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Joint work with François Delarue and René Schott
Transmission mechanism

Transmission of a message (epidemic,...) on the complete graph with $n$ vertices (=servers) and limited resources. Each server has its own random emission capital $K$.

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- At \( t = 0 \), the message is cast from outside to one server, which is turned from inactive to active (if its emission capital \( \geq 1 \)) or exhausted (if its emission capital = 0), though the \( n - 1 \) other servers are inactive.

- At each time \( t \geq 1 \), one of the active servers casts the message, it looses one unit of its own emission capital, and it selects the target at random among the \( n \) servers.
  - If the target is inactive, it discovers the information, it becomes itself active or exhausted according to its own emission capital.
  - If not, this broadcast is unsuccessful and nothing else happens.
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When an active server exhausts its emission capital, it becomes exhausted. Transmission ends at a finite time \( \tau_n \).
Some questions and a few answers

Questions: Asymptotics $n \to \infty$ of

- Exhaustion time $\tau_n$?
- Number of visited nodes $N_n(t)$ at time $t$. Final proportion $N_n(\tau_n)/n$?

- Depend on the distribution $\mu$ of the capital $K$ of a given vertex.

Literature:

- Machado, Machurian, Matzinger 2011: $K = 2$, LLN, CLT
- Kurtz, Lebensztayn, Leichsenring, Machado 2008: continuous-time analogue to $K = \text{Cst}$. ODE and diffusion approximations
  
  - Frog model with finite lifetime:
    - Alves, Lebensztayn, Machado, Martinez 2006: complete graph, asynchronous dynamics
    - Alves, Machado, Popov 2002-02: lattice case, discrete time
    - Kesten, Sidoravicius 2005-06: lattice, shape theorem
Outline

1. The model of information transmission
2. Law of Large Numbers and Fluctuations
3. Large deviations
Markovian dynamics

\( N_n(t) = \) number of servers informed by time \( t \)
\( S_n(t) = \) total emission capital available at \( t \)

Then, \( (N_n(t), S_n(t))_{t \in \mathbb{N}} \) Markov chain in \( \mathbb{N} \times \mathbb{N} \), with

\[
S_n(t+1) = S_n(t) + \begin{cases} 
-1 & \text{w.prob.} \\
K - 1 & \text{w.prob.} 
\end{cases} \begin{cases} 
N_n(t)/n \\
1 - N_n(t)/n 
\end{cases},
\]

where \( K \sim \mu \) independent, and accordingly

\[
N_n(t+1) = \begin{cases} 
N_n(t) \\
N_n(t) + 1 
\end{cases}.
\]

Starts from \( N_n(0) = 1, S_n(0) \sim \mu \).
Absorbed at \( S_n = 0 \), and of course \( N_n = n \). \( \tau_n = \) absorption time.

Event of full transmission \( \text{Trans}_n = \{N_n(\tau_n) = n\} \),
The model of information transmission

Law of Large Numbers and Fluctuations

Large deviations

Probabilistic representations

Key observation: \((N_n(t))_{t \in \mathbb{N}}\) is coupon collector process.

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- \((N_n(t))_{t \in \mathbb{N}}\) a coupon collector process with \(n\) coupons.

\[
N_n(t) = \sum_{i=1}^{n} 1\{T_{i,n} \leq t\}, \quad t = 0, 1, \ldots
\]

with \(T_{1,n} = 0\), \(T_{i,n} = \sum_{j=1}^{i-1} \Delta_{j,n}\), and \(\Delta_{i,n}\), independent, geometrically distributed r.v.'s on \(\mathbb{N}^*\) with success probability \(1 - i/n\) respectively.

- \(R(m) = \sum_{i=1}^{m} K_i\) with \(K_i \sim \mu\) iid and independent of \(N_n\).

- Then,

\[
S_n(t) = R(N_n(t)) - t, \quad t \leq \tau_n := \inf\{t \geq 0 : S_n(t) = 0\}
\]
Second representation:

- **Galton-Watson tree** $T_{GW}$ with offspring distribution $K \sim \mu$
- For each $n \geq 1$, a Coupon Collector with $n$ coupons:

\[
N_n(t) = \sum_{i=1}^{n} 1_{\{T_i, n \leq t\}}, \quad t = 0, 1, \ldots
\]

Pruning the Galton-Watson tree with Coupon Collector yields a construction of the Information process. Coupled with the **same** GW tree for all $n$!!
Pruning construction

Figure 1. Galton-Watson tree represented up to the 4th generation. $t = 10$
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Limit for the information coverage and duration

For $E K > 1$, define $\theta \in (0, \infty)$ as the unique root of the equation

$$\frac{1 - e^{-\theta}}{\theta} = \frac{1}{E K}$$

Extend definition by $\theta = 0$ if $E K \leq 1$, and $\theta = \infty$ if $E K = \infty$. Let also

$$p = 1 - e^{-\theta} \in [0, 1].$$

**Theorem ($E K$ finite or infinite)**

(i) As $n \to \infty$, we have convergence in probability of

$$\frac{\tau_n}{n} \to \theta \mathbf{1}_{\text{Surv}^{GW}}, \quad \frac{N_n(\tau_n)}{n} \to p \mathbf{1}_{\text{Surv}^{GW}}.$$

(ii) If $E K \leq 1$ (and $K \neq 1$), then

$$\mathbb{P} - \lim_{n \to \infty} [\tau_n + 1] = \mathbb{P} - \lim_{n \to \infty} N_n(\tau_n) = Z^{GW}_{\text{tot}}.$$

Csq: $\mathbb{P}(\text{Trans}_n) \to 0$ as $n \to \infty$ whenever $E K$ is finite.

[Decay is exponential if $K$ has a finite exponential moment.]
Sketch of proof:

- Immediate termination of information process is due to extinction of GW tree:
  \[
  \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \mathbb{P}\left( \{ \tau_n \geq n\epsilon \} \Delta \text{Surv}^{GW} \right) = 0.
  \]
  Hence on the survival set, we can focus on times proportional to \( n \).

- It is well known that \( N_n(nt) \sim nr \) with \( r = 1 - e^{-t} \). For \( K \) integrable, \( R(m) \sim mE K \). Hence, a.s.,
  \[
  S_n(nt) - nt \sim n \times [(1 - e^{-t})E K - t],
  \]
  which turns from \( > 0 \) to \( < 0 \) at the above value.
Gaussian fluctuations in the case of a light tail

Theorem \((EK > 1 \text{ and } EK^2 < \infty)\)

(i) As \(n \to \infty\), conditionally on \(\text{Surv}^GW\), we have the convergence in law:

\[
n^{-1/2}(\tau_n - n\theta) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2_{\tau})
\]

\[
\begin{cases}
\sigma^2_{\tau} = [(1 - p)EK - 1]^{-2}[p\sigma^2_K + (EK)^2\sigma_N(\theta)^2] \\
\sigma^2_K = \text{Var}(K), \quad \sigma_N(s)^2 = e^{-s}(1 - e^{-s}) - se^{-2s}.
\end{cases}
\]

(ii) Similarly, conditionally on \(\text{Surv}^GW\), we have the convergence in law:

\[
n^{-1/2}(N_n(\tau_n) - np) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2_p),
\]

with \(\sigma^2_p = [(1 - p)EK - 1]^{-2}[p\sigma^2_K e^{-2\theta} + \sigma_N(\theta)^2].\)

Theorem extends results in Kurtz et al’08, Machado et al’11.
Full transmission and heavy tail

When $\mathbb{E}K = \infty$, we have seen that $N_n(\tau_n)/n \to 1$ conditionally on survival, leaving open the asymptotics of the probability $\mathbb{P}(\text{Trans}_n)$.

$$\lim_n \mathbb{P}(\text{Trans}_n) = 0, 1 \text{ or } \in (0, 1)?$$

Intuitively, large fluctuations of $K$ serve the transmission process.
When $\mathbb{E} K = \infty$, we have seen that $N_n(\tau_n)/n \to 1$ conditionally on survival, leaving open the asymptotics of the probability $\mathbb{P}(\text{Trans}_n)$. Intuitively, large fluctuations of $K$ serve the transmission process. Not so difficult to prove:

**Proposition (Full transmission)**

*If there exist $c > 0$ and $\alpha \in (0, 1)$ such that*

$$\liminf_{\ell \to \infty} \ell^\alpha \mathbb{P}(K \geq \ell) \geq c,$$

*then,*

$$\mathbb{P}(\text{Trans}_n) \xrightarrow{n \to \infty} \mathbb{P}(\text{Surv}^{GW})$$

One of the servers activated during the transmission process has a large enough capital $K$ allowing it to contact all the other servers. Critical case: $K$ belongs to the domain of attraction of a stable law of index 1.
The model of information transmission
Law of Large Numbers and Fluctuations
Large deviations

Full transmission and heavy tail: the critical case

Theorem (Partial transmission)

Assume there exists $c > 0$ such that

$$\mathbb{P}(K \geq \ell) \sim \frac{c}{\ell}, \quad \ell \to \infty.$$ 

Then, as $n \to \infty$,

$$\mathbb{P}(\text{Trans}_n) \rightarrow \mathbb{P}(\text{Surv}^{GW}) \times \left\{ \begin{array}{ll} E(\exp(-e^{-S})), & c = 1, \\ 1, & c > 1, \\ 0, & c < 1, \end{array} \right.$$ 

where $S$ is a totally asymmetric Cauchy variable (stable of index 1).

Erdős-Rényi 1961: time to collect all coupons has a Gumbel limit law. Though $K$ is in the domain of attraction of the stable law.
Sketch of proof:

By Erdös-Rényi 1961,

\[ n^{-1} (T_{n,n} - n \ln n) \xrightarrow{\text{law}} G \]

with a Gumbel limit, \( \mathbb{P}(G \leq x) = e^{-e^{-x}}. \)

From the tail assumption for \( K, \)

\[ n^{-1} (R(n - 1) - cn \ln n) \xrightarrow{\text{law}} S, \]

Now, write

\[
\mathbb{P}(\text{Trans}_n) \simeq \mathbb{P}(\text{Trans}_n \cap \text{Surv}^{GW}) \\
\simeq \mathbb{P}(R(n - 1) \geq T_{n,n}) \times \mathbb{P}(\text{Surv}^{GW}) \\
\simeq \mathbb{P}(S \geq G) \times \mathbb{P}(\text{Surv}^{GW}) \quad \text{(for } c = 1) \\
= E(\exp(-e^{-S})) \times \mathbb{P}(\text{Surv}^{GW})
\]

by independence.
Outline

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Ingredients for large deviations

- Assume $E \exp(a_0 K) < \infty$ for some $a_0 > 0$. By Cramer's theorem $R(n)/n$ obeys a LDP with rate $I$,

$$I(u) = \sup \{ au - \ln E \exp(aK) ; a \in \mathbb{R} \}, \quad u \in [0, \infty),$$

Domain $\text{Dom}(I) = [k_*, k^*] \cap \mathbb{R}$, with

$$k_* = \min \{ k : \mathbb{P}(K = k) > 0 \}, \quad k^* = \sup \{ k : \mathbb{P}(K = k) > 0 \}.$$

- Large deviations principle for the coupon collector from Boucheron, Gamboa, Léonard 2002 and Dupuis, Nuzman, Whiting 2004: for all $t > 0$, $N_n(nt)/n$ obeys a LDP with speed $n$ and rate function $r \in (0, t \wedge 1)$, it is given by

$$J_t(r) = (1 - r) \ln(1 - r) + (t - r) \ln \rho(r, t) + te^{-t \rho(r, t)}, \quad r \in (0, t \wedge 1),$$

where $\rho(r, t)$ denotes the unique solution in $(0, \infty)$ of

$$\frac{1 - e^{-t \rho}}{\rho} = r.$$
Large deviations rate function

Define the function $\mathcal{F} : \mathbb{R}^2_+ \to [0, \infty]$ by

\[
\mathcal{F}(r, t) = \begin{cases} 
    rl(t/r) + J_t(r) & \text{if } r > 0, \\
    \infty & \text{if } r = 0, \ t > 0, \\
    0 & \text{if } r = t = 0.
\end{cases}
\]

Theorem

\[
\frac{1}{n} (N_n(\tau_n), \tau_n)_{n \in \mathbb{N} \setminus \{0\}}, \text{ obeys a LDP with rate function } \mathcal{F} \text{ and speed } n.
\]

Corollary: variational formula for the probability of full transmission.

\[
\text{The probability for all servers to be reached is exponentially small :}
\]

\[
\lim_{n \to +\infty} \frac{1}{n} \ln \mathbb{P}(\text{Trans}_n) = - \inf_{s \geq 0} \left\{ I(\lambda(s)) + (\lambda(s) - 1) \ln(1 - e^{-s}) + \lambda(s)e^{-s} \right\},
\]

with $\lambda(s) = s/(1 - e^{-s})$. The above right-hand side is negative.
Large deviations rate function

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    rl(t/r) + J_t(r) & \text{if } r > 0, \\
    \infty & \text{if } r = 0, t > 0, \\
    0 & \text{if } r = t = 0.
\end{cases}
\]

(i) \( \mathcal{F} \) is lower semi-continuous on \( \mathbb{R}^2_+ \) with compact level sets, and domain

\[
\text{Dom}(\mathcal{F}) = \left\{ (r, t) \in \mathbb{R}^2_+ : 0 < r \leq t \land 1, (k_\ast \vee 1)r \leq t \leq k^* r \right\} \cup \{(0, 0)\}.
\]

\( \mathcal{F} \) is continuous on \( \text{Dom}(\mathcal{F}) \setminus \{(0, 0)\} \).

It is continuous at the origin if and only if \( K \) is bounded.

(ii) When \( EK > 1 \), the function \( \mathcal{F} \) is not convex, as it takes the value 0 at points \((p, \theta)\) and \((0, 0)\), and is positive elsewhere.
A plot of the rate function in the supercritical case

Figure: Rate function $\mathcal{F}$ for $K \sim$ Poisson mean 1.4998. Vanishes at the origin and at $(p, \theta) = (0.5827, 0.8740)$, it is unbounded in neighborhoods of $(0, 0)$ in its domain. For convenience, large values of $\mathcal{F}$ are truncated, and the graph over the domain $r \leq 0.65t$ is not shown. The dark blue strip corresponds to $0.65t \leq r \leq 0.67t$, and the yellow part of the graph to $r \geq 0.67t$. 
Sketch of proof of the Large Deviations Principle

Sketch of proof: Upper bound.

Prove the local upper bound: for $0 < r < t \wedge 1$,

$$\limsup_{\varepsilon, \delta \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \ln \mathbb{P} \left( \frac{N_n(\tau_n)}{n} \in [r-\varepsilon, r+\varepsilon], \frac{\tau_n}{n} \in [t-\delta, t+\delta] \right) \leq -\mathcal{F}(r, t).$$

Consider the case $r < 1 - e^{-t}$, $t/r < EK$, and assume $\varepsilon, \delta$ small. With

$$A = \{N_n(n(t-\delta)) \leq n(r+\varepsilon)\}, \quad B = \{R(n(r-\varepsilon)) \leq n(t+\delta)\},$$

(and $[r \pm \varepsilon] = [r - \varepsilon, r + \varepsilon]$,

$$\mathbb{P} \left( \frac{N_n(\tau_n)}{n} \in [r \pm \varepsilon], \frac{\tau_n}{n} \in [t \pm \delta] \right) \leq \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\leq \exp n\{-J_{t-\delta}(r+\varepsilon) - (r-\varepsilon)I\left(\frac{t+\delta}{r-\varepsilon}\right) + o(1)\}.$$

Use similar arguments in all other cases of $0 < r < t \wedge 1$, and extend by (semi-)continuity to boundary points.

Extend to general closed sets the upper bound for compacts, proving exponential tightness.
Sketch of proof of the LDP

Lower bound is subtle. We show that, for $\mathcal{F}(r, t) < \infty$ and $\eta > 0$, 

$$\liminf_{n \to \infty} \frac{1}{n} \ln P \left( \frac{N_n(\tau_n)}{n} \in [r \pm \eta], \frac{\tau_n}{n} \in [t \pm \eta] \right) \geq -\mathcal{F}(r, t).$$

The above event contains $A \cap A' \cap A'' \cap B \cap B' \cap B''$ where

$A = \{N_n(nt + m\eta) \leq nr + n\varepsilon\}$, \hspace{5mm} $A' = \{N_n(n\varepsilon) = [n\varepsilon] + 1\}$,

$A'' = \{N_n(\ell) \geq \ell \frac{r}{t} + n\varepsilon(1 - \frac{r}{t}) - n\eta'; n\varepsilon \leq \ell \leq nt - m\eta\}$

$\cap \{N_n(nt - m\eta) \leq nr + n\varepsilon\},$

$B = \{R(nr + n\varepsilon) - R(m) \leq nt + n\varepsilon \frac{n}{2}\}$, \hspace{5mm} $B' = \{R(m) \in [n\varepsilon, 2n\varepsilon], Z_{tot}^{GW} \geq n\varepsilon\},$

$B'' = \{R(\ell) - R(m) \geq (\ell - m) \frac{t}{r} - n\eta'; m \leq \ell \leq nr + n\varepsilon\},$

with an integer $m$ ($m \in [1, \varepsilon n]$ close to $\varepsilon n$).

Need to bound $P(A \cap A' \cap A'')$ and $P(B \cap B' \cap B'')$ separately.
Sketch of proof: lower bound

By Markov's property:
\[
\mathbb{P}(A \cap A' \cap A'') \geq \inf_i \mathbb{P}(A|\mathcal{N}_n(nt - m\eta) = i) \mathbb{P}(A''|A') \mathbb{P}(A'),
\]

\[
\text{(infimum over } i = \lceil n(r - \eta r/t) + n\varepsilon(1 - r/t) - m\eta' \rceil, \ldots, \lfloor nr + n\varepsilon \rfloor) \geq \exp\{-n[J_t(r) + \delta(\eta) + \delta'(\varepsilon; \eta)] + o(n)\},
\]

with \(\delta(\eta) \to 0\) as \(\eta \to 0\) and \(\delta'(\varepsilon; \eta) \to 0\) as \(\varepsilon \to 0\) for a given \(\eta > 0\).

Let \(C = \{\tau_n \geq T_{m,n}\},\)

\[
\mathbb{P}(B \cap B' \cap B'') \geq \mathbb{P}(C \cap B \cap B' \cap B'') = \mathbb{P}(C \cap B') \mathbb{P}(B \cap B'')
\]

by independence. Since \(\mathbb{P}(E \cap F) \geq \mathbb{P}(E) - \mathbb{P}(F^c),\)

\[
\mathbb{P}(C \cap B') \geq \mathbb{P}({\text{Surv}}^{GW}) - \mathbb{P}(C^c \cap {\text{Surv}}^{GW}) - \mathbb{P}(R(m) \not\in [n\varepsilon, 2n\varepsilon]),
\]

with the second term \(\to 0\) with \(n\).
We complete the proof when $EK > 1$, which is simpler. Choose

$$m = \left\lfloor \frac{\zeta}{EK} n \varepsilon \right\rfloor, \quad \text{with } \zeta = \min\left(\frac{1 + EK}{2}, 3/2\right).$$

Then,

$$\liminf_{n \to \infty} \mathbb{P}(C \cap B') \geq \mathbb{P}(\text{Surv}^{GW}) > 0,$$

since $EK > 1$. Moreover, by Mogulskii’s Theorem

$$\mathbb{P}(B \cap B'') \geq \exp\{-n(r + \varepsilon)l(t/r) + o(n)\}.$$

Collecting the above, we complete the proof of lower bound in the case $EK > 1$. 
Sketch of proof: lower bound

Case $E K \leq 1$ is more difficult since

- $m$ cannot be chosen as before,
- $P(Surv_{GW}^G) = 0$.

 Perform a tilting to go back to the previous case, and use estimates under the tilted measure.
Etienne à l’assaut de la Montagne Sainte-Geneviève
C’est bien lui!

Bon anniversaire
Etienne!