Efficient Symmetry Breaking Predicates for Quantified Boolean Formulae

Gilles Audemard Said Jabbour Lakhdar Sais

CRIL - Lens - France

Motivations

- Solving Quantified Boolean Formulae (QBF) has become an attractive and important research area
  - Many problems can be reduced to QBF (planning, formal verification...)
  - Many impressive progresses achieved on SAT problem
- Symmetries are present in a lot of real world problems
- Symmetries are commonly used to reduce the search space
  - SAT : [Benhamou 94], [Aloul 02]
  - CSP : [Crawdord 96], [Focacci 01], [Puget 02]...
  - Dedicated workshop since 2001, conference this year

Symmetry and QBF problem

QBF : Quantified Boolean Formulas

- QBF : Quantified Boolean Formulas
- Natural extension of SAT problem
- A QBF formula (in prenex form) contains
  - A matrix (in Conjunctive Normal Form)
  - A prefix $Q_1 X_1 \dots Q_n X_n$ where $Q_i \in \{\exists, \forall\}$ and X_i disjoint subsets of matrix variables

$$
\forall x_1 x_2 \exists x_3 x_4 \quad
\begin{array}{ll}
\neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\
\wedge & \\
\neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\
\wedge & \\
x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\
\wedge & \\
x_2 \vee \neg x_3 \vee \neg x_4 & (c_4)
\end{array}
$$

- QBF problem is PSPACE complete

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg \sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$
\forall x_1 x_2 \exists x_3 x_4 \quad
\begin{array}{ll}
\neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\
\wedge & \\
\neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\
\wedge & \\
x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\
\wedge & \\
x_2 \vee \neg x_3 \vee \neg x_4 & (c_4)
\end{array}
$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \dots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \dots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$\neg x_2$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg \sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\neg x_2 \vee \neg x_1$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\neg x_2 \vee \neg x_1 \vee x_3$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\neg x_2 \vee \neg x_1 \vee x_3 \quad (c_1)$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg \sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{lr} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\forall x_2 x_1 \exists x_3 x_4 \quad \begin{array}{lr} \neg x_2 \vee \neg x_1 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_2 \vee \neg x_1 \vee x_4 & (c_2) \end{array}$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1X_1, \ldots, Q_kX_k\Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\}\sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{lr} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\forall x_2 x_1 \exists x_3 x_4 \quad \begin{array}{lr} \neg x_2 \vee \neg x_1 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_2 \vee \neg x_1 \vee x_4 & (c_2) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & \end{array}$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\} \sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\forall x_2 x_1 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_2 \vee \neg x_1 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_2 \vee \neg x_1 \vee x_4 & (c_2) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

Symmetry and QBF

- See [Audemard 04] for a formal definition and detection algorithm

Definition : Let $\Phi = Q_1 X_1, \ldots, Q_k X_k \Psi$ be a QBF. Let σ be a permutation on Φ literals. σ is a symmetry of Φ if and only if :

1. $\forall x \in Lit(\Phi),\ \sigma(\neg x) = \neg\sigma(x)$ *
2. $\sigma(\Phi) = \Phi$ i.e
   (a) $\sigma(\Psi) = \Psi$ *
   (b) and $\forall i \in \{1, \ldots, k\}\sigma(X_i) = X_i$

- $\sigma = (x_1, x_2)(x_3)(x_4)$ is a symmetry of Φ

$$\forall x_1 x_2 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_1 \vee \neg x_2 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_1 \vee \neg x_2 \vee x_4 & (c_2) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \end{array}$$

$$\forall x_2 x_1 \exists x_3 x_4 \quad \begin{array}{ll} \neg x_2 \vee \neg x_1 \vee x_3 & (c_1) \\ \wedge & \\ \neg x_2 \vee \neg x_1 \vee x_4 & (c_2) \\ \wedge & \\ x_2 \vee \neg x_3 \vee \neg x_4 & (c_4) \\ \wedge & \\ x_1 \vee \neg x_3 \vee \neg x_4 & (c_3) \end{array}$$

Symmetry Breaking Predicates in SAT

- See [Crawford92] and [Aloul02]
- used as a preprocessor
- Let Ψ be a SAT formula and $\sigma = (x_1, y_1) \dots (x_n, y_n)$ a symmetry of Ψ

Symmetry Breaking Predicates (SBP)

$$x_1 \leq y_1$$
$$(x_1 = y_1) \rightarrow (x_2 \leq y_2)$$
$$\dots$$
$$(x_1 = y_1) \dots (x_{n-1} = y_{n-1}) \rightarrow (x_n = y_n)$$

- $SBP(\sigma)$ can be encoded in CNF
- SBP forces isomorphic interpretations to be false
- $\Psi \wedge SBP(\sigma)$ is asymmetric wrt σ

Symmetry Breaking Predicates in SAT

$$\Psi = \begin{array}{l} (\neg x_1 \vee y_1 \vee x_2) \\ (x_1 \vee \neg y_1 \vee y_2) \\ (\neg x_1 \vee \neg y_2) \end{array}$$

- $\sigma = (x_1, y_1)(x_2, y_2)$ is a symmetry for Ψ

$$\begin{array}{l} (x_1 \leq y_1) \\ (x_1 = y_1) \rightarrow (x_2 \leq y_2) \end{array}$$

Symmetry Breaking Predicates in SAT

$$\Psi = \begin{array}{l} (\neg x_1 \vee y_1 \vee x_2) \\ (x_1 \vee \neg y_1 \vee y_2) \\ (\neg x_1 \vee \neg y_2) \end{array}$$

- $\sigma = (x_1, y_1)(x_2, y_2)$ is a symmetry for Ψ

$$\begin{array}{l} (x_1 \leq y_1) \\ (x_1 = y_1) \rightarrow (x_2 \leq y_2) \end{array}$$

$$SBP(\sigma) = (\neg x_1 \vee y_1) \wedge (\neg x_1 \vee \neg x_2 \vee y_2) \wedge (y_1 \vee \neg x_2 \vee y_2)$$

- 6 redundant interpretations are removed (for a total of 16)

Symmetry Breaking Predicates in QBF

$$\Phi = \forall x_1, y_1 \exists x_2, y_2 \quad \begin{array}{l} (\neg x_1 \vee y_1 \vee x_2) \\ (x_1 \vee \neg y_1 \vee y_2) \\ (\neg x_1 \neg y_2) \end{array}$$

- $\sigma = (x_1, y_1)(x_2, y_2)$ is also a symmetry for Φ

$$SBP(\sigma) = (\neg x_1 \vee y_1) \wedge (\neg x_1 \vee \neg x_2 \vee y_2) \wedge (y_1 \vee \neg x_2 \vee y_2)$$

- Problem : $(\neg x_1 \vee y_1)$ contains only universal variables
- $\Phi \wedge SBP(\sigma)$ is invalid whereas Φ is valid ! ! ! !
- Here, we want to force redundant interpretations to be true
- How to generate symmetry breaking predicates for QBF ?

A theorical Approach

Existential Symmetry

- σ is existential if all literals in σ are existential
- Otherwise σ is universal (totally universal if all literals are universal)

Let $\Phi = QX\psi$ be a quantified boolean formula and σ an existential symmetry of ψ. Then Φ and $QX(\psi \wedge sbp_\sigma)$ are equivalent with respect to the validity.

- Classical SBP can be used for existential symmetry
- The challenge concern universal symmetries

Totally Universal Symmetry

- Let $\Phi = QX\Psi$ be a QBF and Let $\sigma = (x_1, y_1)$ a symmetry with x_1, y_1 universal if
- We want to consider the interpretation $\{x_1, \neg y_1\}$ as a model of Φ
- Difficult to find a logical formulation implying $y_1 = true$ when $x_1 = true$
- The main idea is to rewrite Ψ, such that Ψ is true when $x_1 = true$ and $y_1 = false$
- $sbp(\sigma)$ is true iff $x_1 = true$ and $y_1 = false$
- Then $QX\Psi$ and $QX(\Psi \vee \neg sbp(\sigma))$ are equivalent

Universal Symmetry

- An other problem occurs on symmetry containing universal and existential variables
- We need to alternate sub parts of $\Psi \wedge sbp(\sigma)$ and $\Psi \vee sbp(\sigma)$
- We obtain QBF formula which is not in prenex form
- A linear transformation to prenex clausal form is proposed

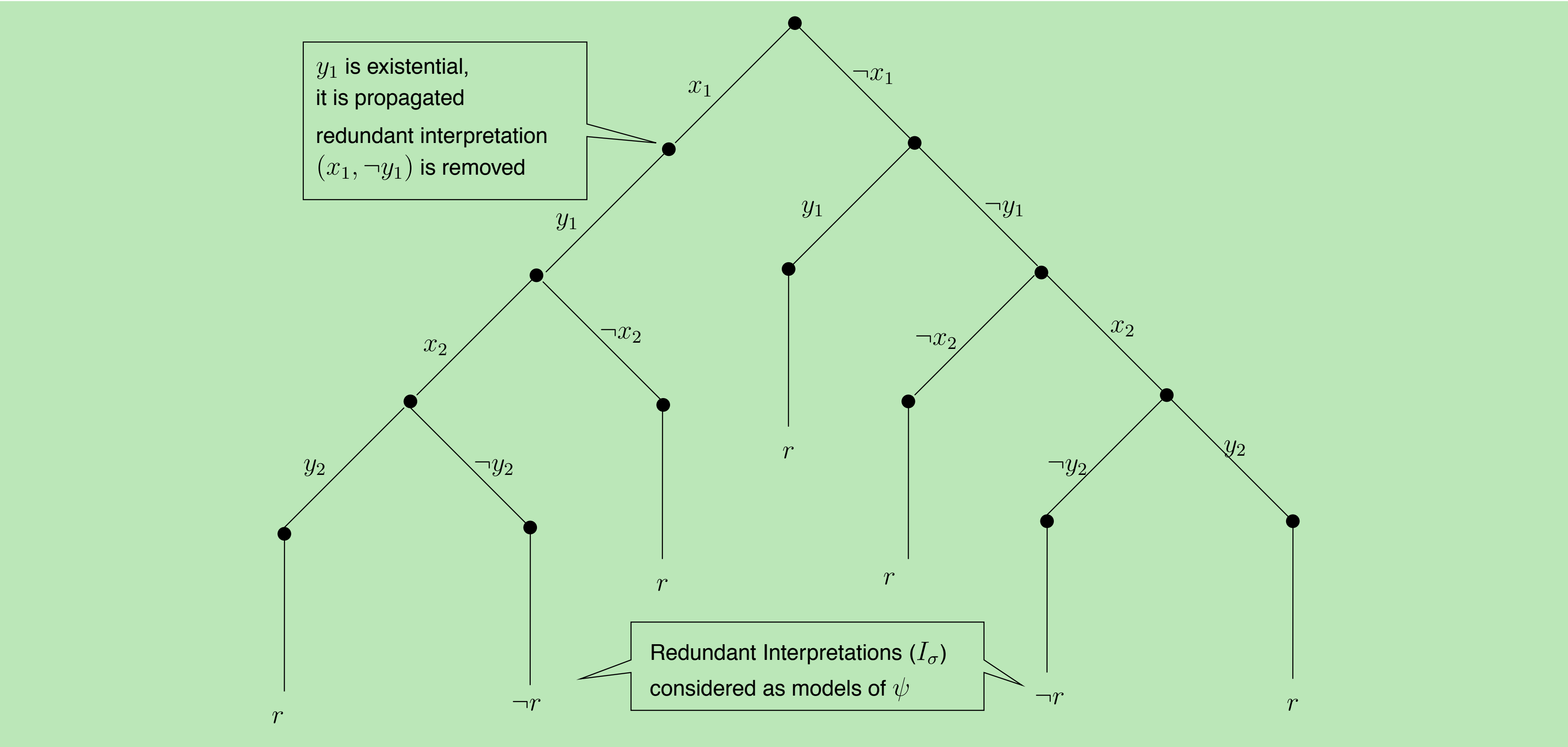
Breaking Symmetries : Clausal form

Clausal form

- Let $\Phi = QX\Psi$ be a QBF and σ a symmetry of Φ
- Some interpretations need to be considered as models of Ψ (noted I_σ
- We want to
  - Propagate symmetrical existential literals thanks to $sbp(\sigma)$
  - consider all interpretations of I_σ as models of Ψ
- To do this :
  - We rewrite Ψ as follow $(\Psi \vee \neg r_\sigma) \wedge f_\sigma(r_\sigma, sbp(\sigma))$
  - f_σ forces r_σ to be false for all interpretations in I_σ and true otherwise
  - f_σ induces propagation on existential cycles

Clausal form : Example

- $\Phi = \exists x_1, y_1 \forall x_2, y_2 \exists x_3, y_3 \Psi$
- $\sigma = \{(x_1, y_1), (x_2, y_2)\}$ a symmetry of Φ
- $\Psi^r = (\Psi \vee \neg r_\sigma) \wedge f_\sigma(r_\sigma, sbp(\sigma))$
- Thanks to f_σ we obtain the following search tree



Experiments

Experiments

- Detection algorithm introduced in [Audemard 04]
- A large panel (411) of symmetric instances from QBFLIB
- Comparison with SEMPROP an efficient QBF solver
  - Φ the original formula
  - Φ^r the asymmetric one
- Time is reported in seconds
- CPU time is limited to 900 seconds

Some results

Instances	U	NB_S	SAT	Φ^r	Φ
biu.mv.xl_ao-p005-IPF02-c02	Y	15	F	–	**1.45**
biu.mv.xl_ao-p005-OPF03-c09	Y	15	T	**17.83**	–
C5315.blif_0.10_1.00_0_1_out_exact	Y	37	T	**788.41**	–
k_path_p-10	Y	4	F	**18.73**	39.48
k_path_p-13	Y	8	F	**203.14**	643.14
ncf_8_16_4_u.4	Y	1	F	**168.39**	176.67
ncf_8_16_4_u.9	Y	1	F	**313.67**	336.21
lut4_3_fAND	Y	11	T	**0.28**	26.74
lut4_AND_fXOR	Y	11	F	**423.40**	–
ken.flash08.C-f4	Y	1	F	438.24	**8.26**
term1.blif_0.10_0.20_0_0_inp_exact	Y	3	F	**151.88**	160.19
TOILET7.1.iv.13	Y	2	F	**12.82**	70.37
k_ph_n-10	N	1	T	**651.07**	–

Global Results

			Φ		Φ^r	
family	NB	U	S	TT	S	TT
fpga	8	Y	6	1834	6	2231.71
biu	23	Y	1	19801	8	13539
toilet_c	53	Y	51	2656	53	374.61
C5315	4	Y	0	3600	1	3488
k_path	40	Y	28	12915	24	15001.79
qshifter	6	Y	6	67	6	40
TOILET	7	Y	6	988	6	926
term1	6	Y	6	174	6	164.13
strategic	100	N	86	13482	86	13477
k_branch	42	N	21	20096	21	20069
k_lin	21	N	5	14553	5	14551
k_grz	37	N	23	15242	24	14997
k_poly	42	N	42	2031	42	2068
toilet_a	22	N	22	12	22	20
TOTAL	411		303	107420	310	100945

Discussion

- Time to generate asymmetric formula is very efficient
- breaking symmetries is very efficient when universal symmetries are present
- symmetries in the innermost quantifier are useless

Conclusion and perspectives

- a technique to break universal and existential symmetries in QBF was presented
- sbp4qbf is a preprocessor that
  - detect symmetries using Nauty
  - generate a new formula which is asymmetric
- Experimentations on other QBF solvers (like skizzo or quantor) are in progress