NONLINEAR ELLIPTIC EQUATIONS WITH
RIGHT HAND SIDE MEASURES

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§ 1 - INTRODUCTION AND STATEMENT OF RESULTS

Let $\Omega$ be an open bounded set of $\mathbb{R}^N$ ($N \geq 1$), $p \in (1, \frac{N}{N-1})$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. We assume that there exist two positive real constants $\alpha$, $\beta$ and a function $h(x) \in L^p(\Omega)$ such that for any $x \in \Omega$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$, and for almost every $x \in \Omega$:

1. $a(x, \alpha, \xi) \geq |\xi|^p$

2. $|a(x, \alpha, \xi)| \leq \beta h(x) + |\alpha|^p \xi + |\eta|^p$

3. $a(x, \alpha, \xi) - a(x, \alpha, \eta)(\xi - \eta) \geq 0$, $\xi \neq \eta$

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The first aim of this paper (Theorem 1) is to obtain a solution of the equation

$$A(u) = -\text{div}(a(x,u,Du)) = f \text{ in } \Omega$$

(4)

$$u = 0 \text{ on } \partial \Omega$$

in the sense of the distributions, when $f$ is a bounded Radon measure on $\Omega$ (i.e., $f \in M(\Omega)$). To be more precise, we will say that $u$ is a weak solution of (4) if it satisfies

$$u \in W^{1,1}_0(\Omega), \quad a(x,u,Du) \in L^1(\Omega)$$

(5)

$$\int_{\Omega} a(x,u,Du) \text{div} v \, dx \leq C\|v\|_{L^q(\Omega)}$$

for any $v \in \mathcal{D}(\Omega)$.

We will prove the following existence theorem.

**THEOREM 1.** Let $f \in M_*(\Omega)$, then there exists a solution $u$ of (5). Furthermore one has $u \in W^{1,q}_0(\Omega)$ for any $q < N/(N-1)$.

The theorem above is already known ([BG1]) if $f$ does not depend on $u$ and under a more technical assumption than (3). It is also well known ([LL1]) if $p > N$, since in this case $M_p(\Omega) \subset W^{1,p}(\Omega)$ and therefore one can use the theory of operators acting between Sobolev spaces in duality.

In (4), we can also consider a lower order term $g(x,u)$ or $g(x,u,Du)$ with a sign condition $g(x,u) > 0$, as it is done in [BG1], [GM1], [GM2], [BG1], [BG2].

**EQUATIONS WITH RIGHT HAND SIDE MEASURES**

We assume $p \in (2 - \frac{N}{N-1}, N)$ which is equivalent to $q > 1$. Thus, if $1 < p < 2 - \frac{N}{N-1}$, $Du$ does not belong to $L^q(\Omega)$ (to overcome this difficulty see the forthcoming paper [K]).

The existence result stated in Theorem 1 is optimal: it is possible to find cases where $f \in M_p(\Omega)$ (for instance, $f = a(x,x,Du)$ or $f \in L^1(\Omega)$ such that the corresponding solution $u$ of (4) does not belong to $W^{1,q}_0(\Omega)$.

The existence result in this limiting case is obtained in the following theorem when $|\log|f|| \in L^1(\Omega)$.

**THEOREM 2.** Let $2 - \frac{N}{N-1} < p < N$. Assume that $|\log|f|| \in L^1(\Omega)$. Then there exists $u \in W^{1,q}_0(\Omega)$, $q = (p-1)N/(N-1)$, solution of (5).

A classical method due to G. Stampacchia ([S]) yields a solution of (4) when $A$ is linear (and $p = 2$), through a duality and a regularity argument in the case $f \in M_p(\Omega)$ or $|\log|f|| \in L^1(\Omega)$.

When $p = N$ the weaker assumption $|\log|f||$ $\in L^1(\Omega)$ implies $f \in W^{1,N}_0(\Omega)$ (see [G]) and therefore the existence of a solution $u \in W^{1,N}_0(\Omega)$ follows from [LL].

Finally let $m = \frac{Np}{Np - N}$ (remark that $m = 1$ implies $m = 2$), by Sobolev Embedding Theorem, $f$ lies in $W^{1,m}(\Omega)$ and again the existence of a solution $u$ follows from [LL]. But if $m = 1$ and $f \in L^m(\Omega)$, the following theorem can be seen as a regularity theorem regarding the solution obtained in Theorem 1.

**THEOREM 2.** Let $2 - \frac{N}{N-1} < p < N$, $1 < m < m = \frac{Np}{Np - N}$ and $f \in L^m(\Omega)$, then there exists a solution $u$ of (5) which belongs to $W^{1,p-1,m}(\Omega)$.
2. ESTIMATES.

In order to prove Theorems 2, 3, in this section we will present estimates for $u$ only depending on $N$, $\alpha$, $p$, $p, \sigma$, $\|f\|_{L^1}$, when $f$ is smooth and $u$ is a solution of (3), given by [LC]. We point out that through this section we will not use assumptions (2), (3).

**ESTIMATE 1.** Let $f \in L^1(\Omega) \cap W^{-1,p'}(\Omega)$, $1 \leq q \leq \frac{N(p-1)}{N-1}$ and $2 \leq p \leq N$. Then there exists a constant $c_1$ (depending only on $N$, $\alpha$, $\sigma$, $p$, $q$, $\|f\|_{L^1}$) such that

$$
\int_{\Omega} |Du|^q \leq c_1.
$$

**ESTIMATE 2.** Let $2 \leq \frac{1}{p} < \frac{1}{N-1}$, $f \in W^{-1,p}(\Omega)$ and $f \log |f| \in L^1(\Omega)$. Then there exists $c_2$ (a constant depending only on $N$, $\alpha$, $\sigma$, $p$, $q$, $\|f\|_{L^1}$) such that

$$
\int_{\Omega} |Du|^q \leq c_2, \quad q = \frac{N(p-1)}{N-1}.
$$

**ESTIMATE 3.** Let $2 \leq \frac{1}{p} < \frac{1}{N-1}$, $m = \frac{BN(p-1)}{N} < m < \frac{N}{p}$, and $f \in L^1(\Omega) \cap W^{-1,p}(\Omega)$. Then there exists a constant $c_3$ (depending only on $N$, $\alpha$, $\sigma$, $p$, $m$, $\|f\|_{L^1}$) such that

$$
\int_{\Omega} |Du|^{m-1} \leq c_3.
$$

Estimate 3 stated above improves Proposition 1 of [BG1] where it was proved that $Du \in L^q(\Omega)$, for any $q < (p-1)m$.
Combining (11) and (12) we get

\[
\int_\Omega |u|^\alpha \leq c_1 \left( \left( \int_{\Omega} |Du|^\beta \right)^{\frac{\alpha}{\beta}} \right) \leq c_1 + c_2 \left( \int_{\Omega} |u|^\alpha \right)^{\frac{\alpha}{\beta}} \cdot \frac{\alpha}{\beta}
\]

We remark that \( \frac{\alpha}{\beta} \cdot \frac{\beta}{p} < 1 \) since \( p < N \).

Thus we have proved the a priori estimate (7).

**PROOF OF ESTIMATE 2** - Let \( q = (p-1)m^* \) \((q < p)\) and \( sq^* \cdot \frac{q}{\beta} > 0\). Using the inequality (10) as before, we have

\[
\int_\Omega |Du|^s \leq \left( \int_{\Omega} |Du|^\beta \right)^{\frac{s}{\beta}} \left( \int_{\Omega} |1 + |u||^\alpha \right)^{\frac{s}{\alpha}} \leq \left( \frac{c_1}{\alpha} \int_{\Omega} |1 + |u||^\alpha \right)^{\frac{s}{\alpha}} \left( \int_{\Omega} |1 + |u||^\alpha \right)^{\frac{s}{\alpha}}
\]

Because we have proved a bound on \( |u| \) in Estimate 1.
\[ L \leq c_1 + c_2 \left( \left( \left\| u \right\|_{H^m} \right)^6 \left( \left\| u_t \right\|_{H^m} \right)^{2/3} \right) \]

Combining (14) and (15) we have \( (q^*) = (1,2m^*) \)

\[ \left\| Du \right\| \leq c_0 + c_2 \left( \left\| u \right\|_{H^m} \right)^{2/3} \left\| u_t \right\|_{H^m} \]

Then Estimate 3 follows by the previous inequality because \( \frac{q}{q^*} > 1 - \frac{2}{m^*} + \frac{2}{p} \). Indeed the choice of \( q \) is such that \( (N-q-Np+Nq)m^* = Nq \). That is \( (N-1)m^*-Nq = (N-p)m^* \), i.e. \( q = (p-1)m^* \). Moreover \( q < p \) follows from the inequality \( m < \theta \).

**Remark 1.** In order to prove the above estimates we have used the test function \( \Phi(x) \) in \([B,G]\), but it is also possible to use \( \Phi(x) \), as in \([B,GV]\), where

\[ \Phi(x) = \frac{1}{9} \frac{\left\| u_t \right\|_{L^3(\Omega)}}{\left\| u \right\|_{L^3(\Omega)}} \]

Then Estimates 1, 2 and 3 can be proved with (respectively) \( s = 1 \), \( s = 1 \), \( s = 1-\frac{2}{2m^*} (pq(p-1)m^*) \).

**§ 3 - PROOFS OF THEOREMS 1, 2, 3.**

In this section we take \( f \in M^1(\Omega) \) and a sequence \( (f_n) \in W^{-1,p} \cap L^1(\Omega) \) converging to \( f \) and such that \( \left\| f_n \right\|_{L^1(\Omega)} \leq \left\| f \right\|_{M^1(\Omega)} \). We then pass to the limit in the equations

\[ u_n \in W^{1,2}_0(\Omega) \]

\[ A(u_n) = f_n \quad \text{and} \quad u_n \to u \quad \text{weakly in} \quad W^{1,2}_0(\Omega) \]

\[ u_n \to u \quad \text{strongly in} \quad L^q(\Omega) \]

\[ u_n \to u \quad \text{a.e. in} \quad \Omega \]

This is not sufficient to pass to the limit in (16). We need, for instance \( D_{Du} \rightarrow D_u \) a.e. This is the context of the following Lemma.

**Lemma 1.** Assume (1), (2), (3) and

\[ f_n \text{ bounded in } L^1(\Omega), \quad f \in L^1(\Omega) \cap W^{-1,2}(\Omega) \]

Then the sequence \( u_n \) defined in (16) is compact in \( W^{1,2}_0(\Omega) \), for any \( q < q^* \).
650

BUCCARDO AND GALLOGLI

END OF THE PROOF OF THEOREM 1 - Combining Lemma 1, (2) and (18) we deduce that

\[(21) \quad a(x, \nabla u, \nabla u) = a(x,u, \nabla u) \quad \text{in} \quad L^1, \quad \forall \varepsilon \in \left[ \frac{N}{1+\varepsilon} \right].\]

and therefore \(u\) is a solution of (4), in the weak sense (5).

PROOF OF THEOREMS 2 AND 3 - The existence results are a consequence of Theorem 1 and Estimate 2 or 3, since

\[a(x, u, \nabla u) = a(x, \nabla u) \quad \text{weakly in} \quad L^{2\rho}(\Omega) \quad \text{or in} \quad L^{\infty}(\Omega)\]

REMARK - With our proof of Theorem 3 the convergence of \(D\nabla u\) in \(L^{(1-2M)^n}\) is an open problem.

Before proving Lemma 1 we recall that \(L^1\)-compactness results for the gradients of a sequence of approximate solutions of nonlinear equations have been obtained in [BMP], [BM], [BG], [LM], and we emphasize that the first result is contained in a pioneering work by Leray-Lions [LL].

In the proof we will need the following standard

LEMMA 2 - Let \((X,T,m)\) a measurable space, such that \(m(X) < \infty\). Let \(\gamma\) be a measurable function, \(\gamma : X \rightarrow [0, \infty]\) such that \(m(\{x \in X, \gamma(x)=0\})=0\). Then for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that

\[\int_A \gamma \, dm \leq \delta\]

implies \(m(A) \leq \varepsilon\).

EQUATIONS WITH RIGHT HAND SIDE MEASURES

PROOF OF LEMMA 1 - By Estimate 1, \((u_n)\) is bounded in \(W^{1,\rho}_0(\Omega)\), for any \(\rho < \infty\).

Then we can assume (for some \(u \in W^{1,\rho}_0(\Omega)\) and for some subsequence still denoted \(u_n\)) that

\[u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,\rho}_0(\Omega)\]

\[u_n \rightarrow u \quad \text{in measure}\]

Our proof relies on the following claim

\[D\nabla u \rightarrow D\nabla u \quad \text{in measure.}\]

In order to prove (24), given \(\lambda > 0\) and \(\varepsilon > 0\) we set for some \(B > 1, k > 0 (a,m \in \mathbb{N})\)

\[E_1 = \{x \in \Omega : |D\nabla u_n(x)| > B \} \cup \{x \in \Omega : |D\nabla u_0(x)| > B \} \cup \{x \in \Omega : |u_n(x) - u_0(x)| > B \} \cup \{x \in \Omega : |u_0(x)| > B \}, \]

\[E_2 = \{x \in \Omega : |u_n(x) - u_0(x)| > k \} \]

\[E_3 = \{x \in \Omega : |u_0(x)| > B, |D\nabla u_0(x)| \leq B, |u_n(x)| \leq B, |u_0(x)| \leq B, |D\nabla u_0(x)| \geq \lambda \}.

Remark that

\[\{x \in \Omega : |D\nabla u_n - u_n(x)| \geq \lambda \} \subset E_1 \cup E_2 \cup E_3.

Since \((u_n)\) and \((D\nabla u_n)\) are bounded in \(L^1(\Omega)\), one has measure \(E_3 \leq \varepsilon\), for \(B\) large enough, independently of \(a, m\). Thus we fix \(B\) in order to have.
mean $E_1 \subset E$.

We now take into account mean $E$. Assumption (3) implies that there exists a real valued function $\gamma(x)$ such that

$$\text{mean}(\{x \in \Omega : \gamma(x) = 0\}) = 0$$

and

$$\sigma(x,\xi,\eta) \cdot [\xi - \eta] \geq \gamma(x), \quad \forall x \in \mathbb{R}, \xi, \eta \in \mathbb{R}^n : ||\xi||, ||\eta|| \leq B,$$

$$||\xi - \eta|| \geq \lambda, \quad \text{a.e.} \quad x \in \Omega.$$

Indeed there exists a subset $C$ of $\Omega$ such that $\text{mean}(C) = 0$ and the function $\sigma(x,\xi,\eta)$ is continuous with respect to $(\xi,\eta)$ for any $x \in \Omega \setminus C$. Then assumption (3) implies that for $x \in \Omega \setminus C$ and $\xi \neq \eta$ one has

$$\sigma(x,\xi,\eta) \cdot [\xi - \eta] > 0.$$

Define

$$K = \{(x,\xi,\eta) \in \mathbb{R}^{3n+3} : ||\xi|| \leq B, \quad ||\eta|| \leq B, \quad ||\xi - \eta|| \geq \lambda\}.$$
and

$$2kM < \delta/3.$$ 

Then we have

$$\gamma(x) < \delta$$

and we can deduce that

$$\int_{\Omega} \gamma(x) < \epsilon$$

independently of $n$ and $m$.

Now we fix such $n$ and thanks to the fact that $u_0$ is a Cauchy sequence in measure, we can choose $n_2$ such that

$$\text{meas } E_2 \leq \epsilon$$

for $m \geq n_2$.

Then the convergence (24) and Estimate 1 yield the desired compactness results.

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EQUATIONS WITH RIGHT HAND SIDE MEASURES


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