty$. This implies $u < 0$ for $r$ large enough and yields a contradiction.

(2) From (51) we get $u^* \leq u^*$, and this implies easily the second inequality of (52).

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APPENDIX

In this appendix we prove several lemmas used in the preceding sections. Let $n \geq 1$.

**Lemma A.1.** Let $\Omega$ be a bounded regular open set of $\mathbb{R}^n$ and let $u \in W^{1,\infty}(\Omega)$ be such that $u \leq 0$ a.e. on $\partial \Omega$ (in the sense of traces) and $\Delta u \leq 0$ in $\Omega$. Then there exists $u_0 \in \mathcal{S}(\Omega)$ such that $u_0 \leq 0$ on $\partial \Omega$, $u_0 \to u$ in $L^2(\Omega)$ and $\Delta u_0 \to \Delta \Omega$ in $L^2(\Omega)$.

**Proof of lemma A.1.** Let $f \in C_0^\infty(\Omega)$ and $g \in C^\infty(\partial \Omega)$ such that

$$f_k \to -\Delta u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad k \to +\infty,$$

$$g_k \to g \quad \text{in} \quad L^2(\partial \Omega), \quad \text{as} \quad k \to +\infty.$$

There exists $u_0 \in C^\infty(\Omega)$ such that

$$\begin{align*}
\Delta u_0 &= f_k \quad \text{on} \quad \Omega, \\
u_0 &= g_k \quad \text{on} \quad \partial \Omega.
\end{align*}$$

**Step 1. Estimate on $u_k$.**

Let $\varphi \in W^{1,\infty}(\Omega)$, $\varphi = 0$ on $\partial \Omega$ (1 < $q$ < $\infty$). One has

$$\varphi \leq u_k \Delta \varphi \leq \varphi \Delta u \leq 0.$$

Thus

$$\int_\Omega f \varphi \, dx \leq -\int_\Omega \Delta u \varphi \, dx = -\int_\Omega u_0 \Delta \varphi \, dx - \int_\Omega \partial \varphi \, dx + \int_{\partial \Omega} g \varphi \, ds.$$

The equation $-\Delta u \leq -|f|^{-1} f$, for $0 < a < 1$

Then

$$\int_\Omega u_0 \Delta \varphi \, dx = -\int_\Omega f_k \varphi \, dx + \int_{\partial \Omega} g \varphi \, ds$$

Let $\varphi \in L^2(\Omega)$, there exists $\varphi \in W^{1,q}(\Omega)$ (1 < $q$ < $\infty$) such that

$$\Delta \varphi = \psi \quad \text{on} \quad \Omega,$$

$$\varphi = 0 \quad \text{on} \quad \partial \Omega,$$

and furthermore one has (cf. [9]):

$$|\psi|_{L^q} < C|\varphi|_{L^q} < C|\varphi|_{\infty}$$

where $C$ denote some "constants" only depending on $\Omega$ and $q$.

Taking some $a > 1$ we conclude (using the Sobolev injection theorem)

$$|\psi|_{L^q} < C|\varphi|_q.$$
LEMMA A.2. Let \( \Omega \) be a bounded regular open set of \( \mathbb{R}^n \). Let \( u \in W^{1,1}(\Omega) \) be such that \( u = 0 \) a.e. on \( \partial \Omega \) and \( -\Delta u \in L^1(\Omega) \). Let \( P \in C^1 \cap L^1(\mathbb{R}) \), \( P = 0 \) on \( \mathbb{R} \), \( P' > 0 \). Then one has
\[
\int_{\Omega} -\Delta u u' \, dx > 0.
\]

**Proof of Lemma A.2.** By Lemma A.1 there exists (a.e. \( h \in \mathbb{N} \subset \mathbb{R}(\Omega) \) such that
\[
\begin{align*}
 u_k & \to u \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad k \to +\infty, \\
-\Delta u_k & \to -\Delta u \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad k \to +\infty, \\
u_k & \leq 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
We clearly have
\[
\int_0^\infty \Delta u_k P(u_k) \, dx > 0.
\]
We can assume (after extracting a subsequence) that
\[
\begin{align*}
u_k & \to \nu \quad \text{a.e.,} \\
-\Delta u_k & \to -\Delta u \quad \text{a.e.,} \\
|\Delta u_k| & \leq g \quad \text{a.e.,} \quad \text{with some} \quad g \in L^1(\Omega).
\end{align*}
\]
By the dominated convergence theorem we conclude that
\[
\int_0^\infty -\Delta u P(u) \, dx > 0.
\]
For \( a, b \in \mathbb{R} \), we set \( \Omega(a, b) = \{ a < |x| < b \} \).

LEMMA A.3. (Compact support lemma—cf. [12, theorem 6.1] and [7]) Let \( 0 < a < 1 \). For all \( 0 < a < b \) and for all \( e > 0 \), there exists \( C(e, a, b) \) such that if
\[
f \in L^1_0(\Omega; \mathbb{R}), f > 0, \quad \text{Supp} f \subset \Omega(a, b), \quad \int \Omega \, dx \leq C(e, a, b),
\]
then the problem
\[
\begin{cases}
-\Delta u + u^+ = f \quad \text{in} \quad \Omega(a, b), \\
u \in L^1(\Omega), \nu > 0, \quad \text{Supp} u \subset \Omega(a - e, b + e)
\end{cases}
\]
has a (unique) solution.

**Proof of Lemma A.3.** Let \( 0 < a < 1 \). We prove Lemma A.3 with, for instance, \( a = 2, b = 3, \text{ and some } 0 < e < 1. \text{(This case contains all the difficulties.)} \)
We set \( \Omega = \Omega(2, 3), B = \{ 1 < |x| < 4 \} \) and \( \Omega_2 = \Omega(2 - e, 3 + e) \).

Let \( f \in L^1_0(\mathbb{R}^n), f > 0, \) \( \text{Supp} f \subset \Omega \), and \( u \) be the unique solution (cf. [10]) of
\[
\begin{cases}
-\Delta u + u^+ = f \quad \text{in} \quad \Omega(2, 3), \\
u \in W^{1,1}(B), \quad u > 0.
\end{cases}
\]
Our aim is to prove that for \( \int_\mathbb{R} u^+ \, dx \) small enough there is \( u \in \mathbb{R}(\Omega) \). We then obtain of course a (unique) solution for (P').

**Step 1. Estimates on \( u.** One has (cf. [10]) \( \int_\mathbb{R} u^+ \, dx \leq \int_\mathbb{R} \, dx \) and then \( \int_\mathbb{R} |\Delta u | \, dx \leq 2 \int_\mathbb{R} f \, dx \), this gives for some \( C_1 \) (only depending on \( n \))
\[
\int_a^b u \, dx \leq c_1 \int_a^b f \, dx.
\]

The equation \(-\Delta u + |u|^\sigma - u = f, \forall \sigma \neq 1\) for \( 0 < \sigma < 1 \).

Since \( \Delta u = u^+ \) on \( \partial \Omega \), a classical "bootstrap" argument gives \( u \in W^{1,2}(B; \Omega) \) (and then, taking \( p > 1, u \in C^{1}(B(\Omega)) \)). Let \( 1 < R_1 < R_2 < 2 < R_1 < R_2 < 4 \). Then there exists \( C_2 \) (only depending on \( n, R_1, R_2, R_1, R_2 \)) such that
\[
\text{Supp}(u), \ R_1 \leq |x| \leq R_2 \quad \text{or} \quad R_2 \leq |x| \leq R_3 \quad \text{then} \quad C_2 \int \, dx.
\]
We take \( R_1 = 1 + e/4, R_2 = 2 - e/4, R_3 = 3 + e/4, R_4 = 4 - e/4 \).

**Step 2. Estimates on \( |x| < R_2 \).** In this step we construct on \( |x| < R_2 \) some radial comparison functions.

Let \( M_1 = \text{Max}(u(x), |x| = R_2) \).

One has, by step 1,
\[
M_1 \leq C_2 \int \, dx.
\]
We set, for some \( A > 0 \) and \( R_4 = R_2 - e/2, \)
\[
\begin{align*}
\alpha(x) & = A(|x| - R_2)^{2(1 - \sigma)} \quad \text{if} \quad R_2 < |x| < R_1, \\
\alpha(x) & = 0 \quad \text{if} \quad 1 < |x| < R_2,
\end{align*}
\]
an easy calculation yields
\[
-\Delta u + \alpha(x) = \left( A^+ - A \frac{2(1 + \alpha)}{(1 - \alpha)^2} \right) (|x| - R_2)^{2(1 - \sigma)}
\]
\[
\frac{A}{(a - 1)} \left( 1 - \frac{2}{1 - a} \right) (|x| - R_3)^{2(1 - \sigma)} \quad \text{if} \quad R_2 < |x| < R_3,
\]
\[
= 0 \quad \text{if} \quad 1 < |x| < R_2.
\]
Since \( R_2 < R_3 < 1 \) and \( 0 < \sigma < 1 \) one has
\[
\frac{1}{|a|} (|x| - R_2)^{2(1 - \sigma)} + \frac{1}{|a|} (|x| - R_3)^{2(1 - \sigma)} \quad \text{if} \quad R_2 < |x| < R_3,
\]
that is
\[
-\Delta u + \alpha(x) = \left( A^+ - A \frac{2(1 + \alpha)}{(1 - \alpha)^2} \right) (|x| - R_2)^{2(1 - \sigma)} + \frac{A}{(a - 1)} \left( 1 - \frac{2}{1 - a} \right) (|x| - R_3)^{2(1 - \sigma)}, \quad \text{if} \quad R_2 < |x| < R_3.
\]
Taking \( A = A, > 0 \) small enough (\( A \), only depends on \( n, \sigma \) and \( e \)) one has
\[
-\Delta u + \alpha(x) \geq 0 \quad \text{on} \quad 1 < |x| < R_2.
\]
Taking also
\[
\int \, dx \leq \frac{1}{C_2} A_\sigma (R_2 - R_1)^{2(1 - \sigma)} = C_2^\sigma > 0,
\]
one has
\[
\alpha(x) \leq M_1 \leq C_2 \int \, dx \leq c(x) \quad \text{if} \quad |x| < R_2.
\]
(A7) and (A9) then give:
\[
\begin{cases}
-\Delta u + \alpha(x) \geq 0 \quad \text{on} \quad 1 < |x| < R_2, \\
u - u \in W^{1,1}(|x| < R_2), \\
u - v \leq 0 \quad \text{on} \quad \{|x| = 1\} \cup \{|x| = R_2\}
\end{cases}
\]
Thus, by Lemma A.2, one has for \( P \in C^{1} \cap L^1(\mathbb{R}) \), \( P = 0 \) on \( \mathbb{R} \), \( P > 0 \) on \( \mathbb{R} \), \( P > 0 \),
\[
\int_{|x| < R_2} (u^+ - v^+)(u - v) \leq 0.
\]
This proves \( u = v \) if \( 1 < |x| < R_2 \).
Then we conclude \( u(x) = 0 \) if \( 1 < |x| < R_1 = 2 - \varepsilon/2 \).

By a similar method (a little bit simpler) we prove that there exists \( C_{**} > 0 \) (only depending on \( a, b, a_1, \varepsilon \)) such that

\[
\int_a^x f dx < C_{**}
\]

(A10)

then

\[ u(x) = 0 \quad \text{if} \quad 3 + \varepsilon/2 - R_1 < |x| < 4. \]

Hence, if \( f \) satisfies (A8) and (A10) (that is \( f_x f / \partial x \leq C = \text{Inf}(C_{***}, C_{**}) \)) one has\( \text{Supp} \ u \subset \Omega_t = \Omega(2 - a, 3 + \varepsilon) \).

This proves lemma A.3 for \( a = 2 \) and \( b = 3 \).

ANALYSIS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS ARISING IN AGE-DEPENDENT EPIDEMIC MODELS

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1. INTRODUCTION

Basically, there are two modes for directly transmitting an infectious disease within a single population: vertical transmission and horizontal transmission. Vertical transmission is defined as the direct transfer of infection from a parent organism to its offspring. Horizontal transmission is any transfer of infection except that which is vertically transmitted. For example, AIDS is both vertically and horizontally transmitted while malaria is horizontally transmitted.

Vertically transmitted diseases have seldom been considered in mathematical models of epidemics. Examples of previous such models are found in Anderson and May [1], Cooke and Busenberg [7], Busenberg and Cooke [3], Busenberg, Cooke and Pozio [4], Fine [10] and Re'gniere [17].

Likewise age-dependent diseases has been presented by Cooke and Busenberg [7] and Dietz [8]. Age-dependence introduces a coupling of age-structure and vertical transmission which can produce novel dynamic behavior.

In this paper, a system of nonlinear integro-differential equations which model an age-dependent epidemic of a disease with vertical transmission is investigated. This model treats the simple \( S \rightarrow I \) type of epidemic in this new setting. Existence and uniqueness are proved under suitable hypotheses and the asymptotic behavior of the system is determined. A renewal theorem is used to study the behavior of the model equations in various pertinent parameter ranges. A numerical method for integrating this system of equations is developed and is used to obtain approximations of its solutions for some special cases which illustrate the results obtained via analytical means. Moreover, numerical integrations of the equations are used to study some phenomena that were not treated analytically.

2. A MODEL OF A VERTICALLY TRANSMITTED DISEASE

Consider an age-structured population of variable size exposed to a disease which is both horizontally and vertically transmitted with the following assumptions on the model.

(a) Let \( s(a, t) \) and \( i(a, t) \), respectively, denote the age-density for susceptibles and infectives of age \( a \) at time \( t \). Then

\[
\int_{a_1}^{a_2} s(a, t) \, da = \text{total number of susceptibles at time } t \text{ of ages between } a_1 \text{ and } a_2.
\]

\[
\int_{a_1}^{a_2} i(a, t) \, da = \text{total number of infectives at time } t \text{ of ages between } a_1 \text{ and } a_2.
\]