On choosing wavelet resolution in image deblurring

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Abstract

In wavelet deblurring of noisy images the finest resolution levels are key parameters which require fine tuning. In this paper we present a blockwise thresholding rule in the Fourier domain to derive near-optimal finest resolution levels for a large class of blurred-image scenarios. Numerical results based on the WaveD fast algorithm of [5] illustrate the improvement over the current WaveD setting.

1 Introduction

In a recent paper [5] the 2D-WaveD algorithm was introduced. WaveD takes full advantages of the Fast Fourier transform and performs deconvolution of noisy \((n \times n)\) images in \(O(n(\log n)^2)\)-steps only. The WaveD method is a non-linear fully adaptive algorithm which enjoys near-optimal properties [6]. This deconvolution algorithm is particularly attractive since it is a non-iterative technique which offers both theoretical and computational advantages. In practice a fine tuning of WaveD key parameters (threshold, resolution level) is required to ensure optimal performances. The optimal choice of the threshold parameter has been addressed at length in [6] and extended to the 2D-setting in [5]. An important issue in the tuning of 2D-WaveD is the choice of the finest resolution level(s). The asymptotic theory of [6] prescribes that the finest resolution level \(j_1\) should satisfy:

\[
2^{j_1} \lesssim (n/\log n)^{1/(1+2\nu)}.
\] (1)

Here \(\nu\) is a decay parameter which depends on the Degree of Ill-Posedness of the blurred-image model (3). While condition (1) can hardly be used in practice, it is very important however. It shows that the more ill-posed a problem is (large \(\nu\)) the sooner the wavelet expansion must stop. This contrasts with the case of direct estimation (\(\nu = 0\)) where there is no need to choose the resolution level and one always set \(j_1 = J - 1\) where \(J = \log_2(n)\) is the largest accessible level [3]. For the deblurring of noisy images one has to take into account the noise level as well as the Degree of Ill-Posedness \(\nu\) when choosing the finest resolution level. In the current version of the WaveD software(s) the finest resolution level has to be fixed by the user. In practice it is preferable to have a data-driven method for choosing WaveD finest resolution level(s). In this paper we present a blockwise thresholding method in the Fourier domain which allows to choose the finest resolution level(s) from the data. Our method agrees with (1) and is adaptive to both the noise level and the Degree of Ill-Posedness (DIP). The paper is organised as follows: in section 2 we review basic definitions associated with the WaveD method. In section 3 we present our method and some simulation results which illustrate the improvement over the current WaveD setting.

2 Preliminaries

2.1 A statistical model for blurred images

We begin with a function \(f\) defined on the unit square \(T = [0,1]^2\), this function represents our image. Let \(g\) be a blurring kernel also defined on \(T\) and let

\[
h(x) := f * g(x) = \int_{T} f(x_1-u_1,x_2-u_2) g(u_1,u_2) du_1 du_2,
\] (2)

represent our blurred image. Suppose we observe \(h\) with some additive noise

\[
Y_n(dx) = h(x)dx + \sigma n^{-1} W(dx), \quad x \in T = [0,1] \times [0,1],
\] (3)

where \(\sigma\) is a positive constant and \(W(\cdot)\) is a 2D Wiener process. In the Fourier domain the model (3) is written as:

\[
ye = f_c g_e + \sigma n^{-1} z_t, \quad t \in \mathbb{Z}^2,
\] (4)

where \(z_t\) are iid complex-valued Gaussian r.v.’s. An illustration of the model (3) is given in Figure 1. See [1],[7] for examples and applications of this model.

2.2 Two-dimensional Meyer wavelet basis

A (1D) Meyer wavelet \(\psi\) is a function whose Fourier transform \(F(\psi) := \tilde{\psi}\) is smooth, [9]. The formula for the construction of \(\tilde{\psi}\) is given in [8]. The Meyer wavelet and associated scaling function \(\phi\) are band limited and we have \(\text{Supp}(\tilde{\phi}(w)) = \{w : |w| \in [0,4\pi/3]\}\) and \(\text{Supp}(\tilde{\psi}(w)) = \{w : |w| \in [2\pi/3,8\pi/3]\}\). Two-dimensional Meyer wavelets (and scaling function) are defined in the the Fourier domain from the 1D-pair \((\phi,\psi)\) by tensorial-product

\[
\tilde{\psi}^1(w_1,w_2) = \tilde{\phi}(w_1)\tilde{\psi}(w_2), \quad \tilde{\psi}^2(w_1,w_2) = \tilde{\psi}(w_1)\tilde{\phi}(w_2),
\]

\[
\tilde{\phi}^1(w_1,w_2) = \tilde{\phi}(w_1)\tilde{\phi}(w_2).
\]
\[ \hat{\psi}^3(w_1, w_2) = \hat{\psi}(w_1)\hat{\psi}(w_2), \hat{\phi}^3(w_1, w_2) = \hat{\phi}(w_1)\hat{\phi}(w_2). \]

In the fast algorithm of [5] the periodised version of the 2D-Meyer basis is computed by sampling \( \hat{\psi}^m \) along a dyadic grid:

\[ \Psi^m_{j,0,0}(\ell) = 2^{-j}\hat{\psi}^m(\ell/(2^j \pi)), \]

see Figure 2 of [5] for an illustration.

### 2.3 The WaveD estimator

We recall the definition of the 2D-WaveD estimator [5]

\[ \hat{f} = \sum_{k \in I_0} \hat{\alpha}_k I_{\{x_i \geq \lambda_{k_i}\}} \Phi_k + \sum_{m=1}^{3} \sum_{k \in I_1} \hat{\beta}_m^m I_{\{|z_m| \geq \lambda^m\}} \Psi^K^m, \]

where \( I_0, I_1 \) are sets of indices; \( I_0 = \{(j_0, k_1, k_2) : 0 \leq k_i \leq 2^{j_0} - 1, i = 1, 2\} \) corresponds to a coarse resolution level \( j_0 \) and \( I_1 = \{(j, k_1, k_2) : 0 \leq k_i \leq 2^j - 1, i = 1, 2, j_0 \leq j \leq j_1\} \) indexes details up to a fine resolution level \( j_1 \). The WaveD coefficients are computed in the Fourier domain

\[ \hat{\beta}_m^m = \text{WaveD}(Y, g, \Psi^m_K) := \sum_{\ell \in C_j^m} \left( \frac{g_\ell}{|g_\ell|} \right) \Psi^K_{\ell, \kappa}, \]

where \( C_j^m = \{\ell : \Psi^K_{\ell, \kappa} \neq 0\} \). Next we set

\[ \tau_j^m = \left( |C_j^m|^{-1} \sum_{\ell \in C_j^m} |g_\ell|^{-2} \right)^{1/2}, \]

where \( |C_j^m| \) denotes the cardinality of \( C_j^m \). The WaveD (level-by-level) thresholds are defined according to each orientation \( m = 1, 2, 3 \) by

\[ \lambda_j^m := \gamma \hat{\sigma} \tau_j^m (\log n)/n, \]

where \( \gamma \) is a constant and \( \hat{\sigma} \) is an estimate of \( \sigma \). The asymptotic theory of [6] prescribes that for “smooth” convolution kernel convolution where \( |g_\ell| \sim (|\ell_1| + |\ell_2|)^{-\nu}, \) the finest resolution level \( j_1 \) should satisfy (1).

### 3 Resolution tuning by blockwise Fourier thresholding

In this section we propose a data-driven method for choosing the finest WaveD resolution level in (5). Our method uses the observed eigen-values \( (\hat{g}_\ell) \) whose inverse is \( \ell^2 \) averaged over dyadic blocks (7). When \( (\tau_j^m)^{-1} \) goes below a certain noise level for some \( j = j_\text{opt} \) we set \( j_1^m := j_\text{opt} \), see Figure 2. We refer to this process as blockwise Fourier thresholding. The blockwise approach allows for different resolutions according to each directions (Horizontal, vertical, diagonal).

#### 3.1 The method

At resolution level \( j \) the inversion-process inflates the noise standard deviation by a factor of \( (\tau_j^m) \) according to each direction \( m = 1, 2, 3 \). The idea is to keep all resolution level such that the price for deconvolution (inverting the eigen-values) is no more than the maximum (allowed) noise level. Extending the method of [2] to the 2D-setting the maximum noise level that we can accept while preserving optimal properties is given by \( (2^{j/2}\hat{\sigma}(\log n)/n)^{-1} \). For a fixed direction \( m \) we choose \( j_1^m \) to be the smallest resolution such that it does not exceed the maximum noise level:

\[ j_1^m = \min \left\{ j \geq 0 : \tau_j^m \geq \left( 2^{j/2} \hat{\sigma}(\log n)/n \right)^{-1} \right\} - 1. \]
In this case Definition (10) shows that $\tilde{\sigma}_j$ is the largest resolution level such that $(\tilde{\sigma}_j)^{-1}$ is below the maximum noise level:

$$\tilde{\sigma}_j = \max \left\{ j \geq 0 : (\tilde{\sigma}_j)^{-1} \leq \left( 2^{j/2} \tilde{\sigma}(\log n)/n \right) \right\}.$$  \hfill (10)

Definition (10) is used for illustration of the blockwise method on Figure 2 with the data-example $(\hat{j}, \hat{j}_1, \hat{j}_2, \hat{j}_3) = (4, 4, 4)$; middle: blockwise level selection $(\hat{j}_1, \hat{j}_2, \hat{j}_3) = (5, 5, 4)$; right: blockwise level selection $(\hat{j}_1, \hat{j}_2, \hat{j}_3) = (5, 5, 5)$.

Alternatively, we can choose $j_m$ to be the largest resolution level such that $(\tau_j)^{-1}$ is below the maximum noise level:

$$j_m = \max \left\{ j \geq 0 : (\tau_j)^{-1} \leq \left( 2^{j/2} \tilde{\sigma}(\log n)/n \right) \right\}.$$  \hfill (10)

Definition (10) is used for illustration of the blockwise method on Figure 2 with the data-example $(\hat{j}, \hat{j}_1, \hat{j}_2, \hat{j}_3) = (5, 5, 4)$; right: blockwise level selection $(\hat{j}_1, \hat{j}_2, \hat{j}_3) = (5, 5, 5)$.

In this case definition (10) shows that $j_m$ satisfies

$$2^{-j_m} \geq \left( 2^{j_m/2} \tilde{\sigma}(\log n)/n \right).$$

which agrees with condition (1). The implementation of the blockwise method (10) is straightforward from the the WaveD2.0 software package which is available at http://www.maths.usyd.edu.au/u/marcr/.

3.2 The results

We present some simulation results to compare the 2D-WaveD with fixed $j_1$ as in [5] to the data-driven level selection method presented in section 3. The 2 methods are referred as fixed and blockwise respectively. We illustrate our results with the well-known 'Camera' image. The original image ($256 \times 256$) is depicted on Figure 1(a). In Figure 1(b) we depicted the model (3) with high noise level $\sigma_{high} = 0.0663$ and a smooth blurring kernel $g$ with DIP $\nu = 1$ as in [5]. In Figure 1(b) the Blurred-Signal-to-Noise-Ratio (BSNR) is $10dB$. In our study we also report performances for medium noise level $\sigma_{med} = 0.0037$ (BSNR=35dB) and high noise level $\sigma_{high} = 0.0003$ (BSNR=60dB). Our numerical results, based on 100 replications of the model (3), are summarised in Table 1. Visual performances in different noise levels are illustrated on Figures 3, 4 and 5. The results speaks for themselves since in all cases the blockwise method outperformed (or equalled) the fixed-method. Only in the situation where the user has a good 'guess' of the best resolution level $j_1$ the 2 method agree, see Figure 5. In cases where the user over-estimate $j_1$ the results of WaveD are very noisy see Figure 3(a). Even in scenarios where the user has a good 'guess' of $j_1$ the blockwise method is more refined since it allows direction-wise choice which may differ from one direction to the other This is the case in Figure 4(b) where blockwise level selection yields $(\hat{j}_1, \hat{j}_2, \hat{j}_3) = (5, 5, 4)$ which gives a a better visual appearance as well as a smaller RMISE than the fixed choice $j_1 = 5$, compare Figure 4(a).

<table>
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<th>Resolution-level</th>
<th>Low-noise</th>
<th>Med.-noise</th>
<th>High-noise</th>
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<td>User-Fixed $j_1 = 4$</td>
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<td>0.0255</td>
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<tr>
<td>User-Fixed $j_1 = 5$</td>
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<td>0.0201</td>
<td>0.0197</td>
</tr>
</tbody>
</table>

3.3 Conclusion

We have presented a data-driven method for choosing resolution level(s) in wavelet deblurring of noisy images. The method is based on a blockwise thresholding rule on the eigen-values. This method automatically adapt to the
Figure 3: WaveD estimates: left arbitrary $\hat{j}_1 = 5$, right blockwise level selection $(\hat{j}_1^1, \hat{j}_1^2, \hat{j}_1^3) = (4, 4, 4)$.

Figure 4: WaveD estimates: left arbitrary $\hat{j}_1 = 5$, right blockwise level selection $(\hat{j}_1^1, \hat{j}_1^2, \hat{j}_1^3) = (5, 5, 4)$. 
degree of ill-posedness of the blurring kernel as well as to the noise level. In 1D-classical Singular Value Decomposition recent results of [2] suggests that this method may be applied in the case of noisy eigenvalues. Extension to the 2D-WaveD setting of the blockwise method with noisy eigen-values is under investigation by the authors.

References


